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Master's Thesis

Conditions for the Existence of the lcs of $\mathcal{EL}$-concepts w.r.t. General *Horn*-$\mathcal{ALC}$ TBoxes

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Declaration of Authorship

I, hereby, confirm that this master's thesis has been written by me. Moreover, I certify that I have not used any additional literature and sources except those, which are cited in my thesis.

Adilzhan Abdrakhmanov

Contents

1 Introduction

In this master's thesis we will investigate the existence of the least common subsumer (*lcs*) of two concept descriptions in *Horn*-$\mathcal{ALC}$ ontologies. There are some practical areas in computer science, where we can apply the *lcs*. For example, the *lcs* can be employed in similarity measures in order to calculate similarities between concept descriptions. Such a similarity measure, which uses the *lcs*, was presented in [6]. Computing concept similarities is widely applied in biomedical ontology alignment. For instance, in the Gene ontology with the help of similarity measures we are able to determine functionally similar genes. Additionally, we can use the *lcs* in order to enrich ontologies with new concept descriptions by appending intermediate concepts [13].
Horn-$\mathcal{ALC}$ ontology is based on description logic. Description logics (DLs) are a family of knowledge representation languages which have structured and well-understood semantics. Correspondingly, DLs are considered as a subfield of knowledge representation. Knowledge representation is a subfield of artificial intelligence. There are several representatives of this family of knowledge representation systems, namely, several DLs with various expressive power and properties [2, 4]. DLs are more expressive than propositional logic. Generally, DLs, which have more expressive power, show higher complexity as well. DLs are widely used in the basis of many important applications, for instance, in areas related to databases, biomedical ontologies, the semantic web, etc.
Knowledge in DL is represented by concept names (unary predicates) and role names (binary predicates). Complex concepts are expressions which built from concept and role names using constructors provided by the corresponding DL. For example, the DL $\mathcal{ALC}$ (attributive language with complement) consist the concept constructors: negation (complement operator), in symbols $\neg$, conjunction ($\sqcap$), disjunction ($\sqcup$), existential restriction ($\exists$) and value restriction ($\forall$). The DL $\mathcal{ALC}$ can be considered as a fragment of first-order logic with two variables.
Horn DLs do not allow non-deterministic constructors, for example, positive disjunction of the form $A \sqsubseteq B \sqcup C$ [8]. We are interested in *Horn* ontologies, since they show lower data complexities than the corresponding non-*Horn* ontologies. Moreover, various existing ontologies are *Horn*, for example, medical ontologies SNOMED CT and GALEN [14]. Because of that, recently in computer science numerous *Horn* ontologies are investigated [8, 14, 15].
The semantics of concepts in DLs is based on the notion of interpretations. An interpretation contains a non-empty set of domain elements and a mapping function such that concept names are mapped to sets of domain elements and roles are mapped to binary relations on this domain. The mapping function can be inductively extended to complex concepts.
An ontology includes a TBox (a set of general concept inclusions) and an ABox (a set of assertions). TBoxes show a hierarchy between concept descriptions. Besides presenting a hierarchy, TBoxes allow reasoning services. Concept subsumption is one of the main reasoning tasks, which allows us to obtain implicit knowledge from explicit one. Subsumption relation gives as an answer, which concept is more general from two given concepts. We denote general concept inclusions as $C \sqsubseteq D$, where C and D can be arbitrary concepts.
Common subsumers are concept descriptions, which subsume all concepts from a given set. In other words, common subsumers generalize a set of concepts into one concept, which includes all commonalities of the initial concepts.

The least common subsumer is a common subsumer, which is the least w.r.t. subsumption relation. The task of defining the *lcs* was studied in many resources, for example, in [3, 5, 13, 16, 17]. Several approximate methods for computing the *lcs* were proposed in [5, 13]. We will consider the case of the existence of the *lcs* of given $\mathcal{EL}$-concepts w.r.t. general TBoxes written in *Horn-*$\mathcal{ALC}$. In this case, the *lcs* employs constructors only from the target DL $\mathcal{EL}$, but w.r.t. general *Horn-*$\mathcal{ALC}$ TBoxes. Note, the target DL is restricted, since in order to define the *lcs* of two given concepts in the target DL with disjunction we just need to take the disjunction of these two concepts. Because of that, we are more interested in the target DL without disjunction and full negation, where the *lcs* captures commonalities of both initial concepts [5].

The *lcs* of two $\mathcal{EL}$-concepts w.r.t. general $\mathcal{EL}$-TBoxes does not need to exist. Conditions for the existence of the *lcs* of two (or more) $\mathcal{EL}$-concepts in the case of general $\mathcal{EL}$-TBoxes were defined by B. Zarrieß and A.-Y. Turhan in [16, 17]. In their works computation of the *lcs* is based on the notions of a canonical model, a characteristic concept and simulation relation. This thesis investigates the question how these conditions for $\mathcal{EL}$-TBoxes can be adapted to *Horn-*$\mathcal{ALC}$-TBoxes. It should be noted, that in the case of general *Horn-*$\mathcal{ALC}$-TBoxes the *lcs* of input $\mathcal{EL}$-concepts may not exist. But there is an opportunity to obtain the role-depth bounded *lcs*.

The rest of this thesis is organized in the following way. In Section 2 the main notions related to DLs are introduced. Definitions of the *lcs*, tree unraveling and characteristic concept are also presented in Section 2. The notion of canonical model w.r.t. normalized *Horn-*$\mathcal{ALC}$-TBox and related lemmas are proposed in Section 3. In this section we provide conditions for the existence of the *lcs* for input $\mathcal{EL}$-concepts as well. In Section 4, a decision procedure for the existence of the *lcs* is shown and computational complexity of this decision procedure is investigated. In the last Section 5, some concluding remarks and future work are discussed.

2 Preliminaries

2.1 $\mathcal{ALC}$

We start with presenting the main notions of DL $\mathcal{ALC}$. In this language, concept descriptions are defined with the help of the following constructors: negation ($\neg$), conjunction ($\sqcap$), disjunction ($\sqcup$), existential restriction ($\exists$) and value restriction ($\forall$).

Definition 2.1. (Syntax of $\mathcal{ALC}$-concept descriptions). Let N_C and N_R be mutually disjoint sets of concept and role names, respectively. *$\mathcal{ALC}$-concept descriptions* are formed by induction:

1. $\top$ (top-concept), $\bot$ (bottom-concept) are $\mathcal{ALC}$-concept descriptions.
2. if $A \in N_C$, then A is an $\mathcal{ALC}$-concept description.
3. if C, D are $\mathcal{ALC}$-concept descriptions and $r \in N_R$, then the following are also $\mathcal{ALC}$-concept descriptions:
   - $\neg C$ (negation),
   - $C \sqcap D$ (conjunction),
   - $C \sqcup D$ (disjunction),
   - $\exists r.C$ (existential restriction),
   - $\forall r.C$ (value restriction).

Notice, we distinguish between concept names N_C and other *complex* concept descriptions. Additionally, we can call $\mathcal{ALC}$-concept description as *$\mathcal{ALC}$-concept* or *concept description* or just *concept*.
For example, complex concepts $\exists r.\exists r.C \sqcap D \sqcap \neg E$ and $\exists r.((C \sqcap D) \sqcup (\forall r.E \sqcap \neg F))$ are $\mathcal{ALC}$-concept descriptions.
The semantics of concept descriptions is based on interpretations, which was explained by F. Baader in [1].

Definition 2.2. (Semantics of $\mathcal{ALC}$-concept descriptions). An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. The *domain* $\Delta^{\mathcal{I}}$ is a non-empty set of elements and an *extension function* $\cdot^{\mathcal{I}}$ is an interpretation function that maps:

1. each concept name $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.
2. each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The extension mapping is extended to $\mathcal{ALC}$-concepts as follows [2]:

- $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}$,
- $\bot^{\mathcal{I}} := \emptyset$,
- $(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$,
- $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$,
- $(C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}$,
- $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \text{there is } e \in \Delta^{\mathcal{I}} : (d, e) \in r^{\mathcal{I}} \text{ and } e \in C^{\mathcal{I}}\}$,

- $(\forall r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \text{for all } e \in \Delta^{\mathcal{I}} : (d, e) \in r^{\mathcal{I}} \text{ implies } e \in C^{\mathcal{I}}\}$.

DL $\mathcal{EL}$ is the fragment of DL $\mathcal{ALC}$, in which only $\top$, conjunctions and existential restrictions are allowed, i.e. $C \sqcap \exists r.\exists r.\exists r.D$ is an example of a complex $\mathcal{EL}$-concept.
Further, we will present the definition of a general concept inclusion (GCI) and TBoxes. Using GCIs we are able to describe the hierarchy of concept descriptions.

Definition 2.3. A *general concept inclusion* (GCI) is of the form $C \sqsubseteq D$, where C and D are concepts. A finite set containing GCIs is called a *TBox*.

In other words, a TBox represents the terminological knowledge by expressing relationship between concept descriptions.

Example 2.4. A simple TBox $\mathcal{T} = \{Male \sqsubseteq Human,\ Female \sqsubseteq Human,\ Male \sqcap Female \sqsubseteq \bot,\ Father \sqcup Mother \sqsubseteq Parent\}$.

We say the interpretation $\mathcal{I}$ *satisfies* the GCI $C \sqsubseteq D$, denoted as $\mathcal{I} \models C \sqsubseteq D$, if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds. It is easy to see, that the following is true: $C \sqsubseteq C$, $\bot \sqsubseteq C$, $C \sqsubseteq \top$; if $C \sqsubseteq D$ then $\exists r.C \sqsubseteq \exists r.D$ and $\forall r.C \sqsubseteq \forall r.D$.
A TBox and an ABox together form an *ontology* (*knowledge base*), where the ABox contains the assertional knowledge.

Definition 2.5. (Model). If the interpretation $\mathcal{I}$ satisfies all the GCIs in a TBox $\mathcal{T}$, then this interpretation $\mathcal{I}$ is called a *model* of the TBox $\mathcal{T}$ and denoted as $\mathcal{I} \models \mathcal{T}$.

Now we are able to define a subsumption relation with respect to a TBox $\mathcal{T}$. It should be noted, that *concept subsumption* is one of the important reasoning tasks, which allows us to obtain implicit knowledge from explicit knowledge.

Definition 2.6. (Subsumption). The concept description C is *subsumed* by the concept description D w.r.t. the TBox $\mathcal{T}$, written as $C \sqsubseteq_{\mathcal{T}} D$, if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all models $\mathcal{I}$ of $\mathcal{T}$.

Notice, the subsumption is a pre-order relation, because it is:

1. reflexive: $E \sqsubseteq_{\mathcal{T}} E$.

2. not antisymmetric: from $E \sqsubseteq_{\mathcal{T}} F$ and $F \sqsubseteq_{\mathcal{T}} E$ it does not follow that E identically equals to F. Since, as you see below, concepts E and F can be equivalent but not syntactically equal.

3. transitive: if $E \sqsubseteq_{\mathcal{T}} F$ and $F \sqsubseteq_{\mathcal{T}} G$, then $E \sqsubseteq_{\mathcal{T}} G$ holds.

The procedure of determining all subsumption relations between concept names is called *classification*, that is the whole TBox is classified. We will also use the notion of concept equivalence ($\equiv$).

Definition 2.7. (Equivalence). Two concept descriptions C and D are *equivalent* w.r.t. the TBox $\mathcal{T}$, denoted as $C \equiv_{\mathcal{T}} D$, if and only if $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$.

It is easy to see, that in this case $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all models $\mathcal{I}$ of the TBox $\mathcal{T}$.
Note, if it is irrelevant (or the TBox $\mathcal{T}$ is empty), we can omit the symbol $\mathcal{T}$ in the signs of subsumption ($\sqsubseteq_{\mathcal{T}}$) and equivalence ($\equiv_{\mathcal{T}}$) relations and merely use $\sqsubseteq$ and $\equiv$ symbols.

Further, we do not distinguish between $C_1 \sqcap C_2$ and $C_2 \sqcap C_1$, since they are equal on semantic level. Similarly, $C_1 \sqcup C_2 \equiv C_2 \sqcup C_1$. This holds also for multiple appearance of the same concept in conjunctions or disjunctions, for instance for the binary case we have $C \sqcap C \equiv C$ and $D \sqcup D \equiv D$. It should be noticed, that $C \sqcap \top \equiv C$ and $D \sqcup \bot \equiv D$.
Now we can add some more axioms in our TBox $\mathcal{T}$ from Example 2.4.

Example 2.8. Let us consider the following TBox $\mathcal{T}_1$:

$$\begin{aligned}
\mathcal{T}_1 = \{ & Male \sqsubseteq Human, \\
& Female \sqsubseteq Human, \\
& Father \equiv Male \sqcap \exists child.Human, \\
& Mother \equiv Female \sqcap \exists child.Human, \\
& Grandparent \equiv Human \sqcap \exists child.\exists child.Human, \\
& Male \sqcap Female \sqsubseteq \bot, \\
& Father \sqcup Mother \sqsubseteq Parent, \\
& Mother_without_daughter \sqsubseteq Mother \sqcap \forall child.Male\}.
\end{aligned}$$

In order to show equivalence of two concepts w.r.t. this TBox $\mathcal{T}_1$, we can write, e.g. $Grandparent \equiv_{\mathcal{T}_1} Human \sqcap \exists child.(Father \sqcup Mother)$.

The least common subsumer is an inference, which generalizes a set of given concepts to one concept. Let us give the formal definition of the *lcs* [3].

Definition 2.9. (Least common subsumer). Let $C_1, C_2, \ldots, C_k$ be concepts and $\mathcal{T}$ a TBox. A concept description D is called the *least common subsumer* of $C_1, C_2, \ldots, C_k$ w.r.t. $\mathcal{T}$, written as $lcs_{\mathcal{T}}(C_1, C_2, \ldots, C_k)$, if the following two conditions are fulfilled:

1. $\forall i, 1 \leq i \leq k, C_i \sqsubseteq_{\mathcal{T}} D$.
2. for all concepts E: if $C_i \sqsubseteq_{\mathcal{T}} E$ is true $\forall i, 1 \leq i \leq k$, then $D \sqsubseteq_{\mathcal{T}} E$.

