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LTCS–Report

Subsumption in Finitely Valued Fuzzy $\mathcal{EL}$

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1 Introduction

Description Logics (DLs) are a family of knowledge representation formalisms that are successfully applied in many application domains. They provide the logical foundation for the Direct Semantics of the standard web ontology language OWL 2.[1] The light-weight DL $\mathcal{EL}$, underlying the OWL 2 EL profile, is of particular interest since all common reasoning problems are polynomial in this logic, and it is used in many prominent biomedical ontologies like SNOMED CT[2] and the Gene Ontology.[3] Knowledge is represented by a set of *general concept inclusions (GCIs)* like

$$\exists \mathsf{hasDisease}.\mathsf{Flu} \sqsubseteq \exists \mathsf{hasSymptom}.\mathsf{Headache} \sqcap \exists \mathsf{hasSymptom}.\mathsf{Fever} \tag{1}$$

which states that every patient with a flu must also show headache and fever as symptoms. Reasoning in $\mathcal{EL}$ is a polynomial problem [2].

An important problem for AI practical applications is to represent and reason with vague or imprecise knowledge in a formal way. Fuzzy Description Logics (FDLs) [22, 15] were introduced with this goal in mind. The main premise of fuzzy logics is the use of more than two truth degrees to allow a more fine-grained analysis of dependencies between concepts. Usually, these degrees are arranged in a total order, or *chain*, in the interval $[0, 1]$. A patient having a body temperature of 37.5 °C can have a degree of fever of 0.5, whereas a temperature of 39.2 °C may be interpreted as a fever with degree of 0.9. Considering the GCI (1), the severity of the symptoms certainly influences the severity of the disease, and thus truth degrees can be transferred between concepts. Depending on the granularity one wants to have, one can choose to allow 10 or 100 truth degrees, or even admit the whole interval $[0, 1]$. Another degree of freedom in FDLs comes from the choice of possible semantics for the logical constructors. The most general semantics are based on *triangular norms (t-norms)* that are used to interpret conjunctions. Among these, the most prominent ones are the *Gödel*, *Łukasiewicz*, and *product* t-norms. All (continuous) t-norms over chains can be expressed as combinations of these three basic ones.

Unfortunately, reasoning in many infinitely valued FDLs becomes undecidable [3, 12]. For a systematic study on this topic, see [6]. On the other hand, every finitely valued FDL that has been recently studied has not only been proved to be decidable, but even to belong to the same complexity class as the corresponding classical DL [8, 9, 10].

A question that naturally arises is whether the finitely valued fuzzy framework always yields the same computational complexity as the corresponding classical formalisms. A common opinion is that everything that can be expressed in finitely valued FDLs can be reduced to the corresponding classical DLs without any serious loss of efficiency. Indeed, although some known direct translations of finitely valued FDLs into classical DLs are exponential [4], more efficient reasoning can be achieved through direct algorithms [8]. The problem of finding a complexity gap between classical and finitely valued logics has already been considered. In [13], the authors analyze different constructors that could cause an increase in the complexity, but no specific answer is found. In [5] it is shown that the Łukasiewicz t-norm is a source of nondeterminism able to cause a significant increase in expressivity in very simple propositional languages. In this work, we build on the methods devised in [5] to show even more dramatic increases in complexity for finitely valued extensions of $\mathcal{EL}$.

The question about the computational complexity of $\mathcal{EL}$ under infinitely valued semantics has been already considered. In [7], reasoning in $\mathcal{EL}$ under semantics including the Łukasiewicz t-norm was proven co-NP-hard, but the proof does not apply to the finitely valued case. In contrast, infinitely valued Gödel semantics do not increase the complexity of reasoning [18].

[1] `http://www.w3.org/TR/owl2-overview/`
[2] `http://www.ihtsdo.org/snomed-ct/`
[3] `http://geneontology.org/`

In this work, we prove that $\mathcal{EL}$ under finitely valued semantics is ExpTime-complete whenever the Łukasiewicz t-norm is included in the semantics. This proves a dichotomy similar to one that exists for infinitely valued FDLs [6] since, for all other finitely valued chains of truth values, reasoning in fuzzy $\mathcal{EL}$ can be shown to be in PTime using the methods from [18]. The relevance of our result goes beyond the computational aspect. Indeed, this is so far the first instance of a finitely valued DL that is more complex than the same language under classical semantics. In this way, we obtain an answer to the open problem whether finitely valued FDLs and classical DLs are equally powerful, at least from a computational complexity point of view. As a side benefit, we obtain the same (ExpTime) lower bound for the complexity of infinitely valued fuzzy extensions of $\mathcal{EL}$ that use the Łukasiewicz t-norm, improving the lower bound from [7].

2 Preliminaries

Fuzzy Description Logics extend classical DLs by allowing more than two truth degrees in the semantics. We first introduce the classes of truth degrees relevant for this paper and then recall the logics $\mathcal{ELU}$ and L-$\mathcal{EL}$.

2.1 Chains of Truth Values

We are working with structures of the form $\mathsf{L} = (\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$, where

- L is subset of $[0,1]$ that contains the extreme elements 0 and 1.
- The *t-norm* $*_\mathsf{L}$ is a binary operator on L that is associative, commutative, monotone in both components, and has 1 as unit element.
- The *residuum* $\Rightarrow_\mathsf{L}$ of $*_\mathsf{L}$ is a binary operator on L that satisfies the following condition for all $x, y, z \in \mathsf{L}$: $x *_\mathsf{L} y \leqslant z$ iff $y \leqslant x \Rightarrow_\mathsf{L} z$.

An *interval* in L is a subset of the form $[a, b] := \{x \in \mathsf{L} \mid a \leqslant x \leqslant b\}$ with $a, b \in \mathsf{L}$. An *idempotent element* in L is an element x such that $x *_\mathsf{L} x = x$. For ease of presentation, we will often identify L and $(\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$ and omit the subscript L if the chain we use is clear from the context.

We consider in particular the two cases where (i) L is defined over the interval $[0,1]$ of real numbers, or (ii) L is a *finite* chain. In the former case, we always make the assumption that the operator $*_\mathsf{L}$ is *continuous* as a function from $[0,1] \times [0,1]$ to $[0,1]$. One reason for this assumption is that it ensures that the residuum is uniquely determined by the t-norm [17]. In case (ii), we similarly assume that $*_\mathsf{L}$ is *smooth*, i.e. for every $x, y, z \in \mathsf{L}$, whenever x and y are direct neighbors in L, with $x < y$, then there is no $w \in \mathsf{L}$ such that $x *_\mathsf{L} z < w < y *_\mathsf{L} z$ [19]. If $*_\mathsf{L}$ is continuous (smooth), then we call L *continuous (smooth)*.

By restricting the algebra of truth values to two elements, the classical Boolean algebra of truth and falsity is obtained: $\mathsf{B} = (\{0,1\}, *_\mathsf{B}, \Rightarrow_\mathsf{B}, 0, 1)$. Here, $*_\mathsf{B}$ and $\Rightarrow_\mathsf{B}$ are the classical conjunction and the material implication respectively.

The most interesting kinds of chains with continuous or smooth t-norms are the ones defined by the Gödel (G), Łukasiewicz ($\mathsf{Ł}$), and product (Π) t-norms. The finitely valued versions of the former two, denoted by $\mathsf{Ł}_n$ and G_n for $n \geqslant 2$, are defined over the n-element total order $0 < \frac{1}{n-1} < \cdots < \frac{n-2}{n-1} < 1$:

- The *(finite) Gödel t-norm* (or *minimum t-norm*)

$$x *_{\mathsf{G}_n} y := x *_{\mathsf{G}} y := \min\{x, y\}$$

  and its residuum

$$x \Rightarrow_{\mathsf{G}_n} y := x \Rightarrow_{\mathsf{G}} := \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

- The *(finite) Łukasiewicz t-norm* $x *_{\mathsf{Ł}_n} y := x *_{\mathsf{Ł}} y := \max\{0, x + y - 1\}$ and its residuum $x \Rightarrow_{\mathsf{Ł}_n} y := x \Rightarrow_{\mathsf{Ł}} y := \min\{1, 1 - x + y\}$.
- The *product t-norm* $x *_{\Pi} y := x \cdot y$ and its residuum

$$x \Rightarrow_{\Pi} y := \begin{cases} 1 & \text{if } x \leqslant y, \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

A finite-valued version of the product t-norm cannot exist since the chain L needs to be closed under the t-norm, but for any $x \in (0, 1)$, the set $\{x^m \mid m \geqslant 0\}$ is infinite.

