Technische Universität Dresden

Fakultät Informatik
MCL Master's Thesis on

The Subsumption Problem of the Fuzzy Description Logic $\mathcal{FL}_0$ with greatest fixed-point semantics

by

José A. Leyva Galano

Supervisor: **Prof.Dr.-Ing. Franz Baader**
Advisor: **Dr. rer. nat. Rafael Peñaloza Nyssen**

Dresden, October 2013

Technische Universität Dresden

Author :	**José A. Leyva Galano**
Matrikel-Nr. :	**3645958**
Title :	**The Subsumption Problem of the Fuzzy Description Logic $\mathcal{FL}_0$ with greatest fixed-point semantics**
Degree :	**Master of Science**
Date of Submission :	

Declaration
Hereby I certify that the thesis has been written by me. Any help that I have received in my research work has been acknowledged. Additionally, I certify that I have not used any auxiliary sources and literature except those cited in the thesis.

José A. Leyva Galano

Abstract

Because of the simplicity in the structure of the Description Logic (DL) $\mathcal{FL}_0$, it has been commonly taken as the first example for the presentation of new notions and their associated problems [1],[2]. Fuzzy Description Logics extend classical Description Logics by allowing truth degrees to deal with imprecise concepts. In this work we show that in the fuzzy DL $\mathcal{FL}_0$ with greatest fixed-point semantics, the subsumption problem can be stated in terms of finite automata. Furthermore, following the new characterization, it was shown that this inference problem lies in the PSPACE-complete complexity class.

Acknowledgements

First of all, I would like to thank Mr. Horn and Mr. Gómez for their constant support, encouragement and infinite patience during the last years. Many thanks go to Stefan Borgwardt and Douglas Helman for checking the main proofs, reading through the draft and pointing out things I needed to correct or clarify. Most important, many thanks to the big and great Leyva-Hevia-Rogosch family for everything. In particular, I am deeply grateful to Linda and Bruno.

Contents

1 Introduction

In this thesis, we obtain a new characterization completely stated in terms of finite automata for the Subsumption Problem in the Fuzzy Description Logic $\mathcal{FL}_0$ with greatest fixed-point semantics (gfp-semantics). As a direct consequence of this characterization, we show that this problem lies in the PSPACE-complete complexity class.

Description Logics (DLs) is the most recent name for a family of knowledge representation formalisms [1, p.43] that can be used for representing knowledge from a given universe. On the one hand, from a syntactic point of view, this form of knowledge representation first requires the identification of the classes (atomic concepts), relations (atomic roles) and objects (individual names) that are basic in the universe that we want to describe. From them, more complex concepts (concept descriptions) can be built inductively with the assistance of the concept constructors that are allowed in each particular Description Logic. From a semantic point of view, every concrete Description Logic comes with a formal (logic–based) semantics that specifies what an interpretation is and how to use it for interpreting (give meaning to) each *concept description* in the given formalism.

In a broad sense, the Subsumption Problem in a DL involves the question: given two concept descriptions, is the first concept *less general than* (subsumed by) the second one in all those *settings* (models), from the given universe, *that are represented by our description* (TBox)? After introducing the Fuzzy DL $\mathcal{FL}_0$, the semantics that we are interested in (i.e, gfp-semantics) and the notion of *less general* that we are going to consider, we obtain a particular instance of this problem in Chapter 2.

As happens in the case of several logics [3], also in the area of Description Logic, automata-based algorithms are frequently used to show decidability and complexity results for basic inference problems (see [2], [4] or [5]). In Chapter 3 we show that, for the particular case of Fuzzy $\mathcal{FL}_0$ with gfp-semantics, Weighted Semi-Automata with word transitions allows us to give an alternative formulation of the subsumption problem. This alternative formulation is written in the language of weighted automata. Starting from it, we make a second transformation of the problem, this time in order to obtain an equivalent characterization of the subsumption problem in terms of Semi-Automata with word transitions. The second characterization involves several tests of language inclusion, where the languages are defined over the structure of a Semi-Automata with word transitions. Along these two transformations we keep the size of the new problems inside of a polynomial growing factor. This, together with the fact that the languages defined over the structure of a semi-automaton with word transitions are regular [2, p.182] and because the inclusion problem of regular languages is known to be in PSPACE-complete [7], allows us to show, in Chapter 4, that the subsumption problem is in PSPACE-complete.

2 Preliminaries

In this chapter we present formal definitions, introduce new notions and show some results that are relevant for the following chapters. Because they are going to be necessary for introducing the semantics that we consider here, we start by recalling some definitions about posets, ordinals and fixed-points. Next, we formalize the syntax and the greatest fixed-point semantics of the Fuzzy Description Logic $\mathcal{FL}_0$. In this step, the notion of model of a TBox is of particular significance as it justifies the presence of some results that explain, in more detail the structure of what we regard here as a model. Then, the inference problem of deciding subsumption w.r.t. greatest fixed-point semantics is introduced, which crucial for the next chapters. Finally, we present the Weighted Semi-Automata with word transitions that, as mentioned, will serve in Chapter 3, as an equivalent representation formalism.

2.1 Order, Ordinals and Fixed-Points

The notions introduced in this section can be found in [8, p. 26-29], in some cases with a slightly different notation.

In mathematics, especially order theory, a partially ordered set formalizes and generalizes the intuitive concept of an ordering of the elements of a set. A poset consists of a set together with a binary relation that indicates that, for certain pairs of elements in the set, one of the elements precedes the other. Such a relation is called a partial order to reflect the fact that not every pair of elements need be related: for some pairs, it may be that neither element precedes the other in the poset.

Definition 2.1.1. A relation R on a set S is a *partial order* if the following conditions are satisfied:
a) xRx for all $x \in S$.
b) xRy and yRx implies $x = y$ for all $x, y \in S$.
c) xRy and yRz implies xRz for all $x, y, z \in S$.

In a partial order, as we show next, the idea of the existence of an element that is bigger (smaller) than a group of elements can be formalized through the notion of upper (lower) bound.

Definition 2.1.2. Let R be a partial order on a set S. Then $u \in S$ is an *upper bound* of a subset X of S if xRu for all $x \in X$. Similarly, $l \in S$ is a *lower bound* of X if lRx for all $x \in X$.

Sometimes it is possible to draw a clear border between the elements of a preorder and their bounds. Next, in the definitions of least upper bound and greatest lower bound the notion of border is captured.

Definition 2.1.3. Let R be a partial order on a set S. Then $u \in S$ is the *least upper bound* of a subset X of S if u is an upper bound of X and, for all upper bound $\bar{u}$ of X, we have that $uR\bar{u}$. Similarly, $l \in S$ is the *greatest lower bound* of a subset X of S if l us a lower bound of X and, for all lower bound $\bar{l}$ of X, we have that $\bar{l}Rl$.

When the *least upper bound* or the *greatest lower bound* of a set X exist, they are unique. Here we denoted them by $\mathrm{lub}(L)$ and $\mathrm{glb}(L)$. Now, by adding two properties to the rather general definition of partial order, we get (as we show later) a quite practical notion of order.

Definition 2.1.4. A partially ordered set L is a *complete lattice* if every subset X of L has a $\mathrm{lub}(X)$ and a $\mathrm{glb}(X)$ in L. We let Top denote the element $\mathrm{lub}(L)$.

The greatest fiexed-point semantics is presented in the next chapter. Informally, here we can say that under this semantics the meaning of imprecise concepts is given by functions with finite domain and image in $[0, 1]$. The following examples (in particular the second) are important steps for showing that the space of interpretations (from a group of concepts) can be regarded as a partially ordered set.

Example 2.1.5. Given a finite set S, the set of functions $F : S \to [0,1]$ together with the following order is a partially ordered set. Let f_1,$f_2 \in F$, $f_1 \preceq f_2$ iff $f_1(x) \leq f_2(x)$ for all $x \in S$.
In addition, for any subset X of F, the functions $f_{inf}(x){:=}\ inf_{f\in X}\{f(x)\}$ and $f_{sup}(x){:=}\ sup_{f\in X}\{f(x)\}$ are the greatest lower bound and the least upper bound of X w.r.t. $\preceq$. This fact is trivial when the set X is finite, if X is infinite it also holds because the real unit interval [0,1] is a complete lattice with respect to the usual order of the reals [9, p.304]. Therefore, the *poset* $(F, \preceq)$ is a complete lattice.

Example 2.1.6. Let F be defined as in the previous example, consider now the n-fold cartesian product $H = F \times \cdots \times F$. The set H is ordered componentwise by the $\preceq_n$ order relation as follows, $(f_1, \ldots, f_n) \preceq_n (h_1, \ldots, h_n)$ iff $f_1 \preceq h_1, \ldots, f_n \preceq h_n$. Greatest lower bounds and least upper bounds with respect to $\preceq_n$ are obtained componentwise from the greatest lower bounds and least upper bounds of each component. Thus, $(H, \preceq_n)$ is a complete lattice.

As its name suggest, the greatest fixed-point semantics is connected with the notion of greatest fixed-point. Through this notion we formalize in the next chapter that only a certain kind of interpretations are going to be relevant for us.

Definition 2.1.7. Let L be a complete lattice, and T: $L \to L$ be a mapping. We say that $f \in L$ is the *greatest fixed-point* of T, denoted as $\mathrm{gfp}(T)$, if f is a fixed-point of T, i.e., $T(f) = f$, and for any other fixed-point g of T we have $f \geq g$.

Mappings that preserve orders are strongly connected with the notion of fixed-point. The following definition formalizes them.

Definition 2.1.8. Let L be a partially ordered set, and let $T\colon\ L \rightarrow L$ be a mapping. Then T is *monotonic* iff for all x, y in L, $x \leq y$ implies $T(x) \leq T(y)$.

The next result is a week form of a theorem due to Tarski [6]. It gives sufficient conditions for the existence of the greatest fixed-point of a monotonic mapping.

