Undecidability of Fuzzy Description Logics

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Abstract

Fuzzy description logics (DLs) have been investigated for over two decades, due to their capacity to formalize and reason with imprecise concepts. Very recently, it has been shown that for several fuzzy DLs, reasoning becomes undecidable. Although the proofs of these results differ in the details of each specific logic considered, they are all based on the same basic idea.

In this paper, we formalize this idea and provide sufficient conditions for proving undecidability of a fuzzy DL. We demonstrate the effectiveness of our approach by strengthening all previously-known undecidability results and providing new ones. In particular, we show that undecidability may arise even if only crisp axioms are considered.

1 Introduction

Description logics (DLs) (Baader et al. 2003) are a family of logic-based knowledge representation formalisms, which can be used to represent the knowledge of an application domain in a formal way. They have been successfully used for the definition of medical ontologies, like SNOMED CT[1] and GALEN,[2] but their main breakthrough arguably was the adoption of the DL-based language OWL (Horrocks, Patel-Schneider, and van Harmelen 2003) as the standard ontology language for the semantic web.

Fuzzy variants of description logics have been introduced to deal with applications where concepts cannot be specified in a precise way. For example, in the medical domain a high body temperature is often a symptom for a disease. When trying to represent this knowledge, it makes sense to see High as a fuzzy concept: there is no precise point where a temperature becomes high, but we know that 36°C belongs to this concept with a lower membership than 39°C. A more detailed description of the use of fuzzy semantics in medical applications can be found in (Molitor and Tresp 2000).

A great variety of fuzzy DLs can be found in the literature (see (Lukasiewicz and Straccia 2008; García-Cerdaña, Armengol, and Esteva 2010) for a survey). In fact, fuzzy DLs have several degrees of freedom for defining their expressiveness. In addition to the choice of concept constructors (such as conjunction $\sqcap$ or existential restriction $\exists$), and the type of axioms allowed (like acyclic concept definitions or general concept inclusions), one must also decide how to interpret the different constructors, through a choice of functions over the domain of fuzzy values $[0, 1]$. These functions are typically determined by a continuous t-norm (like Gödel, Łukasiewicz, or product) that interprets conjunction; there exist uncountably many such t-norms, each with different properties. For example, under the product t-norm semantics, existential- ($\exists$) and value-restrictions ($\forall$) are not interdefinable, while under the Łukasiewicz t-norm they are. Even after fixing the t-norm, one can choose whether to interpret negation by the involutive negation operator, or using the residual negation. An additional level of liberty comes from selecting the class of models over which reasoning is considered: either all models, or so-called witnessed models only (Hájek 2005).

Most existing reasoning algorithms have been developed for the Gödel semantics, either by a reduction to crisp reasoning (Straccia 2001; Bobillo et al. 2009), or by a simple adaptation of the known algorithms for crisp DLs (Stoilos et al. 2005; 2006; Tresp and Molitor 1998). However, methods based on other t-norms have also been explored (Bobillo and Straccia 2007; 2008; 2009; Straccia and Bobillo 2007; Stoilos and Stamou 2009). Usually, these algorithms reason w.r.t. witnessed models.[3]

Very recently, it was shown that the tableaux-based algorithms for logics with semantics based on t-norms other than the Gödel t-norm and allowing general concept inclusions were incorrect (Baader and Peñaloza 2011a; Bobillo, Bou, and Straccia 2011). This raised doubts about the decidability of these logics, and eventually led to a series of undecidability results for fuzzy DLs (Baader and Peñaloza 2011a; 2011b; 2011c; Cerami and Straccia 2011). All these papers, except (Baader and Peñaloza 2011c), focus on one specific fuzzy DL; that is, undecidability is proven for a specific set of constructors, axioms, and underlying semantics. A small generalization is made in (Baader and Peñaloza

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[1] http://www.ihtsdo.org/snomed-ct/

[2] http://www.opengalen.org/

[3] In fact, witnessed models were introduced in (Hájek 2005) to correct the algorithm from (Tresp and Molitor 1998).

2011c), where undecidability is shown for a whole family of t-norms—specifically, all t-norms "starting" with the product t-norm—and two variants of witnessed models.

Abstracting from the particularities of each logic, the proofs of undecidability appearing in (Baader and Peñaloza 2011a; 2011b; 2011c; Cerami and Straccia 2011) follow similar ideas. The goal of this paper is to formalize this idea and give a general description of a proof of undecidability, which can be instantiated to different fuzzy DLs. More precisely, we describe a general proof method based on a reduction from the Post Correspondence Problem and present sufficient conditions for the applicability of this method to a given fuzzy DL.

We demonstrate the effectiveness of our approach by providing several new undecidability results for fuzzy DLs. In particular, we improve the results from (Baader and Peñaloza 2011a; Cerami and Straccia 2011) by showing that a weaker DL suffices for obtaining undecidability, and the results from (Baader and Peñaloza 2011b; 2011c), by allowing a wider family of t-norms. We also prove the first undecidability results for reasoning w.r.t. general models. An interesting outcome of our study is that, for the product t-norm and any t-norm "starting" with the Łukasiewicz t-norm, undecidability can arise even if only crisp axioms are allowed.

Due to a lack of space, some technical details have been left out of this paper. Full proofs and details can be found in the technical report (Borgwardt and Peñaloza 2011c).

2 T-norms and Fuzzy Logic

Fuzzy logics are formalisms introduced to express imprecise or vague information (Hájek 2001). They extend classical logic by interpreting predicates as fuzzy sets over an interpretation domain. Given a non-empty domain $\mathcal{D}$, a *fuzzy set* is a function $F : \mathcal{D} \to [0, 1]$ from $\mathcal{D}$ into the real unit interval $[0, 1]$, with the intuition that an element $\delta \in \mathcal{D}$ belongs to F with *degree* $F(\delta)$. The interpretation of the logical constructors is based on appropriate truth functions that generalize the properties of the connectives of classical logic to the interval $[0, 1]$. The most prominent truth functions used in the fuzzy logic literature are based on t-norms (Klement, Mesiar, and Pap 2000).

A *t-norm* is an associative and commutative binary operator $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ that has 1 as its unit element, and is monotonic, i.e., for every $x, y, z \in [0, 1]$, if $x \leq y$, then $x \otimes z \leq y \otimes z$. If $\otimes$ is a continuous t-norm, then there exists a unique binary operator $\Rightarrow$, called the *residuum*, that satisfies $z \leq x \Rightarrow y$ iff $x \otimes z \leq y$ for every $x, y, z \in [0, 1]$. For every continuous t-norm $\otimes$ and $x, y \in [0, 1]$, we have (i) $x \Rightarrow y = 1$ iff $x \leq y$ and (ii) $1 \Rightarrow y = y$ (Hájek 2001).

Three important continuous t-norms are the Gödel, product and Łukasiewicz t-norms, shown in Table 1.

We say that a t-norm $\otimes$ (a, b)-*contains* the t-norm $\otimes'$, for $0 \leq a < b \leq 1$, if for every $x, y \in [0, 1]$ it holds that

$$(a + (b - a)x) \otimes (a + (b - a)y) = a + (b - a)(x \otimes' y).$$

In this case, if $\Rightarrow$ and $\Rightarrow'$ denote the residua of $\otimes$ and $\otimes'$, respectively, then it also holds that for every $x > y$,

$$(a + (b - a)x) \Rightarrow (a + (b - a)y) = a + (b - a)(x \Rightarrow' y).$$

Name	t-norm ($x \otimes y$)	Residuum ($x \Rightarrow y$)
Gödel	$\min\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
product	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$\min\{1 - x + y, 1\}$

Table 1: Three t-norms and their residua

Moreover, for every $x \in [a, b]$ and $y \notin [a, b]$, we have that $x \otimes y = \min\{x, y\}$. Intuitively, $\otimes$ behaves like a scaled-down version of $\otimes'$ in the interval $[a, b]$, and as the Gödel t-norm if exactly one of the arguments belongs to $[a, b]$.

We say that a t-norm *contains* $\otimes'$ if it (a, b)-contains $\otimes'$ for some $0 \leq a < b \leq 1$. A consequence of the Mostert-Shields Theorem (Mostert and Shields 1957) is that every continuous t-norm $\otimes$ that is not the Gödel t-norm must contain the product or the Łukasiewicz t-norm. Notice that $\otimes$ may contain both t-norms; in fact, it may even contain infinitely many instances of these t-norms over disjoint intervals. For example, the t-norm defined by

$$x \otimes y = \begin{cases} 2xy & \text{if } x, y \in [0, 0.5] \\ \max\{x + y - 1, 0.5\} & \text{if } x, y \in [0.5, 1] \\ \min(x, y) & \text{otherwise,} \end{cases}$$

$(0, 0.5)$-contains the product t-norm, and $(0.5, 1)$-contains the Łukasiewicz t-norm.

We denote the product and Łukasiewicz t-norms by Π and Ł, respectively. In general, a continuous t-norm that is not the Gödel t-norm may contain several instances of the product and Łukasiewicz t-norms. In the following, we always choose and fix a representative, and use the notation $\Pi^{(a,b)}$ to express that the t-norm (a, b)-contains the product t-norm, and similarly for Ł$^{(a,b)}$. Since our constructions differ according to the t-norm, it is important to emphasize that the representative is fixed throughout the whole construction.

Fuzzy logics are sometimes extended with the involutive negation $\sim x := 1 - x$ (Zadeh 1965; Esteva et al. 2000). If $\otimes$ is the Łukasiewicz t-norm, then this operator can be expressed through the equality $\sim x = x \Rightarrow 0$. However, for any other continuous t-norm $\sim$ is not expressible in terms of $\otimes$ and its residuum $\Rightarrow$.

3 Fuzzy Description Logics

Just as classical description logics, fuzzy DLs are based on concepts, which are built from the mutually disjoint sets $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ of *concept names*, *role names*, and *individual names*, respectively, using different constructors. A wide variety of constructors can be found in the literature. For this paper, we consider only the constructors $\top$ (*top*), $\bot$ (*bottom*), $\sqcap$ (*conjunction*), $\to$ (*implication*), $\neg$ (*involutive negation*), $\boxminus$ (*residual negation*), $\exists$ (*existential restriction*), and $\forall$ (*value restriction*). When restricted to classical semantics, this set of constructors corresponds to the crisp DL $\mathcal{ALC}$.

