REMARK: Sometimes it is more appropriate to consider semigroups instead of monoids.

SEMIGROUP: has only a binary associative operation (no unit element required)

SYNTACTIC SEMIGROUP: $\sim_L$ is also a congruence on the free semigroup $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. For $L \subseteq \Sigma^*$ the syntactic semigroup is $\Sigma^+/\sim_L$.

ALTERNATIVELY: The syntactic semigroup is the transition semigroup $\{\delta_w \mid w \in \Sigma^+\}$ of the minimal automaton of L.

S-VARIETY: Class of finite semigroups closed under direct products, homomorphic images, and building sub-semigroups.

NOTE: even if $1 \in S$, the unit element need not be an element of the sub-semigroups of S.

EQUATIONS: S-varieties can also ultimately defined by equations (Prop. 1.20 holds in a semigroup variant 1.20s)

These equations may not contain 1.

NOTE: S-varieties sometimes yield a more fine-grained divison into classes.

Consider the equation $x \cdot y = y$.

Only trivial monoids satisfy this equation: for $m \in M$ we have

$$m = 1_M \circ_M m = 1_M .$$

But non-trivial semigroups satisfy $x \cdot y = y$:

$\cdot$	a	b
a	a	a
b	b	b

$\cdot$ is associative

CLASS OF LANGUAGES: If V is an S-variety, then

$$L(V)_\Sigma = \{ L \subseteq \Sigma^* \mid S_L \in V \}$$

Lem. 1.21 and Prop 1.22 also hold in a semigroup variant 1.21s and 1.22s.