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Investigations on an Action Formalism based on the Description Logic $\mathcal{ALCQIO}$

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Contents

1 Introduction and Motivation

Description logics are used in several applications, where we develop concrete knowledge bases to represent different systems. Because there is a huge variety of description logics with distinct properties, the applications are just as varied. Description logics are used for example in the semantic web as a background for several ontology languages like OWL, for ontology languages used in the Ontology Based Data Access or in medical applications.

But the real motivation to study description logics in fact is the following. Description logics are beautiful. On the one hand they are closely connected to the wide area of modal logics, that can look back to a long tradition. But at the same time description logics are just beginning to emerge and therefore working in this field is vivid and also open to twisted thoughts. One special point about the research on description logics is that it is always working at a border, as for example the border of being as fast as possible for huge data sources. There we have the small description logics, designed to handle large data amounts in reasonable time. Or they work at the border of being as expressive as possible, but still decidable to meet todays requirements on knowledge engineering. Many results on description logics concern the question, whether a systems with specific properties is decidable or computable with reasonable effort.

In summary we can say description logics evolve rapidly to be always close to the technical possibilities of today and therein it is not a closed field. All this together is what makes description logics exciting and special.

1.1 The Description Logics Based Action Formalism

Description Logics have been developed to represent knowledge closer to the human mind than for example first order logics. The description logics based action formalism has been introduced by Franz Baader, Carsten Lutz, Maja Miličić, Ulrike Sattler and Frank Wolter in their article [BLM+05] and the accompanying report [BML+05]. Not only do they introduce the formalism, they also show how to reduce the major inference problem for actions called projection to a well studied problem for description logics, namely ABox consequence. Thereby they determine the limits of this approach—limitations for the underlying description logics and in the construction of the actions. Also they consider only a simple ABox assertion as a consequence in the projection problem.

Since 2005 this theory evolved to stretch the limits for possible applications. One task in this thesis will be to investigate, whether the reduction obtained in the report [BML+05] also works for description logics which contain a so-called universal role. A universal role

is given by a role name and a fixed interpretation. The challenge will be to determine the influence of this role statement on the construction of the reduction.
The other question is to decide whether it is possible to replace the simple ABox assertion by the more expressive concept of a query. We will try to extend the reduction idea to projection with Boolean conjunctive queries. As a result we reduce projection with such queries to entailment with unions of conjunctive queries.

Due to the speed and complexity of the research on description logics it is not easy to mention all the work related to this thesis. There are a lot of books, articles and reports that have influenced my studies. Most of the work will be introduced in the process when it is first referred to. But there are of course books and articles with a major impact on this thesis, I have used almost daily.
First of all there is the already mentioned article "Integrating Description Logics and Action Formalisms: First Results", accompanied by a technical report from Franz Baader, Carsten Lutz, Maja Miličić, Ulrike Sattler and Frank Wolter. Because it is the task of this thesis to develop the there presented ideas further, this article is the one with the greatest influence and by far the most quoted work of this thesis.
The Description Logics Handbook has to be mentioned here, because it is a good start if you want to become familiar with the history and applications of, and reasoning in various description logics. Most of the definitions used in this thesis are based on this book.
Last but not least there is one of the most inspiring books at all—the Handbook of Modal Logics edited by Patrick Blackburn, Johan van Bentheim and Frank Wolter. This book provides inspiration and background knowledge throughout this thesis, because it places description logics in a rich and inspiring context of related work, where topics are treated in a depth that is directly proportional to the weight of this book.

1.2 Overview

What all chapters of this thesis have in common is the direct relation to the article [BLM+05] and the accompanying report [BML+05]. Every chapter makes a different attempt to develop the therein developed description logics based action formalism further.

Subsequent to this introduction we provide the most important basics to understanding description logics and the thereon based action formalism. Several reasoning problems for different situations, whose investigation will be subject of the following chapters, are introduced.

Chapter three focuses on the influence the universal role has on different description logics reasoning tasks. We derive complexity statements for ABox consequence in presence of a universal role and use this results as a base for a reduction from the action inference problem projection to ABox consequence. At the end of the third chapter we pursue some thoughts on the extension of this calculus.

In chapter four we replace the simple ABox assertion by a Boolean conjunctive query. Based on the results from the third chapter and the reduction developed in the report [BML+05], we reduce projection with Boolean conjunctive queries to entailment with unions of conjunctive queries. From this reduction we can derive decidability and complexity statements for projection with Boolean conjunctive queries.

The last chapter then summarizes the thoughts and obtained results of this thesis. Further we will point to possible next steps and the perspectives of a future work.