The following easy observations about the introduced operators will be useful in the proofs. For all $x, y \in \mathsf{L}$ and $T \subseteq \mathsf{L}$, it holds that

- $x *_{\mathsf{L}} y = 1$ iff both $x = 1$ and $y = 1$;
- $\sup T = 1$ iff $1 \in T$;
- $x \Rightarrow_{\mathsf{L}} y = 1$ iff $x \leqslant y$;
- $x \Rightarrow_{\mathsf{L}} y \geqslant y$;
- $x \Rightarrow_{\mathsf{Ł}} 0 = 1 - x$;
- if $\mathsf{L} = \mathsf{Ł}_n$, then $x *_{\mathsf{Ł}_n} y \geqslant \frac{n-2}{n-1}$ iff either $x = 1$ or $y = 1$;
- if $\mathsf{L} = \mathsf{Ł}_n$ and $x < 1$, then $x *_{\mathsf{Ł}_n} \overset{m}{\ldots} *_{\mathsf{Ł}_n} x = 0$ for all $m \geqslant n - 1$;
- if $\mathsf{L} = \mathsf{G}_n$, then $x *_{\mathsf{G}_n} \overset{m}{\ldots} *_{\mathsf{G}_n} x = x$ for all $m \geqslant 1$.

The t-norms defined so far can be used to build all other continuous t-norms over $[0, 1]$, and all smooth t-norms over finite chains, using the following construction.

Definition 2.1. *Let L be a chain, $(\mathsf{L}_i)_{i \in I}$ be a family of chains, and $(\lambda_i)_{i \in I}$ be isomorphisms between intervals $[a_i, b_i] \subseteq \mathsf{L}$ and L_i such that the intersection of any two intervals contains at most one element. L is the* ordinal sum *of the family $(\mathsf{L}_i, \lambda_i)_{i \in I}$ if, for all $x, y \in \mathsf{L}$,*

$$x *_{\mathsf{L}} y = \begin{cases} \lambda_i^{-1}\big(\lambda_i(x) *_{\mathsf{L}_i} \lambda_i(y)\big) & \textit{if } x, y \in (a_i, b_i), \\ \min\{x, y\} & \textit{otherwise.} \end{cases}$$

Every chain over $[0, 1]$ with a continuous t-norm is isomorphic to an ordinal sum of infinite-valued Łukasiewicz and product chains [14, 21]. Similarly, every finite chain with a smooth t-norm is an ordinal sum of chains of the form $\mathsf{Ł}_n$ with $n \geqslant 3$ [20]. All elements that are not contained strictly within one such Łukasiewicz or product component are idempotent and can be thought of as belonging to a (finite) Gödel chain. We say that a (finite or infinite) chain *contains* the Łukasiewicz t-norm if its ordinal sum representation contains at least one Łukasiewicz component; similarly, it *starts with* the Łukasiewicz t-norm if it contains a Łukasiewicz component in an interval $[0, b]$. Note that every chain that contains the Łukasiewicz t-norm can be represented as the ordinal sum of an arbitrary chain L_1 and another chain L_2 that starts with the Łukasiewicz t-norm.

Another way to view these characterizations is to observe that every smooth finite chain is either a Gödel chain or contains at least one finite Łukasiewicz component, and every continuous chain over $[0,1]$ is either a Gödel chain or contains at least one Łukasiewicz or product component. We will use this insight later in our hardness proofs.

2.2 $\mathcal{ELU}$ and L-$\mathcal{EL}$

A *description signature* is a tuple $(\mathsf{N_C}, \mathsf{N_R})$, where $\mathsf{N_C} = \{A, B, \dots\}$ is a countable set of *atomic concepts* or *concept names*, and $\mathsf{N_R} = \{r, s, \dots\}$ is a countable set of *atomic roles* or *role names*. *Complex concepts* in the FDL language L-$\mathcal{EL}$ are built inductively from atomic concepts and roles by means of the following concept constructors:

$$\begin{array}{llll} C, D & \longrightarrow & \top & \text{top} \\ & & A & \text{atomic concept} \\ & & C \sqcap D & \text{conjunction} \\ & & \exists r.C & \text{existential restriction} \end{array}$$

where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. $\mathcal{ELU}$ concepts are formed by adding the option $C \sqcup D$ to the previous rule. In the rest of the paper we will use the abbreviation C^m, $m \geqslant 1$, for the m-ary conjunction; i.e. $C^1 := C$ and $C^{m+1} := C^m \sqcap C$.

There is often no difference between the syntax of classical and fuzzy languages. The differences between both frameworks begin when the *semantics* of concepts and roles is introduced.

2.3 Semantics

In this section we introduce the semantics of concepts, which is what differentiates the many-valued framework from the classical one. Even though, as stressed in Section 2.1, it is enough to restrict the semantics to the two element chain B to obtain the classical semantics, we prefer to define both kinds of semantics to aid understanding (and indeed, writing down) the proofs.

2.3.1 Fuzzy Semantics of L-$\mathcal{EL}$.

Given an arbitrary but fixed chain $\mathsf{L} = (\mathsf{L}, *, \Rightarrow)$, an L-*interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of:

- a nonempty (classical) set $\Delta^{\mathcal{I}}$ (called *domain*), and
- a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns
  - to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$, and
  - to each role name $r \in \mathsf{N_R}$ a fuzzy relation $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$.

The semantics of complex concepts is a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$ inductively defined as follows:

$$\begin{array}{rcl} \top^{\mathcal{I}}(x) & := & 1, \\ (C \sqcap D)^{\mathcal{I}}(x) & := & C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x), \\ (\exists r.C)^{\mathcal{I}}(x) & := & \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y). \end{array}$$

2.3.2 Classical semantics of $\mathcal{ELU}$.

In the classical framework an interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of:

- a nonempty (classical) set $\Delta^{\mathcal{I}}$ (called *domain*), and
- an *interpretation function* $\cdot^{\mathcal{I}}$ that assigns:
  - to each concept name $A \in \mathsf{N_C}$ a crisp set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and
  - to each role name $r \in \mathsf{N_R}$ a crisp relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$,

This function is extended to $\mathcal{ELU}$ concepts by setting

$$\begin{array}{rcl} \top^{\mathcal{I}} & := & \Delta^{\mathcal{I}}, \\ (C \sqcap D)^{\mathcal{I}} & := & C^{\mathcal{I}} \cap D^{\mathcal{I}}, \\ (C \sqcup D)^{\mathcal{I}} & := & C^{\mathcal{I}} \cup D^{\mathcal{I}}, \\ (\exists r.C)^{\mathcal{I}} & := & \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} \colon (x,y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}. \end{array}$$

By replacing the relation $\in$ by its characteristic function $\chi_{\in} \colon \Delta^{\mathcal{I}} \to \{0,1\}$, we obtain a special case of fuzzy semantics. Whenever L is one of the specific chains introduced in the previous section, e.g. $\mathsf{Ł}_n$, then we denote the resulting logic by $\mathsf{Ł}_n$-$\mathcal{EL}$ instead of L-$\mathcal{EL}$.

In infinite chains, it interpretations are often restricted to be *witnessed* [15], which means that for every existential restriction $\exists r.C$ and $x \in \Delta^{\mathcal{I}}$ there is an element $y \in \Delta^{\mathcal{I}}$ that realizes the supremum in the semantics of $\exists r.C$ at x, i.e. we have $(\exists r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)$. Under finite-valued (and classical) semantics this property is always satisfied, and it corresponds to the intuition that an existential restriction actually forces the existence of a single individual that satisfies it, instead of infinitely many that only satisfy the restriction in the limit. We also adopt this restriction in the following.

2.4 Axioms and Reasoning Tasks

In DLs, the domain knowledge is represented by axioms that restrict the class of interpretations under consideration. In the fuzzy framework, these axioms are assigned a minimum degree of truth to which they must be satisfied. Graded *general concept inclusions (GCIs)* are expressions of the form $\langle C \sqsubseteq D \geqslant \ell \rangle$, where $\ell \in \mathsf{L}$. The L-interpretation $\mathcal{I}$ *satisfies* this axiom if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geqslant \ell$ holds for all $x \in \Delta^{\mathcal{I}}$. As usual, a TBox is a finite set of GCIs, and an L-interpretation $\mathcal{I}$ *satisfies* a TBox if it satisfies every axiom in it.