Theorem 2.1.9. *Let L be a complete lattice and let $T\colon\ L \rightarrow L$ be a monotonic mapping. Then T has a greatest fixed-point.*

If the existence of the greatest fixed-point of a mapping is known, it is possible to describe its structure in terms of a (potentially) transfinite iteration of the mapping. The formal presentation of this result involves the concept of ordinal number, which is informally introduced in the next paragraphs.

Intuitively, the *ordinal numbers* are what we use to count with. The first ordinal 0 is defined to be $\emptyset$. Then we define $1 := \{\emptyset\} = \{0\}$, $2 := \{\emptyset, \{\emptyset\}\} = \{0, 1\}$, $3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$ and so on. These are *finite ordinals*; the first *infinite ordinal* is ω, the set of non-negative integers. Ordinals can be ordered as follows: $\alpha < \beta$ iff $\alpha \in \beta$. For example, $1 < 2$ and $n < \omega$ for all finite ordinal n. In addition, the presented ordering is well-founded and linear; thus, any set of ordinals numbers has a least element and a least upper bound.

If α is an ordinal, by $\alpha + 1$ we denote the least ordinal greater than α, which is $\alpha \cup \{\alpha\}$. When an ordinal is of the form $\alpha + 1$ for some α, it is called *succesor ordinal*; otherwise it is a *limit ordinal*. For example, 2 is a succesor ordinal and ω is a limit ordinal. The smallest limit ordinal (apart from 0) is ω; which is followed by $\omega + 1 := \omega \cup \{\omega\}$, $\omega + 2 := (\omega + 1) + 1$, $\omega + 3$ and so on. Regarding the structure of ordinals, on the one hand, we already saw that a succesor ordinal is of the form $\alpha + 1$ for some ordinal α; on the other hand, a limit ordinal can be obtained as the least upper bound of all smaller ordinals, i.e., $\alpha = lub(\{\beta \,|\, \beta < \alpha\})$.

Transfinite induction is the extension of mathematical induction to ordinals. It says that to prove that a property P on the ordinals, holds at all ordinals, it is enough to prove that for all ordinals α : if P(β) holds for all ordinals $\beta < \alpha$, then also P(α) holds.

Definition 2.1.10. Let L be a complete lattice, and $T\colon\ L \rightarrow L$ be a monotonic mapping. Then we define the ordinal powers of T as follows:
$T^0 := Top$
$T^{\gamma+1} := T(T^\gamma)$ If γ is a succesor ordinal
$T^{\gamma+1} := \text{glb}(\{T^\beta |\ \beta < \gamma\})$ If γ is a limit ordinal

The following theorem explains the definition of greatest fixed point with the notion of *iterations*, which is a common term in automata theory. As we will see, many results in the next chapter are consequence of this starting point.

Theorem 2.1.11. *Let L be a complete lattice and let $T\colon\ L \rightarrow L$ be a monotonic mapping. Then, for any ordinal γ, $T^\gamma \geq$ gfp(T). Furthermore, there exists an ordinal β, such that $T^\beta =$ gfp(T).*

2.2 $\mathcal{FL}_0$

Definition 2.2.1. (Concept Description in $\mathcal{FL}_0$)
Let N_C and N_R be disjoint sets of concept names and role names respectively. The set of $\mathcal{FL}_0$-concept descriptions is the smallest set satisfying the following conditions:

- Every atomic concept $A \in N_C$ is a $\mathcal{FL}_0$–concept description.
- If C and D are $\mathcal{FL}_0$–concept descriptions and $r \in N_R$ is an *atomic role*, then $C \sqcap D$ and $\forall r.C$ are also $\mathcal{FL}_0$–concept descriptions.

Here we are interested in the representation of imprecise knowledge through $\mathcal{FL}_0$, that is why a *fuzzy interpretation* of the concepts from this DL will be considered.

Definition 2.2.2. (Fuzzy semantics)
A fuzzy interpretation is a pair $\mathcal{I} := (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the domain and $\cdot^{\mathcal{I}}$ is a function mapping:

- every atomic concept A to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$, and
- every atomic role r to a function $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$

What characterizes the Gödel semantics is how this interpretation is extended to complex concepts as we show in the next definition.

Definition 2.2.3. (Gödel semantics)

$$(A \sqcap B)^{\mathcal{I}}(x) := min(A^{\mathcal{I}}(x), B^{\mathcal{I}}(x))$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

$$(\forall r.A)^{\mathcal{I}}(x) := inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y)\}$$

Lemma 2.2.4. *Let v_1, v_2, v_3 and v_4 be real values in $[0, 1]$. Then, $(v_1 \Rightarrow \min(v_2, v_3)) \geq v_4$ iff $(v_1 \Rightarrow v_2) \geq v_4$ and $(v_1 \Rightarrow v_3) \geq v_4$.*

Proof
Let assume that $\min(v_2, v_3) = v_2$. From the definition of $\Rightarrow$, we have that $(v_1 \Rightarrow v_3) \geq (v_1 \Rightarrow v_2)$. Thus it is inmediate that $(v_1 \Rightarrow v_2) \geq v_4$ iff $(v_1 \Rightarrow v_2) \geq v_4$ and $(v_1 \Rightarrow v_3) \geq v_4$. If $\min(v_2, v_3) = v_3$, from the definition of $\Rightarrow$, we have that $(v_1 \Rightarrow v_2) \geq (v_1 \Rightarrow v_3)$. Thus it is inmediate that $(v_1 \Rightarrow v_3) \geq v_4$ iff $(v_1 \Rightarrow v_2) \geq v_4$ and $(v_1 \Rightarrow v_3) \geq v_4$. □

Lemma 2.2.5. *Let v_1, v_2, v_3 and v_4 be real values in $[0, 1]$. Then, $(v_1 \Rightarrow v_2) \geq v_3$ and $(v_1 \Rightarrow v_2) \geq v_4$ iff $(v_1 \Rightarrow v_2) \geq \max(v_3, v_4)$.*

Proof
Both directions are immediate. □

Lemma 2.2.6. *Given $r \in N_R$ and two $\mathcal{FL}_0$–concept descriptions C and D; for every interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ and $d \in \Delta^{\mathcal{I}}$, $(\forall r.(C \sqcap D))^{\mathcal{I}}(d) = (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}(d)$.*

Proof

For proving that these two terms have the same value, we expand both expressions and then, by a simple case-analysis, the equality becomes clear.

$(\forall r(C \sqcap D))^{\mathcal{I}}(d)$
$= inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow (C \sqcap D)^{\mathcal{I}}(y)\}$
$= min_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow (C \sqcap D)^{\mathcal{I}}(y)\}$ since the set $\Delta^{\mathcal{I}}$ is finite
$= min_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow min\{C^{\mathcal{I}}(y), D^{\mathcal{I}}(y)\}\}$ $(*)$

$(\forall rC \sqcap \forall rD)^{\mathcal{I}}(d)$
$= min\{(\forall rC)^{\mathcal{I}}(d), (\forall rD)^{\mathcal{I}}(d)\}$
$= min\{inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow C^{\mathcal{I}}(y)\}, inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow D^{\mathcal{I}}(y)\}\}$
$= min\{min_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow C^{\mathcal{I}}(y)\}, min_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(d,y) \Rightarrow D^{\mathcal{I}}(y)\}\}$ since the set $\Delta^{\mathcal{I}}$ is finite
$= min_{y \in \Delta^{\mathcal{I}}}\{min\{r^{\mathcal{I}}(d,y) \Rightarrow C^{\mathcal{I}}(y)\ , r^{\mathcal{I}}(d,y) \Rightarrow D^{\mathcal{I}}(y)\}\}$ $(**)$

For every $y \in \Delta^{\mathcal{I}}$, each of the 13 possible order relation among $r^{\mathcal{I}}(d,y)$, $C^{\mathcal{I}}(y)$ and $D^{\mathcal{I}}(y)$ results in the equality of $r^{\mathcal{I}}(d,y) \Rightarrow min\{C^{\mathcal{I}}(y), D^{\mathcal{I}}(y)\}$ and $min\{r^{\mathcal{I}}(d,y) \Rightarrow C^{\mathcal{I}}(y)\ , r^{\mathcal{I}}(d,y) \Rightarrow D^{\mathcal{I}}(y)\}$. Thus we have that $(*)$ and $(**)$ are equal. □

Definition 2.2.7. (*Fuzzy* TBox)
A (labeled) *terminological axiom* is of the form $\langle A \sqsubseteq C, q\rangle$, where A is an atomic concept, C is a concept description and $q \in [0,1]$. A *fuzzy* TBox is a finite set of *terminological axioms*.

In a TBox there are two kinds of atomic concepts: *primitive concepts* and *defined concepts*. An atomic concept B is *defined* if it appears on the left-hand-side of a terminological axiom in the TBox. Otherwise, it is *primitive*.

Definition 2.2.8. (Primitive Interpretation)
Let $P_1, \ldots, P_m$ and $r_1, \ldots, r_l$ be the primitive concepts and roles in a TBox $\mathcal{T}$. A *primitive interpretation* $\mathcal{J}$ consists of a set $\Delta^{\mathcal{J}}$ and a function $\cdot^{\mathcal{J}}$, where $\Delta^{\mathcal{J}}$ is the domain of the primitive interpretation and the function $\cdot^{\mathcal{J}}$ links P_i and r_j with $P_i^{\mathcal{J}} : \Delta^{\mathcal{J}} \rightarrow [0,1]$ and $r_j^{\mathcal{J}} : \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} \rightarrow [0,1]$ respectively.