2 Description Logics Preliminaries

The roots of description logics go back to the 1970s when the connection between *semantic networks* and knowledge representation was investigated. The terminus *description logics* came up already in the 1980s with the introduction of concept descriptions and a TBox formalism, used to abbreviate complex concept descriptions. Since then the theory evolved rapidly, as for example with the *tableau based algorithms* a new algorithmic paradigm for efficiently treating propositionally closed description logics has been introduced. Current research concerns very expressive description logics which are used in applications like the Semantic Web. Otherwise we are interested in basic properties of smaller description logics to be able to handle very large knowledge bases [BCM+03, vHvHLP07].
Description logics consist of concepts, which are the expressions of this language used to build statements. Further essential elements of description logics are individual names, role names and a knowledge base, containing the terminological structure. The expressions of description logics are built from atomic concepts and atomic roles using concept and role constructors. The basic propositionally closed description language is called $\mathcal{ALC}$, which is an acronym for attributive language with complement. It plays an essential role as a basis which can be extended by several constructors, where the resulting language will be named by concatenation of the corresponding letters. Description logics have on the one hand an easy to understand building structure that can be extended in different directions and are on the other hand very powerful. Powerful here means, they are a subset of FOL, extending propositional logics, while still being complete and decidable.
This chapter contains the basic definitions we are going to use through this thesis. We start with the formal definition of the syntax and semantics of description logics and several reasoning problems for these logics. Then we present the description logics based action formalism and two reasoning problems, namely executability and projection for this action formalism.

2.1 The Formal Definition of Description Logics

We start with the formal definition of the syntax and semantics of $\mathcal{ALC}$ and the several constructors to extend the expressiveness of $\mathcal{ALC}$, i.e. nominals, inverses, qualified number restriction and we establish a universal role. Further we give the notion of a TBox and an ABox and specify how they cover the terminological knowledge and the assertions of an application. Afterwards we introduce three reasoning tasks for description logics. Most of the formulations and problems in this sections are inspired by or included from the Handbook [BCM+03] and the article [BLM+05].

2.1.1 Syntax and Semantics

The syntax of $\mathcal{ALC}$ is given by inductively defined concepts and with the participation of additional constructors.

Definition 2.1. Let N_C be a countably infinite set of concept names, N_R be a countably infinite set of role names and N_I be a countably infinite set of individual names, where we understand the sets N_C, N_R and N_I to be disjoint.

The concept descriptions in $\mathcal{ALC}$ are defined as follows:

$$C, D \longrightarrow A \mid \top \mid \bot \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall r.C \mid \exists r.C$$

where $A \in N_C$ is called atomic concept and $r \in N_R$.

$\mathcal{ALC}$ can be extended by the use of various concept constructors. In the following we present the syntax of all fragments $\mathcal{L}$ of the description logic called $\mathcal{ALCQIO}^{\mathcal{U}}$. The availability of the corresponding constructors is indicated by concatenation of letters.

The letter $\mathcal{O}$ indicates the availability of nominals, i.e individuals occurring in concept definitions, where for all $a \in N_I$ there is a concept $\{a\}$ available. The letter $\mathcal{Q}$ represents qualified number restriction. For $C \in N_C$, $r \in N_R$ and $n \in \mathbb{N}$ the following concepts are available

$$C \longrightarrow (\geq n\ r\ C) \mid (\leq n\ r\ C)$$

where the first rule defines at least and the second rule defines at most number restriction. Further the letter $\mathcal{I}$ denotes the availability of inverse roles, where we require $r^- \in N_R$ for each $r \in N_R$. When $\mathcal{ALC}$ is extended by adding a distinguished role element U to N_R this is indicated by the superscript $\mathcal{U}$.

Some concept definitions can be given in terms of others. We first define the boolean standard abbreviations for concepts $A, B \in N_C$:

$$A \to B :\Longleftrightarrow \neg A \sqcup B, \quad A \leftrightarrow B :\Longleftrightarrow (A \to B) \sqcap (B \to A)$$

The next four given concepts are used frequently in description logics.

$$\begin{array}{rlrl} \exists r.C :\Longleftrightarrow & (\geq 1\ r\ C), & \forall r.C :\Longleftrightarrow & (\leq 0\ r\ \neg C), \\ \top :\Longleftrightarrow & A \sqcup \neg A, & \bot :\Longleftrightarrow & \neg\top \end{array}$$

These concepts express value restriction and existential restriction in terms of qualified number restriction. The two concepts below introduce the top and the bottom concept. The next definition introduces the formal semantics for description logics based on interpretations.

Definition 2.2. An interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non empty set, called domain of $\mathcal{I}$ and $.^{\mathcal{I}}$ is the interpretation function, assigning to each concept name $A \in N_C$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to each role name $r \in N_R$ a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The extension of the interpretation function to the previously defined concept descriptions is given in Figure 2.1.