Definition 1 (concepts)**.** *(Complex) concepts* are built inductively from $\mathsf{N_C}$ and $\mathsf{N_R}$ as follows:

Name	$\top$	$\bot$	$\sqcap$	$\to$	$\neg$	$\boxminus$	$\exists$	$\forall$
$\mathcal{EL}$	$\surd$		$\surd$				$\surd$	
$\mathcal{ELC}$	$\surd$		$\surd$		$\surd$		$\surd$	
$\mathfrak{NEL}$	$\surd$		$\surd$			$\surd$	$\surd$	
$\mathcal{AL}$	$\surd$		$\surd$				$\surd$	$\surd$
$\mathcal{ALC}$	$\surd$		$\surd$		$\surd$		$\surd$	$\surd$
$\mathfrak{IAL}$	$\surd$	$\surd$	$\surd$	$\surd$			$\surd$	$\surd$

Table 2: Some relevant DLs and the constructors they allow.

- every concept name $A \in \mathsf{N_C}$ is a concept
- if C, D are concepts and $r \in \mathsf{N_R}$, then $\top$, $\bot$, $C \sqcap D$, $C \to D$, $\neg C$, $\boxminus C$, $\exists r.C$, and $\forall r.C$ are also concepts.

We will use the expression C^n to denote the n-ary conjunction of a concept C with itself; formally, $C^0 := \top$ and $C^{n+1} := C \sqcap C^n$ for every $n \geq 0$.

Different DLs are determined by the choice of constructors used. The DL $\mathcal{EL}$ allows only for the constructors $\top, \sqcap$, and $\exists$. $\mathcal{AL}$ additionally allows value restrictions. Following the notation from (Cerami, García-Cerdaña, and Esteva 2010), the letters $\mathcal{C}$ and $\mathfrak{I}$ express that the involutive negation or implication and bottom constructors are allowed, respectively. $\mathfrak{NEL}$ extends $\mathcal{EL}$ with the residual negation constructor. Table 2 summarizes this nomenclature.

The knowledge of a domain is represented using a set of axioms that express the relationships between individuals, roles, and concepts.

Definition 2 (axioms)**.** An *axiom* is one of the following:

- A *general concept inclusion (GCI)* is of the form $C \sqsubseteq D$ for concepts C and D.[4]
- An *assertion* is of the form $\langle e : C \rhd p\rangle$ or $\langle (d, e) : r \rhd p\rangle$ for a concept C, $r \in \mathsf{N_R}$, $d, e \in \mathsf{N_I}$, and $\rhd \in \{\geq, =\}$. This axiom is called a *crisp assertion* if $p = 1$, an *inequality assertion* if $\rhd$ is $\geq$ and an *equality assertion* if $\rhd$ is $=$.
- A *crisp role axiom* is of the form $\mathsf{crisp}(r)$ for $r \in \mathsf{N_R}$.

An *ontology* is a finite set of axioms. It is called a *classical ontology* if it contains only GCIs and crisp assertions.

As with the choice of the constructors, the axioms influence the expressivity of the logic. Our logics always allow at least classical ontologies. Given a DL $\mathcal{L}$, we will use the subscripts $\geq$, $=$, and c to denote that also inequality assertions, equality assertions, and crisp role axioms are allowed, respectively. For instance, $\mathcal{EL}_{\geq,\mathsf{c}}$ denotes the logic $\mathcal{EL}$ where ontologies can additionally contain inequality assertions and crisp role axioms, but not equality assertions.

Compared to classical DLs, fuzzy DLs have an additional degree of freedom in the selection of their semantics since the interpretation of the constructors depends on the t-norm chosen. Given a DL $\mathcal{L}$ and a continuous t-norm $\otimes$, we obtain the fuzzy DL $\otimes$-$\mathcal{L}$ with the following semantics.

[4]One can also consider fuzzy GCIs $\langle C \sqsubseteq D \geq p\rangle$ (see, e.g. (Straccia 1998)). Since our proofs of undecidability do not require these more general axioms, we do not consider them here.

Definition 3 (semantics)**.** An *interpretation* $\mathcal{I} = (\mathcal{D}^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty *domain* $\mathcal{D}^\mathcal{I}$ and an *interpretation function* $\cdot^\mathcal{I}$ that assigns to every $e \in \mathsf{N_I}$ an element $e^\mathcal{I} \in \mathcal{D}^\mathcal{I}$, to every $A \in \mathsf{N_C}$ a fuzzy set $A^\mathcal{I} : \mathcal{D}^\mathcal{I} \to [0, 1]$, and to every $r \in \mathsf{N_R}$ a fuzzy binary relation $r^\mathcal{I} : \mathcal{D}^\mathcal{I} \times \mathcal{D}^\mathcal{I} \to [0, 1]$.

This function is extended to concepts as follows:

- $\top^\mathcal{I}(x) = 1, \qquad \bot^\mathcal{I}(x) = 0,$
- $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x),$
- $(C \to D)^\mathcal{I}(x) = C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x),$
- $(\neg C)^\mathcal{I}(x) = 1 - C^\mathcal{I}(x), \quad (\boxminus C)^\mathcal{I}(x) = C^\mathcal{I}(x) \Rightarrow 0,$
- $(\exists r.C)^\mathcal{I}(x) = \sup_{y\in\mathcal{D}^\mathcal{I}}(r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)),$
- $(\forall r.C)^\mathcal{I}(x) = \inf_{y\in\mathcal{D}^\mathcal{I}}(r^\mathcal{I}(x, y) \Rightarrow C^\mathcal{I}(y)).$

We say that an interpretation $\mathcal{I}'$ is an *extension* of $\mathcal{I}$ if it has the same domain as $\mathcal{I}$, agrees with $\mathcal{I}$ on the interpretation of $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ and additionally defines values for some new concept names not appearing in $\mathsf{N_C}$.

The reasoning problem that we consider in this paper is ontology consistency; that is, deciding whether there is an interpretation satisfying all the axioms in an ontology.

Definition 4 (consistency)**.** An interpretation $\mathcal{I} = (\mathcal{D}^\mathcal{I}, \cdot^\mathcal{I})$ *satisfies* the GCI $C \sqsubseteq D$ if $C^\mathcal{I}(x) \leq D^\mathcal{I}(x)$ for all $x \in \mathcal{D}^\mathcal{I}$. It *satisfies* the assertion $\langle e : C \rhd p\rangle$ (resp., $\langle (d, e) : r \rhd p\rangle$) if $C^\mathcal{I}(e^\mathcal{I}) \rhd p$ (resp., $r^\mathcal{I}(d^\mathcal{I}, e^\mathcal{I}) \rhd p$). It *satisfies* the crisp role axiom $\mathsf{crisp}(r)$ if $r^\mathcal{I}(x, y) \in \{0, 1\}$ for all $x, y \in \mathcal{D}^\mathcal{I}$. It is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

An ontology is *consistent* if it has a model.

Notice that the GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$ are satisfied iff $C^\mathcal{I}(x) = D^\mathcal{I}(x)$ for every $x \in \mathcal{D}^\mathcal{I}$. It thus makes sense to abbreviate them through the expression $C \equiv D$.

In fuzzy DLs, reasoning is often restricted to a special kind of models, called witnessed models (Hájek 2005; Bobillo and Straccia 2009). An interpretation $\mathcal{I}$ is called *witnessed* if for every concept C, $r \in \mathsf{N_R}$, and $x \in \mathcal{D}^\mathcal{I}$ there exist $y, y' \in \mathcal{D}^\mathcal{I}$ such that

- $(\exists r.C)^\mathcal{I}(x) = r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$, and
- $(\forall r.C)^\mathcal{I}(x) = r^\mathcal{I}(x, y') \Rightarrow C^\mathcal{I}(y')$.

This means that the suprema and infima in the semantics of existential and value restrictions are actually maxima and minima, respectively. Restricting to this kind of models changes the reasoning problem since there exist consistent ontologies that have no witnessed models (Hájek 2005).

We also consider a weaker notion of witnessing, where witnesses are required only for the existential restrictions $\exists r.\top$ evaluated to 1. Formally, $\mathcal{I}$ is called $\top$-*witnessed* if for every $r \in \mathsf{N_R}$ and $x \in \mathcal{D}^\mathcal{I}$ such that $(\exists r.\top)^\mathcal{I}(x) = 1$, there is a $y \in \mathcal{D}^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$. Obviously, every witnessed interpretation is also $\top$-witnessed. We use the subscripts w and $\top$ to denote that reasoning is restricted to witnessed and $\top$-witnessed models, respectively. Thus, $\otimes_\mathsf{w}$-$\mathcal{ELC}$ represents the logic $\otimes$-$\mathcal{ELC}$ restricted to witnessed models.

In general, a fuzzy DL is determined by three parameters: the class $\mathcal{L}$ of constructors and axioms it allows, the t-norm $\otimes$ that describes its semantics, and the class of models x over which reasoning is considered. In the following, we will use the expression $\otimes_\mathsf{x}$-$\mathcal{L}$ to denote an arbitrary fuzzy DL.

Before we present our general framework for proving undecidability, it is worth to relate the fuzzy DLs introduced according to their expressive power. For every choice of constructors $\mathcal{L}$ and t-norm $\otimes$, the inequality concept assertion $\langle e : C \geq q\rangle$ can be expressed in $\otimes$-$\mathcal{L}_=$ using the axioms $\langle e : A = q\rangle, A \sqsubseteq C$, where A is a new concept name. For every t-norm $\otimes$, $\otimes$-$\mathfrak{N}\mathcal{EL}$ is a sublogic of $\otimes$-$\mathfrak{I}\mathcal{EL}$ since $(\boxminus C)^{\mathcal{I}}(x) = (C \to \bot)^{\mathcal{I}}(x)$. It also holds that Ł-$\mathcal{ELC}$, Ł-$\mathfrak{N}\mathcal{EL}$, Ł-$\mathfrak{I}\mathcal{EL}$, Ł-$\mathcal{ALC}$, and Ł-$\mathfrak{I}\mathcal{AL}$ are all equivalent (Hájek 2001): the residual and involutive negation are equivalent and can express implication together with conjunction $(C \to D)^{\mathcal{I}} = \neg(C \sqcap \neg D)^{\mathcal{I}}$, and the duality between value and existential restrictions $(\forall r.C)^{\mathcal{I}} = \neg(\exists r.\neg C)^{\mathcal{I}}$ holds. However, in general these logics have different expressive power; if any t-norm different from Łukasiewicz is used, then $(\neg\exists r.\neg C)^{\mathcal{I}} \neq (\forall r.C)^{\mathcal{I}}$.