Definition 2.2.9. (Extension of an interpretation)
Let $P_1, \ldots, P_m$ and $r_1, \ldots, r_l$ be the primitive concepts and roles contained in a TBox $\mathcal{T}$. An interpretation $\mathcal{I}$ is the *extension* of the primitive interpretation J iff $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$, $P_1^{\mathcal{I}} = P_1^{\mathcal{J}}, \ldots, P_m^{\mathcal{I}} = P_m^{\mathcal{J}}$ and $r_1^{\mathcal{I}} = r_1^{\mathcal{J}}, \ldots, r_l^{\mathcal{I}} = r_l^{\mathcal{J}}$. When this is the case, we could also say that $\mathcal{J}$ is the *primitive interpretation of* $\mathcal{I}$

Definition 2.2.10. (Components of an interpretation)
Assume that $D_1, \ldots, D_n$ is an order of the defined concepts in a TBox $\mathcal{T}$. We say that the interpretation $\mathcal{I}$ *is defined by* $\mathcal{J}$ and f iff $\mathcal{J}$ is the primitive interpretation of $\mathcal{I}$ and $f = (D_1^{\mathcal{I}}, \ldots, D_n^{\mathcal{I}})$.

Clearly, this separation of an interpretation is always possible and unique up to the order of the defined concepts. Here, without loss of generality, we assume that this order is fixed, i.e., for a defined concept B, its order is denoted by $index(B)$. Then, given an interpretation $\mathcal{I}$, defined by $\mathcal{J}$ and f, each defined concept in $\mathcal{T}$ is interpreted as a component of the n-tuple f, i.e., if for a defined concept B we have $index(B) = i$ then $(B)^{\mathcal{I}} = f_i$.

Next, we show that a possible order between two interpretations is propagated into the interpretation of concepts. The notions of order that we use here, for comparing functions ($\preceq$) and tuples of functions ($\preceq_n$), are the same that we introduced in Example 2.1.5 and in Example 2.1.6. The same applies for the nexts sections.

Proposition 2.2.11. *Let $\mathcal{I}_f$ and $\mathcal{I}_g$ be two interpretations defined by $\mathcal{J}$,f and $\mathcal{J}$,g respectively. Then, $f \preceq_n g$ implies that for all $\mathcal{FL}_0$-concept descriptions C, it holds that $C^{\mathcal{I}_f} \preceq C^{\mathcal{I}_g}$.*

Proof

We proceed by induction on the structure of the concept C:

- If C is a primitive concept, $\forall x \in \Delta^{\mathcal{J}}$: $C^{\mathcal{J}}(x) = C^{\mathcal{I}_f}(x) = C^{\mathcal{I}_g}(x)$. Thus $C^{\mathcal{I}_f} \preceq C^{\mathcal{I}_g}$.

- If C is a defined concept, $C^{\mathcal{I}_f} \preceq C^{\mathcal{I}_g}$ by the definition of $\preceq_n$, and because the interpretations $\mathcal{I}_f, \mathcal{I}_g$ are defined by $\mathcal{J}$,f and $\mathcal{J}$,g respectively.

- If $C = \forall r.D$, by induction, the proposition holds for the concept D. Then we have that $\forall x, y \in \Delta^{\mathcal{J}}$: $\{r^{\mathcal{J}}(x,y) \Rightarrow D^{\mathcal{I}_f}(y)\} \leq \{r^{\mathcal{J}}(x,y) \Rightarrow D^{\mathcal{I}_g}(y)\}$, which directly implies that $\forall x \in \Delta^{\mathcal{J}}$: $inf_{y\in\Delta^{\mathcal{J}}}\{r^{\mathcal{J}}(x,y) \Rightarrow D^{\mathcal{I}_f}(y)\} \leq inf_{y\in\Delta^{\mathcal{J}}}\{r^{\mathcal{J}}(x,y) \Rightarrow D^{\mathcal{I}_g}(y)\}$. Thus $(\forall r.D)^{\mathcal{I}_f} \preceq (\forall r.D)^{\mathcal{I}_g}$.

- If $C = D \sqcap E$, by induction, the proposition holds for the concepts D and E. Then we have that $\forall x \in \Delta^{\mathcal{J}}$: $min(D^{\mathcal{I}_f}(x), E^{\mathcal{I}_f}(x)) \leq min(D^{\mathcal{I}_g}(x), E^{\mathcal{I}_g}(x))$. Thus, $(D \sqcap E)^{\mathcal{I}_f} \preceq (D \sqcap E)^{\mathcal{I}_g}$. □

Definition 2.2.12. (*Model* of a TBox)

Let $\mathcal{I}$ be an interpretation. We say that $\mathcal{I}$ *satisfies* the *terminological axiom* $\langle A \sqsubseteq C, q\rangle$ iff $A^{\mathcal{I}}(x) \Rightarrow C^{\mathcal{I}}(x) \geq q$ for every $x \in \Delta^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ iff it satisfies every *terminological axiom* in $\mathcal{T}$.

Definition 2.2.13. ($T_{\mathcal{J}}$)

Let $D_1, \ldots, D_n$ be the defined concepts contained in a TBox $\mathcal{T}$, and let $\mathcal{J}$ be a primitive interpretation. The mapping $T_{\mathcal{J}}$: $F^n \rightarrow F^n$, where $F = [0,1]^{\Delta^{\mathcal{J}}}$, is defined as follows. Let f be an element of F^n and let $\mathcal{I}$ be the interpretation defined by $\mathcal{J}$ and f. Then:
$T_{\mathcal{J}}(f) := (h_1, \ldots, h_n)$, where $h_i(x) := \min_{\langle D_i \sqsubseteq Y, q\rangle \in \mathcal{T}}\{q \Rightarrow Y^{\mathcal{I}}(x)\}$.

Example 2.2.14. (T_J)

Consider the TBox $\mathcal{T} = \{\langle Z \sqsubseteq \forall c.G, 0.7\rangle, \langle K \sqsubseteq \forall c.G, 0.3\rangle, \langle K \sqsubseteq Z, 0.2\rangle\}$ with $index(Z) := 1$, $index(K) := 2$; and the primitive interpretation $\mathcal{J} := (\Delta^{\mathcal{J}}, .^{\mathcal{J}})$, where $\Delta^{\mathcal{J}} = \{d_0, d_1\}$, $G^{\mathcal{J}}(d_0) = 1$, $G^{\mathcal{J}}(d_1) = 0$, $c^{\mathcal{J}}(d_0, d_0) = 1$, $c^{\mathcal{J}}(d_1, d_0) = 1$, $c^{\mathcal{J}}(d_1, d_1) = 0$ and $c^{\mathcal{J}}(d_0, d_1) = 0$. Let the particular function $f = (f_1, f_2)$ be the defined by $f_1(d_i) = f_2(d_i) = 1$, for all $i \in \{0, 1\}$. Let $\mathcal{I}_f$ be the interpretation defined by $\mathcal{J}$ and f, then $T_{\mathcal{J}}(f) = (h_1, h_2)$, where $h_1(x) = (0.7 \Rightarrow (\forall c.G)^{\mathcal{I}_f}(x))$ and $h_2(x) = \min((0.3 \Rightarrow (\forall c.G)^{\mathcal{I}_f}(x)), (0.2 \Rightarrow (Z)^{\mathcal{I}_f}(x)))$.

Lemma 2.2.15. *Let $\mathcal{T}$ be a TBox and let $\mathcal{J}$ be a primitive interpretation. The mapping $T_{\mathcal{J}}$ is monotonic.*

Proof

We need to show that for all $f, g \in F^n$, $f \preceq_n g$ implies $T_{\mathcal{J}}(f) \preceq_n T_{\mathcal{J}}(g)$. In Proposition 2.2.11 it was proved that $f \preceq_n g$ implies $C^{\mathcal{I}_f} \preceq C^{\mathcal{I}_g}$ for every $\mathcal{FL}_0$-concept description C. With this in mind we can see why $T_{\mathcal{J}}$ is monotonic. By definition of $T_{\mathcal{J}}$, we know that $(T_{\mathcal{J}}(f))_i(x) = \min_{\langle D_i \sqsubseteq Y, q\rangle \in \mathcal{T}}\{q \Rightarrow Y^{\mathcal{I}_f}(x)\}$ and $(T_{\mathcal{J}}(g))_i(x) = \min_{\langle D_i \sqsubseteq Y, q\rangle \in \mathcal{T}}\{q \Rightarrow Y^{\mathcal{I}_g}(x)\}$ but, because the Proposition 2.2.11, we know that if $f \preceq_n g$ then $Y^{\mathcal{I}_f} \preceq Y^{\mathcal{I}_g}$ which implies that $(T_{\mathcal{J}}(f))_i(x) = \min_{\langle D_i \sqsubseteq Y, q\rangle \in \mathcal{T}}\{q \Rightarrow Y^{\mathcal{I}_f}(x)\}$ $\leq \min_{\langle D_i \sqsubseteq Y, q\rangle \in \mathcal{T}}\{q \Rightarrow Y^{\mathcal{I}_g}(x)\} = (T_{\mathcal{J}}(g))_i(x)$ and thus $T_{\mathcal{J}}$ is monotonic. □

Proposition 2.2.16. *Let I be a model of a TBox $\mathcal{T}$ and let I be defined by the primitive interpretation $\mathcal{J}$ and the tuple f. Then it holds that $f \preceq_n T_{\mathcal{J}}(f)$.*

Proof
Assume that I is a model of $\mathcal{T}$ and that $f \preceq_n T_{\mathcal{J}}(f)$ does not hold. Then, following the definition of $\preceq_n$, we have that there exist a defined concept C, a number $i \leq n$ and $d \in \Delta^{\mathcal{J}}$ such that $i = index(C)$ in f and $f_i(d) > (T_{\mathcal{J}}(f))_i(d)$. Without loss of generality, we may assume that in the definition of $T_{\mathcal{J}}$ the value $(T_{\mathcal{J}}(f))_i(d)$ is produced by an axiom with the form $\langle C \sqsubseteq Y, q\rangle$. Then we have that $C^{\mathcal{I}}(d) = f_i(d) > (T_{\mathcal{J}}(f))_i(d) = (q \Rightarrow Y^{\mathcal{I}}(d))$. But, from $f_i(d) > (T_{\mathcal{J}}(f))_i(d)$, we get that $(T_{\mathcal{J}}(f))_i(d) < 1$, which means that $q > Y^{\mathcal{I}}(d)$. Then $(q \Rightarrow Y^{\mathcal{I}}(d)) = Y^{\mathcal{I}}(d)$ and $C^{\mathcal{I}}(d) > (Y^{\mathcal{I}}(d))$. Thus $(C^{\mathcal{I}}(d) \Rightarrow Y^{\mathcal{I}}(d)) < q$ and this is means that I can not be a model of $\mathcal{T}$, which is in contradiction with our assumption. □

Proposition 2.2.17. *Let I be a model of a TBox $\mathcal{T}$ and let I be defined by the primitive interpretation $\mathcal{J}$ and the tuple f. Then, for all γ, it holds that $f \preceq_n T_{\mathcal{J}}^{\gamma}$.*

Proof
We proceed by induction on γ. For $\gamma = 0$ the proposition is trivially fulfilled because of the definition of ordinal power. Assuming that for all $\alpha < \gamma$, $f \preceq_n T_{\mathcal{J}}^{\alpha}$ holds, next we show that it also holds for γ.