We consider the problem of deciding whether a concept C is *ℓ-subsumed* by another concept D with respect to a TBox $\mathcal{T}$ for a value $\ell \in \mathsf{L} \setminus \{0\}$. That is, whether every L-interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ also satisfies $\langle C \sqsubseteq D \geqslant \ell \rangle$. In the classical case, we talk simply about *subsumption*, and for $\ell = 1$ the problem simplifies to the question whether $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in all interpretations $\mathcal{I}$ that satisfy $\mathcal{T}$.

We first show that this problem is ExpTime-hard for all finite Łukasiewicz chains with at least three elements. We then use this result in Section 4 to show ExpTime-hardness under any finite chain with only idempotent elements. A matching ExpTime upper bound was shown in [8]. The subsumption problem for G_n-$\mathcal{EL}$ can be shown to be in PTime using the ideas from [18]. In Section 5, we adapt the reduction to show ExpTime-hardness of $\mathsf{Ł}$-$\mathcal{EL}$, and even for every continuous chain over $[0,1]$ containing a Łukasiewicz component (see Definition 2.1).

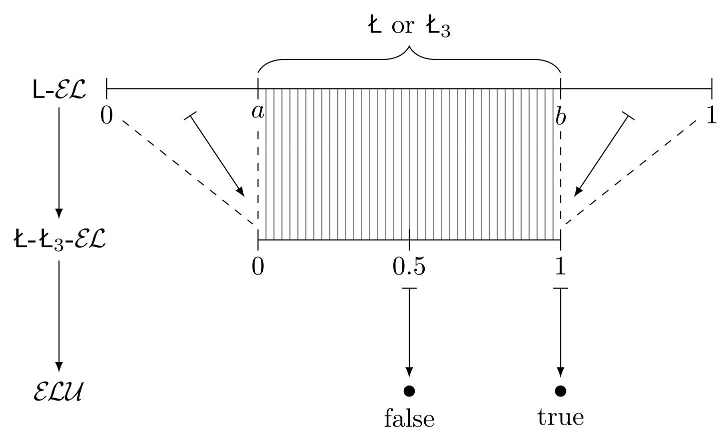

Figure 1: Illustration of the reductions

The idea behind the reductions is illustrated in Figure 1 for chains L containing either an Ł$_3$-component or an infinitely valued Ł-component. To simulate the semantics of $\mathcal{ELU}$, the values 0.5 and 1 in Ł$_3$-$\mathcal{EL}$ (or Ł-$\mathcal{EL}$) are used to simulate the truth values *false* and *true*, respectively. The chain Ł$_3$ (Ł) is then embedded into L as depicted.

3 Finite Łukasiewicz Chains

In this section, we reduce the subsumption problem of the classical DL $\mathcal{ELU}$ to the subsumption problem of Ł$_n$-$\mathcal{EL}$, where $n \geqslant 3$. Concept subsumption in $\mathcal{ELU}$ is an ExpTime-complete problem [2]. This reduction shows that the subsumption problem is ExpTime-hard already for Ł$_3$-$\mathcal{EL}$; i.e., for Łukasiewicz chains containing three truth degrees. For ease of presentation, we omit the subscript Ł$_n$ from $*$ and $\Rightarrow$ in this section.

First note that it suffices to consider subsumption problems between two *concept names* since an $\mathcal{ELU}$ concept C is subsumed by another $\mathcal{ELU}$ concept D w.r.t. an $\mathcal{ELU}$ TBox $\mathcal{T}$ iff the new concept name A is subsumed by the new concept name B w.r.t. $\mathcal{T} \cup \{\langle A \sqsubseteq C\rangle, \langle D \sqsubseteq B\rangle\}$ [2].

Furthermore, we can restrict our considerations to $\mathcal{ELU}$ TBoxes in *normal form*, which only contain axioms of the following forms:

$$\begin{aligned} A_1 \sqcap A_2 &\sqsubseteq B \\ \exists r.A &\sqsubseteq B \\ A &\sqsubseteq \exists r.B \\ A &\sqsubseteq B_1 \sqcup B_2 \end{aligned}$$

where A, A_1, A_2, B, B_1 and B_2 are concept names or $\top$. It was shown in [2] that every $\mathcal{ELU}$ TBox can be polynomially reduced to an equivalent one in normal form.

The main idea of our reduction is to simulate a classical concept name in Ł$_n$-$\mathcal{EL}$ by considering all values below $\frac{n-2}{n-1}$ to be equivalent to 0, and thus only the value 1 can be used to express that a domain element belongs to the concept name. We can then express a classical disjunction of the form $B_1 \sqcup B_2$ by restricting the value of the fuzzy conjunction $B_1 \sqcap B_2$ to be $\geqslant \frac{n-2}{n-1}$ since the latter is equivalent to B_1 *or* B_2 having value 1. Furthermore, we reformulate classical subsumption between C and D as 1-subsumption between C^{n-1} and D^{n-1} since the latter two concepts can take only the values 0 or 1.

More formally, let $n \geqslant 3$, $\mathcal{T}$ be an $\mathcal{ELU}$ TBox in normal form, and C, D be atomic concepts. We construct an Ł_n-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$ such that C is subsumed by D w.r.t. $\mathcal{T}$ if and only if C^{n-1} is subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$. Since $\mathcal{T}$ is in normal form, we can define the reduction ρ_n for each of the four kinds of axioms listed above:

$$\begin{aligned}
\rho_n(A_1 \sqcap A_2 \sqsubseteq B) &:= \langle A_1 \sqcap A_2 \sqsubseteq B \geqslant 1\rangle \\
\rho_n(\exists r.A \sqsubseteq B) &:= \langle \exists r.A \sqsubseteq B \geqslant 1\rangle \\
\rho_n(A \sqsubseteq \exists r.B) &:= \langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \tfrac{1}{n-1}\rangle \\
\rho_n(A \sqsubseteq B_1 \sqcup B_2) &:= \langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \tfrac{n-2}{n-1}\rangle
\end{aligned}$$

Finally, $\rho_n(\mathcal{T}) := \{\rho_n(\alpha) \mid \alpha \in \mathcal{T}\}$. Notice that $\rho_n(\mathcal{T})$ has as many axioms as $\mathcal{T}$, and the size of each axiom is increased by a factor of at most n. Hence, the translation $\rho_n(\mathcal{T})$ can be performed in polynomial time. We show that this translation satisfies the properties described above.

3.1 Soundness

In this subsection we prove that if C is subsumed by D with respect to $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} with respect to the Ł_n-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$. In order to achieve this result, for any Ł_n-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ we define the crisp interpretation $\mathcal{I}_{cr} = (\Delta^{\mathcal{I}_{cr}}, \cdot^{\mathcal{I}_{cr}})$, where:

- $\Delta^{\mathcal{I}_{cr}} := \Delta^{\mathcal{I}}$,
- $x \in A^{\mathcal{I}_{cr}}$ iff $A^{\mathcal{I}}(x) = 1$ for $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$,
- $(x, y) \in r^{\mathcal{I}_{cr}}$ iff $r^{\mathcal{I}}(x, y) = 1$ for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Note that also $x \in \top^{\mathcal{I}_{cr}}$ iff $\top^{\mathcal{I}}(x) = 1$ for all $x \in \Delta^{\mathcal{I}}$. Thus, in the following proofs we can treat $\top$ as a concept name.

Before proving soundness of $\rho_n(\cdot)$ we need to prove that the translation $\cdot_{cr}$ preserves satisfaction of our TBoxes.

Lemma 3.1. *Let $\mathcal{I}$ be an Ł_n-interpretation that satisfies $\rho_n(\mathcal{T})$. Then $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be an Ł_n-interpretation that satisfies $\rho_n(\mathcal{T})$. We will prove case-by-case that $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$.