4 Showing Undecidability

We now describe a general approach for proving that the consistency problem for a fuzzy DL $\otimes_{\mathsf{x}}$-$\mathcal{L}$ is undecidable. This approach is based on a reduction from the undecidable Post correspondence problem (Post 1946).

Definition 5 (PCP)**.** Let $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be a finite set of pairs of words over the alphabet $\Sigma = \{1, \ldots, s\}$ with $s > 1$. The *Post correspondence problem (PCP)* asks whether there is a finite sequence $i_1 \ldots i_k \in \{1, \ldots, n\}^+$ such that $v_{i_1} \ldots v_{i_k} = w_{i_1} \ldots w_{i_k}$. If this sequence exists, it is called a *solution* for $\mathcal{P}$.

We define $\mathcal{N} := \{1, \ldots, n\}$ and for $\nu = i_1 \ldots i_k \in \mathcal{N}^+$, we use the notation $v_\nu = v_{i_1} \ldots v_{i_k}$ and $w_\nu = w_{i_1} \ldots w_{i_k}$.

Let $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be an instance of the PCP. We can represent $\mathcal{P}$ by its *search tree*, which has one node for every $\nu \in \mathcal{N}^*$, where ε represents the root, and νi is the i-th successor of ν, $i \in \mathcal{N}$. Each node ν in this tree is labelled with the words $v_\nu, w_\nu \in \Sigma^*$.

We reduce the PCP to the consistency problem of $\otimes_{\mathsf{x}}$-$\mathcal{L}$ in two steps. We first construct an ontology $\mathcal{O}_{\mathcal{P}}$ that describes the search tree of $\mathcal{P}$ using two designated concept names V, W. More precisely, we will enforce that for every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$ and every $\nu \in \mathcal{N}^*$, there is an $x_\nu \in \mathcal{D}^{\mathcal{I}}$ such that $V^{\mathcal{I}}(x_\nu) = \mathsf{enc}(v_\nu)$ and $W^{\mathcal{I}}(x_\nu) = \mathsf{enc}(w_\nu)$, where $\mathsf{enc} : \Sigma^* \to [0, 1]$ is an injective function that encodes words over Σ into the interval $[0, 1]$ (see Section 4.1).

Once we have encoded the words v_ν and w_ν using V and W, we add axioms that restrict every node to satisfy that $V^{\mathcal{I}}(x_\nu) \neq W^{\mathcal{I}}(x_\nu)$. This will ensure that $\mathcal{P}$ has a solution iff the ontology is inconsistent (see Section 4.2).

Recall that the alphabet Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s + 1$. On the other hand, every natural number n has a unique representation in base $s + 1$, which can be seen as a word over the alphabet $\Sigma_0 := \Sigma \cup \{0\} = \{0, \ldots, s\}$. This is not a bijection since, e.g. the words 001202 and 1202 represent the same number. However, it is a bijection between the set $\Sigma\Sigma_0^*$ and the positive natural numbers. We will in the following interpret the empty word ε as 0, thereby extending this bijection to $\{\varepsilon\} \cup \Sigma\Sigma_0^*$ and all non-negative integers.

In the following constructions and proofs, we will view elements of Σ_0^* both as words and as natural numbers in base $s + 1$. To avoid confusion, we will use the notation $\underline{u}$ to express that u is seen as a word. Thus, for instance, if $s = 3$, then $3 \cdot 2^2 = 30$ (in base 4), but $\underline{3} \cdot \underline{2}^2 = \underline{322}$. Furthermore, $\underline{000}$ is a word of length 3, whereas 000 is simply the number 0. For a word $u = \alpha_1 \cdots \alpha_m$ with $\alpha_i \in \Sigma_0, 1 \leq i \leq m$, we denote as $\overleftarrow{u}$ the word $\alpha_m \cdots \alpha_1 \in \Sigma_0^*$.

Recall that for every $p, q \in [0, 1]$, $p = q$ iff $p \Rightarrow q = 1$ and $q \Rightarrow p = 1$. Thus, $\mathcal{P}$ has no solution iff for every $\nu \in \mathcal{N}^+$ either $\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) < 1$ or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) < 1$ holds. Instead of performing this test directly, we will construct a word whose encoding bounds these residua. Clearly, the precise word and encoding must depend on the t-norm used. The needed properties are formalized by the following definition.

Definition 6 (valid encoding function)**.** $\mathsf{enc} : \Sigma_0^* \to [0, 1]$ is a *valid encoding function* for $\otimes$ if it is injective on $\{\varepsilon\} \cup \Sigma\Sigma_0^*$ and there exist two words $u_\varepsilon, u_+ \in \Sigma_0^*$ such that for every $\nu \in \mathcal{N}^+$ it holds that $v_\nu \neq w_\nu$ iff either

$$\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) \leq \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}) \quad \text{or}$$
$$\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}).$$

For every continuous t-norm $\otimes$ except the Gödel t-norm, we give a valid encoding function, which depends on whether $\otimes$ contains the product or the Łukasiewicz t-norm. If $\otimes$ (a, b)-contains the product t-norm, then we define $\mathsf{enc}(u) = a + (b - a)2^{-u} \in (a, b]$ for every $u \in \Sigma_0^*$. If $\otimes$ is of the form $Ł^{(a,b)}$, then $\mathsf{enc}(u) = a + (b - a)(1 - 0.\overleftarrow{u}) \in (a, b]$.

Lemma 7. *The functions* enc *described above are valid encoding functions.*

Proof. $[\Pi^{(a,b)}]$ Let $v \neq w$ and assume w.l.o.g. that $v < w$. Then $v + 1 \leq w$ and hence $2^{-w} \leq 2^{-(v+1)} \leq 2^{-v}/2$. This implies that

$$\begin{aligned} \mathsf{enc}(v) \Rightarrow \mathsf{enc}(w) &= a + (b - a)2^{-w}/2^{-v} \\ &\leq a + (b - a)/2 = \mathsf{enc}(1) < 1. \end{aligned}$$

Conversely, if $v = w$, then $(\mathsf{enc}(v) \Rightarrow \mathsf{enc}(w)) = 1$ and $(\mathsf{enc}(w) \Rightarrow \mathsf{enc}(v)) = 1$. Thus, $u_\varepsilon = 1$ and $u_+ = \varepsilon$ satisfy the condition of Definition 6.

$[Ł^{(a,b)}]$ Let $k = \max\{|v_i|, |w_i| \mid i \in \mathcal{N}\}$ be the maximal length of a word in $\mathcal{P}$. Then, for every $\nu \in \mathcal{N}^+$, $|v_\nu| \leq |\nu|k$ and $|w_\nu| \leq |\nu|k$. If $v_\nu \neq w_\nu$, these words must differ in one of the first $|\nu|k$ digits. Thus, either

$$\begin{aligned} &\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) \\ &\qquad = a + (b - a)\min\{1, 1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu}\} \\ &\qquad = \min\{b, a + (b - a)(1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu})\} \\ &\qquad \leq a + (b - a)(1 - (s + 1)^{-|\nu|k}) \\ &\qquad = \mathsf{enc}((s + 1)^{|\nu|k}) < 1 \end{aligned}$$

or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}((s + 1)^{|\nu|k})$.[5] If $v_\nu = w_\nu$, then both residua are 1. Thus, $u_\varepsilon = 1$ and $u_+ = \underline{0}^k$ give the desired result. □

[5] We have $(s+1)^{|\nu|k} = \underline{1} \cdot \underline{0}^{|\nu|k}$ and $(s+1)^{-|\nu|k} = 0.\underline{0}^{|\nu|k} \cdot \underline{1}$.

Variants of the above encoding functions and words u_ε, u_+ have been used before to show undecidability of fuzzy description logics based on the product (Baader and Peñaloza 2011c) and Łukasiewicz (Cerami and Straccia 2011) t-norms. For the rest of this paper, enc represents a valid encoding function for $\otimes$.

4.1 Encoding the Search Tree

As a first step for our reduction to the consistency problem in fuzzy DLs, we simulate the search tree for the instance $\mathcal{P}$ using the concept names V, W. Since we will later use this construction to decide whether a solution exists, we designate the concept name M to represent the bound $\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}$ from Definition 6. We use V_i, W_i, M_+ to encode the words v_i, w_i, u_+, and the role names r_i to distinguish the successors in the search tree. We start by constructing the interpretation $\mathcal{I}_\mathcal{P} = (\mathcal{N}^*, \cdot^{\mathcal{I}_\mathcal{P}})$, where $e_0^{\mathcal{I}_\mathcal{P}} = \varepsilon$ and for every $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$,

- $V^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(v_\nu)$, $\;W^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(w_\nu)$,
- $V_i^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(v_i)$, $\;W_i^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(w_i)$,
- $M^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$, $\;M_+^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(u_+)$,
- $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu i) = 1$ and $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu') = 0$ if $\nu' \neq \nu i$.

Since every element of $\mathcal{N}^*$ has exactly one r_i-successor with degree greater than 0, $\mathcal{I}_\mathcal{P}$ is a ($\top$-)witnessed interpretation.

Our aim is to produce an ontology that can only be satisfied by interpretations that "include" the interpretation $\mathcal{I}_\mathcal{P}$, as described by the following property.

Canonical model property ($\mathrm{P}_\triangle$):

> $\otimes_\mathsf{x}$-$\mathcal{L}$ has the *canonical model property* if there is an ontology $\mathcal{O}_\mathcal{P}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ there is a mapping $g : \mathcal{D}^{\mathcal{I}_\mathcal{P}} \to \mathcal{D}^\mathcal{I}$ with
> $$A^{\mathcal{I}_\mathcal{P}}(\nu) = A^\mathcal{I}(g(\nu)) \text{ and } r_i^\mathcal{I}(g(\nu), g(\nu i)) = 1$$
> for every $A \in \{V, W, M, M_+\} \cup \bigcup_{j=1}^n \{V_j, W_j\}, \nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$.

Rather than trying to prove this property directly, we provide several simpler properties that together imply the canonical model property. We will often motivate the constructions using only V and v_ν; however, all the arguments apply analogously to W, w_ν and $M, \underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}$.

