Unification in Description Logics

Part IV: Related work in Modal Logics

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Basic modal systems

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- the Aristotle axiom $\Box(x \to y) \to (\Box x \to \Box y)$.

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- Kripke frame. A pair $F = (W, (R_{p_1}, \ldots, R_{p_n}))$ where:
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- Kripke model. A pair $M = (F, V)$ where V is a valuation of the propositional variables:

$$V : Vars \to 2^W.$$

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- A is valid in a class of Kripke frames K ($K \models A$) iff it is valid in all $F \in K$.
- $L(K)$ is called the modal logic induced by the class of frames K.

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There are modal logics that cannot be obtained from a class of Kripke frames [VB84].

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- Inference problems

$$A_C \text{ is valid in } \mathsf{K}_m \textbf{ iff } C \equiv \top$$

$$C \equiv D \textbf{ iff } A_C \leftrightarrow A_D \text{ is valid in } \mathsf{K}_m.$$

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- Unification type of A: defined w.r.t. $(U_L(A), \leq_L^{Vars(A)})$.

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- For MLs, all variables are eligible to be substituted.

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- In DLs, $\{C_1 \equiv^? D_1, \ldots, C_n \equiv^? D_n\}$ can be transformed into:

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- In uni-modal logics, like K, the previous trick is not possible. However,
$$\sigma \text{ solves } \{A_1, \ldots, A_n\} \text{ iff it solves } \{A_1 \wedge \ldots \wedge A_n\}.$$

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  lower bounds/undecidability of unification transfer to the admissibility problem.

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  Suppose a modal logic L has:

  - finitary unification type.
  - there is an effective algorithm computing a complete set of unifiers for a unification problem A.

Motivation for unification in MLs

Recognizability of admissible rules

Instance: A modal logic L and a rule $\frac{A}{B}$.

Question: Does $\vdash_L \sigma(A)$ implies $\vdash_L \sigma(B)$ for every substitution σ?

How can unification help?

- In certain cases unification can be used to solve the admissibility problem.

  Suppose a modal logic L has:

  - finitary unification type.
  - there is an effective algorithm computing a complete set of unifiers for a unification problem A.

  Then,

  $\frac{A}{B}$ is admissible
  iff
  $\vdash_L \sigma(B)$ for all $\sigma \in U_L(A)$.

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Main open problem

- Unification and admissibility in K. K has unification type zero [Jer15]!

General summary

Unification
Theory

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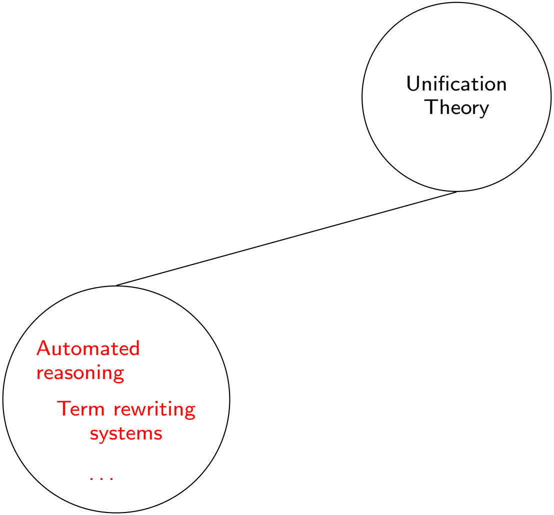

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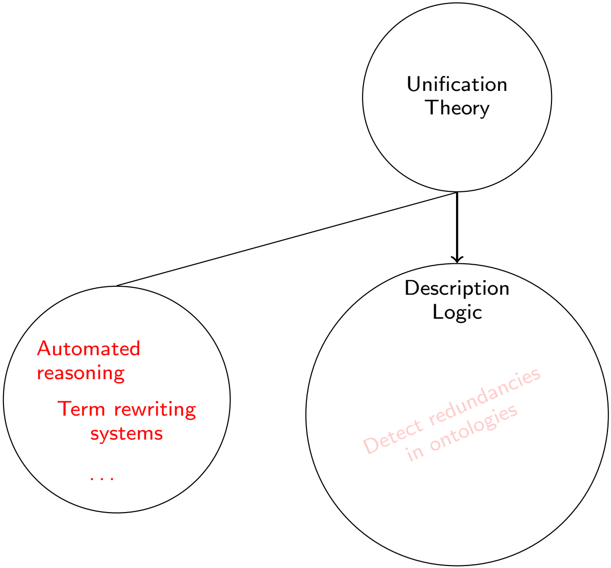