Case: $\gamma > 0$ and limit ordinal
From the assumption, we get that f is a lower bound for the set $X = \{T_{\mathcal{J}}^{\alpha} | \alpha < \gamma\}$. Also, by definition of ordinal power (in the case of limit ordinal) we know that $T_{\mathcal{J}}^{\gamma} = glb(X)$. Now, since f is a lower bound of X and $T_{\mathcal{J}}^{\gamma} = glb(X)$, because of the definition of greatest lower bound, we get that $f \preceq_n T_{\mathcal{J}}^{\gamma}$.

Case: γ is successor ordinal
Without loss of generality we may assume that $\gamma = \beta + 1$. Then, from the assumptions we have that $f \preceq_n T_{\mathcal{J}}^{\beta}$, and because $T_{\mathcal{J}}$ is a monotonic mapping, we get that $T_{\mathcal{J}}(f) \preceq_n T_{\mathcal{J}}(T_{\mathcal{J}}^{\beta})$. This, together with the Proposition 2.2.16, give us that $f \preceq_n T_{\mathcal{J}}(f) \preceq_n T_{\mathcal{J}}(T_{\mathcal{J}}^{\beta})$. Thus, $f \preceq_n T_{\mathcal{J}}^{\gamma}$. □

Proposition 2.2.18. *Let I be a model of a TBox $\mathcal{T}$ and let I be defined by the primitive interpretation $\mathcal{J}$ and the tuple f. Then, there exists a fixed-point f_p of $T_{\mathcal{J}}$ such that $f \preceq_n f_p$.*

Proof
Since we know that $T_{\mathcal{J}}$ is a monotonic mapping and it is defined over a complete lattice, we have that the $gfp(T_{\mathcal{J}})$ exists and for some ordinal β, $gfp(T_{\mathcal{J}}) = T_{\mathcal{J}}^{\beta}$. Finally, following the Proposition 2.2.17, we get that $m \preceq_n T_{\mathcal{J}}^{\beta}$. Thus, $f_p := T_{\mathcal{J}}^{\beta}$. □

Definition 2.2.19. (gfp–semantics)
Let $\mathcal{T}$ be a TBox. The *greatest fixed–point semantics* (gfp–semantics) allows only those models (gfp–models) of $\mathcal{T}$ that are defined by an initial interpretation $\mathcal{J}$ and the *greatest fixed-point* of the mapping $T_{\mathcal{J}}$.

Definition 2.2.20. (gfp-Subsumption of concepts)
Given two concept names C,D and $q \in [0,1]$. We say that C is *subsumed to a degree* q by D w.r.t. a TBox $\mathcal{T}$ (denoted as $\langle C \sqsubseteq_{gfp,\mathcal{T}} D \ , \ q\rangle$) if for every gfp–model $\mathcal{I}$ of $\mathcal{T}$ it holds that $inf_{x\in\Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\} \geq q$.

Then, for deciding (from the definition) whether a concept is subsumed to a certain degree by another concept, it is necessary to make an iteration over the whole space of primitive interpretations. Since this space is infinite, we are not going to do that. Instead, in the next chapter, we present an equivalent characterization for the notion of gfp-Subsumption. Following that, we will provide a short argumentation as to why, for instance, in the Example 2.2.14, it does not hold that $\langle K \sqsubseteq_{gfp,\mathcal{T}} Z \ , \ 0.4\rangle$.

2.3 Automata and Words

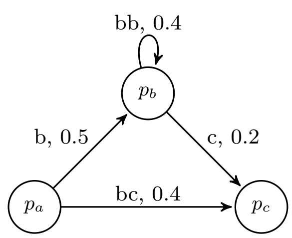
In [2], *semi-automata* with words transitions are used for capturing the effect of the greatest fixed-point semantics in $\mathcal{FL}_0$ (with the possible presence of terminological cylces). Here, for extending this analysis to fuzzy $\mathcal{FL}_0$, weighted semi-automata with words transitions seems to be the suitable formalism.

Let Σ be a finite alphabet. The set of all finite words over Σ is denoted by Σ^* and the empty word by ϵ. A word $W = \sigma_0 \ldots \sigma_{n-1}$ over Σ (of length n) can be regarded as a mapping W from the finite ordinal $n = \{0 \ldots n-1\}$ into Σ, i.e. $W(i) := \sigma_i$.

Definition 2.3.1. A *weighted semi-automaton* (WSA) is a tuple $A = (\Sigma, Q, E, wt)$ where Σ is a finite alphabet, Q is a finite set of states, $E \subseteq Q\times\Sigma^*\times Q$ is a finite set of transitions and wt: $E \to [0,1]$ is a *transition weight function*.

Therefore, a transition connects two states and each connection is labeled by a word in Σ^* and by its corresponding weight from wt.

Example 2.3.2. (A weighted semi-automaton)



$\Sigma = \{b, c\}$
$Q = \{p_a, p_b, p_c\}$
$E = \{(p_a, b, p_b), (p_b, bb, p_b), (p_b, c, p_c), (p_a, bc, p_c)\}$
$wt((p_a, b, p_b)) = 0.5$; $wt((p_b, bb, p_b)) = 0.4$
$wt((p_a, b, p_b)) = 0.2$; $wt((p_a, bc, p_c)) = 0.4$

Let A be a WSA and let p,q be two states of A. A *finite path* from p to q in A is a sequence p_0,U_0,p_1,U_2,...,U_{n-1},p_n, where $p = p_0$, $q = p_n$, and for each i, $1 \leq i \leq n$, (p_{i-1}, U_{i-1}, p_i) is a transition of A. The *label* of this path, is the finite word $U_0U_1 \ldots U_{n-1}$, and the particular case of the empty path from p to p is labeled the empty word ϵ. By $L_A(p, q)$ we denote the set of all finite words that are labels of paths from p to q.

Let p,q be two states in a WSA A, and let W be a finite word; the *set of finite paths* from p to q that are labeled by W is denoted $paths(A, W, p, q)$. The *weight* of a path $l = p_0$,U_0,p_1,U_1,...,U_{n-1},$p_n \in paths(A, W, p, q)$ is defined as the value:

$val_A(l) := \inf(\bigcup_{i=0}^{n-1}\{wt(p_i, U_i, p_{i+1})\})$

The *behavior* of A from p to q on a finite word W is defined as the value:

$val(A_{p,q}, W) := \sup(\bigcup_{l \in paths(A,W,p,q)}\{val_A(l)\})$

Since the image of wt is finite, the infima and suprema in the given definitions are restricted to minimum and maximum. Then, the behavior of a weighted semi-automata can be seen as a function that associtės each word with the weight of its execution between to given states. For instance, in the Example 2.3.2 we can see that $val(A_{p_a,p_c}, \text{“}bc\text{”}) = \max(\min(0.5, 0.2), 0.4) = 0.4$.

Finally, it is necessary to mention that in the next chapters we will also take advantage of the *semi-automaton with words transitions* that was defined in [2, p.181], which is a particular case of the WSA when the transition weight function is not taken into account.

3 Semantics Through Finite Automata

3.1 WSA Associated with a TBox

As a preliminary step, before we can associate a WSA $A_{\mathcal{T}}$ with a TBox $\mathcal{T}$, we need to transform $\mathcal{T}$ into an equivalent TBox $\mathcal{T}'$ such that $\mathcal{T}'$ is in normal form. Here, by normal form we mean that the axioms in $\mathcal{T}'$ do not contain any constructor of concept conjunction.

Definition 3.1.1. (TBox in Normal Form)
We say that a terminological axiom $\langle C \sqsubseteq D, q\rangle$ is in normal form if the concept constructor $\sqcap$ does not occur in D. A TBox is in normal form if it only contains axioms that are in normal form.

Next we explain how to transform a TBox $\mathcal{T}$ into an equivalent TBox $\mathcal{T}'$ which is in normal form. For each axiom $\langle C \sqsubseteq D, q\rangle \in \mathcal{T}$ we proceed in three steps. First, the Lemma 2.2.6 allows us to make expansions in the right-hand side of the axiom in order to express D as a conjunction of concepts that do not contain conjuntion, i.e., every concept $\forall r.(D_1 \sqcap D_2)$ is expanded to $\forall r.D_1 \sqcap \forall r.D_2$. Then, assuming that $\langle C \sqsubseteq D_1 \sqcap D_2 \sqcap \cdots \sqcap D_j, q\rangle$ is the result of the previous step, the Lemma 2.2.4 allows the replacement of this axiom by the j axioms $\{\langle C \sqsubseteq D_i, q\rangle\}$. Next, after this division, there could be groups of axioms that only differ in the weights; if this is the case, the Lemma 2.2.5 allows us to keep, from each group, only the axiom with the biggest weight value. Now we can assume, as we wanted, that all the axioms in $\mathcal{T}'$ are in the form $\langle C \sqsubseteq \forall r_1.\forall r_2.\ldots\forall r_n.B, q\rangle$ where B is an atomic concept. In addition, the prefix $\forall r_1.\forall r_2.\ldots\forall.r_n$ is abbreviated to $\forall W$, where $W = r_1r_2\ldots r_n$ is a word over N_R. In the case that $n = 0$ we write $\forall\epsilon.B$ instead of B. Finally, it is clear that the presented transformations are not computationaly heavy; as they were introduced, the expansion, division and removing of some axioms are all tasks that can be completed in polynomial time with respect to the size of the original TBox.