To ensure that the canonical model property holds, we enforce the encoding of the search tree in an inductive way. First, every model $\mathcal{I}$ must satisfy that $A^{\mathcal{I}_\mathcal{P}}(\varepsilon) = A^\mathcal{I}(e_0^\mathcal{I})$ for every relevant concept name. This makes sure that the root ε of the search tree is properly represented at the individual $g(\varepsilon) := e_0^\mathcal{I}$. Let now $g(\nu)$ be a node where all relevant concept names are interpreted as in $\mathcal{I}_\mathcal{P}$, and $i \in \mathcal{N}$. We need to ensure that there is a node $g(\nu i)$ that also satisfies the property, and $r_i^\mathcal{I}(g(\nu), g(\nu i)) = 1$. We do this in three steps: first, we force the existence of an individual y with $r_i^\mathcal{I}(g(\nu), y) = 1$ and set $g(\nu i) := y$. Then, we compute the value $\mathsf{enc}(v_\nu v_i)$ from $V^\mathcal{I}(g(\nu)) = \mathsf{enc}(v_\nu)$ and $V_i^\mathcal{I}(g(\nu)) = \mathsf{enc}(v_i)$. Finally, we "transfer" this value to the previously created successor; that is, we ensure that $V^\mathcal{I}(g(\nu i)) = \mathsf{enc}(v_\nu v_i)$. The value $V_j^\mathcal{I}(g(\nu))$ for every $j \in \mathcal{N}$ is also transferred to $V_j^\mathcal{I}(g(\nu i))$.

Since the values of V_i, W_i, and M_+ are constant throughout the search tree, we additionally present an alternative approach that simply fixes these values for all $x \in \mathcal{D}^\mathcal{I}$. This has the advantage that the initialization only has to consider the values $\mathsf{enc}(v_\varepsilon) = \mathsf{enc}(w_\varepsilon) = \mathsf{enc}(\varepsilon)$ and $\mathsf{enc}(u_\varepsilon)$.

Each step of the construction described above will be ensured by a property of the underlying logic. These properties, which will be used to produce the ontology $\mathcal{O}_\mathcal{P}$, are described next. For each of the properties, we give examples of fuzzy DLs satisfying it. It is important to notice that the interpretation $\mathcal{I}_\mathcal{P}$ can be extended to a witnessed model of each of the ontologies that we introduce in the following.

The first property ensures the existence of an r-successor of degree 1 for every element of the domain.

Successor property ($\mathrm{P}_\rightarrow$):

> $\otimes_\mathsf{x}$-$\mathcal{L}$ has the *successor property* if for every $r \in \mathsf{N_R}$ there is an ontology $\mathcal{O}_{\exists r}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{\exists r}$ and $x \in \mathcal{D}^\mathcal{I}$ there is a $y \in \mathcal{D}^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$.

Lemma 8. *For every t-norm $\otimes$, $\otimes_\top$-$\mathcal{EL}$ and $\otimes$-$\mathcal{EL}_\mathsf{c}$ satisfy* $\mathrm{P}_\rightarrow$.

Proof. [$\otimes_\top$-$\mathcal{EL}$] Let $\mathcal{O}_{\exists r} := \{\top \sqsubseteq \exists r.\top\}$. Any model $\mathcal{I}$ of this ontology satisfies $(\exists r.\top)^\mathcal{I}(x) = 1$ for every $x \in \mathcal{D}^\mathcal{I}$. Since reasoning is restricted to $\top$-witnessed models, there must be a $y \in \mathcal{D}^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$.

[$\otimes$-$\mathcal{EL}_\mathsf{c}$] We define $\mathcal{O}_{\exists r} := \{\top \sqsubseteq \exists r.\top, \mathsf{crisp}(r)\}$. For any model $\mathcal{I}$ of this ontology and $x \in \mathcal{D}^\mathcal{I}$, we have $(\exists r.\top)^\mathcal{I}(x) = 1$. If $r^\mathcal{I}(x, y) = 0$ for all $y \in \mathcal{D}^\mathcal{I}$, then $(\exists r.\top)^\mathcal{I}(x) = \sup_{y \in \mathcal{D}^\mathcal{I}} r^\mathcal{I}(x, y) \otimes \top^\mathcal{I}(y) = 0 \neq 1$. Since r is crisp, there must be a $y \in \mathcal{D}^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$. □

Given this property, we create r_i-successors for every node $\nu \in \mathcal{N}^*$ with the ontology

$$\mathcal{O}_{\mathcal{P},\rightarrow} := \bigcup_{i \in \mathcal{N}} \mathcal{O}_{\exists r_i}.$$

The concatenation property is satisfied if it is possible to compute the encoding of the concatenation $\underline{u'u}$ from the encodings of two words u and u', where u is constant.

Concatenation property ($\mathrm{P}_\circ$):

> $\otimes_\mathsf{x}$-$\mathcal{L}$ has the *concatenation property* if for all $u \in \Sigma_0^*$ and concepts C, C_u, there is an ontology $\mathcal{O}_{C \circ u}$ and a concept name $D_{C \circ u}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{C \circ u}$ and $x \in \mathcal{D}^\mathcal{I}$, if $C_u^\mathcal{I}(x) = \mathsf{enc}(u)$ and $C^\mathcal{I}(x) = \mathsf{enc}(u')$ for $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, then $D_{C \circ u}^\mathcal{I}(x) = \mathsf{enc}(\underline{u'u})$.

Lemma 9. *For any continuous t-norm $\otimes$ different from the Gödel t-norm, $\otimes$-$\mathcal{EL}$ satisfies* $\mathrm{P}_\circ$.

Proof. The t-norm $\otimes$ must contain either the product or the Łukasiewicz t-norm. We divide the proof depending on the representative chosen for the encoding function.

[$\Pi^{(a,b)}$-$\mathcal{EL}$] Since every word in Σ_0^* is seen as a natural number in base $s+1$, for every $u \in \Sigma_0^*$ and $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, we have $u'(s+1)^{|u|} + u = \underline{u'u}$. We define the ontology

$$\mathcal{O}_{C\circ u} := \{D_{C\circ u} \equiv C^{(s+1)^{|u|}} \sqcap C_u\}.$$

Recall that for every interpretation $\mathcal{I}$ and $x \in \mathcal{D}^{\mathcal{I}}$, if $C^{\mathcal{I}}(x) = a+(b-a)p$, then $(C^m)^{\mathcal{I}}(x) = a+(b-a)p^m$. Let now $\mathcal{I}$ be a model of $\mathcal{O}_{C\circ u}$, $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, and $x \in \mathcal{D}^{\mathcal{I}}$ with $C_u^{\mathcal{I}}(x) = a+(b-a)2^{-u}$ and $C^{\mathcal{I}}(x) = a+(b-a)2^{-u'}$. Since $\mathcal{I}$ satisfies $\mathcal{O}_{C\circ u}$, we have

$$D_{C\circ u}^{\mathcal{I}}(x) = a + (b-a)2^{-\left(u'(s+1)^{|u|}+u\right)} = \mathsf{enc}(\underline{u'u}).$$

[$\text{Ł}^{(a,b)}$-$\mathcal{EL}$] We define the ontology

$$\mathcal{O}_{C\circ u} := \{C'^{(s+1)^{|u|}} \equiv C,\ D_{C\circ u} \equiv C' \sqcap C_u\}.$$

Let $\mathcal{I}$ be a model of $\mathcal{O}_{C\circ u}$, $x \in \mathcal{D}^{\mathcal{I}}$, and $C_u^{\mathcal{I}}(x) = \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$. From the first axiom it follows that

$$(C'^{(s+1)^{|u|}})^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) = a+(b-a)(1-0.\overleftarrow{u'}) \in (a,b].$$

Since $\otimes$ is monotone and (a,b)-contains the Łukasiewicz t-norm, it follows that (i) $C'^{\mathcal{I}}(x) > a$ and (ii) $C'^{\mathcal{I}}(x) \geq b$ iff $C^{\mathcal{I}}(x) = b$, i.e. if u' is the empty word. Recall that, whenever $C'^{\mathcal{I}}(x) \in [a,b]$ for some interpretation $\mathcal{I}$ and $x \in \mathcal{D}^{\mathcal{I}}$, then $((C')^m)^{\mathcal{I}}(x) = \max\{a, m\left(C'^{\mathcal{I}}(x) - b\right) + b\}$ holds. If $C^{\mathcal{I}}(x) < b$, then $C'^{\mathcal{I}}(x) \in (a,b)$ and

$$a+(b-a)(1-0.\overleftarrow{u'}) = \max\{a, (s+1)^{|u|}\left(C'^{\mathcal{I}}(x) - b\right)+b\},$$

and thus $C'^{\mathcal{I}}(x) = a + (b-a)(1-(s+1)^{-|u|}0.\overleftarrow{u'})$ and

$$\begin{aligned} D_{C\circ u}^{\mathcal{I}}(x) &= a + (b-a)(1 - 0.\overleftarrow{u} - (s+1)^{-|u|}0.\overleftarrow{u'}) \\ &= \mathsf{enc}(\underline{u'u}). \end{aligned}$$

Otherwise, u' is the empty word and $C'^{\mathcal{I}}(x) \geq b$. Since $C_u^{\mathcal{I}}(x) \leq b$, we know that $C'^{\mathcal{I}}(x) \otimes C_u^{\mathcal{I}}(x) = C_u^{\mathcal{I}}(x)$ and thus $D_{C\circ u}^{\mathcal{I}}(x) = C_u^{\mathcal{I}}(x) = \mathsf{enc}(u) = \mathsf{enc}(\underline{\varepsilon u})$. ☐

The goal of this property is to ensure that at every node with $V^{\mathcal{I}}(x) = \mathsf{enc}(u)$ for some $u \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$ and $C_{v_i}^{\mathcal{I}}(x) = v_i$, we have $D_{V\circ v_i}^{\mathcal{I}}(x) = \mathsf{enc}(\underline{uv_i})$, and similarly for W, w_i and M, u_+. Thus, we define the ontology

$$\mathcal{O}_{\mathcal{P},\circ} := \bigcup_{i=1}^{n} \left(\mathcal{O}_{V\circ v_i} \cup \mathcal{O}_{W\circ w_i} \cup \mathcal{O}_{M\circ u_+}\right).$$

By construction, the values of $V^{\mathcal{I}}(x)$ and $W^{\mathcal{I}}(x)$ should always be encodings of words $v_\nu, w_\nu \in \Sigma^*$, while $M^{\mathcal{I}}(x)$ might encode words in Σ_0^*. To simplify the notation, we use the concept names V_i, W_i, M_+ instead of $C_{v_i}, C_{w_i}, C_{u_+}$ in this ontology.