Name	Syntax	Semantics
top concept	$\top$	$\Delta^{\mathcal{I}}$
bottom concept	$\bot$	$\emptyset$
negation	$\neg C$	$\Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
value restriction	$\forall r.C$	$\{d \in \Delta^{\mathcal{I}} \mid \forall e.(d,e) \in r^{\mathcal{I}} \rightarrow e \in C^{\mathcal{I}}\}$
existential restriction	$\exists r.C$	$\{d \in \Delta^{\mathcal{I}} \mid \exists e.(d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}$
inverse role	r^-	$\{(d,e) \mid (e,d) \in r^{\mathcal{I}}\}$
universal role	U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
nominal	$\{a\}$	$\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$
at least number restriction	$(\geq n\ r\ C)$	$\{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in r^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}\rvert \geq n\}$
at most number restriction	$(\leq n\ r\ C)$	$\{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in r^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}\rvert \leq n\}$

Figure 2.1: Semantics of concept and role constructors.

Remark 2.3. When determining complexity some results differ whether they are made under the assumption of unary coding of numbers in number restrictions. A natural number $n \in \mathbb{N}$ is encoded in unary by listening n times the digit 1 followed by the delimiter symbol 0 and therefore unary coding uses $n+1$ bits. In comparison coding a number in binary needs $\lfloor \log(n+1) \rfloor + 1$ bits. If we consider a binary coded number, the unary coding of the same number is of exponential size. As a consequence we can assume $(\geq n\ r\ C)$ and $(\leq n\ r\ C)$ to be of the size $n + 1 + |C|$ if numbers are codes in unary. Although one would not really use unary coding, some complexity bounds for reasoning problems in description logics only hold under this requirement.

2.1.2 The Terminological and Assertional Formalism

The formal definition of a concept language is supplemented by the terminological and assertional formalism. It describes the ontology and the initial state of the world. Terminological knowledge is the intentional knowledge of a system making statements how concepts or roles are related to each other. Its characteristics is timelessness, means it is not supposed to change.

In defining the TBox we make a difference between a general TBox and its restriction to an acyclic TBox. Acyclic concept definitions build new concepts based on already existing concepts, neither defined in terms of themselves, nor in terms of other concepts, that indirectly refer to them.

Definition 2.4. Let C, D be concepts, R, S roles. Then

$$C \sqsubseteq D,\ R \sqsubseteq S$$
$$C \equiv D,\ R \equiv S$$

are *terminological axioms*. Axioms of the first kind are called *general concept inclusions*, abbreviated by GCI, whereas axioms of the second kind are called *equalities*. An equality whose left-hand side is an atomic concept is a *concept definition*. A concept name is called *defined concept* if it occurs on the left hand side of a concept definition and *primitive concept* otherwise.

A *general TBox* is a finite set of general concept inclusions and the restriction to a finite set of concept definitions is called *TBox*. Note that a concept definition $C \equiv D$ can be represented by the two GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$. A concept name A *directly uses* a concept name B w.r.t a TBox $\mathcal{T}$ if there is a concept definition $A \equiv C \in \mathcal{T}$ with B occurring in C. Let *uses* be the transitive closure of directly uses. Then a TBox $\mathcal{T}$ contains a *terminological cycle* if there is a concept name that uses itself w.r.t. $\mathcal{T}$. If a TBox containing only concept definitions and thereby no terminological cycle is called *acyclic TBox*.

The interpretation based semantics for description logics, as introduced in Definition 2.2 for concept names, binary relations and concept descriptions can be continued to TBoxes. There are basically three approaches of this continuation that became largely accepted, namely the least and the greatest fixpoint semantics and descriptive semantics. The continuation of semantics to TBoxes has been subject of research in the 1990s, an introduction and comparison by Bernhard Nebel can be found in [Neb91]. For general and acyclic TBoxes we use descriptive semantics, which is very intuitive because it is based on first order logics semantics.

Definition 2.5. An interpretation $\mathcal{I}$ *satisfies a GCI* $A \sqsubseteq C$ if $A^{\mathcal{I}} \sqsubseteq C^{\mathcal{I}}$ and we write $\mathcal{I} \models A \sqsubseteq C$. For concept definitions with the above representation follows $A \equiv C$ if $A^{\mathcal{I}} \equiv C^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ if it satisfies all concept definitions in $\mathcal{T}$, we write $\mathcal{I} \models \mathcal{T}$. The set of all models of a TBox $\mathcal{T}$ is denoted by $\mathcal{M}(\mathcal{T})$.

Individuals, conceptual membership or role relationships are covered by the ABox where the initial properties of and relationships between these individuals are presented. These properties are meant to change and therefore dependency of circumstances and contingency are characteristics of the assertional knowledge. So we introduce the ABox to give an actual snapshot of what is known about the world at a concrete time.

