Axiomatization of General Concept Inclusions from Streams of Interpretations with optional Error Tolerance

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Abstract. We propose applications that utilize the infimum and the supremum of closure operators that are induced by structures occuring in the field of *Description Logics*. More specifically, we consider the closure operators induced by interpretations as well as closure operators induced by TBoxes, and show how we can learn GCIs from streams of interpretations, and how an error-tolerant axiomatization of GCIs from an interpretation guided by a hand-crafted TBox can be achieved.

Keywords: Description Logics · Formal Concept Analysis · Most Specific Consequence · Error Tolerance · General Concept Inclusion · TBox · Interpretation · Model · Stream · Incremental Learning · Automatic Learning

1 Introduction

Description Logics [2, 6–8] are a family of well-founded languages for knowledge representation with a strong logical foundation as well as a widely explored hierarchy of decidability and complexity of common reasoning problems. The several reasoning tasks allow for an automatic deduction of implicit knowledge from given explicitly represented facts and axioms, and many reasoning algorithms have been developed. Description Logics are utilized in many different application domains, and in particular provide the logical underpinning of *Web Ontology Language (OWL)* [16] and its profiles.

An interesting problem is the task of (semi-)automatic generation of terminological axioms, so-called *general concept inclusions (GCIs)*, from given data. For example, in [4, 10] Baader and Distel have generalized the construction of implicational bases from so-called *formal contexts* [12, 15] in the field of *Formal Concept Analysis* [14] to the construction of bases of $\mathcal{EL}^{\perp}$-GCIs from *interpretations* in *Description Logics*. The main difference of the underlying data structures is that interpretations additionally allow the expression of binary relations between objects, which implies a number of technical and theoretical difficulties that have been solved by them. In case of incompleteness of the input data set, a technique of *Attribute Exploration* [11–13] can be utilized to axiomatize implications in a sound and complete manner. This approach furthermore presupposes an expert in the domain of interest that is able to correctly answer all queries posed to her. In [5, 10] Baader and Distel have as well extended this technique from formal contexts to interpretations. A further work in the intersection of *Formal Concept Analysis* and *Description Logics* was developed by Rudolph in [20, 21]. He generalized *Attribute*

Exploration to *Relational Exploration*, a technique that processes an interpretation in a multi-step approach where in each step the role-depth of the involved concept descriptions is increased. In particular, Relational Exploration is a sound and complete deduction calculus for GCIs in the Description Logic $\mathcal{FLE}$. A weakness of the exploration methods is the requirement of an expert that is able to truthfully answer all questions posed to it.

In order theory, *closure operators (clop)* denote mappings in a powerset – or more generally, in a lattice – which are extensive, monotone, and idempotent. Many types of data sets give rise to an induced closure operator in such a way that an implication is valid in the data set if, and only if, it is valid in the closure operator. For example, each formal context (G, M, I) induces the clop $X \mapsto X^{II}$ on the powerset $\wp(M)$, and each interpretation $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ induces the clop $C \mapsto C^{\mathcal{I}\mathcal{I}}$ in the lattice of all $\mathcal{EL}^{\bot}$-concept descriptions ordered by subsumption $\sqsubseteq$ and factorized by equivalence $\equiv$. In a recent paper [19], the author investigated how implicational bases for closure operators can be computed in a parallel manner. Furthermore, it was shown that the set of all clops in a complete lattice constitutes a complete lattice itself, and formulae for computing infima and suprema of clops were provided. In this document, we will introduce a closure operator $C \mapsto C^{\mathcal{T}}$ induced by a TBox $\mathcal{T}$, and will furthermore provide applications for computing bases of GCIs for the infimum as well as the supremum of $C \mapsto C^{\mathcal{T}}$ and $C \mapsto C^{\mathcal{I}\mathcal{I}}$.

This document is structured as follows. Section 2 gives a brief overview on the easy description logic $\mathcal{EL}^{\bot}$. Then, Section 3 cites some related work, and recalls important notions. Section 4 introduces the notion of a most specific consequence with respect to a TBox, and shows how to define a closure operator induced by a TBox. Section 5 then discusses applications of infima and suprema of clops induced by interpretations and TBoxes, and Section 7 draws some conclusions. Note that proofs are not included, but the interested reader can find them in the corresponding technical report [17].