In comparison, a concept description D is called a *common subsumer* of concepts $C_1, C_2, \ldots, C_k$ w.r.t. $\mathcal{T}$, if the first condition in the previous definition holds. We denote a *common subsumer* of $C_1, C_2, \ldots, C_k$ w.r.t. $\mathcal{T}$ as $cs_{\mathcal{T}}(C_1, C_2, \ldots, C_k)$.
For instance, using the TBox $\mathcal{T}_1$ from Example 2.8, we can conclude that $Parent \in cs_{\mathcal{T}_1}(Father, Mother)$ and $Human \in cs_{\mathcal{T}_1}(Father, Mother)$. As you can see, it is possible to have a number of common subsumers. On the contrary, the least common subsumer is unique up to equivalence and for concepts *Father* and *Mother* we obtain only *Parent* as the *lcs* w.r.t. the TBox $\mathcal{T}_1$. Notice, the *lcs* of concepts is unique up to equivalence in DL containing $\sqcap$-constructor [13].
We define the *role-depth* of a concept D, in symbols $rd(D)$, as the maximal nesting number of quantifiers occurring in this concept D. Moreover, if in Definition 2.9 $rd(D) \leq l$ and $rd(E) \leq l$, $l \in \mathbb{N}$, then we call such an approximation of the *lcs* as the role-depth bounded *lcs* and denote it as l-$lcs(C_1, C_2, \ldots, C_k)$. Obviously, in order to claim that two different grandparents have the same kind of grandchild it is sufficient to find an appropriate 2-*lcs* of these grandparents. Otherwise, if there is no suitable 2-*lcs*, then there is no need to check for the least common subsumer with the role-depth more than 2.
Let us now introduce the notion of a concept definition.

Definition 2.10. (Concept definition). A *concept definition* is of the form $A \equiv E$, where $A \in N_C$, i.e. A is an element of the set of concept names, and E is a concept description.

The TBox $\mathcal{T}$ contains a *cyclic definition*, if there exists a finite sequence of concept definitions $A_1 \equiv E_1, A_2 \equiv E_2, \ldots, A_n \equiv E_n$, $n \geq 1$, such that:

1. E_i contains the concept name A_{i-1} for all i, $1 < i \leq n$.

2. the concept name A_n occurs in E_1.

We need this notion for formulating the definition of cyclic TBoxes.

Definition 2.11. (Cyclic TBox). A finite set of concept definitions, which has at least one cyclic definition, is called a *cyclic* TBox.

In other words, if the TBox contains a cyclic definition, then the TBox is cyclic. We presented the notion of cyclic TBoxes, because for cyclic TBoxes the least common subsumer does not always exist.
Let us explore the following cyclic TBox $\mathcal{T} = \{A_1 \sqsubseteq E \sqcap \exists r.A_1,\ A_2 \sqsubseteq E \sqcap \exists r.A_2\}$. Consequently, we derive the infinite $lcs(A_1, A_2) = E \sqcap \exists r.(E \sqcap \exists r.(E \sqcap \exists r.(E \ldots$. In this case of cyclic TBoxes, in order to be able to find the finite least common subsumer we employ the role-depth bounded *lcs*, which is an approximation of the *lcs*.

2.2 Characteristic concept and normal form of *Horn-$\mathcal{ALC}$*

Interpretations can be represented as labeled graphs, such that:

- the elements of $\Delta^{\mathcal{I}}$ are nodes of the graph,

- interpretation of concept names are node labels,

- interpretation of role names are edges of the graph.

We can unravel such a graph, i.e. an interpretation $\mathcal{I}$, into a (possibly infinite) tree starting at a given node $d \in \Delta^{\mathcal{I}}$ as the root. Further in this work we use sometimes a tuple $(\mathcal{I}, d)$ in order to formally denote a pointed interpretation $\mathcal{I}$, where d is a domain element.
A *d-path* in an interpretation $\mathcal{I}$ is a sequence $\rho = d_0 r_1 d_1 r_2 d_2 \ldots$, where $d_0 = d$ and for all i holds that $\{d_i, d_{i+1}\} \subseteq \Delta^{\mathcal{I}}$ and they are nodes connected through the edge $r_{i+1}^{\mathcal{I}}$, viz. $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}}$.

Definition 2.12. (Tree unraveling). The *tree unraveling* of an interpretation $\mathcal{I}$ at a node d is the following interpretation $\mathcal{I}_d$, $d \in \Delta^{\mathcal{I}}$ (for all $A \in N_C$ and for all $r \in N_R$):

- $\Delta^{\mathcal{I}_d} := \{\rho \mid \rho$ is a *d-path* in $\mathcal{I}\}$,

- $A^{\mathcal{I}_d} := \{\sigma d_n \mid \sigma d_n \in \Delta^{\mathcal{I}_d}$ and $d_n \in A^{\mathcal{I}}\}$,

- $r^{\mathcal{I}_d} := \{(\sigma, \sigma r d_n) \mid (\sigma, \sigma r d_n) \in \Delta^{\mathcal{I}_d} \times \Delta^{\mathcal{I}_d}\}$.

The number of role names in σ is called the *length* of σ, in symbols $|\sigma|$. Note, for $\sigma = dr_1d_1 \dots r_ld_l$ we define d_l as $tail(\sigma)$. The interpretation $\mathcal{I}^l_d$ is the finite subtree limited to depth l of the unraveled tree $\mathcal{I}_d$ with the root at d [16].
Now we are able to present an l-characteristic concept, which is based on this finite subtree with limited depth l. Later we will show, that the l-characteristic concept of the product of the canonical models of the input concepts E and F belongs to $cs_{\mathcal{T}}(E, F)$.

Definition 2.13. (Characteristic concept). The *l-characteristic concept* $X^l(\mathcal{I}, d)$ of an interpretation $(\mathcal{I}, d)$ is of the form:

- $X^0(\mathcal{I}, d) := \ \bigsqcap \{A \in N_C \mid d \in A^{\mathcal{I}}\}$,
- $X^l(\mathcal{I}, d) := \ X^0(\mathcal{I}, d) \sqcap \bigsqcap\limits_{r \in N_R} \bigsqcap \{\exists r.X^{l-1}(\mathcal{I}, e) \mid (d, e) \in r^{\mathcal{I}}\}$.

In order to use later the notion of product models we introduce now a product interpretation.

Definition 2.14. (Product interpretation). The *product interpretation*, written as $\mathcal{I}_1 \times \mathcal{I}_2$, of two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ (for all $A \in N_C$ and for all $r \in N_R$) is defined as follows:

- $\Delta^{\mathcal{I}_1 \times \mathcal{I}_2} := \ \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$,
- $A^{\mathcal{I}_1 \times \mathcal{I}_2} := \ \{(d_1, d_2) \mid (d_1, d_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2} \text{ and } d_1 \in A^{\mathcal{I}_1}, d_2 \in A^{\mathcal{I}_2}\}$,
- $r^{\mathcal{I}_1 \times \mathcal{I}_2} := \ \{((d_1, d_2), (e_1, e_2)) \mid ((d_1, d_2), (e_1, e_2)) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2} \times \Delta^{\mathcal{I}_1 \times \mathcal{I}_2} \text{ and }$
$(d_1, e_1) \in r^{\mathcal{I}_1}, (d_2, e_2) \in r^{\mathcal{I}_2}\}$.

Further, for presenting a class of *Horn-$\mathcal{ALC}$* ontologies we will need the notion of polarities. *Positive* (or *negative*) *polarity* of $\mathcal{ALC}$-concept is defined recursively in the following way [14]:

- C is positive in C,
- if C is positive (or negative) in E, then C is positive (or negative) in $E \sqcap F$, $E \sqcup F$, $\exists r.E$, $\forall r.E$, $F \sqsubseteq E$ and C is negative (or positive) in $\neg E$, $E \sqsubseteq F$.

An $\mathcal{ALC}$ ontology is called *Horn-$\mathcal{ALC}$* if no concept of the form $E \sqcup F$ occurs positively and no concept of the form $\neg E$, $\forall r.E$ occurs negatively in this ontology [8]. For example, positive disjunction in the axiom $A \sqsubseteq E \sqcup F$ is not allowed.
Next, *Horn-$\mathcal{ALC}$* ontology is in *normal form* if it has only axioms such as [15]:

$$\bigsqcap_{i=1}^{n} A_i \sqsubseteq B, \ A \sqsubseteq \exists r.B, \ \exists r.A \sqsubseteq B, \ A \sqsubseteq \forall r.B \ or \ \bigsqcap_{i=1}^{n} A_i \sqsubseteq \bot,$$

where A_i, A and B are concept names. It should be noted, the big conjunction $\bigsqcap_{i=1}^{n} A_i$ means $A_1 \sqcap A_2 \sqcap \dots \sqcap A_n$, where A_i is called a conjunct. Further, we write $A_i \in C$ (as for sets), if C is of the form $\bigsqcap_{i=1}^{n} A_i$. Moreover, we use $\bigsqcap A_i$ instead of $\bigsqcap_{i=1}^{n} A_i$, if the range is not really important. It can happen, that $n = 0$ and in this case the empty conjunction is recognized as $\top$.
The set of all concept names, which are contained in a TBox $\mathcal{T}$, we denote as $N_{C,\mathcal{T}}$. Correspondingly, $N_{R,\mathcal{T}}$ is the set of all role names, which occur in a TBox

$\mathcal{T}$. Moreover, we denote $N_{C,L}$ ($N_{R,L}$) the set of all concept names (role names) occurring in a concept L.
If we want to build a canonical model for an $\mathcal{EL}$-concept L w.r.t. a normalized *Horn-$\mathcal{ALC}$* TBox $\mathcal{T}$, then we introduce a fresh concept name A_L, such that $A_L \equiv L$. In this case, $\widehat{\mathcal{T}} = \mathcal{T} \cup \{A_L \equiv L\}$. Since a concept L can be a complex concept, therefore, adding an equivalence $A_L \equiv L$ to a TBox $\mathcal{T}$ can lead to the result that an obtained TBox $\widehat{\mathcal{T}}$ is not in normal form. In order to normalize a TBox $\widehat{\mathcal{T}}$ we should exhaustively apply the following normalization rules:

NR1. $A \equiv B \sqcap C \to \{A_C \equiv C, A \equiv B \sqcap A_C\}$

NR2. $A \equiv \exists r.D \to \{A_D \equiv D, A \equiv \exists r.A_D\}$

NR3. $A \equiv B_1 \sqcap B_2 \to \{A \sqsubseteq B_1, A \sqsubseteq B_2, B_1 \sqcap B_2 \sqsubseteq A\}$

NR4. $A \equiv \exists r.B \to \{A \sqsubseteq \exists r.B, \exists r.B \sqsubseteq A\}$

NR5. $A \equiv B \to \{A \sqsubseteq B, B \sqsubseteq A\}$,

where A, B, B_1 and B_2 are concept names; C and D are complex concepts; A_C and A_D are new concept names.

Exhaustively means that we have to apply these rules until no rule can be applied to a TBox $\widehat{\mathcal{T}}$. Furthermore, using normalization rules proposed above, we should consider them as commutative rules regarding conjunction.
Thus, by applying rules NR1-NR5 we transform a TBox $\widehat{\mathcal{T}}$ to a normalized TBox $\mathcal{T}_L$, $N_{C,\mathcal{T}} \subset N_{C,\mathcal{T}_L}$. If $A \in N_{C,\mathcal{T}_L} \setminus \{N_{C,\mathcal{T}} \cup N_{C,L}\}$, then there exists a concept definition $A \equiv D \in \mathcal{T}_L$ with $N_{C,D} \subseteq N_{C,\mathcal{T}} \cup N_{C,L}$.

3 Finding conditions for the existence of the *lcs*

3.1 Canonical model

For an $\mathcal{EL}$-concept L and a normalized *Horn-$\mathcal{ALC}$* TBox $\mathcal{T}_L$ a *canonical model* $\mathcal{I}_{\mathcal{T}_L}$ is constructed as follows.

Definition 3.1. (Canonical model). Let L be an $\mathcal{EL}$-concept and $\mathcal{T}_L$ be a normalized *Horn-$\mathcal{ALC}$* TBox. Then, an interpretation $\mathcal{I}_{\mathcal{T}_L} = (\Delta^{\mathcal{I}_{\mathcal{T}_L}}, \cdot^{\mathcal{I}_{\mathcal{T}_L}})$ is called a *canonical model* of L and $\mathcal{T}_L$ if holds:

1. $\Delta^{\mathcal{I}_{\mathcal{T}_L}} := \{d_M | M = \bigsqcap A_i, A_i \in N_{C,\mathcal{T}_L}, 1 \leq i \leq n, n \geq 0, M \not\sqsubseteq_{\mathcal{T}_L} \bot\}$.
2. for all $A \in N_{C,\mathcal{T}_L}$: $A^{\mathcal{I}_{\mathcal{T}_L}} := \{d_M | M \sqsubseteq_{\mathcal{T}_L} A\}$.
3. for all $r \in N_{R,\mathcal{T}_L}$: $r^{\mathcal{I}_{\mathcal{T}_L}} := \{(d_M, d_K) | M \sqsubseteq_{\mathcal{T}_L} \exists r.K$ and K is maximal$\}$, where the maximality of a conjunction K means $\nexists N$ such that $M \sqsubseteq_{\mathcal{T}_L} \exists r.N$ and $K \subsetneq N$.