Example 3.1.2. (TBox in Normal Form)
$\mathcal{T} = \{\langle Z \sqsubseteq Z \sqcap \forall c.G, 0.7\rangle, \langle Z \sqsubseteq Z, 0.8\rangle, \langle K \sqsubseteq \forall c.(Z \sqcap G), 0.2\rangle\}$

$\mathcal{T}_1 = \{\langle Z \sqsubseteq Z \sqcap \forall c.G, 0.7\rangle, \langle Z \sqsubseteq Z, 0.8\rangle, \langle K \sqsubseteq \forall c.Z \sqcap \forall c.G, 0.2\rangle\}$

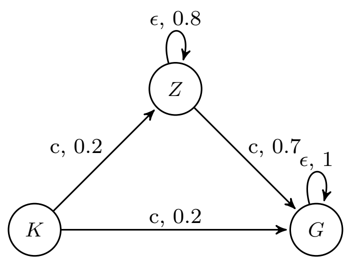
$\mathcal{T}_2 = \{\langle Z \sqsubseteq Z, 0.7\rangle, \langle Z \sqsubseteq \forall c.G, 0.7\rangle, \langle Z \sqsubseteq Z, 0.8\rangle, \langle K \sqsubseteq \forall c.Z, 0.2\rangle\}, \langle K \sqsubseteq \forall c.G, 0.2\rangle\}$

$\mathcal{T}_3 = \{\langle Z \sqsubseteq \forall c.G, 0.7\rangle, \langle Z \sqsubseteq \forall\epsilon.Z, 0.8\rangle, \langle K \sqsubseteq \forall c.Z, 0.2\rangle\}, \langle K \sqsubseteq \forall c.G, 0.2\rangle\}$

$\mathcal{T}' = \mathcal{T}_3$

Definition 3.1.3. (WSA associated with a TBox)
Let $\mathcal{T}$ be a TBox in normal form, the WSA $A_{\mathcal{T}} = (\Sigma, Q, E, wt)$ associated with $\mathcal{T}$ is defined as follows. The alphabet Σ is defined as the set of role names occuring in the axioms of $\mathcal{T}$. The set of states Q is defined as the set of concept names occurring in the axioms of $\mathcal{T}$. Transitions and their weights are determined in the following way: A terminological axiom of the form $\langle C \sqsubseteq \forall W.B, q\rangle$, produce the transition (C, W, B), and $wt((C, W, B)) := q$. Moreover, each primitive concept P produce the transition (P, ϵ, P) with $wt((P, \epsilon, P)) = 1$.

Example 3.1.4. (WSA associated with a TBox)



$\Sigma = \{c\}$
$Q = \{Z, K, G\}$
$E = \{(Z, c, G), (Z, \epsilon, Z), (K, c, Z), (K, c, G), (G, \epsilon, G)\}$
$wt((Z, c, G)) = 0.7$; $wt((Z, \epsilon, Z)) = 0.8$
$wt((K, c, Z)) = 0.2$; $wt((K, c, G)) = 0.2$
$wt((G, c, G)) = 1$

3.2 Characterization of the gfp-Semantics

As we have seen, in the formalization of our semantics, our only interest is in gfp-models. This kinds of models are defined by an initial interpretation $\mathcal{J}$ and the greatest fixed-point of $T_{\mathcal{J}}$. Moreover, by using Theorem 2.1.11, we are allowed to represent the greatest fixed-point of $T_{\mathcal{J}}$ as an ordinal power of $T_{\mathcal{J}}$. Thus, in the next two lemmata, behind some results that involve ordinal powers at the same time, accepted words and behaviors of words, there are the first steps for rewriting the gretest fixed-point semantics in terms of finite automata.

Lemma 3.2.1. *Given a TBox $\mathcal{T}$, let $A_{\mathcal{T}}$ be the corresponding WSA associated with $\mathcal{T}$. Let $\mathcal{I}$ be a gfp-model of $\mathcal{T}$ with primitive interpretation $\mathcal{J}$. Let C be a defined concept ocurring in $\mathcal{T}$, $i = index(C)$ and $d \in \Delta^{\mathcal{I}}$. If there exist a primitive concept P in $\mathcal{T}$, a word $W \in L_{A_{\mathcal{T}}}(C, P)$ and $e \in \Delta^{\mathcal{I}}$ such that: $P^{\mathcal{I}}(e) < q$, $(\forall W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$ and $val(A_{C,P}, W) > P^{\mathcal{I}}(e)$; then there exist an ordinal β such that $(T^{\beta}_{\mathcal{J}})_i(d) \leq P^{\mathcal{I}}(e)$.*

Proof
Consider in $A_{\mathcal{T}}$ the paths from C to P that are labeled by W and their *weight* is equal to $val(A_{C,P}, W)$. Now, among all the possibilities, by $CU_0C_1U_1 \ldots C_mU_mP$ we denote one of them with minimal length. We proceed by induction on m.

For $m = 0$:
We have that $W = U_0$, besides, one of the defining axioms for C is of the form $\langle C \sqsubseteq \forall W.P, val(A_{C,P}, W)\rangle$. From the assumptions of that $(\forall W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$ and $val(A_{C,P}, W) > P^{\mathcal{I}}(e)$ together with the definition of primitive interpretation, we get that $val(A_{C,P}, W) > (\forall W.P)^{\mathcal{I}}(d) = (\forall W.P)^{\mathcal{J}}(d)$. Thus, $(T_{\mathcal{J}}(Top))_i(d) \leq P^{\mathcal{I}}(e)$, and $\beta = 1$.

For $m > 0$:
Since $\forall(W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$, there exist a $\bar{d} \in \Delta^{\mathcal{I}}$ such that $\forall(U_1 \ldots U_m.P)^{\mathcal{I}}(\bar{d}) = P^{\mathcal{I}}(e)$. From the assumptions, $CU_0C_1U_1 \ldots C_mU_mP$ causes that $val(A_{C,P}, W) > P^{\mathcal{I}}(e)$, which implies that $val(A_{C_1,P}, U_1 \ldots U_m) > P^{\mathcal{I}}(e)$. Then, by applying the induction hypothesis we get that $(T^{\lambda}_{\mathcal{J}})_j(\bar{d}) \leq P^{\mathcal{I}}(e)$ for some $\lambda > 0$ (where $j = index(C_1)$). The previous idea, together with the assumption of that $\forall(W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$ and $\forall(U_1 \ldots U_m.P)^{\mathcal{I}}(\bar{d}) = P^{\mathcal{I}}(e)$, implies that $\forall(U_0.C_1)^{\mathcal{I}_{\lambda}}(d) \leq P^{\mathcal{I}}(e)$, where $\mathcal{I}_{\lambda}$ is the interpretation defined by $\mathcal{J}$ and the tuple $T^{\lambda}_{\mathcal{J}}$. Thus, because $val(A_{C,P}, W) > P^{\mathcal{I}}(e)$, we get that for the next ordinal $\lambda + 1$, $(T^{\lambda+1}_{\mathcal{J}})_i(d) \leq P^{\mathcal{I}}(e)$ and $\beta = \lambda + 1$. □

Lemma 3.2.2. *Given a TBox $\mathcal{T}$, let $A_{\mathcal{T}}$ be the corresponding WSA associated with $\mathcal{T}$. Let $\mathcal{I}$ be a gfp-model of $\mathcal{T}$ with primitive interpretation $\mathcal{J}$. Let C be a defined concept ocurring in $\mathcal{T}$, $i = index(C)$, $d \in \Delta^{\mathcal{I}}$ and $q \in [0, 1]$. If there exist an ordinal k such* *that $(T^k_{\mathcal{J}})_i(d) < q$ then there exist a primitive concept P in $\mathcal{T}$, a word $W \in L_{A_\mathcal{T}}(C,P)$ and an individual $e \in \Delta^\mathcal{I}$ such that $P^\mathcal{I}(e) < q$, $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$, $val(A_{C,P}, W) > P^\mathcal{I}(e)$ and $(T^k_{\mathcal{J}})_i(d) \geq P^\mathcal{I}(e)$.*

Proof
We proceed by induction on k:

For $k = 0$
We have that $(Top)_i(d) < q$ which is in contradiction with the definition of Top. Thus, this case can not hold.

For $k > 0$ and succesor ordinal
We have that $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d) < q$. Assume that the concept $\forall W.B$ in the axiom $\langle C \sqsubseteq \forall W.B, v\rangle$ is the responsible for $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d) < q$. If B is a primitive concept, there exists an $e \in \Delta^\mathcal{I}$ such that $B^\mathcal{I}(e) < q$, $(\forall W.B)^\mathcal{I}(d) = B^\mathcal{I}(e)$, $v > B^\mathcal{I}(e)$ and $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d) = (\forall W.B)^\mathcal{I}(d)$. From $v > B^\mathcal{I}(e)$, we have that $val(A_{C,B}, W) > B^\mathcal{I}(e)$, which together with $B^\mathcal{I}(e) < q$, $(\forall W.B)^\mathcal{I}(d) = B^\mathcal{I}(e)$ and $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d) = (\forall W.B)^\mathcal{I}(d) = B^\mathcal{I}(e)$ completes the proof of this case.