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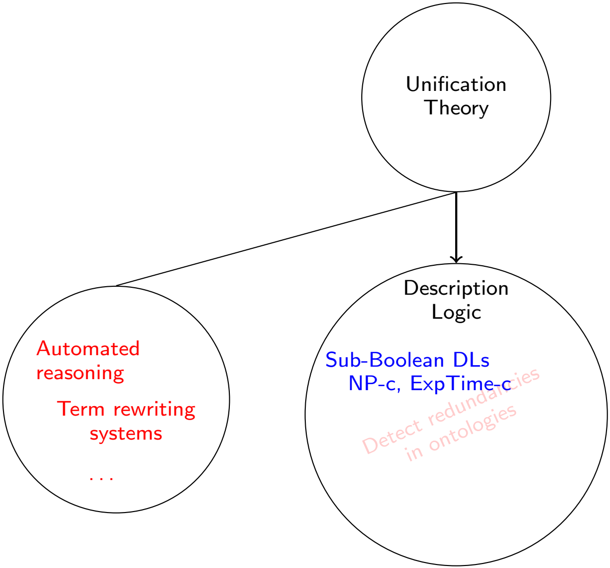

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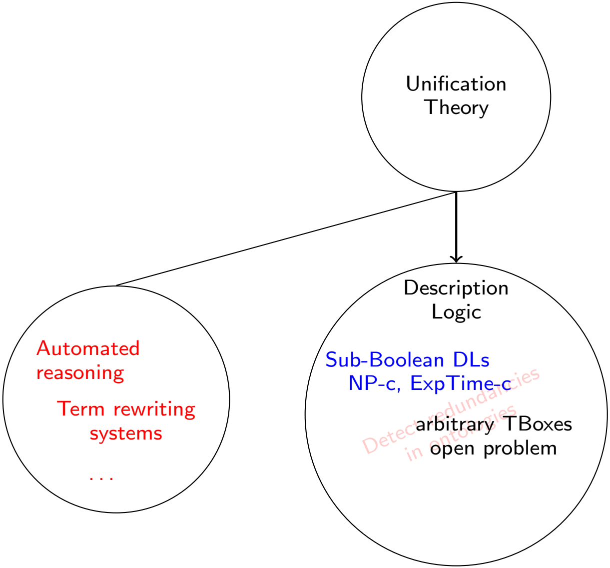

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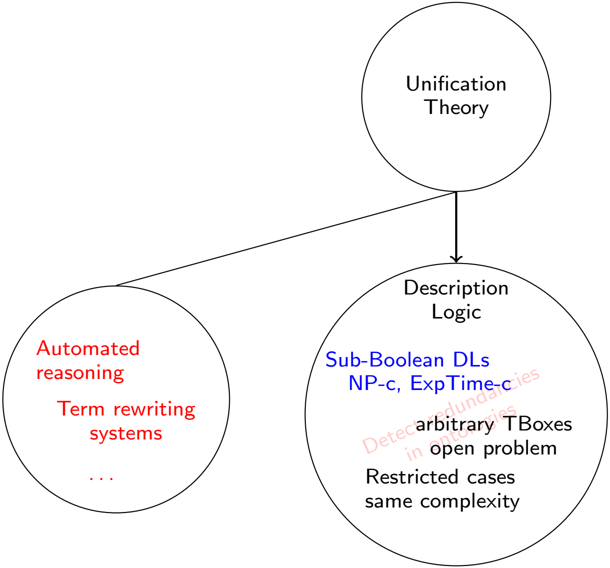