Thus, in order to define a *canonical model* $\mathcal{I}_{\mathcal{T}_L}$ we establish for every conjunction M of concept names from $N_{C,\mathcal{T}_L}$ ($M \not\sqsubseteq_{\mathcal{T}_L} \bot$) a distinguished domain element d_M, i.e. $d_M \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$. Furthermore, this distinguished element $d_M \in M^{\mathcal{I}_{\mathcal{T}_L}}$, what will be formally shown later.
There are at most exponentially many elements d_M in the domain of a canonical model. The computational complexity of deciding whether subsumption holds w.r.t. a general $\mathcal{ALC}$ TBox is ExpTime-complete [2]. Therefore, checking that $M \not\sqsubseteq_{\mathcal{T}_L} \bot$ w.r.t. a normalized *Horn-$\mathcal{ALC}$* TBox $\mathcal{T}_L$ takes at most exponential time. The sizes of $A^{\mathcal{I}_{\mathcal{T}_L}}$ and $r^{\mathcal{I}_{\mathcal{T}_L}}$ are exponentially bounded in the size of the input, checking subsumptions ($M \sqsubseteq_{\mathcal{T}_L} A$ and $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$) is exponential. Since $|N_{C,\mathcal{T}_L}|$ has linear bound in the size of the input (the normalization procedure is linear in the size of the input) and $|N_{R,\mathcal{T}_L}|$ is bounded by a natural number (the normalization procedure does not increase the number of role names), the whole process of building a canonical model runs in exponential time.
We define a set $\mathcal{I}^r_M := \{K \mid M \sqsubseteq_{\mathcal{T}_L} \exists r.K\}$ for each conjunction M such that $d_M \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$. In that case, $K \in \mathcal{I}^r_M$ is maximal in the set $\mathcal{I}^r_M$ if and only if there is no $N \in \mathcal{I}^r_M$ such that $K \subsetneq N$.
A canonical model $\mathcal{I}_{\mathcal{T}_L}$ of L and $\mathcal{T}_L$ is denoted formally as a tuple ($\mathcal{I}_{\mathcal{T}_L}$, d_{A_L}), where d_{A_L} is a domain element. Instead of ($\mathcal{I}_{\mathcal{T}_L}$, d_{A_L}) we will write sometimes only $\mathcal{I}_{\mathcal{T}_L}$. Notice, further in this work when we deal with a canonical model, we mean that this is a canonical model of an $\mathcal{EL}$-concept and a normalized *Horn-$\mathcal{ALC}$* TBox.
Next, in order to clarify the notion of a canonical model let us consider the following example.

Example 3.2. Let a concept $L = A \sqcap \exists r.\exists r.D$ and a TBox $\mathcal{T} = \emptyset$. Therefore,

$$\widehat{\mathcal{T}} = \mathcal{T} \cup \{A_L \equiv A \sqcap \exists r.\exists r.D\}.$$

First, after applying the normalization rule NR1 we have

$$\widehat{\mathcal{T}_1} = \mathcal{T} \cup \{A_L \equiv A \sqcap A_{\exists r \exists r.D}, A_{\exists r \exists r.D} \equiv \exists r.\exists r.D\}.$$

Second, we use the rule NR2 and obtain

$$\widehat{\mathcal{T}_2} = \mathcal{T} \cup \{A_L \equiv A \sqcap A_{\exists r \exists r.D}, A_{\exists r.D} \equiv \exists r.D, A_{\exists r \exists r.D} \equiv \exists r.A_{\exists r.D}\}.$$

Finally, by applying exhaustively rules NR3 and NR4 we obtain a normalized TBox $\mathcal{T}_L$ in the following form

$$\mathcal{T}_L = \{A_L \sqsubseteq A, A_L \sqsubseteq A_{\exists r \exists r.D}, A \sqcap A_{\exists r \exists r.D} \sqsubseteq A_L, A_{\exists r.D} \sqsubseteq \exists r.D, \exists r.D \sqsubseteq A_{\exists r.D}, \\ A_{\exists r \exists r.D} \sqsubseteq \exists r.A_{\exists r.D}, \exists r.A_{\exists r.D} \sqsubseteq A_{\exists r \exists r.D}\}.$$

Thus, $N_{C,\mathcal{T}_L} = \{A_L, A, A_{\exists r \exists r.D}, A_{\exists r.D}, D\}$ and $N_{R,\mathcal{T}_L} = \{r\}$. Afterward, we build a canonical model $\mathcal{I}_{\mathcal{T}_L} = (\Delta^{\mathcal{I}_{\mathcal{T}_L}}, \cdot^{\mathcal{I}_{\mathcal{T}_L}})$ as follows:

- $\Delta^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M = \bigsqcap A_i, A_i \in N_{C,\mathcal{T}_L}, 1 \leq i \leq n, n \geq 0, M \not\sqsubseteq_{\mathcal{T}_L} \bot\} =$
$\{d_\top, d_{A_L}, d_A, d_{A_{\exists r \exists r.D}}, d_{A_{\exists r.D}}, d_D, d_{A_L \sqcap A}, d_{A_L \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A_{\exists r.D}}, d_{A_L \sqcap D},$
$d_{A \sqcap A_{\exists r \exists r.D}}, d_{A \sqcap A_{\exists r.D}}, d_{A \sqcap D}, d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r.D} \sqcap D},$
$d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$

- $A_L^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M \sqsubseteq_{\mathcal{T}_L} A_L\} = \{d_{A_L}, d_{A_L \sqcap A}, d_{A_L \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A_{\exists r.D}}, d_{A_L \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D}}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$
$A^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M \sqsubseteq_{\mathcal{T}_L} A\} = \{d_{A_L}, d_A, d_{A_L \sqcap A}, d_{A_L \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A_{\exists r.D}}, d_{A_L \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D}},$
$d_{A \sqcap A_{\exists r.D}}, d_{A \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D},$
$d_{A_L \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$
$A_{\exists r \exists r.D}^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M \sqsubseteq_{\mathcal{T}_L} A_{\exists r \exists r.D}\} = \{d_{A_L}, d_{A_{\exists r \exists r.D}}, d_{A_L \sqcap A}, d_{A_L \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A_{\exists r.D}},$
$d_{A_L \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D}}, d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap D},$
$d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap D},$
$d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$
$A_{\exists r.D}^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M \sqsubseteq_{\mathcal{T}_L} A_{\exists r.D}\} = \{d_{A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r.D}}, d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$
$D^{\mathcal{I}_{\mathcal{T}_L}} = \{d_M | M \sqsubseteq_{\mathcal{T}_L} D\} = \{d_D, d_{A_L \sqcap D}, d_{A \sqcap D}, d_{A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap D},$
$d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D}, d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D},$
$d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}\}.$

- $r^{\mathcal{I}_{\mathcal{T}_L}} = \{(d_M, d_K) | M \sqsubseteq_{\mathcal{T}_L} \exists r.K$ and $\nexists N$ such that $M \sqsubseteq_{\mathcal{T}_L} \exists r.N$ with $K \subsetneq N\} =$
  $\{(d_{A_{\exists r.D}}, d_D), (d_{A_{\exists r \exists r.D}}, d_{A_{\exists r.D}}), (d_{A_L \sqcap A_{\exists r.D}}, d_D), (d_{A \sqcap A_{\exists r.D}}, d_D), (d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_D),$
  $(d_{A_{\exists r.D} \sqcap D}, d_D), (d_{A_L \sqcap A \sqcap A_{\exists r.D}}, d_D), (d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_D), (d_{A_L \sqcap A_{\exists r.D} \sqcap D}, d_D),$
  $(d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_D), (d_{A \sqcap A_{\exists r.D} \sqcap D}, d_D), (d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_D),$
  $(d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_D), (d_{A_L \sqcap A \sqcap A_{\exists r.D} \sqcap D}, d_D), (d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_D),$
  $(d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_D), (d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_D), (d_{A_L \sqcap A_{\exists r \exists r.D}}, d_{A_{\exists r.D}}),$
  $(d_{A \sqcap A_{\exists r \exists r.D}}, d_{A_{\exists r.D}}), (d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r.D}}), (d_{A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D}}, d_{A_{\exists r.D}}),$
  $(d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r.D}}), (d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r.D}}),$
  $(d_{A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D}}, d_{A_{\exists r.D}}),$
  $(d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A_L \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r.D}}), (d_{A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r.D}}),$
  $(d_{A_L \sqcap A \sqcap A_{\exists r \exists r.D} \sqcap A_{\exists r.D} \sqcap D}, d_{A_{\exists r.D}})\}$.

It is easy to see, that the following Lemma 3.3 is a straightforward outcome of Definition 3.1. Note, in Lemmas 3.3, 3.4 and 3.5 we use $\mathcal{I}_{\mathcal{T}_L}$ as a canonical model of a concept L and a TBox $\mathcal{T}$.

Lemma 3.3. *The following two clauses are true:*

1. *if $d_M \in A^{\mathcal{I}_{\mathcal{T}_L}}$, then $M \sqsubseteq_{\mathcal{T}_L} A$.*
2. *if $(d_M, d_K) \in r^{\mathcal{I}_{\mathcal{T}_L}}$, then $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$, where K is maximal in $\mathcal{I}^r_M$.*

Further, we will show the basic property of an element d_M, namely if an element d_M belongs to the domain of a canonical model $\mathcal{I}_{\mathcal{T}_L}$, then d_M has to belong to $M^{\mathcal{I}_{\mathcal{T}_L}}$.

Lemma 3.4. *If $d_M \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$, then $d_M \in M^{\mathcal{I}_{\mathcal{T}_L}}$.*

Proof. In order to claim $d_M \in M^{\mathcal{I}_{\mathcal{T}_L}}$ it is sufficient to prove that $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$ for all conjuncts $B \in M$.
If $B \in M$, then M is of the form $M = M' \sqcap B$. We can deduce that $M' \sqcap B \sqsubseteq_{\mathcal{T}_L} B$. Therefore, $d_{M' \sqcap B} \in B^{\mathcal{I}_{\mathcal{T}_L}}$ follows immediately from Definition 3.1. □

In a subsequent step, using our Example 3.2, we can check that $d_M \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ implies $d_M \in M^{\mathcal{I}_{\mathcal{T}_L}}$. For instance, $d_A \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ yields $d_A \in A^{\mathcal{I}_{\mathcal{T}_L}}$ and from $d_{A_L \sqcap A \sqcap D} \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ we get $d_{A_L \sqcap A \sqcap D} \in (A_L \sqcap A \sqcap D)^{\mathcal{I}_{\mathcal{T}_L}}$, because $d_{A_L \sqcap A \sqcap D} \in (A_L^{\mathcal{I}_{\mathcal{T}_L}} \cap A^{\mathcal{I}_{\mathcal{T}_L}} \cap D^{\mathcal{I}_{\mathcal{T}_L}})$. Now we claim the following important lemma that a canonical model $\mathcal{I}_{\mathcal{T}_L}$ is indeed a model of a TBox $\mathcal{T}_L$ (and $\mathcal{T}$, respectively).

Lemma 3.5. *$\mathcal{I}_{\mathcal{T}_L} \models \mathcal{T}_L$ ($\mathcal{I}_{\mathcal{T}_L} \models \mathcal{T}$).*

Proof. Let us consider all possible cases of GCIs which can occur in a normalized *Horn-$\mathcal{ALC}$* TBox $\mathcal{T}_L$.

- $\bigsqcap_{i=1}^n A_i \sqsubseteq_{\mathcal{T}_L} B$, $n \geq 0$. For an arbitrary $d_M \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ such that $d_M \in A_i^{\mathcal{I}_{\mathcal{T}_L}}$ for all i, $1 \leq i \leq n$ we have to show $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$. Firstly, from Lemma 3.3, claim 1 we can conclude that $M \sqsubseteq_{\mathcal{T}_L} A_i$ for all i, $1 \leq i \leq n$. Therefore, we derive $M \sqsubseteq_{\mathcal{T}_L} \bigsqcap_{i=1}^n A_i$, $n \geq 1$. Next, since $M \sqsubseteq \top$ we clarify that $M \sqsubseteq_{\mathcal{T}_L} \bigsqcap_{i=1}^n A_i$, where $n \geq 0$. Afterward, by the transitivity of subsumption relations we deduce $M \sqsubseteq_{\mathcal{T}_L} B$. Finally, it holds that $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$ due to Definition 3.1 of canonical models.

- $A \sqsubseteq_{\mathcal{T}_L} \exists r.B$. For an arbitrary $d_M \in A^{\mathcal{I}_{\mathcal{T}_L}}$ we show $d_M \in (\exists r.B)^{\mathcal{I}_{\mathcal{T}_L}}$. By Claim 1 of Lemma 3.3 it follows that $M \sqsubseteq_{\mathcal{T}_L} A$. From $M \sqsubseteq_{\mathcal{T}_L} A$ and $A \sqsubseteq_{\mathcal{T}_L} \exists r.B$ we obtain $M \sqsubseteq_{\mathcal{T}_L} \exists r.B$, i.e. $B \in \mathcal{I}^r_M$. All elements from $\mathcal{I}^r_M$ may contain only concept names from $\mathcal{T}_L$. Because of this, $\mathcal{I}^r_M$ has to be finite. As a result, there exists a maximal K in $\mathcal{I}^r_M$ such that $B \in K$. In consequence, $(d_M, d_K) \in r^{\mathcal{I}_{\mathcal{T}_L}}$ by Definition 3.1, correspondingly, $d_K \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$. Lemma 3.4 implies that $d_K \in K^{\mathcal{I}_{\mathcal{T}_L}}$. Because $B \in K$, then $d_K \in B^{\mathcal{I}_{\mathcal{T}_L}}$. Hence, by the semantics of existential restrictions $d_M \in (\exists r.B)^{\mathcal{I}_{\mathcal{T}_L}}$.

- $\exists r.A \sqsubseteq_{\mathcal{T}_L} B$. For an arbitrary $d_M \in (\exists r.A)^{\mathcal{I}_{\mathcal{T}_L}}$ we have to show $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$. Due to the semantics of existential restrictions, there exists $d_K \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ such that $(d_M, d_K) \in r^{\mathcal{I}_{\mathcal{T}_L}}$ and $d_K \in A^{\mathcal{I}_{\mathcal{T}_L}}$. By Lemma 3.3, claim 2 there exists a maximal $K \in \mathcal{I}^r_M$ such that $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$. Lemma 3.3, claim 1 leads to $K \sqsubseteq_{\mathcal{T}_L} A$. $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$ and $K \sqsubseteq_{\mathcal{T}_L} A$ give us $M \sqsubseteq_{\mathcal{T}_L} \exists r.A$. Thus, from the latter using the given $\exists r.A \sqsubseteq_{\mathcal{T}_L} B$ we derive $M \sqsubseteq_{\mathcal{T}_L} B$. Therefore, $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$ follows immediately from Definition 3.1.