Once we have computed the concatenation of two words, we need to transfer it to the successors of the node, as ensured by the following property.

Transfer property ($\mathrm{P}_{\rightsquigarrow}$):

> $\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *transfer property* if for all concepts C, D and role names r there is an ontology $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D}$ and every $x, y \in \mathcal{D}^{\mathcal{I}}$, if $r^{\mathcal{I}}(x,y) = 1$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$ for some $u \in \Sigma_0^*$, then $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$.

Lemma 10. *For every t-norm $\otimes$, $\otimes$-$\mathcal{AL}$ and $\otimes$-$\mathcal{ELC}$ satisfy* $\mathrm{P}_{\rightsquigarrow}$.

Proof. Notice first that for any model $\mathcal{I}$ of the $\otimes$-$\mathcal{EL}$ axiom $\exists r.D \sqsubseteq C$ and all $x, y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x,y) = 1$ it holds that

$$D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) \leq (\exists r.D)^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x).$$

We now add a restriction ensuring that also $D^{\mathcal{I}}(y) \geq C^{\mathcal{I}}(x)$ holds, depending on the expressivity of the logic used.

[$\otimes$-$\mathcal{AL}$] The axiom $C \sqsubseteq \forall r.D$ restricts every model $\mathcal{I}$ to satisfy that if $r^{\mathcal{I}}(x,y) = 1$, then

$$C^{\mathcal{I}}(x) \leq (\forall r.D)^{\mathcal{I}}(x) \leq r^{\mathcal{I}}(x,y) \Rightarrow D^{\mathcal{I}}(y) = D^{\mathcal{I}}(y).$$

Thus, the ontology $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{C \sqsubseteq \forall r.D, \exists r.D \sqsubseteq C\}$ satisfies the condition.

[$\otimes$-$\mathcal{ELC}$] For a model $\mathcal{I}$ of $\exists r.\neg D \sqsubseteq \neg C$ and $r^{\mathcal{I}}(x,y) = 1$,

$$\begin{aligned} 1 - D^{\mathcal{I}}(y) &= r^{\mathcal{I}}(x,y) \otimes (1 - D^{\mathcal{I}}(y)) \\ &\leq (\exists r.\neg D)^{\mathcal{I}}(x) \leq 1 - C^{\mathcal{I}}(x). \end{aligned}$$

Thus, $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{\exists r.\neg D \sqsubseteq \neg C, \exists r.D \sqsubseteq C\}$ satisfies the required condition. ☐

To ensure that the values of $\mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$, $\mathsf{enc}(u_+)$, $\mathsf{enc}(v_{\nu i})$, and $\mathsf{enc}(v_j)$ for every $j \in \mathcal{N}$ are transfered from x to the successor y_i for every $i \in \mathcal{N}$, we use the ontology

$$\begin{aligned} \mathcal{O}_{\mathcal{P},\rightsquigarrow} := &\bigcup_{i\in\mathcal{N}} \mathcal{O}_{D_{M\circ u_+} \overset{r_i}{\rightsquigarrow} M} \cup \mathcal{O}_{M_+ \overset{r_i}{\rightsquigarrow} M_+} \\ &\cup \bigcup_{i\in\mathcal{N}} \mathcal{O}_{D_{V\circ v_i} \overset{r_i}{\rightsquigarrow} V} \cup \mathcal{O}_{D_{W\circ w_i} \overset{r_i}{\rightsquigarrow} W} \\ &\cup \bigcup_{i,j\in\mathcal{N}} \mathcal{O}_{V_j \overset{r_i}{\rightsquigarrow} V_j} \cup \mathcal{O}_{W_j \overset{r_i}{\rightsquigarrow} W_j}. \end{aligned}$$

The initialization property ensures that the root of the search tree can be encoded.

Initialization property ($\mathrm{P}_{\mathsf{ini}}$):

> $\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *initialization property* if for every concept C, individual name e, and $u \in \Sigma_0^*$ there is an ontology $\mathcal{O}_{C(e)=u}$ such that $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$ for every x-model $\mathcal{I}$ of $\mathcal{O}_{C(e)=u}$.

Lemma 11. *For every t-norm $\otimes$, $\otimes$-$\mathcal{EL}_{=}$ and $\otimes$-$\mathcal{ELC}_{\geq}$ satisfy* $\mathrm{P}_{\mathsf{ini}}$.

Proof. [$\otimes$-$\mathcal{EL}_{=}$] If the equality assertion $\langle e : C = \mathsf{enc}(u)\rangle$ is satisfied by $\mathcal{I}$, then $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$.

[$\otimes$-$\mathcal{ELC}_{\geq}$] We use the two axioms $\langle e : C \geq \mathsf{enc}(u)\rangle$ and $\langle e : \neg C \geq 1 - \mathsf{enc}(u)\rangle$. The first axiom expresses that $C^{\mathcal{I}}(e^{\mathcal{I}}) \geq \mathsf{enc}(u)$, while the second requires that $1 - C^{\mathcal{I}}(e^{\mathcal{I}}) \geq 1 - \mathsf{enc}(u)$, i.e. $C^{\mathcal{I}}(e^{\mathcal{I}}) \leq \mathsf{enc}(u)$, holds. ☐

To initialize the search tree, we need to fix an individual name e_0 at which V and W are both interpreted as the encoding of the empty word and M as the encoding of u_ε. Moreover, we need that M_+ encodes u_+ and every V_i and

W_i encodes the word v_i, w_i, respectively. We thus define the ontology

$$\mathcal{O}_{\mathcal{P},\mathsf{ini}} := \mathcal{O}_{M(e_0)=u_\varepsilon} \cup \mathcal{O}_{M_+(e_0)=u_+} \cup \mathcal{O}_{V(e_0)=\varepsilon} \cup \mathcal{O}_{W(e_0)=\varepsilon} \cup \bigcup_{i=1}^{n} \left(\mathcal{O}_{V_i(e_0)=v_i} \cup \mathcal{O}_{W_i(e_0)=w_i}\right).$$

In some cases, it suffices to consider a weaker version of $\mathrm{P}_{\mathsf{ini}}$, where only the two words ε and u_ε need to be initialized.

Weak initialization property ($\mathrm{P}^w_{\mathsf{ini}}$):

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *weak initialization property* if for every concept C, individual name e, and $u \in \{\varepsilon, u_\varepsilon\}$ there is an ontology $\mathcal{O}_{C(e)=u}$ such that $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$ holds for every x-model $\mathcal{I}$ of $\mathcal{O}_{C(e)=u}$.

Lemma 12. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}^w_{\mathsf{ini}}$.

Proof. We have $\mathsf{enc}(\varepsilon) = 1$ and hence the crisp assertion $\langle e : C \geq 1\rangle$ yields the desired condition for ε. For $u_\varepsilon = 1$, we use the axiom $C \equiv \neg C$, which in particular restricts $C^{\mathcal{I}}(e^{\mathcal{I}}) = 1 - C^{\mathcal{I}}(e^{\mathcal{I}})$ to be $0.5 = \mathsf{enc}(1)$. □

For any logic satisfying $\mathrm{P}^w_{\mathsf{ini}}$, any model of the ontology

$$\mathcal{O}^w_{\mathcal{P},\mathsf{ini}} := \mathcal{O}_{V(e_0)=\varepsilon} \cup \mathcal{O}_{W(e_0)=\varepsilon} \cup \mathcal{O}_{M(e_0)=u_\varepsilon},$$

must contain an individual encoding the values of V, W and M at the root of the search tree of $\mathcal{P}$.

Note that the construction for Π-$\mathcal{ELC}$ works since we know that $u_+ = \varepsilon$, i.e. the value of M is constant. In general, a constant interpretation of a concept name can be enforced through the following property.

Constant property ($\mathrm{P}_=$):

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *constant property* if for every concept name C and word $u \in \Sigma_0^*$ there is an ontology $\mathcal{O}_{C=u}$ such that for every x-model of $\mathcal{O}_{C=u}$ and every $x \in \mathcal{D}^{\mathcal{I}}$ we have $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$.

Lemma 13. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}_=$.

Proof. Consider $\mathcal{O}_{C=u} := \{H \equiv \neg H, C \equiv H^u\}$. From the first axiom it follows that for every model $\mathcal{I}$ of this ontology and $x \in \mathcal{D}^{\mathcal{I}}$, we have $H^{\mathcal{I}}(x) = 1 - H^{\mathcal{I}}(x)$, and thus $H^{\mathcal{I}}(x) = 0.5 = 2^{-1}$. Thus, from the second axiom, $C^{\mathcal{I}}(x) = (2^{-1})^u = 2^{-u} = \mathsf{enc}(u)$. □

The constant values of V_i, W_i, and M_+ are ensured by the ontology

$$\mathcal{O}_{\mathcal{P},=} := \mathcal{O}_{M_+=u_+} \cup \bigcup_{i=1}^{n} \mathcal{O}_{V_i=v_i} \cup \mathcal{O}_{W_i=w_i}.$$

As described before, different combinations of these properties yield the canonical model property.

Theorem 14. *If a logic* $\otimes_{\mathsf{x}}$-$\mathcal{L}$ *satisfies the properties* $\mathrm{P}_\circ$, $\mathrm{P}_{\mathsf{ini}}$, $\mathrm{P}_\to$, *and* $\mathrm{P}_\rightsquigarrow$, *then it also satisfies* $\mathrm{P}_\triangle$.

Proof. We show that $\mathcal{O}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P},\mathsf{ini}} \cup \mathcal{O}_{\mathcal{P},\circ} \cup \mathcal{O}_{\mathcal{P},\to} \cup \mathcal{O}_{\mathcal{P},\rightsquigarrow}$ satisfies the conditions from the definition of $\mathrm{P}_\triangle$. For a model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, we construct the function $g : \mathcal{N}^* \to \mathcal{D}^{\mathcal{I}}$ inductively as follows.

We first set $g(\varepsilon) := e_0^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{O}_{\mathcal{P},\mathsf{ini}}$, we have that $V^{\mathcal{I}}(g(\varepsilon)) = V^{\mathcal{I}}(e_0^{\mathcal{I}}) = \mathsf{enc}(\varepsilon) = V^{\mathcal{I}_{\mathcal{P}}}(\varepsilon)$, and likewise for W, M, M_+, V_i, and W_i for all $i \in \mathcal{N}$.