When B is a defined concept (with $index(B) = j$), there exists an $e \in \Delta^\mathcal{I}$ such that $(T^{k-1}_\mathcal{J})_j(e) < q$, $(\forall W.B)^{\mathcal{I}_{k-1}}(d) = (T^{k-1}_\mathcal{J})_j(e)$ and also $v > (T^{k-1}_\mathcal{J})_j(e)$. Here $\mathcal{I}_{k-1}$ is the interpretation defined by the primitive interpretation $\mathcal{J}$ and the tuple $T^{k-1}_\mathcal{J}$. In the particular case in which $W = \epsilon$, from $(\forall W.B)^{\mathcal{I}_{k-1}}(d) = (T^{k-1}_\mathcal{J})_j(e)$, it is not hard to see that $e = d$. Now we can apply the induction hypothesis to $(T^{k-1}_\mathcal{J})_j(e) < q$; then we know that there exist a primitive concept $\bar{P}$, a word $\bar{W} \in L_{A_\mathcal{T}}(B, \bar{P})$, and an individual $\bar{e} \in \Delta^\mathcal{I}$ such that $\bar{P}^\mathcal{I}(\bar{e}) < q$, $(\forall \bar{W}.\bar{P})^\mathcal{I}(e) = \bar{P}^\mathcal{I}(\bar{e})$, $val(A_{B,\bar{P}}, \bar{W}) > \bar{P}^\mathcal{I}(\bar{e})$ and $(T^{k-1}_\mathcal{J})_j(e) \geq \bar{P}^\mathcal{I}(\bar{e})$. Under the previous assumptions, we first notice (as we will show) that $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) \leq \bar{P}^\mathcal{I}(\bar{e})$. Then, by e' we denote the element of the domain that satisfies $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) = \bar{P}^\mathcal{I}(e')$. We now show that $\bar{P}^\mathcal{I}(e') < q$, $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) = \bar{P}^\mathcal{I}(e')$, $val(A_{C,\bar{P}}, W\bar{W}) > \bar{P}^\mathcal{I}(e')$ and $(T^k_\mathcal{J})_i(d) \geq \bar{P}^\mathcal{I}(e')$.

For doing that, first we see why $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) \leq \bar{P}^\mathcal{I}(\bar{e})$ holds: As mentioned before, if $W = \epsilon$ we have that $e = d$. Thus, $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) = (\forall W\bar{W}.\bar{P})^\mathcal{I}(e) = (\forall \bar{W}.\bar{P})^\mathcal{I}(e) = \bar{P}^\mathcal{I}(\bar{e})$. In the case that $W \neq \epsilon$ (with length n), from the assumptions we have that $(\forall W.B)^{\mathcal{I}_{k-1}}(d) = (T^{k-1}_\mathcal{J})_j(e)$. Then, after unfolding the left part in the previous equality we get that there exist $d_0 = d, \ldots, d_n = e$ such that $W^\mathcal{I}_0(d, d_1) \Rightarrow (\cdots \Rightarrow (W^\mathcal{I}_{n-1}(d_{n-1}, e) \Rightarrow (T^{k-1}_\mathcal{J})_j(e)) \ldots) = (T^{k-1}_\mathcal{J})_j(e)$. But, since from the induction step we know that $(T^{k-1}_\mathcal{J})_j(e) \geq \bar{P}^\mathcal{I}(\bar{e})$, it follows that $W^\mathcal{I}_0(d, d_1) \Rightarrow (\cdots \Rightarrow (W^\mathcal{I}_{n-1}(d_{n-1}, e) \Rightarrow (\forall \bar{W}.\bar{P})^\mathcal{I}(e)) \ldots) = (\forall \bar{W}.\bar{P})^\mathcal{I}(e)$, which together with $(\forall \bar{W}.\bar{P})^\mathcal{I}(e) = \bar{P}^\mathcal{I}(\bar{e})$ makes clear what we wanted to show i.e., that $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) \leq \bar{P}^\mathcal{I}(\bar{e})$.

Now, since $\bar{P}^\mathcal{I}(e') \leq \bar{P}^\mathcal{I}(\bar{e})$, we have that $\bar{P}^\mathcal{I}(e') < q$. From the definition of e', we get directly that $(\forall W\bar{W}.\bar{P})^\mathcal{I}(d) = \bar{P}^\mathcal{I}(e')$. Following the assumptions and the induction step we know that $v > (T^{k-1}_\mathcal{J})_j(e)$, $(T^{k-1}_\mathcal{J})_j(e) \geq \bar{P}^\mathcal{I}(\bar{e})$, $val(A_{B,\bar{P}}, \bar{W}) > \bar{P}^\mathcal{I}(\bar{e})$; then because $\bar{P}^\mathcal{I}(e') \leq \bar{P}^\mathcal{I}(\bar{e})$ it is not hard to see that $val(A_{C,\bar{P}}, W\bar{W}) > \bar{P}^\mathcal{I}(e')$. Moreover, since $\forall W.B$ is the responsible for $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d) < q$, we have that $(T_\mathcal{J}(T^{k-1}_\mathcal{J}))_i(d)$ $= (\forall W.B)^{\mathcal{I}_{k-1}}(d) = (T^{k-1}_\mathcal{J})_j(e) \geq \bar{P}^\mathcal{I}(\bar{e}) \geq \bar{P}^\mathcal{I}(e')$. With the last argument this part of the proof is already completed.

For $k > 0$ and limit ordinal
As before, we have to show that there exist a primitive concept P, a word $W \in L_{A_\mathcal{T}}(C, P)$ and $e \in \Delta^\mathcal{I}$ such that $P^\mathcal{I}(e) < q$, $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$, $val(A_{C,P}, W) > P^\mathcal{I}(e)$ and $(T^k_\mathcal{J})_i(d) \geq P^\mathcal{I}(e)$.

If we assume that $(T^k_\mathcal{J})_i(d)<q$, it implies the existence of at least an ordinal α $(0 < \alpha < k)$ such that $(T^\alpha_\mathcal{J})_i(d)<q$. This is true because k is a limit ordinal, and then $(T^k_\mathcal{J})_i$ is defined as the glb$\{(T^\alpha_\mathcal{J})_i; \alpha < k\}$. Now that we know that an ordinal α $(0 < \alpha < k)$ exists, such that $(T^\alpha_\mathcal{J})_i(d)<q$, we will consider not only this, but all the ordinals η $(0 < \eta < k)$ such that $(T^\eta_\mathcal{J})_i(d)<q$. Since all of them are smaller than k, we can apply the induction hypothesis to each $(T^\eta_\mathcal{J})_i(d)<q$. Then we have that for each η, there exist a primitive concept P_η , a word $W_\eta \in L_{A_\mathcal{T}}(C, P_\eta)$, and also an individual e_η such that $P^\mathcal{I}_\eta(e_\eta) < q$, $(\forall W_\eta.P_\eta)^\mathcal{I}(d) = P^\mathcal{I}_\eta(e_\eta)$, $val(A_{C,P_\eta}, W_\eta) > P^\mathcal{I}_\eta(e_\eta)$ and $(T^\eta_\mathcal{J})_i(d) \geq P^\mathcal{I}_\eta(e_\eta)$.

Now we keep our attention in the ordinal β with smallest value of $P^\mathcal{I}_\beta(e_\beta)$ (if there are more than one sharing the same value we take any of them) among all of the values $P^\mathcal{I}_\eta(e_\eta)$. This action is always possible because even when the set of all the ordinals η could be infinite, the set of primitive concepts is finite and also the set of individuals is finite (we only consider finite domains). Now we can make $P := P_\beta$, $W := W_\beta$ and $e := e_\beta$. From the application of the induction hypothesis to $(T^\beta_\mathcal{J})_i(d)<q$, it follows that $P^\mathcal{I}(e) < q$, $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$ and $val(A_{C,P}, W) > P^\mathcal{I}(e)$. For showing that also $(T^k_\mathcal{J})_i(d) \geq P^\mathcal{I}(e)$ holds, we notice that because of the form in which β was defined, we get that for all η $(T^\eta_\mathcal{J})_i(d) \geq P^\mathcal{I}_\beta(e_\beta)$. Which means, this time because the definition of η, that for all α $(0 < \alpha < k)$, $(T^\alpha_\mathcal{J})_i(d) \geq P^\mathcal{I}_\beta(e_\beta)$. From the last statement, and because k is a limit ordinal, and we know that $(T^k_\mathcal{J})_i$ is defined as the glb$\{(T^\alpha_\mathcal{J})_i; \alpha < k\}$, we have that $(T^k_\mathcal{J})_i(d) \geq P^\mathcal{I}(e)$. Thus, the proof is completed.□

Now, in the following proposition we make use of the two previous lemmata for describing the conditions under which the degree of membership of an individual d in a concept C is greater or equal to a certain value q. Later, this result will be crucial for proving that subsumption can be reduced to inclusion of regular languages.

Proposition 3.2.3. *Let $\mathcal{T}$ be a TBox and let $A_\mathcal{T}$ be the corresponding WSA associated with $\mathcal{T}$. Let $\mathcal{I}$ be a gfp-model of $\mathcal{T}$ and let C be a concept name ocurring in $\mathcal{T}$. For each $d \in \Delta^\mathcal{I}$ and $q \in [0, 1]$, the following are equivalent.*

1) $C^\mathcal{I}(d) \geq q$

2) *For all primitive concepts P, for all words $W \in L_{A_\mathcal{T}}(C, P)$ and all individuals $e \in \Delta^\mathcal{I}$: $P^\mathcal{I}(e) < q$ and $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$ implies $val(A_{C,P}, W) \leq P^\mathcal{I}(e)$.*

Proof
1) $\Rightarrow$ 2) Assume that 2) does not hold, i.e., there exist a primitive concept P, a word $W \in L_{A_\mathcal{T}}(C, P)$ and $e \in \Delta^\mathcal{I}$ such that: $P^\mathcal{I}(e) < q$, $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$ and $val(A_{C,P}, W) > P^\mathcal{I}(e)$.

