Unification in Description Logics

Part III: Unification in the DL $\mathcal{FL}_0$

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ESSLLI'19

Riga, August 2019

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  - Reasoning in $\mathcal{FL}_0$ has an interesting connection to formal language problems.
  - The unification problem corresponds to unification in ACUIh.

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- Let $\mathsf{N_C} = \{A_1, \ldots, A_k\}$. Then, every pair of concepts C, D can be represented as:

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  where L_i, R_i are finite languages over $\mathsf{N_R}$.

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Characterization of subsumption [BN01]

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$$A_2\colon\ \{rr\} \cup \{rs.r\} = \{rsr, rr\} \cup \{r.r\}$$

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A_1: $\{r\} \cup \{rs.\varepsilon, rs.s\} = \{rss\} \cup \{r.\epsilon, r.s\}$

A_2: $\{rr\} \cup \{rs.r\} = \{rsr, rr\} \cup \{r.r\}$

Fixing A_1 is idependent of fixing A_2, and vice versa!

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Example (continuation)

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has a solution **iff** for all $1 \leq i \leq k$ the equation:

$$L_i \cup L_1^*.X_{1,i} \cup \ldots \cup L_m^*.X_{1,m}$$
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Why finite?

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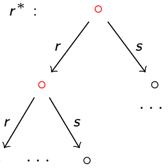

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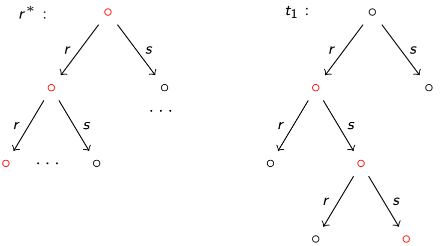

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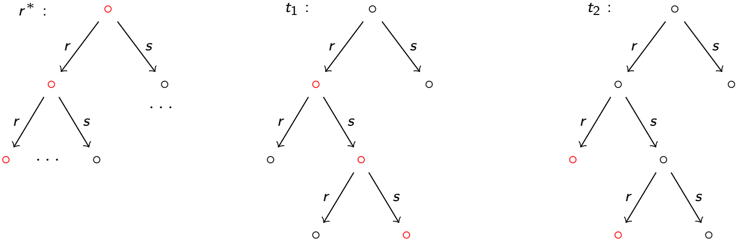

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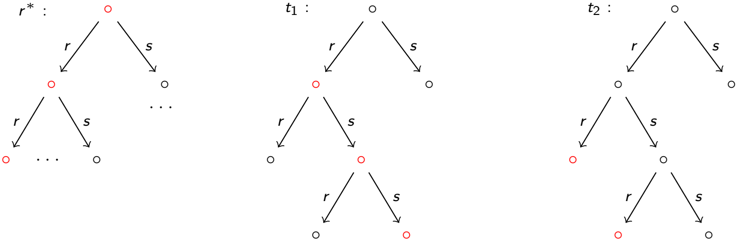

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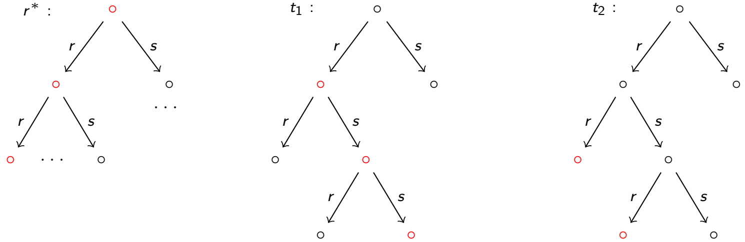

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- $\mathcal{A}_\mathcal{E}$ accepts exactly the trees representing solutions of an equation.