- Consider an axiom of the form $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$ and $x \in A_1^{\mathcal{I}_{cr}} \cap A_2^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $A_1^{\mathcal{I}}(x) = 1$ and $A_2^{\mathcal{I}}(x) = 1$. Hence $(A_1 \sqcap A_2)^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $B^{\mathcal{I}}(x) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we get $x \in B^{\mathcal{I}_{cr}}$.
- Consider an axiom of the form $\exists r.A \sqsubseteq B \in \mathcal{T}$ and $x \in (\exists r.A)^{\mathcal{I}_{cr}}$. Hence there exists an element $y \in \Delta^{\mathcal{I}_{cr}}$ such that $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $r^{\mathcal{I}}(x, y) = 1$ and $A^{\mathcal{I}}(y) = 1$. Hence $\sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * A^{\mathcal{I}}(z) = r^{\mathcal{I}}(x, y) * A^{\mathcal{I}}(y) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, we get $B^{\mathcal{I}}(x) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we conclude that $x \in B^{\mathcal{I}_{cr}}$.
- Consider an axiom of the form $A \sqsubseteq \exists r.B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $((\exists r.B)^{n-1})^{\mathcal{I}}(x) \geqslant \frac{1}{n-1}$, and since $((\exists r.B)^{n-1})^{\mathcal{I}}(x) \in \{0, 1\}$, we obtain $((\exists r.B)^{n-1})^{\mathcal{I}}(x) = 1$, that is,

$$1 = (\exists r.B)^{\mathcal{I}}(x) = \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * B^{\mathcal{I}}(z).$$

Therefore there exists $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x,y) = 1$ and $B^{\mathcal{I}}(y) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we have $(x,y) \in r^{\mathcal{I}_{cr}}$ and $y \in B^{\mathcal{I}_{cr}}$, and hence $x \in (\exists r.B)^{\mathcal{I}_{cr}}$.

- Consider an axiom of the form $A \sqsubseteq B_1 \sqcup B_2 \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $(B_1 \sqcap B_2)^{\mathcal{I}}(x) \geqslant \frac{n-2}{n-1}$. Hence either $B_1^{\mathcal{I}}(x) = 1$ or $B_2^{\mathcal{I}}(x) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we have that either $x \in B_1^{\mathcal{I}_{cr}}$ or $x \in B_2^{\mathcal{I}_{cr}}$. □

Now we are ready to prove the following proposition.

Proposition 3.2. *If C is subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be an $Ł_n$-interpretation satisfying $\rho_n(\mathcal{T})$ and $x \in \Delta^{\mathcal{I}}$ such that $(C^{n-1})^{\mathcal{I}}(x) > 0$. Hence $(C^{n-1})^{\mathcal{I}}(x) = 1$ and thus $C^{\mathcal{I}}(x) = 1$. By the definition of $\mathcal{I}_{cr}$, we have $x \in C^{\mathcal{I}_{cr}}$. By Lemma 3.1 we know that $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$, and thus we get $x \in D^{\mathcal{I}_{cr}}$ by assumption. Again by the definition of $\mathcal{I}_{cr}$, we obtain $D^{\mathcal{I}}(x) = 1$ and therefore $(D^{n-1})^{\mathcal{I}}(x) = 1$. Hence $(C^{n-1})^{\mathcal{I}}(x) \Rightarrow (D^{n-1})^{\mathcal{I}}(x) = 1$, that is, C^{n-1} is 1-subsumed by D^{n-1} with respect to $\rho_n(\mathcal{T})$. □

3.2 Completeness

In this subsection we prove that if C is not subsumed by D with respect to $\mathcal{T}$, then C^{n-1} is not 1-subsumed by D^{n-1} with respect to the $Ł_n$-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$. In order to achieve this result, we define for any crisp interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an $Ł_n$-interpretation $\mathcal{I}_n = (\Delta^{\mathcal{I}_n}, \cdot^{\mathcal{I}_n})$, where:

- $\Delta^{\mathcal{I}_n} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_n}(x) := 1$ if $x \in A^{\mathcal{I}}$ and $A^{\mathcal{I}_n}(x) := \frac{n-2}{n-1}$ otherwise, for every $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$,
- $r^{\mathcal{I}_n}(x,y) := 1$ if $(x,y) \in r^{\mathcal{I}}$ and $r^{\mathcal{I}_n}(x,y) := \frac{n-2}{n-1}$ otherwise, for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Again, $\top$ behaves exactly like the concept names since $\top^{\mathcal{I}_n}(x)$ is always 1.

Before proving completeness of $\rho_n(\cdot)$ we need to prove that the translation $\cdot_n$ preserves satisfiability of TBoxes. This will be proved in the following lemma.

Lemma 3.3. *If a classical interpretation $\mathcal{I}$ satisfies $\mathcal{T}$, then $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be a crisp interpretation that satisfies $\mathcal{T}$. We will prove case-by-case that $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$.

- Consider an axiom of the form $\langle A_1 \sqcap A_2 \sqsubseteq B \geqslant 1 \rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) = 1$, then both $A_1^{\mathcal{I}_n}(x) = 1$ and $A_2^{\mathcal{I}_n}(x) = 1$. By the definition of $\mathcal{I}_n$, we have that $x \in A_1^{\mathcal{I}} \cap A_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this yields $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we get $B^{\mathcal{I}_n}(x) = 1$.

  In the case that $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) < 1$, we have $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) \leqslant \frac{n-2}{n-1} \leqslant B^{\mathcal{I}_n}(x)$ by the definition of $\mathcal{I}_n$, and thus also $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) \Rightarrow B^{\mathcal{I}_n}(x) = 1$.

- Consider an axiom of the form $\langle \exists r.A \sqsubseteq B \geqslant 1\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $(\exists r.A)^{\mathcal{I}_n}(x) = 1$, then $\sup_{z\in\Delta^{\mathcal{I}_n}} r^{\mathcal{I}_n}(x,z) * A^{\mathcal{I}_n}(z) = 1$. This means that there exists $y \in \Delta^{\mathcal{I}_n}$ such that $r^{\mathcal{I}_n}(x,y) = 1$ and $A^{\mathcal{I}_n}(y) = 1$. By the definition of $\mathcal{I}_n$, we know that $(x,y) \in r^{\mathcal{I}}$ and $y \in A^{\mathcal{I}}$. Hence $x \in (\exists r.A)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we have that $B^{\mathcal{I}_n} = 1$.

  Otherwise, we have $(\exists r.A)^{\mathcal{I}_n}(x) \Rightarrow B^{\mathcal{I}_n}(x) = 1$ as in the previous case.

- Consider an axiom of the form $\langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \frac{1}{n-1}\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = 0$, then

$$1 > (\exists r.B)^{\mathcal{I}_n}(x) = \sup_{z\in\Delta^{\mathcal{I}_n}} r^{\mathcal{I}_n}(x,z) * B^{\mathcal{I}_n}(z).$$

  Therefore every $y \in \Delta^{\mathcal{I}_n}$ must satisfy either $r^{\mathcal{I}_n}(x,y) < 1$ or $B^{\mathcal{I}_n}(y) < 1$. By the definition of $\mathcal{I}_n$, for all $y \in \Delta^{\mathcal{I}}$ we have either $(x,y) \notin r^{\mathcal{I}}$ or $y \notin B^{\mathcal{I}}$, and hence $x \notin (\exists r.B)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we have $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Hence $A^{\mathcal{I}_n}(x) \Rightarrow ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = \frac{1}{n-1}$.

  In the case that $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) > 0$, we also get

$$A^{\mathcal{I}_n}(x) \Rightarrow ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) \geqslant ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) \geqslant \tfrac{1}{n-1}.$$

- Consider an axiom of the form $\langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \frac{n-2}{n-1}\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) < \frac{n-2}{n-1}$, then $B_1^{\mathcal{I}_n}(x) = B_2^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. By the definition of $\mathcal{I}_n$, we have that $x \notin B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this implies that $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we have that $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Since by the definition of $\mathcal{I}_n$ and supposition we have $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-3}{n-1}$, we can conclude that $A^{\mathcal{I}_n}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$.

  In the case that $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \frac{n-2}{n-1}$, we also have

$$A^{\mathcal{I}_n}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \tfrac{n-2}{n-1}.$$

$\square$

Now we are ready to prove the following proposition.

Proposition 3.4. *If C is not subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is not 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be a crisp interpretation satisfying $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. By Lemma 3.3, we know that $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$. Moreover, by the definition of $\mathcal{I}_n$, we have $C^{\mathcal{I}_n}(x) = 1$ and $D^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Hence $(C^{n-1})^{\mathcal{I}_n}(x) = 1$ and $(D^{n-1})^{\mathcal{I}_n}(x) = 0$, and therefore $(C^{n-1})^{\mathcal{I}_n}(x) \Rightarrow (D^{n-1})^{\mathcal{I}_n}(x) = 0 < 1$. $\square$

We thus have the following.