First we take care of the case in wich C is a primitive concept. From the Definition 3.1.3, we know that (in $A_\mathcal{T}$) states representing primitive concepts are not reachable from each other. Then we get that $C = P$, $W = \epsilon$ and from $(\forall W.P)^\mathcal{I}(d) = P^\mathcal{I}(e)$

we have that $(P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$. Thus, since $P^{\mathcal{I}}(e) < q$, it also holds that $P^{\mathcal{I}}(d) < q$, which means that 1) does not hold.

If C is a defined concept and $i = index(C)$, from the Lemma 3.2.1 we have that there exist an ordinal β such that $(T^{\beta}_{\mathcal{J}})_i(d) \leq P^{\mathcal{I}}(e)$, where $\mathcal{J}$ is the primitive interpretation of $\mathcal{I}$. But, from Theorem 2.1.11 we know that gfp$(T_{\mathcal{J}}) \preceq_n T^{\beta}_{\mathcal{J}}$. Thus, from the definition of $\preceq_n$ and $\preceq$ we get that $C^{\mathcal{I}}(d) \leq P^{\mathcal{I}}(e) < q$, which means that 1) does not hold.

2) $\Rightarrow$ 1) Assume that 1) does not hold, i.e., $C^{\mathcal{I}}(d) < q$. Again, first we take care of the case in wich C is a primitive concept. Now, by taking $P := C$, $W := \epsilon$, $e := d$, we have that $P^{\mathcal{I}}(e) < q$, $(\forall W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$ and $val(A_{C,P}, W) = val(A_{P,P}, W) = 1 > P^{\mathcal{I}}(e)$, which means that 2) does not hold.

If C is a defined concept and $i = index(C)$, assume that $\mathcal{J}$ is the primitive interpretation of $\mathcal{I}$. Then, from the assumption of $C^{\mathcal{I}}(d) < q$, we know that there exist an ordinal $k \geq 0$ such that $(T^{k}_{\mathcal{J}})_i(d) < q$. But, by applying the Lemma 3.2.2 we get that there exist a primitive concept P in $\mathcal{T}$, a word $W \in L_{A_{\mathcal{T}}}(C, P)$ and an individual $e \in \Delta^{\mathcal{I}}$ such that $P^{\mathcal{I}}(e) < q$, $(\forall W.P)^{\mathcal{I}}(d) = P^{\mathcal{I}}(e)$ and $val(A_{C,P}, W) > P^{\mathcal{I}}(e)$. Thus, 2) does not hold. □

The following lemma makes use of the previous proposition for expressing the gfp-subsumption problem in terms of finite weighted automata.

Lemma 3.2.4. *Let $\mathcal{T}$ be a TBox and let $A_{\mathcal{T}}$ be the corresponding WSA associated with $\mathcal{T}$. Let C, D be atomic concepts occurring in $\mathcal{T}$ and $q \in [0, 1]$, the following are equivalent:*

i) $\langle C \sqsubseteq_{gfp,\mathcal{T}} D \ , \ q \rangle$

ii) For all primitive concepts P and words $W \in \Sigma^*$: $\min(q, val(A_{D,P}, W)) \leq val(A_{C,P}, W)$

Proof

i) $\Rightarrow$ *ii)* Assume that there exists a primitive concept P and a word W such that $\min(q, val(A_{D,P}, W)) > val(A_{C,P}, W)$. Next we show that there exists a gfp–model $\mathcal{I}$ of $\mathcal{T}$ that makes $\langle C \sqsubseteq_{gfp,\mathcal{T}} D \ , \ q \rangle$ to be false. The primitive interpretation $\mathcal{J}$ is defined as follows:

$\Delta^{\mathcal{J}} := \{d_0, d_1, \ldots, d_n\}$ where n is the length of W.

If a role name r takes place in the position i of the word W, $x = d_i$ and $y = d_{i+1}$ then $r^{\mathcal{J}}(x, y) := 1$, in other case $r^{\mathcal{J}}(x, y) := 0$.

$Q^{\mathcal{J}}(x) := 1$ for all primitive concepts $Q \neq P$ and $x \in \Delta^{\mathcal{J}}$.

$P^{\mathcal{J}}(x) := \frac{\min(q, val(A_{D,P}, W)) + val(A_{C,P}, W)}{2}$ if $x = d_n$.

$P^{\mathcal{J}}(x) := 1$ if $x \neq d_n$.

Let $\mathcal{I}$ be the gfp–model defined by $\mathcal{J}$. From $\min(q, val(A_{D,P}, W)) > val(A_{C,P}, W)$, it follows that $P^{\mathcal{I}}(d_n) < q$. Thus, d_n is the only individual with a degree of membership into a primitive concept smaller than q. From the definition of $\mathcal{J}$, it is not hard to see that for all words V, $\forall V.P^{\mathcal{I}}(d_0) = P^{\mathcal{I}}(d_n)$ iff $V = W$. Then, we know by Proposition 3.2.3 that $C^{\mathcal{I}}(d_0) \geq q$ and $D^{\mathcal{I}}(d_0) < q$ which directly implies that $\langle C \sqsubseteq_{gfp,\mathcal{T}} D$, $q\rangle$ does not hold.

ii) $\Rightarrow$ *i)* Assume that $\langle C \sqsubseteq_{gfp,\mathcal{T}} D$, $q\rangle$ does not hold, it means that there exists a gfp–model $\mathcal{I}$ of $\mathcal{T}$, and an individual $d \in \Delta^{\mathcal{I}}$, such that $C^{\mathcal{I}}(d) > D^{\mathcal{I}}(d)$ and $D^{\mathcal{I}}(d) < q$. In addition, assume that for all primitive concepts P and words $W \in \Sigma^*$, $\min(q, val(A_{D,P}, W)) \leq val(A_{C,P}, W)$. Next we show that following the previous assumptions we can reach a contradiction.

Since from the assumptions we have that $C^{\mathcal{I}}(d) > D^{\mathcal{I}}(d)$ and $D^{\mathcal{I}}(d) < q$, then we know that there exist a $\hat{q} \in [0, 1]$ such that $C^{\mathcal{I}}(d) > \hat{q} > D^{\mathcal{I}}(d)$ and $\hat{q} < q$. Thus, by the application of the Proposition 3.2.3 to $\hat{q} > D^{\mathcal{I}}(d)$ we get that there exist a primitive concept $\hat{P}$, a word $\hat{W} \in L_{A_{\mathcal{T}}}(D, \hat{P})$ and an individual $\hat{e}$ such that $\hat{P}^{\mathcal{I}}(\hat{e}) < \hat{q}$, $(\forall \hat{W}.\hat{P})^{\mathcal{I}}(d) = \hat{P}^{\mathcal{I}}(\hat{e})$ and $val(A_{D,\hat{P}}, \hat{W}) > \hat{P}^{\mathcal{I}}(\hat{e})$. But, since $\hat{P}^{\mathcal{I}}(\hat{e}) < \hat{q}$ and $\hat{q} < q$ we get that $\min(q, val(A_{D,\hat{P}}, \hat{W})) > \hat{P}^{\mathcal{I}}(\hat{e})$. This, together with the last of the assumptions, results in $val(A_{C,\hat{P}}, \hat{W}) > \hat{P}^{\mathcal{I}}(\hat{e})$.

Then, in the case of $\hat{W} \in L_{A_{\mathcal{T}}}(C, \hat{P})$, since $\hat{P}^{\mathcal{I}}(\hat{e}) < \hat{q}$, $(\forall \hat{W}.\hat{P})^{\mathcal{I}}(d) = \hat{P}^{\mathcal{I}}(\hat{e})$ and $val(A_{C,\hat{P}}, \hat{W}) > \hat{P}^{\mathcal{I}}(\hat{e})$, we get by Proposition 3.2.3 that $C^{\mathcal{I}}(d) < \hat{q}$ which is in contradiction with $C^{\mathcal{I}}(d) > \hat{q} > D^{\mathcal{I}}(d)$. If $\hat{W} \notin L_{A_{\mathcal{T}}}(C, \hat{P})$, then $val(A_{C,\hat{P}}, \hat{W}) = 0$ which is in contradiction with $val(A_{C,\hat{P}}, \hat{W}) > \hat{P}^{\mathcal{I}}(\hat{e})$. □

Now we can reconsider the following. In the WSA A associated with the TBox from the Example 2.2.14, it holds that $\min(0.4, val(A_{Z,G}, c)) = \min(0.4, 0.7) = 0.4 > val(A_{K,G}, c) = 0.3$. Thus, taking into account the Lemma 3.2.4, we get directly that $\langle K \sqsubseteq_{gfp,\mathcal{T}} Z$, $0.4\rangle$ does not hold.

Definition 3.2.5. Let $A = (Q, \Sigma, E, wt)$ be a WSA automaton; $q_1, q_2 \in Q$ and $p \in [0, 1]$. By $A_{\geq p}$ we denote the semi-automaton with words transitions (Q, Σ, E'), where $E' := \{e \in E | \ \ wt(e) \geq p\}$.