2 The Description Logic $\mathcal{EL}^{\bot}$

The syntax and semantics of the light-weight description logic $\mathcal{EL}^{\bot}$ are introduced as follows. Throughout the whole document assume that (N_C, N_R) is a signature, i.e., N_C is a set of *concept names*, and N_R is a set of *role names*. An $\mathcal{EL}^{\bot}$*-concept description* is a term that is constructed by means of the following inductive rule:

$$C ::= \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C.$$

A *general concept inclusion* (abbr. *GCI*) is an expression $C \sqsubseteq D$ where both the *premise* C as well as the *conclusion* D are $\mathcal{EL}^{\bot}$-concept descriptions. A *TBox* is a set of GCIs.

The *role depth* $\mathsf{rd}(C)$ of an $\mathcal{EL}^{\bot}$-concept description C is inductively defined as follows:

$$\mathsf{rd}(\bot) := \mathsf{rd}(\top) := \mathsf{rd}(A) := 0, \quad \mathsf{rd}(C \sqcap D) := \mathsf{rd}(C) \vee \mathsf{rd}(D),$$
$$\text{and} \quad \mathsf{rd}(\exists\, r.\, C) := 1 + \mathsf{rd}(C).$$

An *interpretation* $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, called the *domain*, and an *extension function* $\cdot^{\mathcal{I}}$ that maps concept names $A \in N_C$ to subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and maps role names $r \in N_R$ to binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Then, the extension function is canonically extended to all $\mathcal{EL}^{\bot}$-concept descriptions by the following definitions:

$$\bot^{\mathcal{I}} := \emptyset, \quad \top^{\mathcal{I}} := \Delta^{\mathcal{I}}, \quad (C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}},$$
$$\text{and} \quad (\exists\, r.\, C)^{\mathcal{I}} := \{\, d \in \Delta^{\mathcal{I}} \mid \exists\, e \in \Delta^{\mathcal{I}} \colon (d, e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}} \,\}.$$

A GCI $C \sqsubseteq D$ is *valid* in $\mathcal{I}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. We then also refer to $\mathcal{I}$ as a *model* of $C \sqsubseteq D$, and denote this by $\mathcal{I} \models C \sqsubseteq D$. Furthermore, $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, symbolized as $\mathcal{I} \models \mathcal{T}$, if each GCI in $\mathcal{T}$ is valid in $\mathcal{I}$. The entailment relation is lifted to TBoxes as follows: A GCI $C \sqsubseteq D$ is *entailed* by a TBox $\mathcal{T}$, denoted as $\mathcal{T} \models C \sqsubseteq D$, if each model of $\mathcal{T}$ is a model of $C \sqsubseteq D$, too. We then also say that C is *subsumed* by D with respect to $\mathcal{T}$. A TBox $\mathcal{T}$ *entails* a TBox $\mathcal{U}$, symbolized as $\mathcal{T} \models \mathcal{U}$, if $\mathcal{T}$ entails each GCI in $\mathcal{U}$, or equivalently if each model of $\mathcal{T}$ is also a model of $\mathcal{U}$. Two $\mathcal{EL}^{\bot}$-concept descriptions C and D are *equivalent* with respect to $\mathcal{T}$, and we shall write $\mathcal{T} \models C \equiv D$, if $\mathcal{T} \models \{C \sqsubseteq D, D \sqsubseteq C\}$. In case $\mathcal{T} = \emptyset$ we may ommit the prefix "$\emptyset \models$". However, then we have to carefully interpret an expression $C \sqsubseteq D$ – it either just denotes a general concept inclusion, i.e., an axiom, without stating where it is valid; or it expresses that C is subsumed by D (w.r.t. $\emptyset$), i.e., $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ is satisfied in all interpretations $\mathcal{I}$. An analogous hint applies to concept equivalences $C \equiv D$.