Theorem 3.5. *For any $n \geqslant 3$, deciding ℓ-subsumption with respect to a TBox in Ł$_n$-$\mathcal{EL}$ is* ExpTime*-complete.*

Proof. The result follows from the above reduction and the fact that the subsumption problem with respect to a TBox for the language $\mathcal{ELU}$ is ExpTime-hard [2]. The ExpTime upper bound was shown in [8] for the more expressive language Ł$_n$-$\mathcal{ALC}$. $\square$

4 Arbitrary Finite Chains

We now show that the above hardness result can be transferred to almost all logics of the form L-$\mathcal{EL}$ where L is a finite chain. The exception of course being the finite chains using the minimum as t-norm—this case can be shown to be tractable as in [18].

As detailed in Section 2, any chain L that is not of this form must contain a finite Łukasiewicz chain in an interval $[a, b]$ with at least three elements. This is the basis of our reduction to the result from the previous section. More formally, we reduce the subsumption problem in Ł_n-$\mathcal{EL}$, where n is the cardinality of $[a, b]$, to the subsumption problem in L-$\mathcal{EL}$.

In the following, let $\mathcal{T}$ be a TBox in Ł_n-$\mathcal{EL}$, $\ell \in \text{Ł}_n \setminus \{0\}$, and A, B two concept names for which we want to check whether A is ℓ-subsumed by B w.r.t. $\mathcal{T}$. We extend the bijection $\lambda\colon [a, b] \to \text{Ł}_n$ as follows to the whole chain L:

- $\lambda(x) := 0$ if $x < a$ and
- $\lambda(x) := 1$ if $x > b$.

We also make use of the inverse $\lambda^{-1}\colon \text{Ł}_n \to 2^{\mathsf{L}}$ of this function, for which we in particular have $\lambda^{-1}(0) = [0, a]$ and $\lambda^{-1}(1) = [b, 1]$. When we sometimes treat $\lambda^{-1}(x)$ as a single value, we implicitly refer to the original bijection $\lambda^{-1}\colon \text{Ł}_n \to [a, b]$. A useful property of λ and λ^{-1} is the compatibility with all relevant operations (at least in the interval $[a, 1]$), as shown in the following two lemmata.

Lemma 4.1. *For all $p, q \in \mathsf{L}$, we have*

- $\lambda(p *_{\mathsf{L}} q) = \lambda(p) *_{\text{Ł}_n} \lambda(q)$, *and*
- *if $q \geqslant a$, then $\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q)$.*

Proof. If both $p > b$ and $q > b$, then we have $\lambda(p) = \lambda(q) = 1$ and $p *_{\mathsf{L}} q \geqslant b$, and thus $\lambda(p *_{\mathsf{L}} q) = 1 = 1 *_{\text{Ł}_n} 1 = \lambda(p) *_{\text{Ł}_n} \lambda(q)$. If either $p < a$ or $q < a$, then $\lambda(p) = 0$ or $\lambda(q) = 0$, respectively. Since then also $p *_{\mathsf{L}} q < a$, we obtain $\lambda(p *_{\mathsf{L}} q) = 0 = \lambda(p) *_{\text{Ł}_n} \lambda(q)$. If neither of these two cases applies, then we have $p *_{\mathsf{L}} q \in [a, b]$ and $\lambda(p *_{\mathsf{L}} q) = \lambda(p) *_{\text{Ł}_n} \lambda(q)$ since L contains Ł_n in $[a, b]$.

For the second claim, we consider the following cases.

- If $p \leqslant q$, then by the monotonicity of λ we get $\lambda(p) \leqslant \lambda(q)$, and thus
$$\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(1) = 1 = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q).$$
- If $b \geqslant p > q \geqslant a$, then the claim follows directly from the fact that L contains Ł_n in $[a, b]$.
- If $p \geqslant b \geqslant q \geqslant a$ and $p > q$, then $\lambda(p) = 1$ and $p \Rightarrow_{\mathsf{L}} q = q \geqslant a$, and thus $\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(q) = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q)$.
- Finally, if $p > q \geqslant b$, then $p \Rightarrow_{\mathsf{L}} q \geqslant q \geqslant b$, and hence $\lambda(p) = \lambda(q) = 1$ and $\lambda(p \Rightarrow_{\mathsf{L}} q) = 1 = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q)$. ∎

Lemma 4.2. *For all $p, q \in \text{Ł}_n$, $p' \in \lambda^{-1}(p) \cap [a, 1]$, and $q' \in \lambda^{-1}(q) \cap [a, 1]$, we have*

- $p' *_{\mathsf{L}} q' \in \lambda^{-1}(p *_{\text{Ł}_n} q) \cap [a, 1]$, *and*
- $p' \Rightarrow_{\mathsf{L}} q' \in \lambda^{-1}(p \Rightarrow_{\text{Ł}_n} q) \cap [a, 1]$.

Proof. If $p < 1$ or $q < 1$, then we have $p' = \lambda^{-1}(p)$ or $q' = \lambda^{-1}(q)$, respectively. Furthermore, we know that $p *_{\text{Ł}_n} q < 1$ and $\lambda^{-1}(p *_{\text{Ł}_n} q) \cap [a,1]$ contains a single element. Since L contains Ł_n in $[a,b]$ and all elements above b act as neutral elements for the elements in $[a,b]$ w.r.t. $*_{\mathsf{L}}$, we have $p' *_{\mathsf{L}} q' = \lambda^{-1}(p *_{\text{Ł}_n} q) \cap [a,1]$. In the case that $p = q = 1$, we have $p' \in [b,1]$ and $q' \in [b,1]$, and therefore also $p' *_{\mathsf{L}} q' \in [b,1] = \lambda^{-1}(1) = \lambda^{-1}(p *_{\text{Ł}_n} q)$.

For the second claim, we again make a case analysis on p and q.

- If $p = q = 1$, then both p' and q' are contained in $[b,1]$. By the properties of ordinal sums, we also have $p' \Rightarrow_{\mathsf{L}} q' \in [b,1] = \lambda^{-1}(1) = \lambda^{-1}(p \Rightarrow_{\text{Ł}_n} q)$.
- If $p \leqslant q$, but not $p = q = 1$, then we know that $p' \leqslant q'$ by the monotonicity of λ^{-1} and the fact that $p' < b$. Thus, $p' \Rightarrow_{\mathsf{L}} q' = 1 \in \lambda^{-1}(1) = \lambda^{-1}(p \Rightarrow_{\text{Ł}_n} q)$.
- If $1 = p > q$, then $p' \Rightarrow_{\mathsf{L}} q' = q' \in \lambda^{-1}(q) \cap [a,1] = \lambda^{-1}(p \Rightarrow_{\text{Ł}_n} q) \cap [a,1]$.
- Finally, if $1 > p > q$, then the claim follows directly from the fact that L contains Ł_n in $[a,b]$. ☐

We now define the new TBox $\mathcal{T}'$ in L-$\mathcal{EL}$ as follows:

$$\mathcal{T}' := \{\langle C \sqsubseteq D \geqslant \lambda^{-1}(p)\rangle, \langle \top \sqsubseteq D \geqslant a\rangle \mid \langle C \sqsubseteq D \geqslant p\rangle \in \mathcal{T}\} \cup \{\langle \top \sqsubseteq B \geqslant a\rangle\}.$$

Recall that B is one of the concept names for which we want to check subsumption in Ł_n-$\mathcal{EL}$.

4.1 Soundness

We first prove that if A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is also ℓ-subsumed by B w.r.t. $\mathcal{T}$. For this purpose, we consider an Ł_n-interpretation $\mathcal{I}$ and define an L-interpretation $\mathcal{I}_L$ as follows:

- $\Delta^{\mathcal{I}_L} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_L}(x) := \lambda^{-1}(A^{\mathcal{I}}(x))$ for all $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, and
- $r^{\mathcal{I}_L}(x,y) := \lambda^{-1}(r^{\mathcal{I}}(x,y))$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Lemma 4.3. *If $\mathcal{I}$ is an Ł_n-model of $\mathcal{T}$, then $\mathcal{I}_L$ is an L-model of $\mathcal{T}'$.*

Proof. The axioms of the form $\langle \top \sqsubseteq A \geqslant a\rangle$ for $A \in \mathsf{N_C}$ are satisfied by the definition of $\mathcal{I}_L$. For the remaining claim, we show that $C^{\mathcal{I}_L}(x) \in \lambda^{-1}(C^{\mathcal{I}}(x)) \cap [a,1]$ holds for all concepts C and $x \in \Delta^{\mathcal{I}}$ by induction on the structure of C. For all concept names, this holds by the definition of $\mathcal{I}_L$, and for and conjunctions, this is a consequence of Lemma 4.2. We also have $\top^{\mathcal{I}_L}(x) = 1 \in \lambda^{-1}(\top^{\mathcal{I}}(x)) \cap [a,1]$.