- $A \sqsubseteq_{\mathcal{T}_L} \forall r.B$. For an arbitrary $d_M \in A^{\mathcal{I}_{\mathcal{T}_L}}$ we show that $d_M \in (\forall r.B)^{\mathcal{I}_{\mathcal{T}_L}}$, namely for all $d_K \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ it holds: if $(d_M, d_K) \in r^{\mathcal{I}_{\mathcal{T}_L}}$, then $d_K \in B^{\mathcal{I}_{\mathcal{T}_L}}$. Due to Lemma 3.3, claim 2 there exists a maximal K from the set $\mathcal{I}^r_M$ such that $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$. From $d_M \in A^{\mathcal{I}_{\mathcal{T}_L}}$ using Lemma 3.3, claim 1 we deduce $M \sqsubseteq_{\mathcal{T}_L} A$. Consequently, due to the transitivity of subsumption relations it follows $M \sqsubseteq_{\mathcal{T}_L} \forall r.B$. From the latter and $M \sqsubseteq_{\mathcal{T}_L} \exists r.K$ we have $M \sqsubseteq_{\mathcal{T}_L} \exists r.(K \sqcap B)$, that is $(K \sqcap B) \in \mathcal{I}^r_M$ by the definition of $\mathcal{I}^r_M$. Since K is maximal in $\mathcal{I}^r_M$, hence, $B \in K$. On the other hand, Lemma 3.4 brings us to the conclusion that $d_K \in K^{\mathcal{I}_{\mathcal{T}_L}}$. Lastly, from $B \in K$ and $d_K \in K^{\mathcal{I}_{\mathcal{T}_L}}$ we obtain $d_K \in B^{\mathcal{I}_{\mathcal{T}_L}}$.

□

3.2 Properties of canonical models, simulation and characteristic concepts

Afterward, we will widely use the notion of a *simulation* relation and *simulation-equivalence*. For instance, if a canonical model $(\mathcal{I}_{\mathcal{T}_G}, d_G)$ is simulation-equivalent to the product model $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$, then we can claim that a concept G is the *lcs* of the concepts E and F. Let us start with the first notion of a *simulation* relation.

Definition 3.6. (Simulation). Let $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ and $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$ be two interpretations and $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$. A non-empty relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is called a *simulation relation* from $\mathcal{I}_1$ to $\mathcal{I}_2$ if both of the following statements are fulfilled:

- for all $A \in N_C$ and all $(d_1, d_2) \in S$ if $d_1 \in A^{\mathcal{I}_1}$ then $d_2 \in A^{\mathcal{I}_2}$,

- for all $r \in N_R$, all $(d_1, d_2) \in S$ and all $e_1 \in \Delta^{\mathcal{I}_1}$ such that $(d_1, e_1) \in r^{\mathcal{I}_1}$ there exists $e_2 \in \Delta^{\mathcal{I}_2}$ with $(d_2, e_2) \in r^{\mathcal{I}_2}$ and $(e_1, e_2) \in S$.

If there exists a *simulation* relation S from $\mathcal{I}_1$ to $\mathcal{I}_2$ such that $(d_1, d_2) \in S$, then we say that an interpretation $(\mathcal{I}_1, d_1)$ is *simulated* by an interpretation $(\mathcal{I}_2, d_2)$ (or $(\mathcal{I}_2, d_2)$ *simulates* $(\mathcal{I}_1, d_1)$), in symbols $(\mathcal{I}_1, d_1) \lesssim (\mathcal{I}_2, d_2)$.

Definition 3.7. (Simulation-equivalence). Interpretations $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are called *simulation-equivalent*, denoted as $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, if it satisfies that $(\mathcal{I}_1, d_1) \lesssim (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \lesssim (\mathcal{I}_1, d_1)$, simultaneously.

Next, we introduce the notion of a *maximal simulation*. A simulation S from $\mathcal{I}_1$ to $\mathcal{I}_2$ is *maximal*, if and only if for all simulations S_i from $\mathcal{I}_1$ to $\mathcal{I}_2$ it holds that $S_i \subseteq S$ for all i.
Further, Lemma 3.8 shows an expected interconnection between a simulation and the product interpretation, viz. if an interpretation $\mathcal{J}$ is simulated by two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, then an interpretation $\mathcal{J}$ is also simulated by the product of these two interpretations $\mathcal{I}_1 \times \mathcal{I}_2$ [10, 17].

Lemma 3.8. *Assume that $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ and $(\mathcal{J}, d)$ are interpretations. $(\mathcal{J}, d) \lesssim (\mathcal{I}_1, d_1)$ and $(\mathcal{J}, d) \lesssim (\mathcal{I}_2, d_2)$ imply $(\mathcal{J}, d) \lesssim (\mathcal{I}_1 \times \mathcal{I}_2, (d_1, d_2))$.*

There exists the following relationship between models of the TBox $\mathcal{T}$ and a canonical model, which was constructed w.r.t. the TBox $\mathcal{T}$.

Lemma 3.9. *Let $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$ be a canonical model. For all models $\mathcal{I}$ of the TBox $\mathcal{T}$ and all $d \in \Delta^{\mathcal{I}}$ it is true that $d \in L^{\mathcal{I}}$ if and only if $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}, d)$.*

Proof. We will prove this lemma in two directions.

1. Given $d \in L^{\mathcal{I}}$, we have to show that $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}, d)$. We establish a relation $S \subseteq \Delta^{\mathcal{I}_{\mathcal{T}_L}} \times \Delta^{\mathcal{I}}$ such that $(d_M, e) \in S$ if and only if $e \in M^{\mathcal{I}}$.

   Assume $(d_M, e) \in S$, $M = A_1 \sqcap \cdots \sqcap A_n$, hence, it holds:

   (a) if $A_i \in N_{C,\mathcal{T}} \cup N_{C,L}$, then $e \in A_i^{\mathcal{I}}$,

   (b) if $A_i \notin N_{C,\mathcal{T}} \cup N_{C,L}$, then A_i is A_D, where $A_D \equiv_{\mathcal{T}_L} D$ and $e \in D^{\mathcal{I}}$ $(D \in N_{C,\mathcal{T}} \cup N_{C,L})$.

   Let us now check that this relation S is a simulation.

   - If $d_M \in B^{\mathcal{I}_{\mathcal{T}_L}}$, then by Lemma 3.3, claim 1 we derive that $M \sqsubseteq_{\mathcal{T}_L} B$. Next, if any $A_D \in M$ such that $A_D \notin N_{C,\mathcal{T}} \cup N_{C,L}$, i.e. there exists a concept definition $A_D \equiv_{\mathcal{T}_L} D$, then we replace A_D with D, where $D \in N_{C,\mathcal{T}} \cup N_{C,L}$. Finally, by replacing all $A_i \in N_{C,\mathcal{T}_L} \setminus \{N_{C,\mathcal{T}} \cup N_{C,L}\}$ in M we obtain $\widetilde{M}$, which contains only $\widetilde{A_i} \in N_{C,\mathcal{T}} \cup N_{C,L}$. In the same way, if $B \notin N_{C,\mathcal{T}} \cup N_{C,L}$, we replace B by $\widetilde{B} \in N_{C,\mathcal{T}} \cup N_{C,L}$, $B \equiv_{\mathcal{T}_L} \widetilde{B}$. Due to this replacement and $M \sqsubseteq_{\mathcal{T}_L} B$ we have that $\widetilde{M} \sqsubseteq_{\mathcal{T}} \widetilde{B}$.

     Since $e \in \widetilde{M}^{\mathcal{I}}$ follows from our assumption ((a),(b)) and $\mathcal{I}$ is a model of the TBox $\mathcal{T}$, namely $\mathcal{I} \models \widetilde{M} \sqsubseteq_{\mathcal{T}} \widetilde{B}$, we deduce that $e \in \widetilde{B}^{\mathcal{I}}$, $B \equiv_{\mathcal{T}_L} \widetilde{B}$.
   - If $(d_M, d_{M'}) \in r^{\mathcal{I}_{\mathcal{T}_L}}$, then Lemma 3.3, claim 2 implies $M \sqsubseteq_{\mathcal{T}_L} \exists r.M'$. Analogously, by replacing in M and M' all concept names from $N_{C,\mathcal{T}_L} \setminus \{N_{C,\mathcal{T}} \cup N_{C,L}\}$ by their equivalences from $N_{C,\mathcal{T}} \cup N_{C,L}$, we get $\widetilde{M}$ and $\widetilde{M'}$. The consequence is $\widetilde{M} \sqsubseteq_{\mathcal{T}} \exists r.\widetilde{M'}$.

     Therefore, $e \in \widetilde{M}^{\mathcal{I}}$ (from (a),(b)) leads to $e \in (\exists r.\widetilde{M'})^{\mathcal{I}}$, because $\mathcal{I}$ is a model. Next, $e \in (\exists r.\widetilde{M'})^{\mathcal{I}}$ brings us to the existence of $e' \in \Delta^{\mathcal{I}}$ such that $(e, e') \in r^{\mathcal{I}}$ and $e' \in \widetilde{M'}^{\mathcal{I}}$. Lastly, from the latter due to the construction of a relation S and an equivalence $M' \equiv_{\mathcal{T}_L} \widetilde{M'}$ we obtain that $(d_{M'}, e') \in S$.

Thus, both requirements of a simulation are satisfied, S is a simulation relation from $\mathcal{I}_{\mathcal{T}_L}$ to $\mathcal{I}$. Recall, $d \in L^{\mathcal{I}}$, i.e. $(d_{A_L}, d) \in S$. To sum up, we can conclude that $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}, d)$.

2. Let us show converse direction. Given $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}, d)$, to prove $d \in L^{\mathcal{I}}$. Due to Definition 3.6 of a simulation S from $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}, d)$ we have $(d_{A_L}, d) \in S$. $\mathcal{I}_{\mathcal{T}_L}$ is a canonical model, $d_{A_L} \in \Delta^{\mathcal{I}_{\mathcal{T}_L}}$, therefore, it follows $d_{A_L} \in A_L^{\mathcal{I}_{\mathcal{T}_L}}$. To summarise, since $(d_{A_L}, d) \in S$, S is a simulation from $\mathcal{I}_{\mathcal{T}_L}$ to $\mathcal{I}$, then $d_{A_L} \in A_L^{\mathcal{I}_{\mathcal{T}_L}}$ and $A_L \equiv_{\mathcal{T}_L} L$ give us $d \in L^{\mathcal{I}}$.

□

We will use the similar feature of canonical models that was presented in [12]. In a subsequent step, we formulate several interconnections between subsumption and canonical models, e.g. if a concept E is subsumed by a concept F w.r.t. $\mathcal{T}$, then a canonical model $\mathcal{I}_{\mathcal{T}_F}$ is simulated by a canonical model $\mathcal{I}_{\mathcal{T}_E}$.

Lemma 3.10. *Let E, F be $\mathcal{EL}$-concepts. There is equivalence of the following claims:*

1. $E \sqsubseteq_{\mathcal{T}} F$.
2. $d_E \in F^{\mathcal{I}_{\mathcal{T}_E}}$.
3. $(\mathcal{I}_{\mathcal{T}_F}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_E)$.

Proof. We will confirm these statements pairwise.

(1) ⇒ (2). It was shown in Lemma 3.5 that $\mathcal{I}_{\mathcal{T}_E}$ is a model of the TBox $\mathcal{T}$. Therefore, from $d_E \in E^{\mathcal{I}_{\mathcal{T}_E}}$ and $E \sqsubseteq_{\mathcal{T}} F$ we can derive that $d_E \in F^{\mathcal{I}_{\mathcal{T}_E}}$.

(2) ⇒ (3). Obviously, it follows from the previous Lemma 3.9.

(3) ⇒ (1). Assume $\mathcal{I}$ is an arbitrary model of the TBox $\mathcal{T}$ and $d \in E^{\mathcal{I}}$. Due to Lemma 3.9 $(\mathcal{I}_{\mathcal{T}_E}, d_E) \lesssim (\mathcal{I}, d)$. Simulation relations are transitive. Thus, $(\mathcal{I}_{\mathcal{T}_F}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_E)$ and $(\mathcal{I}_{\mathcal{T}_E}, d_E) \lesssim (\mathcal{I}, d)$ lead to $(\mathcal{I}_{\mathcal{T}_F}, d_F) \lesssim (\mathcal{I}, d)$. Lemma 3.9 implies $d \in F^{\mathcal{I}}$.

□

Next, we prove the following lemma, which is similar to the corresponding lemma from [12].

Lemma 3.11. *Let E, H be $\mathcal{EL}$-concepts. Then for all $d_F \in \Delta^{\mathcal{I}_{\mathcal{T}_E}} \cap \Delta^{\mathcal{I}_{\mathcal{T}_H}}$ it holds that $(\mathcal{I}_{\mathcal{T}_E}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_H}, d_F)$.*

Proof. Assume $d_F \in \Delta^{\mathcal{I}_{\mathcal{T}_E}} \cap \Delta^{\mathcal{I}_{\mathcal{T}_H}}$. We establish a relation $S \subseteq \Delta^{\mathcal{I}_{\mathcal{T}_E}} \times \Delta^{\mathcal{I}_{\mathcal{T}_H}}$ as follows: $(d_G, d_G) \in S$ if and only if $d_G \in \Delta^{\mathcal{I}_{\mathcal{T}_F}}$. Therefore, $(d_F, d_F) \in S$, because $d_F \in \Delta^{\mathcal{I}_{\mathcal{T}_F}}$. Since S is a simulation relation, we derive $(\mathcal{I}_{\mathcal{T}_E}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_H}, d_F)$. □

It should be noted, that $(\mathcal{I}_{\mathcal{T}_E}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_H}, d_F)$ and $(\mathcal{I}_{\mathcal{T}_H}, d_F) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_F)$ for all concepts E, H and all $d_F \in \Delta^{\mathcal{I}_{\mathcal{T}_E}} \cap \Delta^{\mathcal{I}_{\mathcal{T}_H}}$. Consequently, we can formulate the following corollary.

