Unification in Description Logics

Part I: Introduction

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Chair of Automata Theory

TECHNISCHE UNIVERSITÄT DRESDEN

ESSLLI'19

Riga, August 2019

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- A little bit more formal/general,

Equational theory. Let E by a set of identities between first-order terms. The equational theory defined by $=_E$ consists of all identities $s = t$ that can be "derived" from E.

E-unification problem. $\Gamma := \{s_1 =^?_E t_1, \ldots, s_n =^?_E t_n\}$. A substitution σ is an E-unifier of Γ if

$$\sigma(s_i) =_E \sigma(t_i), \text{ for all } 1 \leq i \leq n.$$

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  - minimal complete sets of unifiers may not exist (we will later see)

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- Description Logics: detecting redundancies in ontologies.
- Modal Logics: special case of recognizability of admissible inference rules.

Description Logics

dl.kr.org

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Concept descriptions: built using the concept/role constructors provided by a DL.

$$\text{Human} \sqcap \text{Athlete} \sqcap \exists \text{wears}.(\text{Helmet} \sqcup \text{Cap}) \sqcap$$
$$\exists \text{plays}.\text{Baseball} \sqcap \neg \text{Lazy} \sqcap \forall \text{owns\_bat}.\text{Shiny}$$

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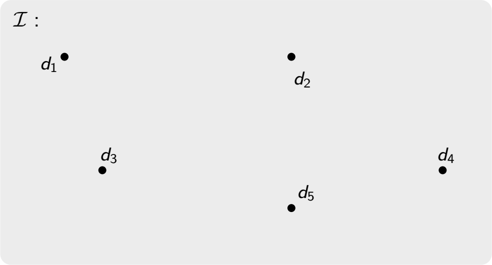

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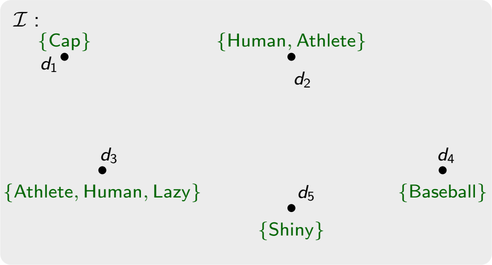

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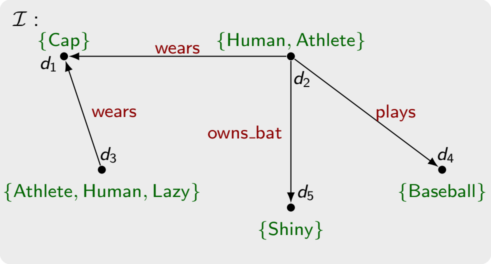

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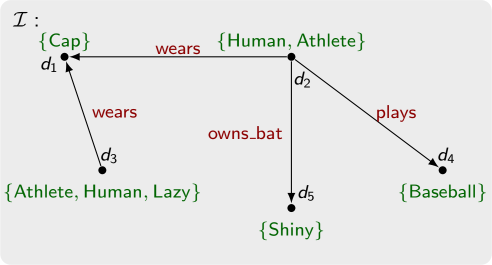

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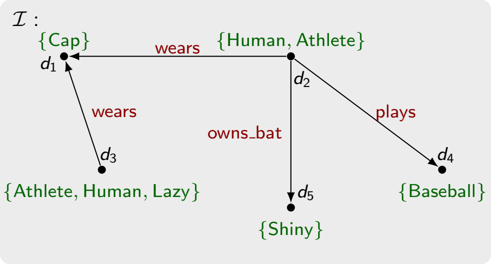

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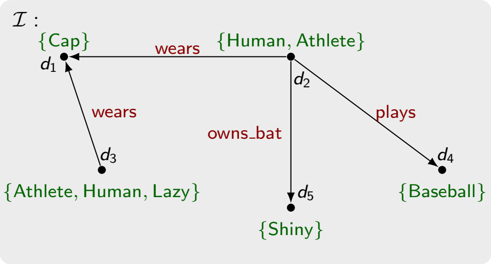

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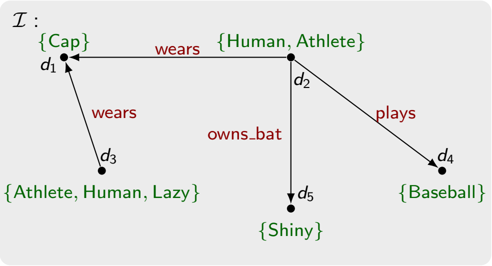

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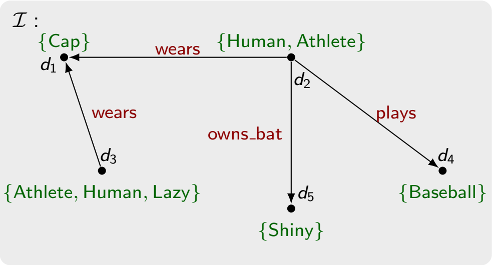