Detour to finite tree automata

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Let Σ be a finite alphabet such that each $f \in \Sigma$ has a rank $\mathrm{rk}(f) \geq 0$. A finite Σ-tree is a mapping $t : \mathrm{dom}(t) \to \Sigma$ such that:

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Let Σ be a finite alphabet such that each $f \in \Sigma$ has a rank $\mathsf{rk}(f) \geq 0$. A finite Σ-tree is a mapping $t : \mathsf{dom}(t) \to \Sigma$ such that:

- $\mathsf{dom}(t)$ is a finite subset of $\{1, \ldots, \max(\mathsf{rk}(f))\}^*$,
- $\varepsilon \in \mathsf{dom}(t)$,
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Leaves of t must be mapped to constants, i.e., arity 0.

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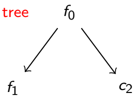
Let $\Sigma = \{f_0^{(2)}, f_1^{(2)}, c_0^{(0)}, c_1^{(0)}\}$

not a tree



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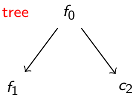
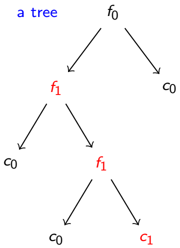


a tree



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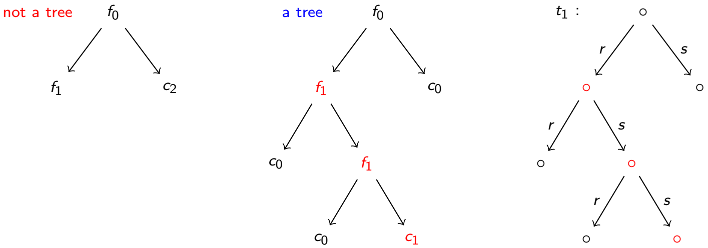

Detour to finite tree automata

Detour to finite tree automata

Definition 12 (Root-to-frontier tree automata)

A root-to-frontier tree automaton (RFA) on Σ-trees is a tuple $\mathcal{A} = (\Sigma, Q, I, \Delta, F)$ where:

- Q is a finite set of states.
- $I \subseteq Q$ is the set of initial states.
- Δ is a transition function s.t.:
$$\forall f \in \Sigma,\ \mathrm{rk}(f) = n > 0\colon\ \Delta(f) \subseteq Q \times Q^n.$$
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Tree language accepted by $\mathcal{A}$:

$$\mathcal{L}(\mathcal{A}) = \{t \mid \mathcal{A} \text{ has a successful run on } t\}.$$

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- Acceptance condition:
  - $F(c_0) = \{q_{\mathrm{N}}\}$ and $F(c_1) = \{q_{\mathrm{N}}, q_\varepsilon\}$.

Solving linear equations

Recall: we want to construct an RFA $\mathcal{A}_{\mathcal{E}}$ accepting exactly the "trees" solving

$$L_i \cup L_1^*.X_{1,i} \cup \ldots \cup L_m^*.X_{1,m}$$
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Property

$$X_{1,i}, \ldots, X_{1,m} \text{ solves } \mathcal{E}$$
iff
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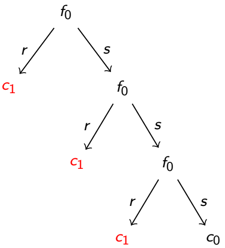

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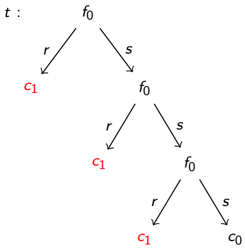

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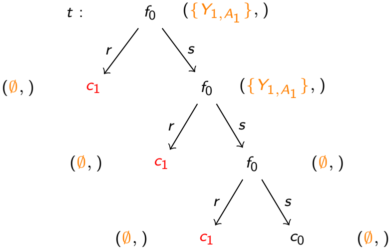

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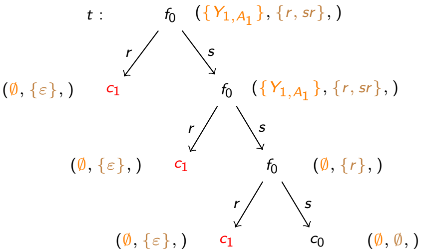