Proposition 3.2.6. *Let $A = (Q, \Sigma, E, wt)$ be a WSA, $q_1, q_2 \in Q$ and $p \in (0, 1]$. Then, for all words $W \in \Sigma^*$, $W \in L_{A_{\geq p}}(q_1, q_2)$ iff $val(A_{q_1,q_2}, W) \geq p$.*

Proof
$W \in L_{A_{\geq p}}(q_1, q_2)$ iff there exists in $A_{\geq p}$ a path l from q_1 to q_2 which is labeled by W. By definition Definition 3.2.5 this is possible iff there exists in A, a path $l' \in paths(A, W, q_1, q_2)$ such that $val_A(l') \geq p$, which can happen iff $val(A_{q_1,q_2}, W) \geq p$. □

Let $\mathcal{T}$ be a TBox, by *weights*$(\mathcal{T})$ we denote the set of weights that occur in the terminological axioms of $\mathcal{T}$. Formally, weights$(\mathcal{T}) := \{p \ | \ \langle C \sqsubseteq D, p\rangle \in \mathcal{T}\}$

Lemma 3.2.7. *Let $\mathcal{T}$ be a TBox and let A be the corresponding WSA associated with $\mathcal{T}$. Let C and D be atomic concepts occurring in $\mathcal{T}$ and $q \in [0, 1]$, the following are equivalent:*

i)For all primitive concepts P and words $W \in \Sigma^*$: $\min(q, val(A_{D,P}, W)) \leq val(A_{C,P}, W)$

ii) For all primitive concepts P and for all $p \in$weights$(\mathcal{T}) \cup \{q\}$ such that $0 < p \leq q$, it holds that $L_{A_{\geq p}}(D, P) \subseteq L_{A_{\geq p}}(C, P)$.

Proof

i) $\Rightarrow$ *ii)*

Assume that *ii)* does not hold. This means that there exist a primitive concept P and a positive weight $p \leq q$ such that $L_{A_{\geq p}}(D, P) \nsubseteq L_{A_{\geq p}}(C, P)$. Then we know that there exist a word W, such that $W \in L_{A_{\geq p}}(D, P)$ and $W \notin L_{A_{\geq p}}(C, P)$, which by Proposition 3.2.6 implies that $val(A_{D,P}, W) \geq p$ and $val(A_{C,P}, W) < p$. Thus, $\min(q, val(A_{D,P}, W)) \geq p > val(A_{C,P}, W)$, which means that *i)* does not hold.

ii) $\Rightarrow$ *i)*

Assume that *i)* does not hold. Then, there exists a primitive concept P and a word W such that $\min(q, val(A_{D,P}, W)) > val(A_{C,P}, W)$. Now we make $p := \min(q, val(A_{D,P}, W))$, and by applying the Proposition 3.2.6 it results that, $W \in L_{A_{\geq p}}(D, P)$ and $W \notin L_{A_{\geq p}}(C, P)$. Thus, *ii)* does not hold. $\square$

4 Complexity results

This small chapter is devoted to the analysis of the computational complexity of deciding whether a concept is subsumed to a certain degree by another concept (see Definition 2.2.20).

4.1 The gfp-Subsumption Problem

Next we rewrite the notion of subsumption of concepts as a decision procedure. In our analysis we need to make use of an analogous complexity result that was shown for the crisp Description Logic $\mathcal{FL}_0$ in [2, p.191]. Here, we make a clear distintion between the crisp and the fuzzy cases by the name of each problem, i.e., GFP–SUBSUMPTION and FUZZY GFP–SUBSUMPTION.

Problem: GFP–SUBSUMPTION
Input: A TBox $\mathcal{T}$ and two atomic concepts C, D.
Question: Does $C \sqsubseteq_{gfp,\mathcal{T}} D$ hold?

Problem: FUZZY GFP–SUBSUMPTION
Input: A TBox $\mathcal{T}$, two atomic concepts C, D and a value $q \in [0, 1]$.
Question: Does $\langle C \sqsubseteq_{gfp,\mathcal{T}} D\ , q\rangle$ hold?

The proof of the following theorem can be found in [2, p.207]. In the proof of the reduction from GFP–SUBSUMPTION to FUZZY GFP–SUBSUMPTION we will make use of it .

Theorem 4.1.1. *Let $\mathcal{T}$ be a terminology and let $A_{\mathcal{T}}$ be the corresponding semi-automaton. Let C,D be concept names occurring in $\mathcal{T}$. Subsumption in $\mathcal{T}$ can be reduced to inclusion of regular languages defined by $A_{\mathcal{T}}$. More precisely,*

$$C \sqsubseteq_{gfp,\mathcal{T}} D \text{ iff } L(D,P) \subseteq L(C,P) \text{ for all primitive concepts } P.$$

In the following theorem the complexity of the FUZZY GFP-SUBSUMPTION problem is stated. From it we can see that from a computaional complexity point of view, we can regard FUZZY GFP-SUBSUMPTION and GFP–SUBSUMPTION as equivalent problems.

Theorem 4.1.2. *The FUZZY GFP-SUBSUMPTION problem is PSPACE-complete.*

Proof
We proceed in two steps: First we show that the FUZZY GFP–SUBSUMPTION problem is in PSPACE. After that, the PSPACE-hardness is shown by a reduction from the GFP–SUBSUMPTION problem, which is known to be in PSPACE-complete [2, p.191].

1) We have seen that by the combination of Lemma 3.2.4 and Lemma 3.2.7, the FUZZY GFP-SUBSUMPTION problem can be reduced to inclusion of regular languages (defined by nondeterministic automata) in polynomial time. Since it is well-known that the inclusion problem for regular languages defined in terms of nondeterministic automata is PSPACE-complete [7, p.265], we also have that FUZZY GFP-SUBSUMPTION is in PSPACE.

2) Let the TBox $\mathcal{T}_0$ together with the atomic concepts C_0, D_0 be an instance of

the GFP- SUBSUMPTION problem. We construct an instance of FUZZY GFP-SUBSUMPTION as follows:

$\mathcal{T} := \{\langle A \sqsubseteq B, 1\rangle |$ where $A = B$ is an axiom in $\mathcal{T}_0\}$
$C := C_0$
$D := D_0$
$q := 1$

It is clear that this construction indeed creates an instance of the FUZZY GFP-SUBSUMPTION problem and it can be done in time polynomial in the size of the original instance of the GFP- SUBSUMPTION problem. Next we present two claims. The first one is a direct result from the previous transformation and from the definitions of semi-automaton associated with a TBox. The equivalence of the introduced instances is shown in the second claim.

Claim 1: Let $A_{\mathcal{T}_0}$ and $A_{\mathcal{T}}$ be the corresponding semi-automaton of $\mathcal{T}_0$ and $\mathcal{T}$ respectively. Let $\hat{D}$ be an atomic concept of $\mathcal{T}_0$ and let P be a primitive concepts of $\mathcal{T}_0$. For a word W, it holds that $W \in L(\hat{D}, P)$ in $A_{\mathcal{T}_0}$ if and only if $val(A_{\hat{D},P}, W) = 1$ in $A_{\mathcal{T}}$.

Claim 2: $C \sqsubseteq_{gfp,\mathcal{T}_0} D$ iff $\langle C \sqsubseteq_{gfp,\mathcal{T}} D,\, 1\rangle$.
($\Rightarrow$)Assume that $\langle C \sqsubseteq_{gfp,\mathcal{T}} D$, $1\rangle$ does not hold. Because of the Lemma 3.2.4, we know that there exist a primitive concept P and a word W such that $\min(1, val(A_{D,P}, W))$ $> val(A_{C,P}, W)$. Now, since the only possible weights in $A_{\mathcal{T}}$ are 0 or 1, we get that $val(A_{C,P}, W) = 0$ and $val(A_{D,P}, W) = 1$. Thus, by applying the Claim 1 to C,P and W we get that $W \notin L(C, P)$; and by applying the Claim 1 to D,P and W we get that $W \in L(D, P)$. Then, the last argument together with the Theorem 4.1.1, give us that $C \sqsubseteq_{gfp,\mathcal{T}_0} D$ does not hold.
($\Leftarrow$) Assume that $C \sqsubseteq_{gfp,\mathcal{T}_0} D$ does not hold; then because of the Theorem 4.1.1 we know that there exist a primitive concept P and a word W such that $W \notin L(C, P)$ and $W \in L(D, P)$. Now, by applying the Claim 1 to C,P and W we get that $val(A_{C,P}, W) \neq 1$, which actually means that $val(A_{C,P}, W) = 0$. In the same way, by applying the Claim 1 to D,P and W we get that $val(A_{D,P}, W) = 1$. Thus, since $val(A_{C,P}, W) = 0$ and $val(A_{D,P}, W) = 1$, following the Lemma 3.2.4, we have that $\langle C \sqsubseteq_{gfp,\mathcal{T}} D$, $1\rangle$ does not hold. □

5 Conclusions

In this thesis, we have considered the Subsumption Problem in the Fuzzy Description Logic $\mathcal{FL}_0$ w.r.t. greatest fixed-point semantics. The main objective was to investigate whether it is possible to obtain for this problem, an alternative characterization completely stated in terms of finite automata. In addition, the computational complexity of this problem was investigated.

In Chapter 2 we gave a positive answer to the first question. To that purpose, we proceeded in three steps: First, weighted automata over finite words was introduced as the suitable formalism for the representation of TBoxes. This means that for a given TBox we showed how to build a weighted automaton such that weighted runs in the automaton correspond to dependencies of concepts in the TBox. Next, the conditions under which the degree of membership of an individual in to a concept is greater than a certain value was formally described. This was a useful tool for the last step where it was shown that the Subsumption Problem in the Fuzzy $\mathcal{FL}_0$ can be reduced to several tests of language inclusion. More specifically, from a given instance of the Subsumption Problem we have a TBox, from this TBox we build an associated weighted automaton and over the structure of this automaton we define the language inclusion tests that we are interested in.

Starting from the new characterization it was also possible to show, in Chapter 3, that the Subsumption Problem in the Fuzzy Description Logic $\mathcal{FL}_0$ is in PSPACE-complete. On the one hand, the PSPACE-hardness was proved by a reduction from the crisp version of our problem, which is known to be in PSPACE-complete [2]. On the other hand, the main idea underlying the "in PSPACE" proof relies on the possibility of deciding inclusion of weighted automata through the inclusion of non-weighted automata, which is known to be in PSPACE [7, p.265].

Finally, despite the computational complexity of the Subsumption Problem, we also get some benefit. Because the given characterization is completely explained in terms of inclusion of languages, which is a well known problem from automata theory, any practical optimization developed for this problem is also inherited by our approach.

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