It is readily verified that the *subsumption* $\sqsubseteq$ constitutes a quasi-order on the set $\mathcal{EL}^{\bot}(N_C, N_R)$ of all $\mathcal{EL}^{\bot}$-concept descriptions over the signature (N_C, N_R). Hence, the quotient of $\mathcal{EL}^{\bot}(N_C, N_R)$ with respect to the induced *equivalence* $\equiv$ is a partially ordered set (a *poset*). (In the following we will not distinguish between the equivalence classes and their representatives.) The poset is even a bounded lattice. Of course, $\bot$ is the smallest element, and $\top$ is the greatest element. Furthermore, the conjunction $\sqcap$ corresponds to the *infimum* operation, and the least common subsumer mapping $\vee$ corresponds to the *supremum* operation. Remark that the *least common subsumer* (abbr. *lcs*) $C \vee D$ of two $\mathcal{EL}^{\bot}$-concept descriptions C and D is (up to equivalence) uniquely defined by the following conditions: 1. $C \sqsubseteq C \vee D$ as well as $D \sqsubseteq C \vee D$, and 2. for each $\mathcal{EL}^{\bot}$-concept description E, if $C \sqsubseteq E$ and $D \sqsubseteq E$, then $C \vee D \sqsubseteq E$. It is easy to see that the equivalence $\equiv$ is compatible with both $\sqcap$ and $\vee$. In the sequel of this document, we shall denote this bounded lattice by $\boldsymbol{\mathcal{EL}}^{\bot}(N_C, N_R) \coloneqq {}^{(\mathcal{EL}^{\bot}(N_C, N_R), \sqsubseteq)}/_{\equiv}$. For a role-depth bound $\delta \in \mathbb{N}$, $\mathcal{EL}^{\bot}(N_C, N_R)\restriction_{\delta}$ is the set of all $\mathcal{EL}^{\bot}$-concept descriptions with a role depth of at most δ, and accordingly $\boldsymbol{\mathcal{EL}}^{\bot}(N_C, N_R)\restriction_{\delta} \coloneqq {}^{(\mathcal{EL}^{\bot}(N_C, N_R)\restriction_{\delta}, \sqsubseteq)}/_{\equiv}$ symbolizes the corresponding bounded lattice of (equivalence classes of) $\mathcal{EL}^{\bot}$-concept descriptions. Note that $\boldsymbol{\mathcal{EL}}^{\bot}(N_C, N_R)\restriction_{\delta}$ is complete if the underlying signature (N_C, N_R) is finite.

3 Related Work

Baader and Distel introduced a technique for the axiomatization of general concept inclusions that are valid in a given interpretation. More specifically, they interconnected Formal Concept Analysis with Description Logics by defining so-called *induced* formal contexts such that its canonical base can directly be converted into a base of GCIs for the underlying interpretation. However, there was no possibility to include existing knowledge. For the case where a set of GCIs valid in the given interpretation is available, a solution for the computation of a *relative* base of GCIs has been proposed in [18]. However, it was not clear how to proceed in presence of an interpretation $\mathcal{I}$ and a TBox $\mathcal{T}$ where $\mathcal{I} \not\models \mathcal{T}$. This problem will be tackled in the following section.

Beforehand, we recall the notion of a model-based most specific concept description. Let $\mathcal{I}$ be an interpretation, and consider a set $X \subseteq \Delta^{\mathcal{I}}$ as well as a role-depth bound $\delta \in \mathbb{N}$. Then an $\mathcal{EL}^{\bot}$-concept description C is called a *role-depth-bounded model-based most specific concept description* (abbr. *mmsc*) of X with respect to $\mathcal{I}$ and δ if the following statements hold: 1. $\mathsf{rd}(C) \leq \delta$, 2. $X \subseteq C^{\mathcal{I}}$, and 3. for all concept descriptions

D, if $X \subseteq D^{\mathcal{I}}$, then $\emptyset \models C \sqsubseteq D$. As an immediate consequence of the definition we infer that the mmsc of X w.r.t. $\mathcal{I}$ and δ is unique up to equivalence. Hence, we may speak of *the* mmsc, and denote it by $X^{\mathcal{I}_\delta}$.

4 Most Specific Consequences with respect to a TBox

For a given $\mathcal{EL}^\perp$-TBox $\mathcal{T}$, and an $\mathcal{EL}^\perp$-concept description C, one may ask for a concept description D which is *most specific* with respect to the condition that C is subsumed by D w.r.t. $\mathcal{T}$. Such a concept description is called *most specific consequence*, or *most specific subsumer* (abbr. *mss*). Note that Distel has investigated a dual notion in [10, Chapter 7], namely that of a *minimal possible consequence*, which he utilized to constitute an algorithm for the exploration of ontologies, called *ABox Exploration*. To emphasize this duality, it is reasonable to use the name of a *minimal certain consequence*.

In this section, the notion of a most specific consequence shall be formally introduced, and necessary as well as sufficient conditions for its existence will be explored. Furthermore, we investigate its relationship to entailment with respect to a TBox. The next section then provides at least one application that utilizes most specific consequences to construct a base of GCIs that are both valid in a given interpretation as well as are entailed by a given TBox.