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- suffixes left-hand side

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$$t: \quad \begin{tikzcd}
 & f_0\ (\{Y_{1,A_1}\},\{r,sr\},\{ssr,r\}) \arrow[dl,"r"'] \arrow[dr,"s"] & & & \\
(\emptyset,\{\varepsilon\},\{\varepsilon\})\ c_1 & & f_0\ (\{Y_{1,A_1}\},\{r,sr\},\{sr,r\}) \arrow[dl,"r"'] \arrow[dr,"s"] & & \\
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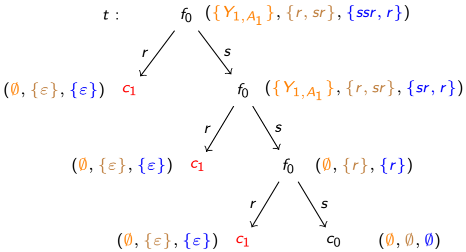

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Third and second sets are sets of suffixes occurring in $\overline{\mathcal{E}}$.

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- Set of states: $Q := 2^{M} \times 2^{SL} \times 2^{SR}$.

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Complexity of solving linear equations

Lemma 13

The following are equivalent

1. $\mathcal{L}(A_{\mathcal{E}}) \neq \emptyset$.
2. there are finite sets $Y_{1,i}, \ldots, Y_{1,m}$ such that:

$$\overline{L_i} \cup Y_{1,i}.\overline{L_1^*} \cup \ldots \cup Y_{1,m}.\overline{L_m^*}$$
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Algorithm

1. Γ is translated into k linear equations of size polynomial.
2. Build an automaton $\mathcal{A}_{\mathcal{E}}$ for each equation. The size is exponential w.r.t. Γ!
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Lemma 14 [BN01]

Unification in $\mathcal{FL}_0$, ACUIh and solving linear equations are in ExpTime.

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Theorem 15 [BN01]

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Unification in $\mathcal{FL}_{reg}$

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$\mathcal{FL}_{reg}$ extends $\mathcal{FL}_0$ with complex roles:

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Semantics of complex roles:

$$\begin{aligned}
\varepsilon^{\mathcal{I}} &:= \{(d,d) \mid d \in \Delta^{\mathcal{I}}\}\\
\emptyset^{\mathcal{I}} &:= \emptyset\\
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$\{r\} \circ (\{s\} \cup \{r\}) \circ \{s\}^* \rightarrow$ pairs (d,e) such that d reaches e through a word in $r.(s|r).s^*$.

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A complex role can be seen as a regular expression/language!

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Theorem 16 [BK01]

Unification in $\mathcal{FL}_{reg}$ is ExpTime-complete.

Matching with respect to general $\mathcal{FL}_0$ TBoxes

Deciding matching w.r.t. the empty TBox

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Matching is a unification problem of the form:

$$\forall L_1.A_1 \sqcap \ldots \sqcap \forall L_k.A_k \sqcap \forall L_1^*.X_1 \sqcap \ldots \sqcap \forall L_m^*.X_m$$
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Lemma 17 [BN01]

An $\mathcal{FL}_0$-matching problem has a solution iff the following is a solution:

$$\sigma(X_i) = \bigcap_{u \in L_i^*} u^{-1}.R_0, \text{ where } u^{-1}.R_0 := \{v \mid uv \in R_0\}.$$

Decidable in PTime.

Characterization of subsumption in $\mathcal{FL}_0$ - TBox

In the presence of a non-empty TBox, a new characterization is needed:

$$\mathcal{L}_{\mathcal{T}}(C) := \{(w, A) \in \mathsf{N_R}^* \times \mathsf{N_C} \mid C \sqsubseteq_{\mathcal{T}} \forall w.A\}$$

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Let $\mathcal{T}$ be an $\mathcal{FL}_0$ TBox and C, D $\mathcal{FL}_0$ concepts. Then, $C \sqsubseteq_{\mathcal{T}} D$ iff $\mathcal{L}_{\mathcal{T}}(D) \subseteq \mathcal{L}_{\mathcal{T}}(C)$.