Corollary 3.12. *For all concepts H and all $d_F \in \Delta^{\mathcal{I}_{\mathcal{T}_E}} \cap \Delta^{\mathcal{I}_{\mathcal{T}_H}}$ it holds that $(\mathcal{I}_{\mathcal{T}_E}, d_F) \sim (\mathcal{I}_{\mathcal{T}_H}, d_F)$.*

Now we are able to introduce Lemma 3.13, which demonstrates some attributes of products of canonical models.

Lemma 3.13. *Assume $\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}$ and $\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}$ are product interpretations of canonical models such that $(d_C, d_D) \in \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}} \cap \Delta^{\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}}$. The following two statements are fulfilled:*

1. *$X^i(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_C, d_D)) \equiv X^i(\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}, (d_C, d_D))$ for every $i \in \mathbb{N}$.*
2. *if Q is a concept, then it holds that $(d_C, d_D) \in Q^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$ if and only if $C \sqsubseteq_{\mathcal{T}} Q$ and $D \sqsubseteq_{\mathcal{T}} Q$.*

Proof. Let us consider these statements one by one.

1. From Corollary 3.12 it follows $X^i(\mathcal{I}_{\mathcal{T}_E}, d_C) \equiv X^i(\mathcal{I}_{\mathcal{T}_G}, d_C)$ and $X^i(\mathcal{I}_{\mathcal{T}_F}, d_D) \equiv X^i(\mathcal{I}_{\mathcal{T}_H}, d_D)$ for every i. As a result, we deduce that $X^i(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_C, d_D)) \equiv X^i(\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}, (d_C, d_D))$ for each i.
2. $(d_C, d_D) \in Q^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$ iff $d_C \in Q^{\mathcal{I}_{\mathcal{T}_E}}$ and $d_D \in Q^{\mathcal{I}_{\mathcal{T}_F}}$ iff (by Lemma 3.10) $C \sqsubseteq_{\mathcal{T}} Q$ and $D \sqsubseteq_{\mathcal{T}} Q$.

□

Further, Lemma 3.14 implies that the l-characteristic concept of the product of the canonical models of the input concepts $(X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)))$ gives us the l-$lcs_{\mathcal{T}}(E, F)$ [16]. Because $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is a common subsumer of concepts E and F, $rd(X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))) \leq l$.

Lemma 3.14. *Let $l \in \mathbb{N}$ be an arbitrary natural number.*

1. *$X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \in cs_{\mathcal{T}}(E, F)$.*
2. *If $G \in cs_{\mathcal{T}}(E, F)$ and $rd(G) \leq l$, then $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} G$.*

Proof. The proof of this lemma is similar to the corresponding proof, which was shown by B. Zarrieß and A.-Y. Turhan in [17].

1. We start with confirming the first statement and use an induction on a natural number l.
   - Induction base $l = 0$. From Definition 2.13 of the characteristic concept it follows that

     $$X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) = \bigsqcap \{A \in N_C \mid (d_E, d_F) \in A^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}\}.$$

     As you can see from the right side of the previous formula for all $A \in N_C$, which occur in this right side, it is true that $(d_E, d_F) \in A^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$. From the latter statement using Definition 2.14 of the product interpretation we conclude that $d_E \in A^{\mathcal{I}_{\mathcal{T}_E}}$ and $d_F \in A^{\mathcal{I}_{\mathcal{T}_F}}$. Lemma 3.10 implies $E \sqsubseteq_{\mathcal{T}} A$ and $F \sqsubseteq_{\mathcal{T}} A$, which hold for all concept names A occurring in the (big) conjunction (in the right-hand side of the formula). Hence, we already have what we need, namely

     $$E \sqsubseteq_{\mathcal{T}} X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)), F \sqsubseteq_{\mathcal{T}} X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)),$$

     i.e. $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is a common subsumer of E and F.

- $l > 0$. Again, from Definition 2.13 it follows that

$$X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) = X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqcap \bigsqcap_{r \in N_R} \bigsqcap \{\exists r.X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)) \mid ((d_E, d_F), (d_G, d_H)) \in r^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}\}.$$

By the induction hypothesis we obtain

$$G \sqsubseteq_{\mathcal{T}} X^{l-1}(\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}, (d_G, d_H)),$$
$$H \sqsubseteq_{\mathcal{T}} X^{l-1}(\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}, (d_G, d_H)).$$

Since Lemma 3.13, claim 1 yields
$X^{l-1}(\mathcal{I}_{\mathcal{T}_G} \times \mathcal{I}_{\mathcal{T}_H}, (d_G, d_H)) \equiv X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H))$, then

$$G \sqsubseteq_{\mathcal{T}} X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)),$$
$$H \sqsubseteq_{\mathcal{T}} X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)).$$

Next, using Lemma 3.10, we derive the following

$$d_G \in (X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_G}},$$
$$d_H \in (X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_H}}.$$

By Corollary 3.12 we have that $(\mathcal{I}_{\mathcal{T}_G}, d_G) \sim (\mathcal{I}_{\mathcal{T}_E}, d_G)$, $(\mathcal{I}_{\mathcal{T}_H}, d_H) \sim (\mathcal{I}_{\mathcal{T}_F}, d_H)$. Therefore,

$$d_G \in (X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_E}},$$
$$d_H \in (X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_F}}.$$

Accordingly, $(d_G, d_H) \in (X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$ due to Definition 2.14 of a product interpretation.
Recall, $((d_E, d_F), (d_G, d_H)) \in r^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$, hence we can conclude

$$(d_E, d_F) \in (\exists r.X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)))^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}.$$

Lastly, from Lemma 3.13, claim 2 it follows

$$E \sqsubseteq_{\mathcal{T}} \exists r.X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)),$$
$$F \sqsubseteq_{\mathcal{T}} \exists r.X^{l-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_G, d_H)).$$

Thus, using the conclusion for $l = 0$ we have proved the first claim of this lemma, namely, $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \in cs_{\mathcal{T}}(E, F)$.

2. In order to confirm the second statement, we apply an induction on $rd(G)$, where $G \in cs_{\mathcal{T}}(E, F)$, $rd(G) \leq l$.

   - Induction base $rd(G) = 0$. $G \in cs_{\mathcal{T}}(E, F)$, i.e. $E \sqsubseteq_{\mathcal{T}} G$, $F \sqsubseteq_{\mathcal{T}} G$ and from Lemma 3.10 we deduce $d_E \in G^{\mathcal{I}_{\mathcal{T}_E}}$, $d_F \in G^{\mathcal{I}_{\mathcal{T}_F}}$. Because the role-depth of G equals to zero, $G = \bigsqcap A_i$. Consequently, $d_E \in (\bigsqcap A_i)^{\mathcal{I}_{\mathcal{T}_E}}$

and $d_F \in (\bigsqcap A_i)^{\mathcal{I}_{\mathcal{T}_F}}$, viz. for all i it holds $d_E \in A_i^{\mathcal{I}_{\mathcal{T}_E}}$, $d_F \in A_i^{\mathcal{I}_{\mathcal{T}_F}}$ and $(d_E, d_F) \in A_i^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$. Recall,

$$X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) = \bigsqcap \{A \in N_C \mid (d_E, d_F) \in A^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}\}$$

due to Definition 2.13 of the characteristic concept. Therefore, we can consider G as a conjunct in $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$, correspondingly, $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is subsumed by G w.r.t. $\mathcal{T}$. Since we have $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is a conjunct in $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$, then $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} G$.

- $rd(G) > 0$. Assume $rd(G) = k$ and $G = \bigsqcap A_i \sqcap \exists r_1.G'_1 \sqcap \cdots \sqcap \exists r_n.G'_n$. From the induction base it holds $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} \bigsqcap A_i$.
  Let for an arbitrary j, $1 \leq j \leq n$ consider a conjunct $\exists r_j.G'_j$ from G, accordingly, it follows that $G \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$. It is given, that G is a common subsumer of E and F w.r.t. $\mathcal{T}$. In other words, $E \sqsubseteq_{\mathcal{T}} G$ and $F \sqsubseteq_{\mathcal{T}} G$, consequently, $E \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$ and $F \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$ because of transitivity of subsumption relations. Further, Lemma 3.10 implies $d_E \in (\exists r_j.G'_j)^{\mathcal{I}_{\mathcal{T}_E}}$ and $d_F \in (\exists r_j.G'_j)^{\mathcal{I}_{\mathcal{T}_F}}$.
  Next, due to the semantics of concept descriptions there exists $d_{E'}$ ($d_{F'}$) such that $(d_E, d_{E'}) \in r_j^{\mathcal{I}_{\mathcal{T}_E}}$ ($(d_F, d_{F'}) \in r_j^{\mathcal{I}_{\mathcal{T}_F}}$) and $d_{E'} \in (G'_j)^{\mathcal{I}_{\mathcal{T}_E}}$ ($d_{F'} \in (G'_j)^{\mathcal{I}_{\mathcal{T}_F}}$). Now from $d_{E'} \in (G'_j)^{\mathcal{I}_{\mathcal{T}_E}}$ ($d_{F'} \in (G'_j)^{\mathcal{I}_{\mathcal{T}_F}}$) using Lemma 3.9 we obtain $(\mathcal{I}_{G'_j,\mathcal{T}}, d_{G'_j}) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_{E'})$ ($(\mathcal{I}_{G'_j,\mathcal{T}}, d_{G'_j}) \lesssim (\mathcal{I}_{\mathcal{T}_F}, d_{F'})$). By Corollary 3.12 we have $(\mathcal{I}_{\mathcal{T}_E}, d_{E'}) \sim (\mathcal{I}_{\mathcal{T}_{E'}}, d_{E'})$ ($(\mathcal{I}_{\mathcal{T}_F}, d_{F'}) \sim (\mathcal{I}_{\mathcal{T}_{F'}}, d_{F'})$).
  Thus, $(\mathcal{I}_{G'_j,\mathcal{T}}, d_{G'_j}) \lesssim (\mathcal{I}_{\mathcal{T}_{E'}}, d_{E'})$ ($(\mathcal{I}_{G'_j,\mathcal{T}}, d_{G'_j}) \lesssim (\mathcal{I}_{\mathcal{T}_{F'}}, d_{F'})$). Therefore, due to Lemma 3.10 it follows $E' \sqsubseteq_{\mathcal{T}} G'_j$ ($F' \sqsubseteq_{\mathcal{T}} G'_j$).
  Further, the induction hypothesis leads to

  $$X^{k-1}(\mathcal{I}_{\mathcal{T}_{E'}} \times \mathcal{I}_{\mathcal{T}_{F'}}, (d_{E'}, d_{F'})) \sqsubseteq_{\mathcal{T}} G'_j.$$

  Using Lemma 3.13 we derive that

  $$X^{k-1}(\mathcal{I}_{\mathcal{T}_{E'}} \times \mathcal{I}_{\mathcal{T}_{F'}}, (d_{E'}, d_{F'})) \equiv X^{k-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{E'}, d_{F'})).$$

  To sum up, we have $X^{k-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{E'}, d_{F'})) \sqsubseteq_{\mathcal{T}} G'_j$. Correspondingly, $\exists r_j.X^{k-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{E'}, d_{F'})) \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$. $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ contains as a conjunct $\exists r_j.X^{k-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{E'}, d_{F'}))$ by definition, i.e.

  $$X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} \exists r_j.X^{k-1}(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{E'}, d_{F'})).$$

  The result is $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$. Recall, j is an arbitrary number from 1 to n, therefore for all j, $1 \leq j \leq n$ it holds that $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} \exists r_j.G'_j$.
  The latter together with $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} \bigsqcap A_i$ yields finally that $X^k(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sqsubseteq_{\mathcal{T}} G$.

$\square$

In other words, the previous Lemma 3.14 means that in order to obtain the set of all *lcs*-candidates for the $lcs_{\mathcal{T}}(E,F)$ it is sufficient to take the set of the *l*-characteristic concepts of $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ for all l. Notice, the set of *lcs*-candidates of concepts E and F is a set of all common subsumers of these concepts E and F. Next, we can assert the following corollary, which shows a condition of the existence of the least common subsumer [16].

Corollary 3.15. *The* $lcs_{\mathcal{T}}(E,F)$ *exists iff there is an* $l \in \mathbb{N}$ *and for all* $n \in \mathbb{N}$ *holds that* l-$lcs_{\mathcal{T}}(E,F) \sqsubseteq_{\mathcal{T}} n$-$lcs_{\mathcal{T}}(E,F)$.

Because we have infinitely many n, it is impossible to test the subsumption in Corollary 3.15 in finite time. For this reason, we do not know whether the l-$lcs_{\mathcal{T}}(E,F)$ is the least common subsumer or just a common subsumer.

3.3 Conditions for the existence of the *lcs*

The significant feature of characteristic concepts and depth-limited subtrees of the unraveled tree $\mathcal{I}$ was presented in [10, 17].

Lemma 3.16. *Let* $(\mathcal{I}, d_1)$ *and* $(\mathcal{J}, d_2)$ *be two interpretations. The following holds:* $d_2 \in (X^l(\mathcal{I}, d_1))^{\mathcal{J}}$ *iff* $(\mathcal{I}^l_{d_1}, d_1) \lesssim (\mathcal{J}, d_2)$.

Next, Lemma 3.17 shows that there is a simulation relation between the tree unraveling of the product model $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ with the limited length and the canonical model of the l-$lcs_{\mathcal{T}}(E,F)$.

Lemma 3.17. *Let* $\mathcal{J}^l_{(d_E,d_F)}$ *be the tree unraveling of* $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ *rooted in* (d_E, d_F) *with the limited depth l and L is the* l-$lcs_{\mathcal{T}}(E,F)$*. Then* $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$.

Proof. Lemma 3.14 implies that $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \in cs_{\mathcal{T}}(E,F)$. Because the role-depth of $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is less or equal to l, we can conclude that $L \sqsubseteq_{\mathcal{T}} X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. The latter statement leads to the following: $d_{A_L} \in (X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)))^{\mathcal{I}_{\mathcal{T}_L}}$ by Lemma 3.10. As a result, using Lemma 3.16 we obtain $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$ as required. □

Altogether, we are able to formulate a condition whether a common subsumer is the least common subsumer. In order to a concept G was the least common subsumer of the concepts E, F its canonical model $(\mathcal{I}_{\mathcal{T}_G}, d_G)$ has to be in a simulation-equivalence relation with the product model $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$.