It remains to show the claim for an existential restriction $\exists r.C$, assuming that it already holds for C. Again by Lemma 4.2 and the definition of $\mathcal{I}_L$, we know that for all $y \in \Delta^{\mathcal{I}}$ we have $r^{\mathcal{I}_L}(x,y) *_{\mathsf{L}} C^{\mathcal{I}_L}(y) \in \lambda^{-1}(r^{\mathcal{I}}(x,y) *_{\text{Ł}_n} C^{\mathcal{I}}(y)) \cap [a,1]$. Since L is finite and $(\exists r.C)^{\mathcal{I}_L}(x)$ is the supremum of all these values, it is an element of $[b,1]$ iff one of the values $r^{\mathcal{I}}(x,y) *_{\text{Ł}_n} C^{\mathcal{I}}(y)$ is 1, and then

$$(\exists r.C)^{\mathcal{I}_L}(x) \in [b,1] = \lambda^{-1}(1) = \lambda^{-1}((\exists r.C)^{\mathcal{I}}(x)).$$

Otherwise, none of these values is 1 and we get

$$(\exists r.C)^{\mathcal{I}_L}(x) = \lambda^{-1}\Big(\sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) *_{\text{Ł}_n} C^{\mathcal{I}}(y)\Big) = \lambda^{-1}((\exists r.C)^{\mathcal{I}}(x)) \in [a,b)$$

by the monotonicity of λ^{-1} when restricted to $[a,b]$. This concludes the proof of the claim.

The claim immediately shows that the axioms of the form $\langle \top \sqsubseteq D \geqslant a\rangle$ in $\mathcal{T}'$ are satisfied by $\mathcal{I}_L$. Consider now an axiom of the form $\langle C \sqsubseteq D \geqslant \lambda^{-1}(p)\rangle$ in $\mathcal{T}'$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we have $C^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} D^{\mathcal{I}}(x) \geqslant p$ for all $x \in \Delta^{\mathcal{I}}$, and thus we get

$$C^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} D^{\mathcal{I}_L}(x) \in \lambda^{-1}(C^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} D^{\mathcal{I}}(x)) \subseteq [\lambda^{-1}(p), 1]$$

by the above claim, Lemma 4.2, and monotonicity of λ^{-1}. ☐

Lemma 4.4. *If A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is ℓ-subsumed by B w.r.t. $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be an Ł_n-model of $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$ such that $A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x) < \ell$. By Lemma 4.3, $\mathcal{I}_L$ is an L-model of $\mathcal{T}'$. By the definition of $\mathcal{I}_L$, we know that both $A^{\mathcal{I}_L}(x)$ and $B^{\mathcal{I}_L}(x)$ satisfy the preconditions of Lemma 4.2. This yields that

$$A^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}_L}(x) \in \lambda^{-1}(A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x)) \cap [a,1].$$

By assumption, we know that the latter set cannot be $[b,1]$, and thus it must be a singleton. By the strict monotonicity of λ^{-1} when restricted to $[a,b]$, we conclude that

$$A^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}_L}(x) = \lambda^{-1}(A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x)) < \lambda^{-1}(\ell).$$ ☐

4.2 Completeness

We now start with an L-interpretation $\mathcal{I}$ and construct an Ł_n-interpretation $\mathcal{I}_n$ as follows:

- $\Delta^{\mathcal{I}_n} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_n}(x) := \lambda(A^{\mathcal{I}}(x))$ for all $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, and
- $r^{\mathcal{I}_n}(x,y) := \lambda(r^{\mathcal{I}}(x,y))$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Lemma 4.5. *If $\mathcal{I}$ is an L-model of $\mathcal{T}'$, then $\mathcal{I}_n$ is an Ł_n-model of $\mathcal{T}$.*

Proof. We first show the auxiliary claim that $C^{\mathcal{I}_n}(x) = \lambda(C^{\mathcal{I}}(x))$ holds for all concepts C and $x \in \Delta^{\mathcal{I}}$ by induction on the structure of C. For all concept names, this holds by the definition of $\mathcal{I}_n$. For conjunctions, it follows directly from Lemma 4.1. We also know that $\top^{\mathcal{I}_n}(x) = 1 = \lambda(1) = \lambda(\top^{\mathcal{I}}(x))$.

Consider now an existential restriction $\exists r.C$ and assume that the claim holds for C. By the definition of $\mathcal{I}_n$ and Lemma 4.1, we know that $r^{\mathcal{I}_n}(x,y) *_{\text{Ł}_n} C^{\mathcal{I}_n}(y) = \lambda(r^{\mathcal{I}}(x,y) *_{\mathsf{L}} C^{\mathcal{I}}(y))$ holds for all $y \in \Delta^{\mathcal{I}}$. Since $(\exists r.C)^{\mathcal{I}_n}(x)$ is the supremum of all these values, L is finite, and λ is monotone, we have

$$(\exists r.C)^{\mathcal{I}_n}(x) = \lambda\Big(\sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) *_{\mathsf{L}} C^{\mathcal{I}}(y)\Big) = \lambda((\exists r.C)^{\mathcal{I}}(x)),$$

which concludes the proof of the claim.

Consider now an axiom $\langle C \sqsubseteq D \geqslant p\rangle$ in $\mathcal{T}$. Since $\mathcal{I}$ is a model of $\mathcal{T}'$, we have $D^{\mathcal{I}}(x) \geqslant a$ and $C^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} D^{\mathcal{I}}(x) \geqslant \lambda^{-1}(p)$ for all $x \in \Delta^{\mathcal{I}}$. We conclude that

$$C^{\mathcal{I}_n}(x) \Rightarrow_{\text{Ł}_n} D^{\mathcal{I}_n}(x) = \lambda(C^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} D^{\mathcal{I}}(x)) \geqslant \lambda(\lambda^{-1}(p))) = p$$

by Lemma 4.1, the above claim, and monotonicity of λ. ☐

Lemma 4.6. *If A is ℓ-subsumed by B w.r.t. $\mathcal{T}$, then A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$.*

Proof. Consider an L-model $\mathcal{I}$ of $\mathcal{T}'$ with $A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}}(x) < \lambda^{-1}(\ell)$ for some $x \in \Delta^{\mathcal{I}}$. By Lemma 4.5, $\mathcal{I}_n$ is a model of $\mathcal{T}$. By the definition of $\mathcal{T}'$, we know that $B^{\mathcal{I}}(x) \geqslant a$. Thus, Lemma 4.1 yields $A^{\mathcal{I}_n}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}_n}(x) = \lambda(A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}}(x))$. Since $\ell > 0$ and λ is strictly monotone in $[a,b]$, this residuum is strictly smaller than $\lambda(\lambda^{-1}(\ell)) = \ell$. $\square$

The main result of this section now follows from Theorem 3.5 and the fact that subsumption in L-$\mathcal{EL}$ for a finite chain L can be decided in ExpTime using the algorithm from [8].

Theorem 4.7. *Let L be a finite chain that is not of the form G_n. Then deciding ℓ-subsumption with respect to a TBox in L-$\mathcal{EL}$ is* ExpTime*-complete.*

In contrast, subsumption in G_n-$\mathcal{EL}$ for any $n \geqslant 2$ can be shown to be decidable in PTime using the approach from [2, 18].

5 The Infinite Łukasiewicz T-norm

Finally, we show ExpTime-hardness for fuzzy $\mathcal{EL}$ also under the infinite Łukasiewicz t-norm, and even all continuous t-norms containing a Łukasiewicz component (see Definition 2.1). By [7, Theorem 13], it suffices to show this for all t-norms *starting* with the Łukasiewicz t-norm. We thus consider an infinite chain L over $[0,1]$ with a continuous t-norm $*$ that is isomorphic to the infinite-valued Łukasiewicz t-norm in an interval $[0,b]$ with $b \in (0,1]$. We denote by $\Rightarrow$ the residuum of L.