As an exemplary TBox, consider $\mathcal{T} \coloneqq \{A \sqsubseteq \exists r.\, A\}$. It can be readily verified that for each $n \in \mathbb{N}$, the concept description $(\exists r.)^n A$ is a *consequence* (i.e., a subsumer) of A with respect to $\mathcal{T}$. However, $(\exists r.)^{n+1} A$ is more specific than $(\exists r.)^n A$, and thus a most specific consequence of A w.r.t. $\mathcal{T}$ does not exist in the description logic $\mathcal{EL}^\perp$ with *descriptive semantics* (as introduced in Section 2). There are two solutions to tackle this problem of existence of most specific consequences. The first one is to use the extension of $\mathcal{EL}^\perp$ with *greatest fixpoint semantics*. This extension has been extensively studied in [1, 3, 10], and in particular it has been shown that this extension can handle terminological cycles (as present in the given TBox $\mathcal{T}$ above). In particular, $\mathcal{EL}^\perp_{\mathsf{gfp}}$-concept descriptions are pairs $C \coloneqq (A_C, \mathcal{T}_C)$ where $\mathcal{T}_C$ is a TBox, and A_C is a defined concept name of $\mathcal{T}_C$. We do not introduce the full machinery of the semantics of $\mathcal{EL}^\perp_{\mathsf{gfp}}$ here, but rather refer the interested reader to [1, 3]. However, it can be shown that the most specific consequence of the example above indeed exists in $\mathcal{EL}^\perp_{\mathsf{gfp}}$, and is given by $(A, \mathcal{T})$. It is straight-forward to claim that most specific consequences always exist in $\mathcal{EL}^\perp_{\mathsf{gfp}}$, but nevertheless a corresponding proof is outstanding.

Another solution for ensuring the existence of most specific consequences is to *restrict the role-depth* of the concept descriptions under consideration, as this has been done in [9] to ensure the existence of model-based most specific concept descriptions in $\mathcal{EL}^\perp$ with descriptive semantics. We introduce the following definition.

Definition 1. *Let $\mathcal{T}$ be an $\mathcal{EL}^\perp$-TBox, C an $\mathcal{EL}^\perp$-concept description, and $\delta \in \mathbb{N}$ a* role-depth bound*. Then an $\mathcal{EL}^\perp$-concept description D is called a* role-depth-bounded most specific consequence *of C with respect to $\mathcal{T}$ and δ if it satisfies the following conditions: 1. $\mathsf{rd}(D) \leq \delta$, 2. $\mathcal{T} \models C \sqsubseteq D$, and 3. for each $\mathcal{EL}^\perp$-concept description E, if $\mathsf{rd}(E) \leq \delta$ and $\mathcal{T} \models C \sqsubseteq E$, then $\emptyset \models D \sqsubseteq E$.*

Provided that such role-depth-bounded most specific consequences exist, they are unique up to equivalence, and hence we may then speak of *the* most specific consequence, and we shall denote it by $C^{\mathcal{T}_\delta}$ for a given concept description C, a TBox $\mathcal{T}$, and a role-depth bound δ. By Definition 1, $\mathcal{T} \models C \sqsubseteq C^{\mathcal{T}_\delta}$. Furthermore, from $\mathcal{T} \models C \sqsubseteq C$ we conclude that $\emptyset \models C^{\mathcal{T}_\delta} \sqsubseteq C$. In summary, $\mathcal{T} \models C \equiv C^{\mathcal{T}_\delta}$.

Lemma 2. *Role-depth-bounded most specific consequences always exist, provided that the signature is finite.*

Lemma 3. *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot$-TBox such that D has a role depth of at most δ. Then the following statements are equivalent:*

1. $\mathcal{T} \models C \sqsubseteq D$.
2. $\emptyset \models C^{\mathcal{T}_\delta} \sqsubseteq D$.
3. $\{\, E \sqsubseteq E^{\mathcal{T}_\delta} \mid E \in \mathcal{EL}^\bot(N_C, N_R)\restriction_\delta \,\} \models C \sqsubseteq D$.

If all conclusions of GCIs in $\mathcal{T}$ have role depths not exceeding δ, then furthermore the following statement is equivalent to the previous ones:

4. $\{\, E \sqsubseteq E^{\mathcal{T}_\delta} \mid \exists F \colon E \sqsubseteq F \in \mathcal{T} \,\} \models C \sqsubseteq D$.