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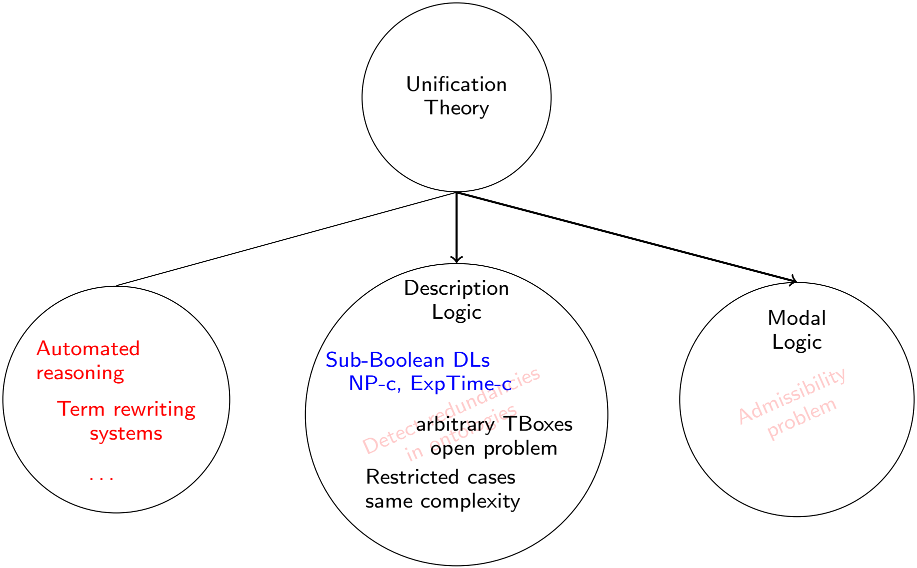

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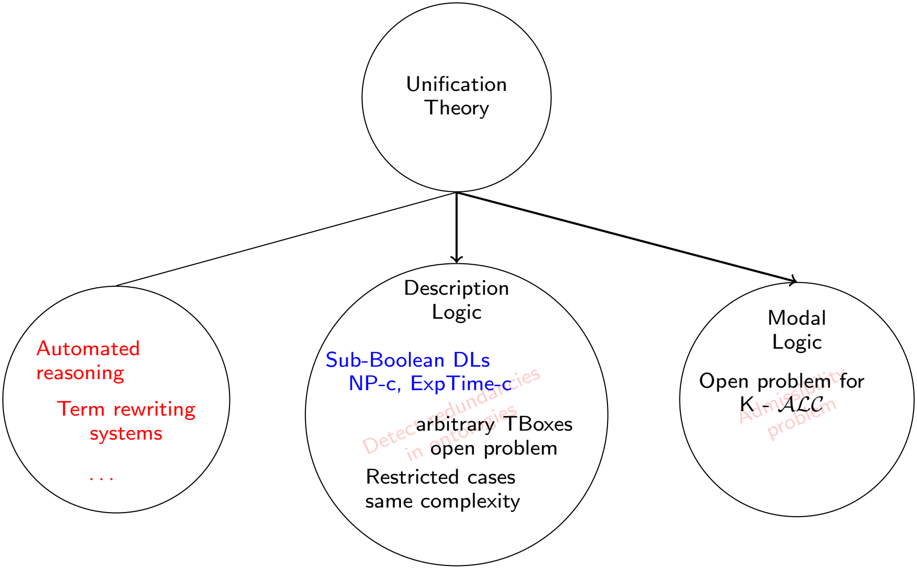

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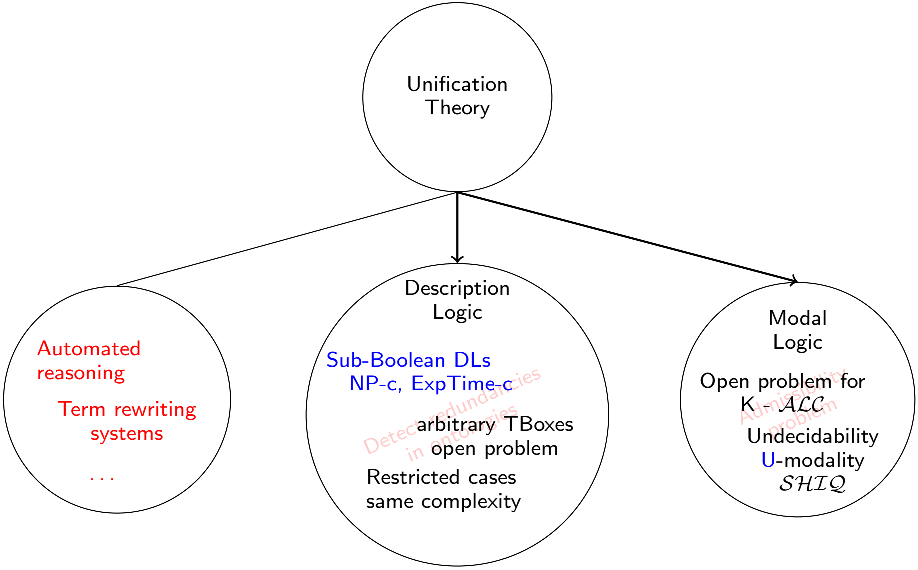

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