The reduction is again from the subsumption problem in $\mathcal{ELU}$, and is very similar to the one in Section 3 for Ł_3-$\mathcal{EL}$. However, we additionally have to ensure that all relevant concepts can only take the values $\frac{b}{2}$ or $\geqslant b$. Given a concept C, let $\mathcal{T}_C$ be the L-$\mathcal{EL}$ TBox

$$\mathcal{T}_C := \{\langle C^2 \sqsubseteq C^3 \geqslant 1\rangle,\ \langle \top \sqsubseteq C \geqslant \tfrac{b}{2}\rangle\}.$$

Every model $\mathcal{I}$ of this TBox must satisfy $C^{\mathcal{I}}(x) \geqslant \frac{b}{2}$ for every $x \in \Delta^{\mathcal{I}}$ due to the second axiom. The first axiom additionally guarantees that $C^{\mathcal{I}}(x) \notin (\frac{b}{2}, b)$ holds: if $\frac{b}{2} < C^{\mathcal{I}}(x) < b$, then $(C^2)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) + C^{\mathcal{I}}(x) - b > 0$, and thus $(C^3)^{\mathcal{I}}(x) < (C^2)^{\mathcal{I}}(x)$, violating the axiom.

Similar to the reduction in Section 3, we will use the truth degree $\frac{b}{2} \in \mathsf{L}$ to stand for "false" in $\mathcal{ELU}$ and any degree greater or equal to b to represent "true." Consider now the mapping $\rho_{\text{Ł}}$ defined on the axioms of $\mathcal{T}$ as follows:

$$\begin{aligned}
\rho_{\text{Ł}}(A_1 \sqcap A_2 \sqsubseteq B) &:= \langle A_1 \sqcap A_2 \sqsubseteq B \geqslant b\rangle\\
\rho_{\text{Ł}}(\exists r.A \sqsubseteq B) &:= \langle \exists r.A \sqsubseteq B \geqslant b\rangle\\
\rho_{\text{Ł}}(A \sqsubseteq \exists r.B) &:= \langle A \sqsubseteq (\exists r.B)^2 \geqslant \tfrac{b}{2}\rangle\\
\rho_{\text{Ł}}(A \sqsubseteq B_1 \sqcup B_2) &:= \langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \tfrac{b}{2}\rangle
\end{aligned}$$

Given an $\mathcal{ELU}$ TBox $\mathcal{T}$ in normal form, let $\mathsf{AC}(\mathcal{T})$ be the set of all concept names and existential restrictions appearing in $\mathcal{T}$. We extend the mapping $\rho_{\text{Ł}}$ to $\mathcal{ELU}$ TBoxes as follows:

$$\rho_{\text{Ł}}(\mathcal{T}) := \{\rho_{\text{Ł}}(C \sqsubseteq D) \mid C \sqsubseteq D \in \mathcal{T}\} \cup \bigcup_{C \in \mathsf{AC}(\mathcal{T})} \mathcal{T}_C.$$

The following proofs are very similar to those of Section 3.

5.1 Soundness

Given an L-interpretation $\mathcal{I}$, we define the crisp interpretation $\mathcal{I}_{cr}$ as follows:

- $\Delta^{\mathcal{I}_{cr}} := \Delta^{\mathcal{I}}$,
- $x \in A^{\mathcal{I}_{cr}}$ iff $A^{\mathcal{I}}(x) \geqslant b$ for every concept name A and $x \in \Delta^{\mathcal{I}}$,
- $(x, y) \in r^{\mathcal{I}_{cr}}$ iff $r^{\mathcal{I}}(x, y) \geqslant b$ for every role name r and $x, y \in \Delta^{\mathcal{I}}$.

Lemma 5.1. *If $\mathcal{I}$ satisfies $\rho_{\text{Ł}}(\mathcal{T})$, then $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$.*

Proof.

- Consider an axiom of the form $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$ and $x \in A_1^{\mathcal{I}_{cr}} \cap A_2^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $A_1^{\mathcal{I}}(x) \geqslant b$ and $A_2^{\mathcal{I}}(x) \geqslant b$. Since b is idempotent w.r.t. $*$, also $(A_1 \sqcap A_2)^{\mathcal{I}}(x) \geqslant b$. Since $\mathcal{I}$ satisfies $\rho_{\text{Ł}}(\mathcal{T})$, this implies that $B^{\mathcal{I}}(x) \geqslant b$, and thus $x \in B^{\mathcal{I}_{cr}}$.
- Consider an axiom of the form $\exists r.A \sqsubseteq B \in \mathcal{T}$ and $x \in (\exists r.A)^{\mathcal{I}_{cr}}$. There must exist an element $y \in \Delta^{\mathcal{I}_{cr}}$ such that $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $r^{\mathcal{I}}(x, y) \geqslant b$ and $A^{\mathcal{I}}(y) \geqslant b$. Hence $\sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * A^{\mathcal{I}}(z) \geqslant r^{\mathcal{I}}(x, y) * A^{\mathcal{I}}(y) \geqslant b$. Since $\mathcal{I}$ satisfies $\rho_{\text{Ł}}(\mathcal{T})$, we get $B^{\mathcal{I}}(x) \geqslant b$, and thus $x \in B^{\mathcal{I}_{cr}}$.
- If $A \sqsubseteq \exists r.B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$, then by the definition of $\mathcal{I}_{cr}$, we have $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $((\exists r.B)^2)^{\mathcal{I}}(x) \geqslant \frac{b}{2}$. By the axioms in $\mathcal{T}_{\exists r.B} \subseteq \rho_{\text{Ł}}(\mathcal{T})$, we know that either $(\exists r.B)^{\mathcal{I}}(x) = \frac{b}{2}$ or $(\exists r.B)^{\mathcal{I}}(x) \geqslant b$, and thus $((\exists r.B)^2)^{\mathcal{I}}(x) \geqslant b$. Thus, we must have also $(\exists r.B)^{\mathcal{I}}(x) \geqslant b$. Since $\mathcal{I}$ is witnessed, this means that there exists a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) \geqslant b$ and $B^{\mathcal{I}}(y) \geqslant b$. Again by the definition of $\mathcal{I}_{cr}$, we have $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in B^{\mathcal{I}_{cr}}$, and hence $x \in (\exists r.B)^{\mathcal{I}_{cr}}$.
- For $A \sqsubseteq B_1 \sqcup B_2 \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$, we know that $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $(B_1 \sqcap B_2)^{\mathcal{I}}(x) \geqslant \frac{b}{2}$. By the axioms in $\mathcal{T}_B$, this implies that either $B_1^{\mathcal{I}}(x) \geqslant b$ or $B_2^{\mathcal{I}}(x) \geqslant b$, and thus $x \in B_1^{\mathcal{I}_{cr}}$ or $x \in B_2^{\mathcal{I}_{cr}}$. □

Lemma 5.2. *If C is subsumed by D w.r.t. $\mathcal{T}$, then C^2 is b-subsumed by D^2 w.r.t. $\rho_{\text{Ł}}(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be an L-interpretation satisfying $\rho_{\text{Ł}}(\mathcal{T})$ and $x \in \Delta^{\mathcal{I}}$ such that $(C^2)^{\mathcal{I}}(x) > 0$. Hence $(C^2)^{\mathcal{I}}(x) \geqslant b$, and thus also $C^{\mathcal{I}}(x) \geqslant b$. By the definition of $\mathcal{I}_{cr}$, we have $x \in C^{\mathcal{I}_{cr}}$. By Lemma 5.1 we know that $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$, and thus we get $x \in D^{\mathcal{I}_{cr}}$ by assumption. Again by the definition of $\mathcal{I}_{cr}$, we obtain $D^{\mathcal{I}}(x) \geqslant b$, and therefore $(D^2)^{\mathcal{I}}(x) \geqslant b$. Hence $(C^2)^{\mathcal{I}}(x) \Rightarrow (D^2)^{\mathcal{I}}(x) \geqslant b$, that is, C^2 is b-subsumed by D^2 with respect to $\rho_{\text{Ł}}(\mathcal{T})$. □

5.2 Completeness

Given a crisp interpretation $\mathcal{I}$, we define the L-interpretation $\mathcal{I}_{\mathsf{L}}$ as follows:

- $\Delta^{\mathcal{I}_{\mathsf{L}}} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_{\mathsf{L}}}(x) := b$ if $x \in A^{\mathcal{I}}$ and $A^{\mathcal{I}_{\mathsf{L}}} := \frac{b}{2}$ otherwise, for every concept name A and $x \in \Delta^{\mathcal{I}}$,
- $r^{\mathcal{I}_{\mathsf{L}}}(x, y) := b$ if $(x, y) \in r^{\mathcal{I}}$ and $r^{\mathcal{I}_{\mathsf{L}}}(x, y) := \frac{b}{2}$ otherwise, for every role name r and $x, y \in \Delta^{\mathcal{I}}$.