Let now ν be such that $g(\nu)$ has already been defined and $V^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(v_\nu)$, $V_i^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(v_i)$. $\mathcal{I}$ being a model of $\mathcal{O}_{\mathcal{P},\circ}$ ensures that $D^{\mathcal{I}}_{V\circ v_i} = \mathsf{enc}(v_{\nu i})$. Since $\mathcal{I}$ satisfies $\mathcal{O}_{\mathcal{P},\to}$, for each $i \in \mathcal{N}$ there must be a $y_i \in \mathcal{D}^{\mathcal{I}}$ with $r_i^{\mathcal{I}}(g(\nu), y_i) = 1$. Define now $g(\nu i) := y_i$. $\mathcal{O}_{\mathcal{P},\rightsquigarrow}$ ensures that $V^{\mathcal{I}}(g(\nu i)) = D^{\mathcal{I}}_{V\circ v_i}(g(\nu)) = \mathsf{enc}(v_{\nu i}) = V^{\mathcal{I}_{\mathcal{P}}}(\nu i)$ and $V_i^{\mathcal{I}}(g(\nu i)) = \mathsf{enc}(v_i) = V_i^{\mathcal{I}_{\mathcal{P}}}(\nu i)$ for all $i \in \mathcal{N}$, and analogously for W, W_i and M, M_+. □

From this theorem and Lemmata 8 to 11, we obtain the following result.

Corollary 15. *If* $\otimes$ *is a continuous t-norm, but not the Gödel t-norm, then the logics* $\otimes_\top$-$\mathcal{AL}_=$, $\otimes$-$\mathcal{AL}_{=,\mathsf{c}}$, $\otimes_\top$-$\mathcal{ELC}_\geq$, *and* $\otimes$-$\mathcal{ELC}_{\geq,\mathsf{c}}$ *satisfy* $\mathrm{P}_\triangle$.

Alternatively, we can substitute $\mathrm{P}_{\mathsf{ini}}$ with the properties $\mathrm{P}^w_{\mathsf{ini}}$ and $\mathrm{P}_=$ and still obtain the canonical model property. The proof of this is analogous to that of Theorem 14, using the ontology $\mathcal{O}_{\mathcal{P}} := \mathcal{O}^w_{\mathcal{P},\mathsf{ini}} \cup \mathcal{O}_{\mathcal{P},=} \cup \mathcal{O}_{\mathcal{P},\circ} \cup \mathcal{O}_{\mathcal{P},\to} \cup \mathcal{O}_{\mathcal{P},\rightsquigarrow}$.

Theorem 16. *If* $\otimes_{\mathsf{x}}$-$\mathcal{L}$ *satisfies the properties* $\mathrm{P}_\circ$, $\mathrm{P}^w_{\mathsf{ini}}$, $\mathrm{P}_=$, $\mathrm{P}_\to$, *and* $\mathrm{P}_\rightsquigarrow$, *then it also satisfies* $\mathrm{P}_\triangle$.

With the help of Lemmata 8 to 13, we now obtain the following result.

Corollary 17. *The logics* $\Pi_\top$-$\mathcal{ELC}$ *and* Π-$\mathcal{ELC}_{\mathsf{c}}$ *satisfy* $\mathrm{P}_\triangle$.

It is a simple task to verify that the interpretation $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of the ontology $\mathcal{O}_{\mathcal{P}}$ in all the cases described. We only need to use a unique new concept name for every auxiliary concept name appearing in the different ontologies. In fact, the values of these auxiliary concept names at each node ν are uniquely determined by the values of the concept names V, W, V_i, W_i, M, M_+ in ν. Moreover, since every ν has exactly one r_i-successor with degree greater than 0 for every $i \in \mathcal{N}$, it follows that $\mathcal{I}_{\mathcal{P}}$ can be extended to a witnessed model of $\mathcal{O}_{\mathcal{P}}$.

We now use the property $\mathrm{P}_\triangle$ to prove undecidability of a fuzzy DL. The idea is to extend $\mathcal{O}_{\mathcal{P}}$ so that every model $\mathcal{I}$ must satisfy $V^{\mathcal{I}}(g(\nu)) \neq W^{\mathcal{I}}(g(\nu))$ for every $\nu \in \mathcal{N}^+$, thus obtaining an ontology that is consistent if and only if $\mathcal{P}$ has no solution.

4.2 Finding a Solution

For the rest of this section, we assume that $\otimes_{\mathsf{x}}$-$\mathcal{L}$ satisfies $\mathrm{P}_\triangle$ and for any given model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, g denotes the function mapping the nodes of $\mathcal{I}_{\mathcal{P}}$ to nodes in $\mathcal{I}$ given by the property. Furthermore, we assume that $\mathcal{I}_{\mathcal{P}}$ can be extended to an x-model of $\mathcal{O}_{\mathcal{P}}$. These assumptions have been shown to hold for a variety of fuzzy DLs in the previous section.

The key to showing undecidability of $\otimes_{\mathsf{x}}$-$\mathcal{L}$ is to be able to express the restriction that V and W encode different words at every non-root node $\nu \in \mathcal{N}^+$ of the search tree. Since enc is a valid encoding function, and M encodes the word $\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}$ at every $\nu \in \mathcal{N}^*$, it suffices to check whether, for all $\nu \in \mathcal{N}^+$, either $(V \to W)^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$ or $(W \to V)^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$ (recall Definition 6). This can easily be done in every logic that has the implication constructor $\to$. However, this constructor is not necessary in general to show undecidability.

Solution property ($\mathrm{P}_{\neq}$)**:**

A logic $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfying $\mathrm{P}_\triangle$ has the *solution property* if there is an ontology $\mathcal{O}_{V\neq W}$ such that

1. For every x-model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ and $\nu \in \mathcal{N}^+$, either

$$V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu)) \quad \text{or}$$
$$W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu)).$$

2. If for every $\nu \in \mathcal{N}^+$ we have either

$$V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu) \quad \text{or}$$
$$W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu),$$

then $\mathcal{I}_\mathcal{P}$ can be extended to a model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$.

Lemma 18. *Let $\otimes$ be a continuous t-norm $\otimes$ different from the Gödel t-norm and $\mathcal{L}$ contain either $\mathfrak{IAL}$ or $\mathcal{ELC}$. If $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfies $\mathrm{P}_\triangle$ and $\mathcal{I}_\mathcal{P}$ can be extended to an x-model of $\mathcal{O}_\mathcal{P}$, then $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfies $\mathrm{P}_{\neq}$.*

Proof. We divide the proof according to the underlying DL.

[$\mathfrak{IAL}$] We define $\mathcal{O}_{V\neq W}$ as

$$\{\top \sqsubseteq \forall r_i.(((V \to W) \sqcap (W \to V)) \to M) \mid i \in \mathcal{N}\}.$$

This ontology is satisfied by $\mathcal{I}$ iff for every $x, y \in \mathcal{D}^\mathcal{I}$ and every $i \in \mathcal{N}$ we have

$$r_i^\mathcal{I}(x,y) \leq ((V \to W) \sqcap (W \to V))^\mathcal{I}(y) \Rightarrow M^\mathcal{I}(y).$$

Let now $\mathcal{I}$ be an x-model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$. Since at least one of $(V \to W)^\mathcal{I}(g(\nu i))$, $(W \to V)^\mathcal{I}(g(\nu i))$ must be 1 and $r_i^\mathcal{I}(g(\nu), g(\nu i)) = 1$ for every $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$, then it holds that either $V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$ or $W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$.

For the second condition, consider an extension $\mathcal{I}$ of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$ and assume that it violates $\mathcal{O}_{V\neq W}$. Thus, there are $\nu \in \mathcal{N}^*$, $i \in \mathcal{N}$ such that

$$1 > (\forall r_i.(((V \to W) \sqcap (W \to V)) \to M))^{\mathcal{I}_\mathcal{P}}(\nu).$$

Since νi is the only r_i-successor of ν, this implies that

$$\begin{aligned} &M^{\mathcal{I}_\mathcal{P}}(\nu i) \\ &< (V^{\mathcal{I}_\mathcal{P}}(\nu i) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu i)) \otimes (W^{\mathcal{I}_\mathcal{P}}(\nu i) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu i)) \\ &\leq \min\{V^{\mathcal{I}_\mathcal{P}}(\nu i) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu i), W^{\mathcal{I}_\mathcal{P}}(\nu i) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu i)\}. \end{aligned}$$

[$\mathcal{ELC}$] Consider the ontology

$$\begin{aligned} \mathcal{O}_{V\neq W} := &\{\exists r_i.\neg Y \sqsubseteq \bot \mid 1 \leq i \leq n\} \cup &(1)\\ &\{X \sqsubseteq X \sqcap X, \top \sqsubseteq \neg(X \sqcap \neg X), \langle e_0 : \neg Y \geq 1\rangle, \\ &Y \sqcap X \sqcap V \sqsubseteq Y \sqcap X \sqcap W \sqcap M, &(2)\\ &Y \sqcap \neg X \sqcap W \sqsubseteq Y \sqcap \neg X \sqcap V \sqcap M\}. &(3) \end{aligned}$$

Every model of the axioms in (1) has to satisfy that every r_i-successor with degree 1 must belong to Y with degree 1, for every $i \in \mathcal{N}$. In particular, this means that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ and every $\nu \in \mathcal{N}^+$, we have $Y^\mathcal{I}(g(\nu)) = 1$. The next axiom ensures that for every $x \in \mathcal{D}^\mathcal{I}$, $X^\mathcal{I}(x) \leq X^\mathcal{I}(x) \otimes X^\mathcal{I}(x)$, and hence, $X^\mathcal{I}(x)$ must be an idempotent element w.r.t. $\otimes$. In particular, this means that $(X \sqcap \neg X)^\mathcal{I}(x) = \min\{X^\mathcal{I}(x), 1 - X^\mathcal{I}(x)\}$ (Klement, Mesiar, and Pap 2000), and from the second axiom it follows that $X^\mathcal{I}(x) \in \{0,1\}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ and $\nu \in \mathcal{N}^+$. If $X^\mathcal{I}(g(\nu)) = 1$, then from axiom (2) it follows that $V^\mathcal{I}(g(\nu)) \leq W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu))$. We consider two cases, according to the representative chosen in $\otimes$.