Theorem 3.18. *A concept G is the least common subsumer of the concepts E and F if and only if* $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sim (\mathcal{I}_{\mathcal{T}_G}, d_G)$.

Proof. (*sketch*). We outline this theorem in two directions.

1. Let us take an arbitrary H from the set of common subsumers of the concepts E and F. From Lemma 3.10 it follows that $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_E)$ and $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_F}, d_F)$. Therefore, $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. Accordingly, $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$ because of transitivity of simulation relations. Lemma 3.10 brings us to $G \sqsubseteq_{\mathcal{T}} H$. As a consequence, a common subsumer G is the $lcs(E,F)$ w.r.t. $\mathcal{T}$.

2. Now we sketch another direction. Again, we use $\mathcal{J}_{(d_E,d_F)}$ as the tree unraveling of $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. Corollary 3.15 yields that G is more specific concept to $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ for every l. The latter implies that for each l there is a simulation from $\mathcal{J}^l_{(d_E,d_F)}$ to $(\mathcal{I}_{\mathcal{T}_G}, d_G)$, where $\mathcal{J}^l_{(d_E,d_F)}$ is the subtree of $\mathcal{J}_{(d_E,d_F)}$ limited to length l. For every l we take a maximal simulation relation in $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$. Let $\sigma \in \Delta^{\mathcal{J}_{(d_E,d_F)}}$ and $|\sigma| = n$, where n is an arbitrary number. In every subtree $\mathcal{J}^i_{(d_E,d_F)}$, $i \geq n$ we have that σ is simulated by a maximal set of elements of this subtree. We denote such a set as $S_i(\sigma)$. Since an infinite sequence $(S_{n+j}(\sigma))_{j=0,1,...}$ does not grow with respect to $\supseteq$-relation, finally we have to find a fixpoint set for σ. Considering the union of fixpoint sets leads to $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$.

□

Proof. (*overall*). The complete proof is also done in both directions.

1. Let $G \in cs_{\mathcal{T}}(E, F)$ and $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, to show $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$. We consider an arbitrary concept H such that $E \sqsubseteq_{\mathcal{T}} H$, $F \sqsubseteq_{\mathcal{T}} H$. Consequently, using Lemma 3.10 we obtain that $(\mathcal{I}_{\mathcal{T}_H}, d_H)$ is simulated by $(\mathcal{I}_{\mathcal{T}_E}, d_E)$ and $(\mathcal{I}_{\mathcal{T}_F}, d_F)$. Next, by Lemma 3.8 we conclude that $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_E)$ and $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_F}, d_F)$ yield $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. Recall, $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, namely $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$. Altogether, $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, i.e. $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$. From the latter it follows that $G \sqsubseteq_{\mathcal{T}} H$ by Lemma 3.10. Thus, for an arbitrary $H \in cs_{\mathcal{T}}(E, F)$ it holds that $G \sqsubseteq_{\mathcal{T}} H$ or, in other words, $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$.

2. Now we confirm the second direction. Given $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$. We show $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sim (\mathcal{I}_{\mathcal{T}_G}, d_G)$ in two steps.

   I. $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$, that is $E \sqsubseteq_{\mathcal{T}} G$ and $F \sqsubseteq_{\mathcal{T}} G$. From Lemma 3.10 it follows that $(\mathcal{I}_{\mathcal{T}_G}, d_G) \lesssim (\mathcal{I}_{\mathcal{T}_E}, d_E)$ and $(\mathcal{I}_{\mathcal{T}_G}, d_G) \lesssim (\mathcal{I}_{\mathcal{T}_F}, d_F)$. Using Lemma 3.8 we derive that $(\mathcal{I}_{\mathcal{T}_G}, d_G) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$.

   II. It remains to show that $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$. Recall, the tree unraveling of the product model $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is denoted as $\mathcal{J}_{(d_E,d_F)}$. For all $A \in N_C$ and for all $r \in N_R$, $l = 0, 1, \ldots$ we represent $\mathcal{J}_{(d_E,d_F)}$ in the following way:

      a) $\Delta^{\mathcal{J}_{(d_E,d_F)}} = \bigcup_l \Delta^{\mathcal{J}^l_{(d_E,d_F)}}$,

      b) $A^{\mathcal{J}_{(d_E,d_F)}} = \bigcup_l A^{\mathcal{J}^l_{(d_E,d_F)}}$,

      c) $r^{\mathcal{J}_{(d_E,d_F)}} = \bigcup_l r^{\mathcal{J}^l_{(d_E,d_F)}}$.

   Assume that for an arbitrary $l \in \mathbb{N}$ a concept L is the l-lcs of E and F w.r.t. $\mathcal{T}$. Then, from Lemma 3.17 it follows $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$. Recall, $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$ is given, which with $L \equiv_{\mathcal{T}} l\text{-}lcs_{\mathcal{T}}(E, F)$ implies $G \sqsubseteq_{\mathcal{T}} L$. Due to Lemma 3.10 we can rewrite the latter subsumption as $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$.

Thus, we have $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$ and $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, i.e. $\mathcal{J}^l_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$ for an arbitrary $l \in \mathbb{N}$. As a result, for finite $\mathcal{J}_{(d_E,d_F)}$ it holds that $\mathcal{J}_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, because there exists an $l_0 \in \mathbb{N}$ with $\mathcal{J}_{(d_E,d_F)} = \mathcal{J}^{l_0}_{(d_E,d_F)}$. Hence, if the tree unraveling is finite, then $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$.

Let us now consider an infinite case of $\mathcal{J}_{(d_E,d_F)}$. For every $i \in \mathbb{N}$ we can construct a maximal simulation relation $S_i \subseteq \Delta^{\mathcal{J}^i_{(d_E,d_F)}} \times \Delta^{\mathcal{I}_{\mathcal{T}_G}}$ from $\mathcal{J}^i_{(d_E,d_F)}$ to $(\mathcal{I}_{\mathcal{T}_G}, d_G)$, where $((d_E, d_F), d_G) \in S_i$. Therefore, we obtain an infinite sequence $S_0, S_1, \ldots$, since $\mathcal{J}_{(d_E,d_F)}$ has an infinite number of subtrees.

Assume σ with an arbitrary length $n \in \mathbb{N}$ is an element of $\Delta^{\mathcal{J}_{(d_E,d_F)}}$, i.e. $\sigma \in \Delta^{\mathcal{J}_{(d_E,d_F)}}$, $|\sigma| = n$. Clearly, in this case every subtree $\mathcal{J}^i_{(d_E,d_F)}$, $i \geq n$ contains σ, namely for all $i \geq n$ there exists $(\sigma, d) \in S_i$, where $d \in \Delta^{\mathcal{I}_{\mathcal{T}_G}}$.

Further, we are interested in pairs, which we can find in the maximal simulation relation S_i. We define $S_i(\sigma) := (\{\sigma\} \times \Delta^{\mathcal{I}_{\mathcal{T}_G}}) \cap S_i$. Note, $S_i(\sigma) \neq \emptyset$ for all i. Next, we prove the following proposition.

Proposition 3.19. *If $\sigma \in \Delta^{\mathcal{J}_{(d_E,d_F)}}$, $|\sigma| = n$, then there is an inclusion such that: $S_n(\sigma) \supseteq S_{n+1}(\sigma) \supseteq S_{n+2}(\sigma) \ldots$.*

Proof. We will apply induction on $i \geq n$ and confirm the following: $S_n(\sigma) \supseteq S_{n+1}(\sigma) \supseteq \cdots \supseteq S_{i-1}(\sigma) \supseteq S_i(\sigma)$.

For the induction base $i = n$ we have $S_n(\sigma) \supseteq S_n(\sigma)$, which is trivial.

Let now $i > n$, $(\sigma, d) \in S_i(\sigma)$ and the maximal simulation relation $S_i \subseteq \Delta^{\mathcal{J}^i_{(d_E,d_F)}} \times \Delta^{\mathcal{I}_{\mathcal{T}_G}}$. In a subsequent step, we construct a restriction of S_i as follows: $S'_i := (\Delta^{\mathcal{J}^{i-1}_{(d_E,d_F)}} \times \Delta^{\mathcal{I}_{\mathcal{T}_G}}) \cap S_i$. As you can see, one member in pairs of S'_i is an element of $\mathcal{J}^{i-1}_{(d_E,d_F)}$, i.e. limited to length $i-1$. Moreover, S'_i is a simulation relation from $\mathcal{J}^{i-1}_{(d_E,d_F)}$ to $(\mathcal{I}_{\mathcal{T}_G}, d_G)$. Because the simulation S_{i-1} is maximal, we can conclude that $S'_i \subseteq S_{i-1}$. To summarise, $S_i(\sigma) \subseteq S'_i \subseteq S_{i-1}$ and $(\sigma, d) \in S_i(\sigma)$ yield $(\sigma, d) \in S_{i-1}(\sigma)$, that is $S_{i-1}(\sigma) \supseteq S_i(\sigma)$. Finally, the induction hypothesis leads to $S_n(\sigma) \supseteq S_{n+1}(\sigma) \supseteq \cdots \supseteq S_{i-1}(\sigma) \supseteq S_i(\sigma)$. □

Thus, by this proposition we have shown the significant property that with growing i the set $S_i(\sigma)$ decreases or remains the same.

As a consequence, it has to be a number $t \in \mathbb{N}$:

$$S_t(\sigma) = \bigcap_{j \geq |\sigma|}^{\infty} S_j(\sigma). \tag{$*$}$$

Next, we show that $S \subseteq \Delta^{\mathcal{J}_{(d_E,d_F)}} \times \Delta^{\mathcal{I}_{\mathcal{T}_G}}$ defined in the following form $S := \bigcup_{\sigma \in \Delta^{\mathcal{J}_{(d_E,d_F)}}} \left(\bigcap_{j \geq |\sigma|}^{\infty} S_j(\sigma) \right)$ is a simulation relation from $\mathcal{J}_{(d_E,d_F)}$ to $(\mathcal{I}_{\mathcal{T}_G}, d_G)$. Since $((d_E, d_F), d_G) \in S_i$ for all $i \in \mathbb{N}$, then $((d_E, d_F), d_G)$ is also contained in S. Lastly, we confirm that a relation S is fulfilled to both statements of the definition of simulation.

A. Assume $(\sigma, d) \in S$ and $\sigma \in A^{\mathcal{J}_{(d_E,d_F)}}$. As a result, for some $n \in \mathbb{N}$ it holds that $(\sigma, d) \in S_n$. By (b), $\sigma \in A^{\mathcal{J}_{(d_E,d_F)}}$ implies $\sigma \in A^{\mathcal{J}^n_{(d_E,d_F)}}$. Therefore, $d \in A^{\mathcal{I}_{\mathcal{T}_G}}$ due to $\mathcal{J}^n_{(d_E,d_F)} \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$. Thus, $(\sigma, d) \in S$ and $\sigma \in A^{\mathcal{J}_{(d_E,d_F)}}$, $A \in N_C$ bring us to $d \in A^{\mathcal{I}_{\mathcal{T}_G}}$.

B. Assume $(\sigma, d) \in S$ with $(\sigma, \sigma re) \in r^{\mathcal{J}_{(d_E,d_F)}}$. From $(*)$ we can derive that there exist t_1 and t_2 such that $S_{t_1}(\sigma) = \bigcap_{j \geq |\sigma|}^{\infty} S_j(\sigma)$, $S_{t_2}(\sigma re) = \bigcap_{j \geq |\sigma re|}^{\infty} S_j(\sigma re)$ and $S_{t_2}(\sigma re) \subseteq S$. W.l.o.g. consider $t_2 \geq t_1$ ($S_{t_2}(\sigma) = S_{t_1}(\sigma)$). Using (c), from $(\sigma, \sigma re) \in r^{\mathcal{J}_{(d_E,d_F)}}$ we deduce $(\sigma, \sigma re) \in r^{\mathcal{J}^{t_2}_{(d_E,d_F)}}$. Since S_{t_2} is a simulation from $\mathcal{J}^{t_2}_{(d_E,d_F)}$ to $(\mathcal{I}_{\mathcal{T}_G}, d_G)$, then $(\sigma, d) \in S_{t_2}$ together with $(\sigma, \sigma re) \in r^{\mathcal{J}^{t_2}_{(d_E,d_F)}}$ leads to the existence d' such that $(d, d') \in r^{\mathcal{I}_{\mathcal{T}_G}}$ and $(\sigma re, d') \in S_{t_2}(\sigma re)$. Recall, $S_{t_2}(\sigma re) \subseteq S$, hence $(\sigma re, d') \in S$. To sum up briefly, $(\sigma, d) \in S$ and $(\sigma, \sigma re) \in r^{\mathcal{J}_{(d_E,d_F)}}$ imply that there exists d' such that $(d, d') \in r^{\mathcal{I}_{\mathcal{T}_G}}$ with $(\sigma re, d') \in S$.

Altogether, it was obtained that $(\mathcal{I}_{\mathcal{T}_G}, d_G) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ together with $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_G)$, that is we have achieved the following conclusion $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \sim (\mathcal{I}_{\mathcal{T}_G}, d_G)$ (if $G \equiv_{\mathcal{T}} lcs_{\mathcal{T}}(E, F)$).

□

With the help of the previous Theorem 3.18 we can rewrite Corollary 3.15 in the following form.

Corollary 3.20. *The $lcs_{\mathcal{T}}(E, F)$ exists if and only if there is an $l \in \mathbb{N}$ and it holds that the product model $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ is simulated by the canonical model of $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$, i.e. $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F)) \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$.*

It should be noted, that in Corollary 3.20 our $l \in \mathbb{N}$ is unbounded, therefore we need to investigate a condition of the existence of the *lcs* further.

4 An upper bound for the role-depth limited *lcs*

We start with introducing the notion of synchronous elements. Elements $(d_G, d_H) \in \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$ are called *synchronous* elements, if $G = H$ holds. If elements (d_G, d_H) are not synchronous, then they are *asynchronous* elements.
Let us consider the following lemma.