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Back to the example

$$\begin{array}{c} C := \forall\{r, s\}.A \sqcap \forall\{s\}.B \\ \equiv^? \\ D := \forall\{rr\}.A \sqcap \forall\{r, s\}.X_1 \sqcap \forall\{s\}.X_2 \end{array} \qquad \mathcal{T} = \{A \sqsubseteq \forall r.A, \forall s.B \sqsubseteq A\}$$

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solution: $\sigma(X_1) := A$ and $\sigma(X_2) := B$

$$\mathcal{L}_\mathcal{T}(C, A) = \{s\} \cup r^* = \mathcal{L}_\mathcal{T}(\sigma(D), A)$$

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Workaround: consider $\mathcal{FL}_{reg}$ concept descriptions.

Deciding the existence of an $\mathcal{FL}_{reg}$-matcher

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The following is true

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1. $C \sqsubseteq_{\mathcal{T}} E$
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Checking that $\widehat{\sigma}$ is a matcher

1. $C \sqsubseteq_{\mathcal{T}} E$ 2. $E \sqcap \forall L_1.\widehat{\sigma}(X_1) \sqcap \ldots \sqcap \forall L_n.\widehat{\sigma}(X_n) \sqsubseteq_{\mathcal{T}} C$.

This can be done in exponential time using automata on infinite trees [BGM18].

Deciding the existence of an $\mathcal{FL}_0$-matcher

By construction of $\widehat{L}_{i,j}$

$$C \equiv^?_{\mathcal{T}} E \sqcap \forall L_1.X_1 \sqcap \ldots \sqcap \forall L_n.X_n \text{ has an } \mathcal{FL}_0 \text{ matcher}$$

iff

1. $C \sqsubseteq E$
2. There are finite languages $L_{i,j} \subseteq \widehat{L}_{i,j}$ such that:

$$E \sqcap \forall L_1.\sigma(X_1) \sqcap \ldots \sqcap \forall L_n.\sigma(X_n) \sqsubseteq_{\mathcal{T}} C$$

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Applying the Compactness Theorem

$$E \sqcap \forall L_1.\widehat{\sigma}(X_1) \sqcap \ldots \sqcap \forall L_n.\widehat{\sigma}(X_n) \sqsubseteq_\mathcal{T} C$$

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Consequences

- There exists an $\mathcal{FL}_0$-matcher **iff** there is an $\mathcal{FL}_{reg}$-matcher.
- Deciding the existing of an $\mathcal{FL}_0$-matcher is ExpTime-complete.

Summary - Unification in the DL $\mathcal{FL}_0$

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- In the presence of a general TBox.
  - Decidability is an open problem.
  - It is only known that it is ExpTime-complete for the special case of matching (non-constructive proof).

References I

Franz Baader, Oliver Fernandez Gil, and Pavlos Marantidis.
Matching in the description logic FL0 with respect to general tboxes.
In *LPAR-22. 22nd International Conference on Logic for Programming, Artificial Intelligence and Reasoning, Awassa, Ethiopia, 16-21 November 2018*, volume 57 of *EPiC Series in Computing*, pages 76–94. EasyChair, 2018.

Franz Baader and Ralf Küsters.
Unification in a description logic with transitive closure of roles.
In *Logic for Programming, Artificial Intelligence, and Reasoning, 8th International Conference, LPAR 2001, Havana, Cuba, December 3-7, 2001, Proceedings*, volume 2250 of *Lecture Notes in Computer Science*, pages 217–232. Springer, 2001.

Franz Baader and Paliath Narendran.
Unification of Concept Terms in Description Logics.
J. Symb. Comput., 31(3):277–305, 2001.

Helmut Seidl.
Haskell overloading is dexptime-complete.
Inf. Process. Lett., 52(2):57–60, 1994.