[$\Pi^{(a,b)}$] We know that $W^\mathcal{I}(g(\nu)) = \mathsf{enc}(w_\nu) > a$ and $M^\mathcal{I}(g(\nu)) = \mathsf{enc}(1) < b$. Thus, for all $m' > M^\mathcal{I}(g(\nu))$,

$$W^\mathcal{I}(g(\nu)) \otimes m' > W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu)) \geq V^\mathcal{I}(g(\nu)).$$

[$\text{Ł}^{(a,b)}$] Since the length of w_ν is bounded by $|\nu|k$, we have

$$\begin{aligned} &W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu)) \\ &\quad = a + (b-a)\max\{0, 1 - 0.\overleftarrow{w_\nu} - (0.\underline{0}^{|\nu|k} \cdot \underline{1})\} \\ &\quad = a + (b-a)(1 - 0.\overleftarrow{w_\nu} - (0.\underline{0}^{|\nu|k} \cdot \underline{1})) \in (a,b). \end{aligned}$$

Thus, for every $m' > M^\mathcal{I}(g(\nu))$,

$$W^\mathcal{I}(g(\nu)) \otimes m' > W^\mathcal{I}(g(\nu)) \otimes M^\mathcal{I}(g(\nu)) \geq V^\mathcal{I}(g(\nu)).$$

In both cases, since $W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu))$ equals

$$\sup\{z \in [0,1] \mid W^\mathcal{I}(g(\nu)) \otimes z \leq V^\mathcal{I}(g(\nu))\},$$

we have $W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$. Using an analogous argument, if $X^\mathcal{I}(g(\nu)) = 0$, then axiom (3) yields $V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$.

To show the second point of $\mathrm{P}_{\neq}$, consider an extension $\mathcal{I}$ of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$, which exists by assumption. We show that $\mathcal{I}$ can be extended to a model of $\mathcal{O}_{V\neq W}$. We first set $Y^\mathcal{I}(\nu) = 1$ for every $\nu \in \mathcal{N}^+$ and $X^\mathcal{I}(\varepsilon) = Y^\mathcal{I}(\varepsilon) = 0$. It remains to find values for $X^\mathcal{I}(\nu)$ for $\nu \in \mathcal{N}^+$.

By assumption, we know that one of the two residua $V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu)$ and $W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)$ is smaller than or equal to $M^{\mathcal{I}_\mathcal{P}}(\nu) < 1$. However, one of them must be equal to 1. If $V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu) = 1$ and $W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$, then we set $X^\mathcal{I}(\nu) = 1$, which trivially satisfies axiom (3) at ν. By definition of the residuum, this implies that $W^{\mathcal{I}_\mathcal{P}}(\nu) \otimes m' > V^{\mathcal{I}_\mathcal{P}}(\nu)$ for all $m' > M^{\mathcal{I}_\mathcal{P}}(\nu)$. Since $\otimes$ is continuous and monotone, this means that $V^{\mathcal{I}_\mathcal{P}}(\nu) \leq W^{\mathcal{I}_\mathcal{P}}(\nu) \otimes M^{\mathcal{I}_\mathcal{P}}(\nu)$, i.e. axiom (2) is also satisfied at ν.

If the other residuum is equal to 1, we set $X^\mathcal{I}(\nu) = 0$ and use dual arguments to show that axioms (2) and (3) are satisfied at ν. We have thus constructed an extension of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_{V\neq W}$. ☐

If a fuzzy DL satisfies the property $\mathrm{P}_{\neq}$, then consistency of ontologies is undecidable.

Theorem 19. *Let $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfy $\mathrm{P}_{\neq}$. Then $\mathcal{P}$ has a solution iff $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ is inconsistent.*

Proof. If $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ is inconsistent, then in particular no extension of $\mathcal{I}_\mathcal{P}$ can satisfy this ontology. By $\mathrm{P}_{\neq}$, there must be a $\nu \in \mathcal{N}^+$ such that both $V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu)$ and $W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)$ are greater than $M^{\mathcal{I}_\mathcal{P}}(\nu)$. By Definition 6 and by $\mathrm{P}_\triangle$, we have $\mathsf{enc}(v_\nu) = \mathsf{enc}(w_\nu)$, i.e. $\mathcal{P}$ has a solution.

Assume now that $\mathcal{O}_\mathcal{P}\cup\mathcal{O}_{V\neq W}$ has a model $\mathcal{I}$. By $\mathrm{P}_{\neq}$, for every $\nu \in \mathcal{N}^+$ either $V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$ or $W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu)) \leq M^\mathcal{I}(g(\nu))$. By $\mathrm{P}_\triangle$, it follows that $\mathsf{enc}(v_\nu) = V^\mathcal{I}(g(\nu)) \neq W^\mathcal{I}(g(\nu)) = \mathsf{enc}(w_\nu)$, and thus $v_\nu \neq w_\nu$ for all $\nu \in \mathcal{N}^+$, i.e. $\mathcal{P}$ has no solution. □

Together with Corollaries 15 and 17, we obtain the following undecidability results.

Corollary 20. *For every continuous t-norm different from the Gödel t-norm, ontology consistency is undecidable in the logics* $\otimes_\top$-$\mathfrak{IAL}_=$, $\otimes$-$\mathfrak{IAL}_{=,\mathsf{c}}$, $\otimes_\top$-$\mathcal{ELC}_\geq$, $\otimes$-$\mathcal{ELC}_{\geq,\mathsf{c}}$, $\Pi_\top$-$\mathcal{ELC}$, *and* Π-$\mathcal{ELC}_\mathsf{c}$.

Since every extension of $\mathcal{I}_\mathcal{P}$ is witnessed, from these results it also follows that ontology consistency in the logics $\otimes_\mathsf{w}$-$\mathfrak{IAL}_=$, $\otimes_\mathsf{w}$-$\mathcal{ELC}_\geq$, and Π_w-$\mathcal{ELC}$ is undecidable.

4.3 Undecidability of Ł$^{(0,b)}$-$\mathfrak{NEL}$

For the logic Π-$\mathcal{ELC}$, we were able to exploit the involutive negation and obtain undecidability of consistency of *classical* ontologies; that is, no membership degrees other than 1 are required to appear in the axioms. The same idea can be applied to show that ontology consistency is also undecidable in Ł-$\mathcal{ELC}$, which is equivalent to Ł-$\mathfrak{NEL}$. We show a stronger result: consistency in Ł$_\top^{(0,b)}$-$\mathfrak{NEL}$ and Ł$^{(0,b)}$-$\mathfrak{NEL}_\mathsf{c}$ for $b > 0$ is undecidable. The t-norms $(0, b)$-containing the Łukasiewicz t-norm cover an important family of t-norms, known as the Mayor-Torrens t-norms that have been studied in the literature (Klement, Mesiar, and Pap 2000).

If $\otimes$ $(0, b)$-contains the Łukasiewicz t-norm, then for every $x \in (0, b]$ we have that $x \Rightarrow 0 = b - x$; that is, the residual negation yields a "local involutive negation" over the interval $[0, b]$. Thus, the concept $\boxminus C$ will be interpreted as the local involutive negation of the interpretation of C, whenever the latter is in this interval. Moreover, if $0 \leq D^\mathcal{I}(x) < C^\mathcal{I}(x) \leq b$, then

$$\begin{aligned}(\boxminus(C \sqcap \boxminus D))^\mathcal{I}(x) &= b - (C^\mathcal{I}(x) + (b - D^\mathcal{I}(x)) - b)\\ &= b - C^\mathcal{I}(x) + D^\mathcal{I}(x) = (C \to D)^\mathcal{I}(x).\end{aligned}$$

Thus, we abbreviate $\boxminus(C \sqcap \boxminus D)$ as $C \rightharpoonup D$. Additionally, $\bot$ can be expressed by $\boxminus\top$.

We encode a word $u \in \Sigma_0^*$ by $\mathsf{enc}(u) = b(0.\overleftarrow{u})$. The proof that this is indeed a valid encoding function uses similar arguments to the case for Ł$^{(a,b)}$ of Lemma 7.

Let $\mathcal{P}$ be an instance of the PCP as before and assume that $v_\nu \neq w_\nu$ for some $\nu \in \mathcal{N}^+$. Then these words must differ in one of the first $|\nu|k$ digits, and thus either

$$\begin{aligned}\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) &= b \min\{1, 1 - 0.\overleftarrow{v_\nu} + 0.\overleftarrow{w_\nu}\}\\ &\leq b(1 - (s+1)^{-|\nu|k})\\ &= \mathsf{enc}(\underline{\varepsilon} \cdot \underline{s}^{|\nu|k})\end{aligned}$$

or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}(\underline{\varepsilon} \cdot \underline{s}^{|\nu|k}) < 1$. Conversely, if $v_\nu = w_\nu$, then both residua are 1. Thus, the words $u_\varepsilon = \varepsilon$ and $u_+ = \underline{s}^k$ satisfy the condition of Definition 6.

We will employ Theorem 16 to show that the logics Ł$_\top^{(0,b)}$-$\mathfrak{NEL}$ and Ł$^{(0,b)}$-$\mathfrak{NEL}_\mathsf{c}$ satisfy the canonical model property. Thus, we need to prove that they satisfy $\mathrm{P}_\to$, $\mathrm{P}_\circ$, $\mathrm{P}_\rightsquigarrow$, $\mathrm{P}_{\mathsf{ini}}^w$, and $\mathrm{P}_=$. By Lemma 8, they satisfy the successor property. We now show that Ł$^{(0,b)}$-$\mathfrak{NEL}$ satisfies the rest of the properties.

Concatenation property Analogous to Lemma 9, the axioms $(\boxminus C')^{(s+1)^{|u|}} \equiv \boxminus C$ and $\boxminus D_{C\circ u} \equiv (\boxminus C') \sqcap (\boxminus C_u)$ yield the concatenation of words represented by C with the constant word u.