Lemma 4.1. *If* $(d_{A_G}, d_{A_G}) \in \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$*, then* $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_G}, d_{A_G})) \sim (\mathcal{I}_{\mathcal{T}_G}, d_{A_G})$.

Proof. Let us determine a relation S_1 as

$S_1 := \{((d_H, d_{H'}), d_H) \mid (d_H, d_{H'}) \in \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}, d_H \in \Delta^{\mathcal{I}_{\mathcal{T}_G}}\}$, *where* $((d_{A_G}, d_{A_G}), d_{A_G}) \in S_1$.

A relation S_2 is defined in the following way:

$S_2 := \{(d_H, (d_H, d_H)) \mid d_H \in \Delta^{\mathcal{I}_{\mathcal{T}_G}}, (d_H, d_H) \in \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}\}$, *where* $(d_{A_G}, (d_{A_G}, d_{A_G})) \in S_2$.

Both requirements of Definition 3.6 of a simulation are fulfilled by relations S_1 and S_2. From the latter and $((d_{A_G}, d_{A_G}), d_{A_G}) \in S_1$ we have $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_G}, d_{A_G})) \lesssim (\mathcal{I}_{\mathcal{T}_G}, d_{A_G})$. In the same way, because S_2 is a simulation relation, which contains $(d_{A_G}, (d_{A_G}, d_{A_G}))$, we conclude that $(\mathcal{I}_{\mathcal{T}_G}, d_{A_G}) \lesssim (\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_G}, d_{A_G}))$. □

Further, as usual, $\mathcal{J}_{(d_{A_E}, d_{A_F})}$ is used as the tree unraveling of $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$. Moreover, $\mathcal{J}^l_{(d_{A_E}, d_{A_F})}$ is the finite subtree of $\mathcal{J}_{(d_{A_E}, d_{A_F})}$ rooted in (d_{A_E}, d_{A_F}) and limited to depth l.
In a subsequent step, the canonical model $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$, where a concept $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$, is represented in the form of a subtree of $\mathcal{J}_{(d_{A_E}, d_{A_F})}$. In order to make such a representation of $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$ we add new tree models at depth l to $\mathcal{J}^l_{(d_{A_E}, d_{A_F})}$ and finally obtain an interpretation, which we denote as $\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$. This extended interpretation $\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$ should be a model of the TBox $\mathcal{T}$, and $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L}) \sim \widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$.

Further, we will call extended elements $\sigma \in \Delta^{\mathcal{J}^l_{(d_{A_E}, d_{A_F})}}$, $|\sigma| = l$, as *stubs*. Correspondingly, the set of extended elements from $\Delta^{\mathcal{J}^l_{(d_{A_E}, d_{A_F})}}$ is called the set of stubs, in symbols $stubs(\mathcal{J}^l_{(d_{A_E}, d_{A_F})})$.

Definition 4.2. Let $\sigma = (d_{A_E}, d_{A_F}) r_1 \ldots r_l (d_P, d_Q)$, respectively, $|\sigma| = l$. We define $M := \bigsqcap \{ A \mid \sigma \in A^{\mathcal{J}^l_{(d_{A_E}, d_{A_F})}}\}$. Then, $\sigma \in stubs(\mathcal{J}^l_{(d_{A_E}, d_{A_F})})$ if the following is true: $\sigma \in M^{\mathcal{J}^l_{(d_{A_E}, d_{A_F})}}$ implies $M \sqsubseteq_{\mathcal{T}} \exists r.K$, where K is a conjunction of concept names and K is maximal in $\mathcal{I}^r_M$.

Recall, K is maximal if there does not exist N such that $M \sqsubseteq_{\mathcal{T}} \exists r.N$ and $K \subsetneq N$.
The set of trees, which we add to each stub σ, is denoted as $TR(\sigma)$. Let σ be contained in $stubs(\mathcal{J}^l_{(d_{A_E}, d_{A_F})})$ and $(\mathcal{I}_{\mathcal{T}_{\exists r.K}}, d_{\exists r.K})$ is the canonical model. $\mathcal{J}_{\sigma r(d_K, d_K)}$ is a subtree of $\mathcal{J}_{(d_{A_E}, d_{A_F})}$, $\sigma r(d_K, d_K) \in \Delta^{\mathcal{J}_{(d_{A_E}, d_{A_F})}}$ due to construction of the tree unraveling of the product model. Using Lemma 4.1 we have that $\mathcal{J}_{\sigma r(d_K, d_K)} \sim (\mathcal{I}_{\mathcal{T}_{\exists r.K}}, d_{\exists r.K})$. That is $\mathcal{J}_{\sigma r(d_K, d_K)} \in TR(\sigma)$.
Next, we clarify that the obtained interpretation $\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$ is in the following form:

$$\Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} := \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cup \bigcup_{\sigma \in stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})} \bigcup_{\mathcal{J} \in TR(\sigma)} \Delta^{\mathcal{J}},$$

$$\text{for all } A \in N_C\text{: } A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} := A^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cup \bigcup_{\sigma \in stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})} \bigcup_{\mathcal{J} \in TR(\sigma)} A^{\mathcal{J}},$$

$$\text{for all } r \in N_R\text{: } r^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} := r^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cup \bigcup_{\sigma \in stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})} \bigcup_{\mathcal{J} \in TR(\sigma)} r^{\mathcal{J}}.$$

There is the following relation between an extended interpretation and a functional simulation.

Lemma 4.3. *Let $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$. For all $\sigma \in \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cap \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ if $\sigma \in G^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, then $SR(\sigma) \in G^{\mathcal{I}_{\mathcal{T}_L}}$, where G is a concept and $SR \subseteq \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \times \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ is a functional simulation relation such that $SR((d_{A_E}, d_{A_F})) = d_{A_L}$.*

Proof. There is a simulation SR from $\mathcal{J}^l_{(d_{A_E},d_{A_F})}$ to $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$, which fulfills:

- $SR \subseteq \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \times \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ is functional and $SR((d_{A_E}, d_{A_F})) = d_{A_L}$,
- if σ belongs to $stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})$ and fulfills the claim of Definition 4.2, then by this definition $\sigma \in M^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}}$ and $M \sqsubseteq_{\mathcal{T}} \exists r.K$, K is maximal. From the latter and $\sigma \lesssim SR(\sigma)$ due to Definition 3.6 of a simulation relation we obtain $SR(\sigma) \in (\exists r.K)^{\mathcal{I}_{\mathcal{T}_L}}$. As a result, $\mathcal{J}_{\sigma r(d_K,d_K)} \lesssim SR(\sigma)$, where $\mathcal{J}_{\sigma r(d_K,d_K)}$ is the tree from $TR(\sigma)$.

A simulation $SR(\sigma)$ simulates all tree unravelings, which we add to an element $\sigma \in stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})$. Therefore, we can extend a simulation $SR \subseteq \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \times \Delta^{\mathcal{I}_{\mathcal{T}_L}}$ by a simulation $\widetilde{S} \subseteq \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} \times \Delta^{\mathcal{I}_{\mathcal{T}_L}}$, where $\widetilde{S}((d_{A_E}, d_{A_F})) = d_{A_L}$. For all stubs $\sigma \in \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cap \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ an extended simulation $\widetilde{S}$ is functional.
Thus, $(\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}, \sigma) \lesssim (\mathcal{I}_{\mathcal{T}_L}, SR(\sigma))$ for $\sigma \in \Delta^{\mathcal{J}^l_{(d_{A_E},d_{A_F})}} \cap \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Next, if an element $\sigma \in G^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, then $(\mathcal{I}_{\emptyset_G}, d_{A_G}) \lesssim (\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}, \sigma)$ by Lemma 3.9. Since the subsumption relation is transitive, we derive $(\mathcal{I}_{\emptyset_G}, d_{A_G}) \lesssim (\mathcal{I}_{\mathcal{T}_L}, SR(\sigma))$. In conclusion, Lemma 3.9 implies $SR(\sigma) \in G^{\mathcal{I}_{\mathcal{T}_L}}$. □

Further, we will prove that the extended interpretation $\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$ is indeed a model of the TBox $\mathcal{T}$.

Lemma 4.4. *$\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})} \models \mathcal{T}$, $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$.*

Proof. Assume $\sigma \in C^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, $C \sqsubseteq_{\mathcal{T}} D$, then we have to confirm that $\sigma \in D^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Let us consider all possible cases for an element σ from $\Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ and all possible GCIs in a normalized TBox $\mathcal{T}$, written in *Horn-$\mathcal{ALC}$*.

1. Case $|\sigma| < l$.

- $\bigsqcap A_i \sqsubseteq_{\mathcal{T}} B$. $\sigma \in (\bigsqcap A_i)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ implies that $\sigma \in (\bigsqcap A_i)^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Consequently, $\sigma \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$, because $\mathcal{J}_{(d_{A_E},d_{A_F})}$ is a model of $\mathcal{T}$.

  For all $\sigma \in \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ and $A \in N_C$ we have

  $$\sigma \in A^{\mathcal{J}_{(d_{A_E},d_{A_F})}} \Longleftrightarrow \sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}. \tag{**}$$

  As a result, from $\sigma \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$ it follows that $\sigma \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

- $A \sqsubseteq_{\mathcal{T}} \exists r.B$. From $\sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ we derive $\sigma \in A^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Since $\mathcal{J}_{(d_{A_E},d_{A_F})}$ is a model of $\mathcal{T}$ we can conclude that $\sigma \in (\exists r.B)^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Let $\sigma = (d_{A_E}, d_{A_F})r_1 \ldots r_i(d_P, d_Q)$, $i < l$. In this case we can deduce $P \sqsubseteq_{\mathcal{T}} \exists r.B$ and $Q \sqsubseteq_{\mathcal{T}} \exists r.B$, respectively. Then $\sigma r(d_B, d_B) \in \Delta^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$, $\sigma r(d_B, d_B) \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$, $|\sigma r(d_B, d_B)| \leq l$, $\sigma r(d_B, d_B) \in \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Therefore, by (**) it holds that $\sigma r(d_B, d_B) \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, i.e. $\sigma \in (\exists r.B)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

- $\exists r.A \sqsubseteq_{\mathcal{T}} B$. $\sigma \in (\exists r.A)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ leads to $\sigma \in (\exists r.A)^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Again, $\mathcal{J}_{(d_{A_E},d_{A_F})}$ is a model of $\mathcal{T}$, $\exists r.A \sqsubseteq_{\mathcal{T}} B$, hence $\sigma \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Using (**) we obtain that σ is contained in $B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

- $A \sqsubseteq_{\mathcal{T}} \forall r.B$. From $\sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ it follows $\sigma \in A^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Let (d_G, d_H) be an r-successor of σ. We have to show $\sigma r(d_G, d_H) \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. $\sigma r(d_G, d_H) \in \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, $|\sigma r(d_G, d_H)| \leq l$. The consequence is then $\sigma r(d_G, d_H) \in \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} \subseteq \Delta^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Because $\mathcal{J}_{(d_{A_E},d_{A_F})}$ is a model of $\mathcal{T}$ and using $\sigma \in A^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$ we derive that $\sigma r(d_G, d_H) \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Thus, $\sigma r(d_G, d_H) \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ by (**).

2. Case $|\sigma| = l$, $\sigma \in stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})$. Then σ fulfills the claim of Definition 4.2.

   - $\bigsqcap A_i \sqsubseteq_{\mathcal{T}} B$. We assumed that $\sigma \in C^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, $C = \bigsqcap A_i$. Analogously to the first item of the first case ($|\sigma| < l$, $\bigsqcap A_i \sqsubseteq_{\mathcal{T}} B$) we conclude that $\sigma \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

   - $A \sqsubseteq_{\mathcal{T}} \exists r.B$. Let $\sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, $M = \bigsqcap B'$ such that $\sigma \in B'^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Consequently, we have $A \in M$, $M \sqsubseteq_{\mathcal{T}} A$. By assumption $A \sqsubseteq_{\mathcal{T}} \exists r.B$. As a result, it follows that $M \sqsubseteq_{\mathcal{T}} \exists r.B$. There exists a maximal K with $B \in K$ and $M \sqsubseteq_{\mathcal{T}} \exists r.K$. Notice, we append the tree model $\mathcal{J}_{\sigma r(d_K,d_K)}$ to $\mathcal{J}^l_{(d_{A_E},d_{A_F})}$, while we build $\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$, such that the element σ belongs to $(\exists r.K)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Since K is maximal and $B \in K$, $\sigma \in (\exists r.B)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

   - $\exists r.A \sqsubseteq_{\mathcal{T}} B$. We assume $\sigma \in (\exists r.A)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, then $\sigma \in (\exists r.A)^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$ and $\sigma \in B^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$, because $\mathcal{J}_{(d_{A_E},d_{A_F})}$ is a model of $\mathcal{T}$. Therefore, it holds that $\sigma \in B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ due to (**).

- $A \sqsubseteq_{\mathcal{T}} \forall r.B$. Let $\sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$, $M = \bigsqcap B'$ such that $\sigma \in B'^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. Hence, it follows $A \in M$, $M \sqsubseteq_{\mathcal{T}} A$. Next, it holds $M \sqsubseteq_{\mathcal{T}} \exists r.K$, where K is maximal. Then $\sigma \in (\exists r.K)^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$. $B \in K$, because K is maximal, i.e. $\sigma r(d_K, d_K)$ has to be contained in $B^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

3. Case $|\sigma| = l$, $\sigma \notin stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})$.

   In this case, by construction of $\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$ we cannot have any successors of σ in $\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$. It means that C can be only of the form $C = \bigsqcap A_i$. From $\sigma \notin stubs(\mathcal{J}^l_{(d_{A_E},d_{A_F})})$ we obtain that D can be only of the form $D = \bigsqcap B_j$. Thus, the only possible case for subsumption is $\bigsqcap A_i \sqsubseteq_{\mathcal{T}} B$.

   - $\bigsqcap A_i \sqsubseteq_{\mathcal{T}} B$. Recall, for all $A \in N_C$ and $\sigma \in \Delta^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ the following is true: $\sigma \in A^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}} \iff \sigma \in A^{\mathcal{J}_{(d_{A_E},d_{A_F})}}$. Therefore, from $\sigma \in C^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$ we are able to conclude that σ is contained in $D^{\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}}$.