Corollary 4. *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot$-TBox such that $\mathcal{T} \models C \sqsubseteq D$, and both C and D have role depths not exceeding δ. Then for each $\mathcal{EL}^\bot$-concept description E, if $\emptyset \models E^{\mathcal{T}_\delta} \sqsubseteq C$, then $\emptyset \models E^{\mathcal{T}_\delta} \sqsubseteq D$.*

This document does not include a method for the computation of most specific consequences, and leaves this problem open for future research. However, we have shown that their existence is guaranteed in the case of descriptive semantics when the candidate concept descriptions are restricted in their role depth. In particular, the notion defined in Definition 1 always exists. As a next step, we provide a technique that allows for the computation of a TBox from a stream of interpretations and that utilizes those most specific consequences.

Lemma 5. *The mapping $C \mapsto C^{\mathcal{T}_\delta}$ is a closure operator in the dual of $\mathcal{EL}^\bot(N_C, N_R)\restriction_\delta$.*

5 Axiomatization of General Concept Inclusions from Streams of Interpretations

Consider a setting where a stream of interpretations $\mathcal{I}_n$, $n \in \mathbb{N}$, can be observed, and furthermore for each time point $n \in \mathbb{N}$, a TBox $\mathcal{T}_n$ shall be constructed such that for each GCI $C \sqsubseteq D$, $\mathcal{T} \models C \sqsubseteq D$ if, and only if, $\mathcal{I}_k \models C \sqsubseteq D$ for all previous time points $k \leq n$. Of course, for the initial moment $n = 0$, we may simply compute $\mathcal{T}_0$ as a base of GCIs for $\mathcal{I}_0$, utilizing the approach of Baader and Distel [4, 10]. For the following moments $n \geq 1$, we may of course also construct a base of GCIs for the disjoint union of the interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$. However, since the method requires the construction of so-called *induced contexts* the size of which may be exponential in the size of the domain of the interpretation, this technique could possibly be infeasible for late time points. Furthermore, it requires the storing of all interpretations observed so far. We want to present another technique for solving the above mentioned task. Please note that this problem was already addressed in [18] for the case of $\mathcal{I}_{n+1} \models \mathcal{T}_n$. Here, we propose a solution that circumvents this rather restrictive precondition.

The infimum of $\cdot^{\mathcal{T}}$ and $\cdot^{\mathcal{I}\mathcal{I}}$ is the greatest closure operator below both $\cdot^{\mathcal{T}}$ and $\cdot^{\mathcal{I}\mathcal{I}}$. The following lemma recalls an important property of infima of clops, specifically tailored to the case of implications of concept descriptions, i.e., of general concept inclusions as they are more commonly called.

Lemma 6. *Let $\mathcal{I}$ be an interpretation, $\mathcal{T}$ an $\mathcal{EL}^\perp$-TBox, and $C \sqsubseteq D$ a general concept inclusion such that both premise and conclusion have a role-depth not exceeding δ. Then the following statements are equivalent:*

1. *$C \sqsubseteq D$ is both valid in $\mathcal{I}$ as well as entailed by $\mathcal{T}$.*
2. *$\emptyset \models \{C^{\mathcal{II}_\delta} \sqsubseteq D, C^{\mathcal{T}_\delta} \sqsubseteq D\}$.*
3. *$\emptyset \models C^{\mathcal{II}_\delta} \vee C^{\mathcal{T}_\delta} \sqsubseteq D$.*
4. *$C \sqsubseteq D$ is valid in the infimum of $C \mapsto C^{\mathcal{II}_\delta}$ and $C \mapsto C^{\mathcal{T}_\delta}$.*

As a conclusion, we infer the following incremental method for the computation of a sequence of TBoxes from a sequence of interpretations:

1. Upon availability of the first observed interpretation $\mathcal{I}_0$, compute its canonical base of GCIs, as proposed by Baader and Distel in [4, 10]. If only concept descriptions up to a certain role depth shall be considered, then the variant described by Borchmann, Distel, et al., in [9] is sufficient.
2. For each new interpretation $\mathcal{I}_{n+1}$, compute the canonical base of the infimum of the clops that are induced by the current TBox $\mathcal{T}_n$ as well as by the newly observed interpretation $\mathcal{I}_{n+1}$.