Lemma 5.3. *If $\mathcal{I}$ satisfies $\mathcal{T}$, then $\mathcal{I}_{\mathsf{L}}$ satisfies $\rho_{\text{Ł}}(\mathcal{T})$.*

Proof. The TBoxes $\mathcal{T}_C$ for $C \in \mathsf{AC}(\mathcal{T})$ are satisfied by the definition of $\mathcal{I}_\mathsf{L}$. In particular, the values for existential restrictions $(\exists r.A)^{\mathcal{I}_\mathsf{L}}(x)$ are computed as suprema of values of the form $r^{\mathcal{I}_\mathsf{L}}(x,y) * A^{\mathcal{I}_\mathsf{L}}(y)$, where each of the operands is either $\frac{b}{2}$ or b.

- Consider an axiom of the form $\langle A_1 \sqcap A_2 \sqsubseteq B \geqslant b\rangle \in \rho_{\text{Ł}}(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_\mathsf{L}}$. If $(A_1 \sqcap A_2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant b$, then both $A_1^{\mathcal{I}_\mathsf{L}}(x) = b$ and $A_2^{\mathcal{I}_\mathsf{L}}(x) = b$. By the definition of $\mathcal{I}_\mathsf{L}$, we have that $x \in A_1^{\mathcal{I}} \cap A_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this yields $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_\mathsf{L}$, we get $B^{\mathcal{I}_\mathsf{L}}(x) = b$.

  In the case that $(A_1 \sqcap A_2)^{\mathcal{I}_\mathsf{L}}(x) < b$, we have $(A_1 \sqcap A_2)^{\mathcal{I}_\mathsf{L}}(x) \leqslant \frac{b}{2} \leqslant B^{\mathcal{I}_\mathsf{L}}(x)$ by the definition of $\mathcal{I}_\mathsf{L}$, and thus also $(A_1 \sqcap A_2)^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow B^{\mathcal{I}_\mathsf{L}}(x) = 1 \geqslant b$.

- Consider an axiom of the form $\langle \exists r.A \sqsubseteq B \geqslant b\rangle \in \rho_{\text{Ł}}(\mathcal{T})$ and $x \in \Delta^{\mathcal{I}_\mathsf{L}}$. If $(\exists r.A)^{\mathcal{I}_\mathsf{L}}(x) \geqslant b$, then there exists a $y \in \Delta^{\mathcal{I}_\mathsf{L}}$ such that $r^{\mathcal{I}_\mathsf{L}}(x,y) = b$ and $A^{\mathcal{I}_\mathsf{L}}(y) = b$. By the definition of $\mathcal{I}_\mathsf{L}$, we know that $(x,y) \in r^{\mathcal{I}}$ and $y \in A^{\mathcal{I}}$, and hence $x \in (\exists r.A)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_\mathsf{L}$, we have that $B^{\mathcal{I}_\mathsf{L}} = b$.

  Otherwise, we have $(\exists r.A)^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow B^{\mathcal{I}_\mathsf{L}}(x) = 1$ as in the previous case.

- Consider an axiom of the form $\langle A \sqsubseteq (\exists r.B)^2 \geqslant \frac{b}{2}\rangle \in \rho_{\text{Ł}}(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_\mathsf{L}}$. If $((\exists r.B)^2)^{\mathcal{I}_\mathsf{L}}(x) = 0$, then

  $$b > (\exists r.B)^{\mathcal{I}_\mathsf{L}}(x) = \sup_{z \in \Delta^{\mathcal{I}_\mathsf{L}}} r^{\mathcal{I}_\mathsf{L}}(x,z) * B^{\mathcal{I}_\mathsf{L}}(z).$$

  Therefore every $y \in \Delta^{\mathcal{I}_\mathsf{L}}$ must satisfy either $r^{\mathcal{I}_\mathsf{L}}(x,y) < b$ or $B^{\mathcal{I}_\mathsf{L}}(y) < b$. By the definition of $\mathcal{I}_\mathsf{L}$, for all $y \in \Delta^{\mathcal{I}}$ we have either $(x,y) \notin r^{\mathcal{I}}$ or $y \notin B^{\mathcal{I}}$, and hence $x \notin (\exists r.B)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_\mathsf{L}$, we have $A^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$. Hence $A^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow ((\exists r.B)^2)^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$.

  In the case that $((\exists r.B)^2)^{\mathcal{I}_\mathsf{L}}(x) > 0$, we also get

  $$A^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow ((\exists r.B)^2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant ((\exists r.B)^2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant \tfrac{b}{2}.$$

- Consider an axiom of the form $\langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \frac{b}{2}\rangle \in \rho_{\text{Ł}}(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_{cr}}$. If $(B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) < \frac{b}{2}$, then $B_1^{\mathcal{I}_\mathsf{L}}(x) = B_2^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$. By the definition of $\mathcal{I}_\mathsf{L}$, we have that $x \notin B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this implies that $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_\mathsf{L}$, we have that $A^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$. Since by the definition of $\mathcal{I}_\mathsf{L}$ and supposition we have $(B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) = 0$, we can conclude that $A^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$.

  In the case that $(B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant \frac{b}{2}$, we also have

  $$A^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant (B_1 \sqcap B_2)^{\mathcal{I}_\mathsf{L}}(x) \geqslant \tfrac{b}{2}.$$

□

Lemma 5.4. *If C is not subsumed by D w.r.t. $\mathcal{T}$, then C^2 is not b-subsumed by D^2 w.r.t. $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be a crisp interpretation satisfying $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. By Lemma 5.3, we know that $\mathcal{I}_\mathsf{L}$ satisfies $\rho_{\text{Ł}}(\mathcal{T})$. Moreover, by the definition of $\mathcal{I}_\mathsf{L}$, we have $C^{\mathcal{I}_\mathsf{L}}(x) = b$ and $D^{\mathcal{I}_\mathsf{L}}(x) = \frac{b}{2}$. Hence $(C^2)^{\mathcal{I}_\mathsf{L}}(x) = b$ and $(D^2)^{\mathcal{I}_\mathsf{L}}(x) = 0$, and therefore $(C^2)^{\mathcal{I}_\mathsf{L}}(x) \Rightarrow (D^2)^{\mathcal{I}_\mathsf{L}}(x) = 0 < b$. □

From the previous arguments, we see that for any continuous chain L that starts with Łukasiewicz, subsumption in L-$\mathcal{EL}$ is ExpTime-hard. As shown in [7, Theorem 13], if L is the ordinal sum of L_1 and L_2 over the intervals $[0,a]$ and $[a,1]$, respectively, for some $a \in (0,1)$, then subsumption in L-$\mathcal{EL}$ is at least as hard as subsumption in L_2-$\mathcal{EL}$. Additionally, every chain L that contains a Łukasiewicz component can be described as such an ordinal sum, where L_2 starts with Łukasiewicz. This means that the ExpTime-hardness holds for all such continuous chains.

Theorem 5.5. *If* L *is defined using any continuous t-norm over* $[0, 1]$ *containing a Łukasiewicz component, then deciding* ℓ*-subsumption with respect to a TBox in* L-$\mathcal{EL}$ *is* ExpTime*-hard.*

This improves the lower bound of co-NP for this problem from [7]. However, it is unknown whether a similar lower bound holds for the product t-norm (and continuous t-norms containing several product components). An upper bound is known only for the case of the infinite Gödel t-norm, where subsumption is PTime-complete [18].

6 Conclusions

We have shown that reasoning in finitely valued extensions of fuzzy $\mathcal{EL}$ becomes exponentially harder than in classical $\mathcal{EL}$ even if only one additional truth value interpreted under Łukasiewicz semantics is considered. This provides the first example of a finitely valued DL that exhibits an increased complexity compared to the underlying classical DL. The same complexity lower bound holds for any infinitely valued t-norm over $[0, 1]$ that contains a Łukasiewicz component.

Although these problems are ExpTime-complete, we believe that subsumption in finitely valued extensions of $\mathcal{EL}$ can be solved more efficiently than by the algorithms developed for expressive finitely valued DLs [8, 9]. We plan to look at suitable adaptations of consequence-based algorithms for classical DLs [2, 16]. On the theoretical side, we will investigate whether other inexpressive DLs like $\mathcal{FL}_0$ [1] or *DL-Lite* [11] also exhibit an increase in complexity under Łukasiewicz semantics. We will also study the effect of the product semantics on the complexity of these logics.

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