4. Case $|\sigma| > l$. The element σ has to belong to a tree model, which we appended in order to extend the subtree $\mathcal{J}^l_{(d_{A_E},d_{A_F})}$. Notice, this appended tree is a model of the TBox $\mathcal{T}$.

□

In the next lemma we show that the extended interpretation $\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$ is simulation-equivalent to the canonical model $(\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$.

Lemma 4.5. *$\widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})} \sim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$, where $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$.*

Proof. The claim follows from Lemmas 4.3 and 4.4. □

In the following lemma we claim that if the *lcs* exists, then its role-depth is polynomially bounded in the size of the product model. More precisely, it is bounded by a quadratic polynomial in the size of the product model. This bound is similar to the role-depth bound of the *lcs* of two $\mathcal{EL}$-concepts w.r.t. an $\mathcal{EL}$ TBox [17].

Lemma 4.6. *Let $k = |\Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}|$. The existence of the $lcs_{\mathcal{T}}(E, F)$ implies the following: $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim \widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$, where $l = k^2 + 1$.*

Proof. Assume there exists $G = lcs_{\mathcal{T}}(E, F)$ such that $rd(G) = n$. Then, using Corollary 3.20 we have $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim (\mathcal{I}_{\mathcal{T}_N}, d_{A_N})$, where a concept $N = X^n(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$. On the other hand, $(\mathcal{I}_{\mathcal{T}_N}, d_{A_N}) \sim \widetilde{\mathcal{J}}^n_{(d_{A_E},d_{A_F})}$ by Lemma 4.5. Thus, $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim \widetilde{\mathcal{J}}^n_{(d_{A_E},d_{A_F})}$ due to transitivity of a simulation relation. Next, we consider two possible cases, namely $n \leq l$ and $n > l$.

- $n \leq l$. In this case we obtain that $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim \widetilde{\mathcal{J}}^l_{(d_{A_E},d_{A_F})}$.

- $n > l$. $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim \widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ implies that there is a simulation relation $Z \subseteq \Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}} \times \Delta^{\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}}$, where $Z((d_{A_E}, d_{A_F})) = (d_{A_E}, d_{A_F})$. In other words, each path $\rho = d_0 r_1 d_1 r_2 d_2 \ldots$ in $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$, where $d_0 = (d_{A_E}, d_{A_F})$, is simulated in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ by a path $\rho_n = \sigma_0 r_1 \sigma_1 r_2 \sigma_2 \ldots$, where $\sigma_0 = (d_{A_E}, d_{A_F})$ as well.

  Notice, by assumption $|\Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}| = k$, therefore, the number of (pairwise) distinct pairs from $\Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}$ is k^2. If the length of an asynchronous prefix of ρ_n is less or equals to k^2, then we are done with this restriction. Otherwise, if the length of an asynchronous prefix of ρ_n is more than k^2, then some pairs (at least one) are repeated several times (at least one). Our goal is using ρ_n to build in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ a simulating path ρ^i_n, which maximal asynchronous prefix contains only pairwise different pairs. Correspondingly, the length of an asynchronous prefix of ρ^i_n will be less or equals to k^2. Further, we will construct such ρ^i_n.

  If the length of an asynchronous prefix of ρ_n is more than k^2, then there exist i' and j' with $(d_{i'}, tail(\sigma_{i'})) = (d_{j'}, tail(\sigma_{j'}))$ in an asynchronous part. W.l.o.g. let $i' < j'$. $(d_0, tail(\sigma_0)), (d_1, tail(\sigma_1)), \ldots, (d_{i'}, tail(\sigma_{i'}))$ are pairwise different. From $d_{i'} = d_{j'}$ and the fact that $\sigma_{j'}$ simulates $d_{j'}$ by the simulation Z, we derive that the subpath $d_{i'} r_{i'+1} d_{i'+1} r_{i'+2} d_{i'+2} \ldots$ from ρ is simulated in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ by a corresponding sequence $\delta^0_n = \sigma_{j'} r_{j'+1} \sigma'_{j'+1} r_{j'+2} \sigma'_{j'+2} \ldots$. The successors of $\sigma_{j'}$ in δ^0_n may differ from the successors of $\sigma_{j'}$ in ρ_n. Recall, $tail(\sigma_{i'}) = tail(\sigma_{j'})$, $\sigma_{i'}$ and $\sigma_{j'}$ ($|\sigma_{i'}| < |\sigma_{j'}|$) represent the same element in $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$. As a consequence, there exists a subpath $\sigma_{i'} r_{i'+1} \widetilde{\sigma}_{i'+1} r_{i'+2} \widetilde{\sigma}_{i'+2} \ldots$ in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ such that $tail(\widetilde{\sigma}_{i'+1}) = tail(\sigma'_{j'+1})$, $tail(\widetilde{\sigma}_{i'+2}) = tail(\sigma'_{j'+2}), \ldots$, where $\sigma'_{j'+1}, \sigma'_{j'+2}, \ldots$ are successors of $\sigma_{j'}$ in a path δ^0_n. Hence, there is a simulating sequence ρ^1_n in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ such that $\rho^1_n = \sigma_0 r_1 \sigma_1 r_2 \sigma_2 \ldots \sigma_{i'} r_{i'+1} \widetilde{\sigma}_{i'+1} r_{i'+2} \widetilde{\sigma}_{i'+2} \ldots$, $\sigma_0 = (d_{A_E}, d_{A_F})$.

  If the length of an asynchronous prefix of ρ^1_n is less or equals to k^2, then we are done. Otherwise, in this asynchronous prefix there exist i'' and j'' with $(d_{i''}, tail(\sigma_{i''})) = (d_{j''}, tail(\sigma_{j''}))$. W.l.o.g. let $i'' < j''$. Since the pairs $(d_0, tail(\sigma_0)), (d_1, tail(\sigma_1)), \ldots, (d_{i'}, tail(\sigma_{i'}))$ are already pairwise different, we conclude that $j'' > i'$. Further, $\sigma_{j'}$ has a successor $\sigma'_{j''}$ in δ^0_n such that $d_{i''}$ is simulated by $\sigma'_{j''}$. Thus, the subsequence $d_{i''} r_{i''+1} d_{i''+1} r_{i''+2} d_{i''+2} \ldots$ from ρ is simulated in $\widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ by a corresponding subpath δ^1_n, which starts in $\sigma'_{j''}$. Because $|\sigma_{j'}| < |\sigma'_{j''}|$ we can apply such replacement steps a finite number of times. As a result, there is a finite i such that the length of an asynchronous part of ρ^i_n is less or equals to k^2. Notice, $\rho^i_n \in \widetilde{\mathcal{J}}^n_{(d_{A_E}, d_{A_F})}$ and ρ^i_n starts in $\sigma_0 = (d_{A_E}, d_{A_F})$.

  Next, we demonstrate that the obtained simulating path $\rho^i_n \in \widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$. Let $\hat{\sigma}_{k^2+1}$ be $k^2 + 1$ element of ρ^i_n. Since an asynchronous part of ρ^i_n equals to k^2 (or less), therefore $\hat{\sigma}_{k^2+1}$ is synchronous, i.e. $tail(\hat{\sigma}_{k^2+1}) = (d_H, d_H)$. It follows that subsequence $\hat{\sigma}_{k^2+1} r_{k^2+1} \hat{\sigma}_{k^2+2} r_{k^2+2} \ldots$ of ρ^i_n is contained in $\mathcal{I}_{\mathcal{T}_H}$. A concept H is a conjunction of concept names, hence $rd(H) = 0$. A part

$\sigma_0 r_1 \sigma_1 r_2 \sigma_2 \ldots \hat{\sigma}_{l-1} r_l \hat{\sigma}_l$ of ρ^i_n belongs to $\mathcal{J}^l_{(d_{A_E}, d_{A_F})}$. Because $\hat{\sigma}_{k^2+1} \in H^{\mathcal{J}^l_{(d_{A_E}, d_{A_F})}}$, then $\hat{\sigma}_{k^2+1} \in H^{\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}}$. Recall, it was shown in Lemma 4.4 that $\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$ is a model of the TBox $\mathcal{T}$. Consequently, $(\mathcal{I}_{\mathcal{T}_H}, d_H) \lesssim (\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}, \hat{\sigma}_{k^2+1})$ due to Lemma 3.9. On the other hand, the tree unraveling of $(\mathcal{I}_{\mathcal{T}_H}, d_H)$ contains synchronous elements, which we add to $\mathcal{J}^l_{(d_{A_E}, d_{A_F})}$ at position $\hat{\sigma}_{k^2+1}$, i.e. $(\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}, \hat{\sigma}_{k^2+1}) \lesssim (\mathcal{I}_{\mathcal{T}_H}, d_H)$. Thus, $\rho^i_n \in \widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$. Since ρ^i_n is a simulating path of a path ρ from $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$, then there is a simulation relation from $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$ to $\widetilde{\mathcal{J}}^l_{(d_{A_E}, d_{A_F})}$.

□

Thus, if the $lcs_{\mathcal{T}}(E, F)$ exists, then $rd(lcs_{\mathcal{T}}(E, F)) \leq k^2 + 1$, where $k = |\Delta^{\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}}|$. Notice, in the case of the *lcs* of two $\mathcal{EL}$-concepts w.r.t. an $\mathcal{EL}$ TBox $rd(lcs_{\mathcal{T}}(E, F)) \leq k^2 + 1 + rd(H)$ [17], whereas in our case $rd(H) = 0$.
Next theorem shows that the problem of the existence of the *lcs* is decidable. We can check the existence of the $lcs_{\mathcal{T}}(E, F)$ w.r.t. *Horn-$\mathcal{ALC}$* TBox $\mathcal{T}$ in exponential time. Moreover, a decision procedure for the existence of the *lcs* also gives us the *lcs* itself, if the *lcs* exists.

Theorem 4.7. *The task of deciding the existence of the lcs of given $\mathcal{EL}$-concepts E and F w.r.t. a Horn-$\mathcal{ALC}$ TBox $\mathcal{T}$ is decidable. The role-depth of the lcs, if the lcs exists, is exponentially bounded by the size of the input. Deciding the existence of the lcs can be done in triple exponential time.*

Proof. The construction of canonical models $\mathcal{I}_{\mathcal{T}_E}$ and $\mathcal{I}_{\mathcal{T}_F}$ terminates in time at most exponential. Next step, we obtain a number l using the corresponding formulae from Lemma 4.6. Further, we compute $L = X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F}))$. This process runs in time exponential in the depth l, which can be already exponential. Therefore, the computation of L is at most double exponential. Next, we build the canonical model $\mathcal{I}_{\mathcal{T}_L}$. Since the computation of L takes already double exponential time (in the worst case), the construction of the canonical model $\mathcal{I}_{\mathcal{T}_L}$ is at most triple exponential. In a subsequent step, we verify whether $\mathcal{I}_{\mathcal{T}_L}$ satisfies $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$. This verification test runs in time polynomial in the largest size of considering graphs, i.e. in triple exponential time (in the worst case). Consequently, triple exponential time is upper bound. Because, if $(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_{A_E}, d_{A_F})) \lesssim (\mathcal{I}_{\mathcal{T}_L}, d_{A_L})$ is true, then due to Corollary 3.20 we have that the $lcs_{\mathcal{T}}(E, F)$ exists, namely $lcs_{\mathcal{T}}(E, F) = L$. Otherwise, using Lemma 4.6 we conclude that the $lcs_{\mathcal{T}}(E, F)$ does not exist. □

5 Conclusions

After introducing the basic definitions of the description logic $\mathcal{ALC}$ we defined the notion of a canonical model (of an $\mathcal{EL}$-concept w.r.t. a normalized *Horn-$\mathcal{ALC}$* TBox) and several related lemmas. Ideas of these lemmas were inherited from the literature. It was confirmed that presented canonical model is indeed a model of a normalized TBox, written in the *Horn-$\mathcal{ALC}$* DL. Then, it was shown that the l-characteristic concept of the product of the canonical models of given concepts is a common subsumer of these concepts, namely, the *l-lcs*.
Since the *lcs* w.r.t. general *Horn-$\mathcal{ALC}$* TBoxes may not exist, we had to identify conditions for the existence of the *lcs*. It was proven that the *lcs* exists iff there is a number l such that the canonical model of the l-characteristic concept of the product model simulates this product model. Afterward, we confirmed that if the *lcs* exists, then its role-depth has a polynomial bound. Using this obtained bound we defined the *l-lcs*, which is actually the *lcs*, if the *lcs* exists. If the *lcs* does not exist, then we can consider this *l-lcs* as an approximation, which generalizes the given concepts, but it is not the least generalization [13]. Finally, it was shown that the existence of the *lcs* can be checked in triple exponential time.
There are some possible directions for future work. Firstly, how to adapt the devised conditions for the existence of the most specific concept (*msc*) for the considered setting. In other words, to find conditions for the existence of the *msc* (in the target DL $\mathcal{EL}$) w.r.t. general TBoxes, written in the DL *Horn-$\mathcal{ALC}$*. Secondly, to extend the target DL $\mathcal{EL}$ for the *lcs* and the *msc* to more expressive target DLs, for example, to DL $\mathcal{EL}$ with atomic negation or to DL $\mathcal{ALE}$ ($\mathcal{EL}$ with atomic negation and value restriction). Recall, we are more interested in the target DL, which does not allow disjunction (and full negation), otherwise, $lcs(E, F) = E \sqcup F$ (or $lcs(E, F) = \neg(\neg E \sqcap \neg F)$). In the target DL $\mathcal{EL}$ with atomic negation we have to consider conjunction of concept names and negated concept names in $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. In the target DL $\mathcal{ALE}$ besides conjunction of concept names and negated concept names in $X^0(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$ we have to deal with value restrictions in $X^l(\mathcal{I}_{\mathcal{T}_E} \times \mathcal{I}_{\mathcal{T}_F}, (d_E, d_F))$. Finally, there is a challenge to investigate under which conditions the *lcs* and the *msc* exist w.r.t. general TBoxes, written in more expressive DLs, at least in more expressive *Horn* DLs.

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