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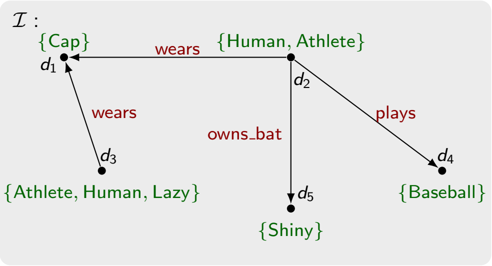

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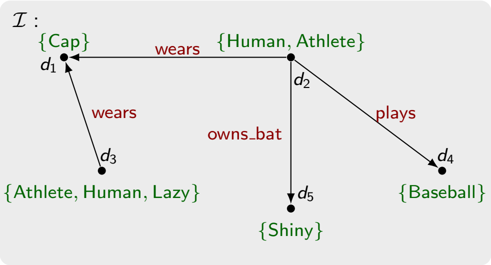

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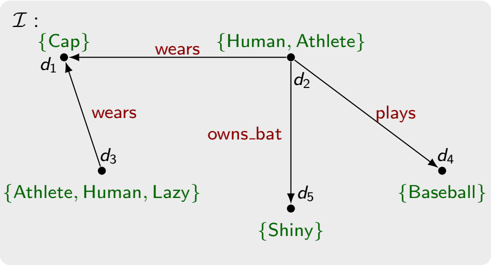

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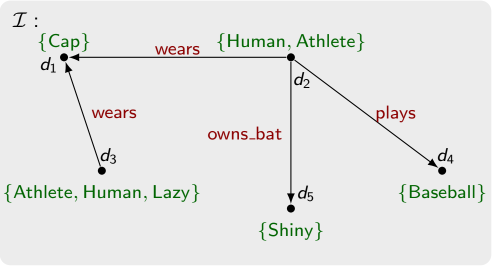

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Semantics

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Assertional knowledge (knowledge about concrete situations)

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Terminological knowledge (general knowledge about the domain)

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Entailments of $\mathcal{K}$: *Pedro throws FastBall*, ...

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- Concept satisfiability.
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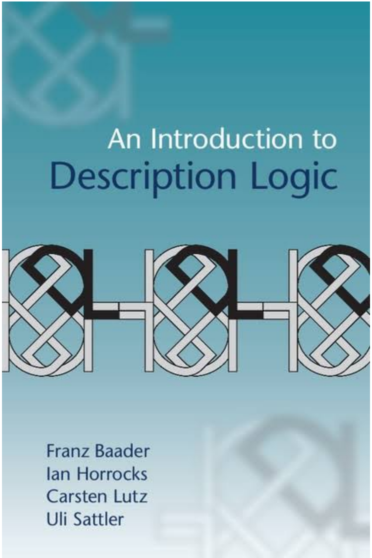
Non-Standard Inferences

- Most specific generalizations.
- Least common subsumer.
- Unification.
- ...

More on DLs...


An Introduction to
Description Logic
Franz Baader
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Unification in Description Logics

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Unification of concept descriptions

- Semi-automated process: suggests possible candidates to ontology engineers.

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- But they are, in presence of background knowledge (TBox) containing the GCI:

$$\exists \text{finding}.\exists \text{severity}.\text{Severe} \sqsubseteq \exists \text{status}.\text{Emergency}$$

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Definition 1 ($\mathcal{L}$-unification)

An $\mathcal{L}$-unification problem is of the form:

$$\Gamma := \{C_1 \equiv^? D_1, \ldots, C_n \equiv^? D_n\}.$$

A substitution σ is a unifier of Γ if

$$\sigma(C_i) \equiv \sigma(D_i), \text{ for all } 1 \leq i \leq n.$$

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Definition 2 ($\mathcal{L}$-unification w.r.t. a general TBox)

Let $\mathcal{T}$ be a general TBox that is ground. An $\mathcal{L}$-unification problem w.r.t. $\mathcal{T}$ is of the form:

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The decision problem

$\mathcal{L}$-Unification Decision Problem

Instance:	A ground general TBox $\mathcal{T}$ and an $\mathcal{L}$-unification problem Γ.
Question:	Is there a unifier σ of Γ w.r.t. $\mathcal{T}$?

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- zero **iff** it does not have a minimal complete $\mathcal{M}$.

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concept descriptions over $\mathsf{N_R} = \{r_1, \ldots, r_n\}$ ⟶ terms over Σ

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concept var. $\rightarrow$ variable symbols
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$\forall r_i$ satisfies
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$\forall r_i(C \sqcap D) \equiv \forall r_i.C \sqcap \forall r_i.D$
↓
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- Undecidability results for very expressive DLs: transferred from research in Modal Logics.

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