Transfer property If $C^\mathcal{I}(x) = \mathsf{enc}(w)$, $w \in \Sigma^*$, then $C^\mathcal{I}(x) < b$, and thus for every model $\mathcal{I}$ of $\exists r.(\boxminus D) \sqsubseteq \boxminus C$ if $r^\mathcal{I}(x, y) = 1$ then

$$b - C^\mathcal{I}(x) = (\boxminus C)^\mathcal{I}(x) \geq (\exists r.(\boxminus D))^\mathcal{I}(x) \geq (\boxminus D)^\mathcal{I}(y).$$

If $D^\mathcal{I}(y) < C^\mathcal{I}(x) < b$, then

$$(\boxminus D)^\mathcal{I}(y) = b - D^\mathcal{I}(y) > b - C^\mathcal{I}(x),$$

which yields a contradiction; hence $C^\mathcal{I}(x) \leq D^\mathcal{I}(y)$ must hold. Together with the first part of the proof of Lemma 10, we have that the ontology

$$\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{\exists r.(\boxminus D) \sqsubseteq \boxminus C, \exists r.D \sqsubseteq C\}$$

yields the transfer property.

Weak initialization property The assertion $\langle e : \boxminus C \geq 1\rangle$ initializes the value $\mathsf{enc}(u_\varepsilon) = \mathsf{enc}(\varepsilon) = 0$.

Constant property We have to restrict the value of a concept C to $\mathsf{enc}(u)$ for some word $u \in \Sigma_0^*$. For $u = \varepsilon$, the axiom $C \sqsubseteq \bot$ suffices. If $u \in \Sigma_0^+$, we employ the ontology

$$\mathcal{O}_{C=u} := \{H^{(s+1)^{|u|}} \equiv \boxminus H^{(s+1)^{|u|}},\ \boxminus C \equiv H^{2\overleftarrow{u}}\}.$$

If an interpretation $\mathcal{I}$ satisfies the first axiom, then for every $x \in \mathcal{D}^\mathcal{I}$ we have $-b = 2(s + 1)^{|u|}(H^\mathcal{I}(x) - b)$; that is $H^\mathcal{I}(x) = b - \frac{b}{2(s+1)^{|u|}}$. From the second axiom it follows that

$$(\boxminus C)^\mathcal{I}(x) = \max\left\{0, 2\overleftarrow{u}\left(-\tfrac{b}{2(s+1)^{|u|}}\right) + b\right\}.$$

Since $\frac{\overleftarrow{u}}{(s+1)^{|u|}} = 0.\overleftarrow{u} < 1$, we obtain

$$(\boxminus C)^\mathcal{I}(x) = b - b(0.\overleftarrow{u}) = b - \mathsf{enc}(u).$$

Since $\mathsf{enc}(u) < b$, we have $0 < (\boxminus C)^\mathcal{I}(x) < b$, and thus $0 < C^\mathcal{I}(x) < b$ and $(\boxminus C)^\mathcal{I}(x) = b - C^\mathcal{I}(x)$. From this, we obtain that $C^\mathcal{I}(x) = \mathsf{enc}(u)$.

One can easily extend $\mathcal{I}_\mathcal{P}$ to a model of the ontology $\mathcal{O}_\mathcal{P}$ that results from the above definitions. By Theorem 16, Ł$_\top^{(0,b)}$-$\mathfrak{NEL}$ and Ł$^{(0,b)}$-$\mathfrak{NEL}_\mathsf{c}$ satisfy the canonical model property. It remains to show that the solution property holds.

	$\mathfrak{NEL}$	$\mathfrak{IAL}$	$\mathcal{ELC}$
classical	$Ł^{(0,b)}$	$Ł^{(0,b)}$	Π, Ł
$\geq$	$Ł^{(0,b)}$	$Ł^{(0,b)}$	$\otimes$
$=$	$Ł^{(0,b)}$	$\otimes$	$\otimes$

Table 3: A summary of the undecidability results.

Lemma 21. *The logics* $Ł_{\top}^{(0,b)}$-$\mathfrak{NEL}$ *and* $Ł^{(0,b)}$-$\mathfrak{NEL}_{\mathsf{c}}$ *satisfy* $\mathrm{P}_{\neq}$.

Proof. Consider the following ontology $\mathcal{O}_{V\neq W}$:

$\{\exists r_i.\boxminus((((V \rightharpoonup W)\sqcap(W \rightharpoonup V)) \rightharpoonup M) \sqsubseteq \bot \mid i \in \mathcal{N}\}$.

In any model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V\neq W}$ and for every $\nu \in \mathcal{N}^+$,

$$(\boxminus(((V \rightharpoonup W)\sqcap(W \rightharpoonup V)) \rightharpoonup M))^{\mathcal{I}}(g(\nu)) = 0,$$

and thus, $(((V \rightharpoonup W)\sqcap(W \rightharpoonup V)) \rightharpoonup M)^{\mathcal{I}}(g(\nu)) \geq b$.

If $V^{\mathcal{I}}(g(\nu)) \leq W^{\mathcal{I}}(g(\nu))$, then either $M^{\mathcal{I}}(g(\nu)) \geq b$ or $M^{\mathcal{I}}(g(\nu)) \geq (W \rightharpoonup V)^{\mathcal{I}}(g(\nu))$. The former is impossible since $M^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(\underline{s}^{|\nu|k}) < b$. We also know that $V^{\mathcal{I}}(g(\nu)) < b$ and $W^{\mathcal{I}}(g(\nu)) < b$, and thus

$$W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu)).$$

Similarly, if $W^{\mathcal{I}}(g(\nu)) \leq V^{\mathcal{I}}(g(\nu))$, then we have $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$.

Consider now an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$ and assume that it violates $\mathcal{O}_{V\neq W}$. Then there must be $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$ such that

$$(\boxminus(((V \rightharpoonup W)\sqcap(W \rightharpoonup V)) \rightharpoonup M))^{\mathcal{I}}(\nu i) > 0,$$

and thus $(V \rightharpoonup W)^{\mathcal{I}}(\nu i) \otimes (W \rightharpoonup V)^{\mathcal{I}}(\nu i) > M^{\mathcal{I}}(\nu i)$. As above, the value $(V \rightharpoonup W)^{\mathcal{I}}(\nu i) \otimes (W \rightharpoonup V)^{\mathcal{I}}(\nu i)$ is either $V^{\mathcal{I}}(\nu i) \Rightarrow W^{\mathcal{I}}(\nu i)$ or $W^{\mathcal{I}}(\nu i) \Rightarrow V^{\mathcal{I}}(\nu i)$. Thus, both $V^{\mathcal{I}}(\nu i) \Rightarrow W^{\mathcal{I}}(\nu i)$ and $W^{\mathcal{I}}(\nu i) \Rightarrow V^{\mathcal{I}}(\nu i)$ must be greater than $M^{\mathcal{I}}(\nu i)$, contradicting the assumption. □

This theorem shows that consistency is undecidable in $Ł_{\top}^{(0,b)}$-$\mathfrak{NEL}$ and $Ł^{(0,b)}$-$\mathfrak{NEL}_{\mathsf{c}}$. Undecidability of $Ł_{\mathsf{w}}^{(0,b)}$-$\mathfrak{NEL}$ follows from the same arguments since every extension of $\mathcal{I}_{\mathcal{P}}$ is witnessed. Notice that $Ł^{(0,1)}$-$\mathfrak{NEL}$ is a sublogic of Ł-$\mathfrak{IEL}$, which is equivalent to Ł-$\mathcal{ELC}$. Thus, consistency in Ł-$\mathcal{ELC}$ is also undecidable.

5 Conclusions

We have presented a framework for showing undecidability of consistency in fuzzy description logics and have successfully applied this framework to numerous fuzzy DLs. Table 3 summarizes the undecidability results. Every cell represents a combination of constructors and axioms. The entry in a cell denotes the largest family of t-norms for which we have shown undecidability of the resulting fuzzy DL with ($\top$-)witnessed models or with general models if crisp role axioms are allowed. Here, $\otimes$ represents all continuous t-norms except the Gödel t-norm.

Our results strengthen all previously known undecidability results for fuzzy DLs in several ways. For all previous results, ontologies required *fuzzy GCIs* of the form $\langle C \sqsubseteq D \geq q\rangle$. More precisely, it was shown that

- Π_{w}-$\mathcal{ALC}_{\geq}$ (with some additional axioms) (Baader and Peñaloza 2011a),
- $\Pi_{\mathsf{w}}^{(0,b)}$-$\mathfrak{IAL}_{=}$ (Baader and Peñaloza 2011c), and
- $Ł_{\mathsf{w}}$-$\mathcal{ELC}_{\geq}$ (Cerami and Straccia 2011)

extended with fuzzy GCIs are undecidable. For the first and last case, we were able to show that classical ontologies suffice to get undecidability. We find these results especially interesting, since they show that it is the underlying semantics, and not the expressivity of the axioms, that yields undecidability. In the second case, we extended the class of t-norms for which the logic is undecidable to cover all continuous t-norms, except the Gödel t-norm.

The decision problem considered in this paper, ontology consistency, is usually studied in crisp DLs because other reasoning problems (like concept satisfiability or subsumption between concepts) can be reduced to it, but a converse reduction is not possible using only the constructors of $\otimes$-$\mathfrak{IALC}$. It is thus natural to ask whether these other problems are also undecidable. Our proofs of undecidability w.r.t. classical ontologies (first row of Table 3) use a set of GCIs and a set of crisp concept assertions using a fixed individual name. It follows that concept satisfiability in $Ł^{(0,b)}$-$\mathfrak{NEL}$, Π-$\mathcal{ELC}$, and Ł-$\mathcal{ELC}$ is undecidable w.r.t. ($\top$-)witnessed models, and w.r.t. general models if crisp role axioms are allowed. Without GCIs, the problem is decidable in $\otimes$-$\mathfrak{IAL}$ for any continuous t-norm $\otimes$ (Hájek 2005). We will continue studying the decidability of these reasoning problems in different fuzzy DLs.

To the best of our knowledge, we have presented the first undecidability results w.r.t. general models, which were obtained with the help of crisp role axioms. Crisp roles are a desirable feature for many application domains and have been considered e.g. in the `fuzzyDL` reasoner[6] or (Vaneková and Vojtás 2010).

In the future, we will continue studying the problem of reasoning w.r.t. general models, and consider also reasoning in other classes of models like finite or strongly witnessed models, for which only a few undecidability results exist (Baader and Peñaloza 2011c). We also want to find decidable classes of fuzzy DLs, beyond the simple restrictions to finitely many fuzzy values (Borgwardt and Peñaloza 2011a; 2011b; Bobillo and Straccia 2011) or to acyclic and unfoldable terminologies (Bobillo, Bou, and Straccia 2011).

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