It is readily verified that – by construction – for each time point $n \in \mathbb{N}$, the TBox $\mathcal{T}_n$ entails a GCI $C \sqsubseteq D$ if, and only if, $C \sqsubseteq D$ is valid in the interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$.

6 Error-Tolerant Axiomatization of General Concept Inclusions from Interpretations

Assume that we were given an interpretation $\mathcal{I}$ as well as a TBox $\mathcal{T}$ such that $\mathcal{I}$ contains observations that may be possibly faulty due to inaccurate generation methods, and that $\mathcal{T}$ is certainly valid in the domain of interest, e.g., as it has been hand-crafted by experts. In particular, we assume that $\mathcal{I}$ is not a model of $\mathcal{T}$, i.e., that at least one domain element in $\mathcal{I}$ exists which serves as a counterexample against at least one GCI from $\mathcal{T}$. However, we are expected to axiomatize terminological knowledge from $\mathcal{I}$, which is valid in the unknown domain of interest. As a solution, we will construct the implicational base of the supremum of the clops that are induced by $\mathcal{I}$, and by $\mathcal{T}$, respectively. It is then ensured that those implications are considered which are valid for all those domain elements of $\mathcal{I}$ that respect the GCIs in $\mathcal{T}$, i.e., that we axiomatize general concept inclusions from $\mathcal{I}$ that are compatible with the axioms contained in $\mathcal{T}$. In a certain sense this yields a method for an error correction in $\mathcal{I}$ when learning GCIs. We will describe a short motivating example. Define a TBox $\mathcal{T}$ and an interpretation $\mathcal{I}$ as follows:

$$N_C := \{\mathsf{Person}, \mathsf{Car}, \mathsf{Wheel}\}, \quad N_R := \{\mathsf{child}\}$$
$$\mathcal{T} := \{\exists\, \mathsf{child}.\, \top \sqsubseteq \mathsf{Person}, \mathsf{Person} \sqcap \mathsf{Car} \sqsubseteq \bot\}$$

$\mathcal{I}$:

Consider the GCI $\mathsf{Car} \sqsubseteq \exists\, \mathsf{child}.\, \mathsf{Wheel}$. Of course, it is valid in $\mathcal{I}$, and furthermore is contained in the canonical base of $\mathcal{I}$ when applying the construction from [4, 10].

We can show that the above mentioned GCI is also valid $\mathcal{II}_\delta \curlyvee \mathcal{T}_\delta$ for $\delta \geq 1$:

$$\begin{aligned}\mathsf{Car}^{\mathcal{II}_\delta} &\equiv \mathsf{Car} \sqcap \exists\, \mathsf{child}.\, \mathsf{Wheel} \\ (\mathsf{Car} \sqcap \exists\, \mathsf{child}.\, \mathsf{Wheel})^{\mathcal{T}_\delta} &\equiv \mathsf{Car} \sqcap \exists\, \mathsf{child}.\, \mathsf{Wheel} \sqcap \mathsf{Person} \sqcap \bot \equiv \bot,\end{aligned}$$

The closure of Car with respect to the supremum is the least common subsumer of those concept descriptions that are closures of both $C \mapsto C^{\mathcal{II}_\delta}$ and $C \mapsto C^{\mathcal{T}_\delta}$, as well as are subsumed by Car. It is easy to see that this supremum closure can be computed by an exhaustive repeated application of both closure operators until a fixpoint is reached. As we have seen above, the fixpoint $\bot$ is reached after the first iteration, and hence $\bot$ is the supremum-closure.

However, the GCI is a consequence of the more specific valid GCI $\mathsf{Car} \sqsubseteq \bot$, and hence would not have been axiomatized upon construction of the canonical base. In particular, we see that d is not compatible with $\mathcal{T}$ – in contrast to the other domain elements e, f, and g. Eventually, Car is a pseudo-closure of the supremum, and hence the canonical base contains the axiom expressing the non-existence of cars.

7 Conclusion

We have defined the notion of most specific consequences with respect to TBoxes, and considered the corresponding closure operator. We have investigated the interplay of this closure operator induced by a given TBox with the closure operator induced by an interpretation – more specifically, we have shown how their infimum can be utilized for learning from streams of interpretations, and have motivated how their supremum can be used for an error-tolerant axiomatization of general concept inclusions from interpretations in the presence of a hand-crafted TBox that indicates errors in the observed interpretation.

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