Definition 2.6. Let $a, b \in N_I$, C a concept, $r \in N_R$ then

$$C(a), \quad r(a,b), \quad \neg r(a,b)$$

are *ABox assertions*, where assertions of the first kind are called concept assertions and the other two kinds of assertions are called positive or negated role assertion respectively. An assertion of the form $A(a)$, $\neg A(a)$ or $r(a,b)$, $\neg r(a,b)$, where $A \in N_C$ a concept name and $r \in N_R \setminus \{U\}$ a role name, is called *literal*. In conclusion an *ABox* $\mathcal{A}$ is a finite set of assertions.

To assign a semantics to ABoxes, we have to extend the interpretations introduced in Definition 2.2 to individual names. Note that because the ABox describes an initial state, the semantics of ABoxes usually follows the open world assumption. It is represented by many different interpretations, namely all its models. This means the absence of information in an ABox only indicates a lack of knowledge, so an ABox can have several models which may vary in content.

Definition 2.7. An Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ maps each individual name to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. As usual in description logics we adopt the unique name assumption which states $a \neq b \Rightarrow a^{\mathcal{I}} \neq b^{\mathcal{I}}$ and is abbreviated by UNA.

The interpretation $\mathcal{I}$ *satisfies the concept assertion* $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, we write $\mathcal{I} \models C(a)$. Further $\mathcal{I}$ *satisfies* the role assertion $r(a, b)$, if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ and we write $\mathcal{I} \models r(a, b)$. The interpretation $\mathcal{I}$ *satisfies* the negated role assertion $\neg r(a, b)$ if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin r^{\mathcal{I}}$, in this case we write $\mathcal{I} \models \neg r(a, b)$. An interpretation $\mathcal{I}$ is a *model* of an ABox $\mathcal{A}$, abbreviated by $\mathcal{I} \models \mathcal{A}$, if $\mathcal{I}$ satisfies all assertions in $\mathcal{A}$. The set of all models of an ABox $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$.

Mostly one is interested in the interaction between the terminological and the assertional knowledge. So these both parts together form the so-called knowledge base.

Definition 2.8. For a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$ we set $\mathcal{K} := (\mathcal{A}, \mathcal{T})$ and call $\mathcal{K}$ a *knowledge base*. We introduced the continuation of semantics introduced on concepts and roles to the ABox and TBox by descriptive semantics. If an interpretation $\mathcal{I}$ is is a common model of the knowledge base $\mathcal{K} := (\mathcal{A}, \mathcal{T})$ then we say $\mathcal{I}$ is a *model of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$*.

Remark 2.9. Descriptive semantics admits classic negation, i.e. an ABox assertion φ is mapped to the double negation $\neg\neg\varphi$. For every model $\mathcal{I}$ of a knowledge base $\mathcal{K}$ we have $\mathcal{I} \models \varphi \iff \mathcal{I} \models \neg\neg\varphi$.

The following example illustrates the different components of description logics. It will be about a circle of friends connected by a friendship relationship, introduced also to be continued later.

Example 2.10. For the sets of individuals, roles and concept names let

$$\{Paul, Anna, Selma, Tom, Karl\} :\subset N_I,$$

$$\{friend, U\} :\subset N_R,$$
$$\{human,\ happy\} :\subset N_C.$$

The TBox contains a single definition stating, that those humans who have a human friend are of the defined concept *happy*.

$$\mathcal{T} := \{happy \equiv human \sqcap \exists friend.human\}$$

The ABox introduces the connection through the role friendships and membership in the concept humans for the individuals.

$$\begin{aligned} \mathcal{A} \ := \ & \{friend(Paul, Anna), \neg friend(Anna, Selma), \\ & human(Paul), human(Anna), human(Selma), human(Tom)\} \end{aligned}$$

Let us consider the following interpretation.

$$\Delta^{\mathcal{I}} := \{\text{Paul, Anna, Selma, Tom, Karl, Henri, Toto}\}$$

where the interpretation function is defined in the following way:

$$\begin{aligned} human^{\mathcal{I}} &:= \{\text{Paul, Anna, Selma, Tom, Henri}\} \\ friend^{\mathcal{I}} &:= \{(\text{Paul}, \text{Anna}), (\text{Anna}, \text{Paul}), (\text{Paul}, \text{Henri}), (\text{Henri}, \text{Paul}), \} \\ happy^{\mathcal{I}} &:= \{\text{Anna}, \text{Paul}, \text{Henry}\} \\ U^{\mathcal{I}} &:= \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}. \end{aligned}$$

For a description logics with nominals available let

$$Paul^{\mathcal{I}} := \text{Paul},\ Selma^{\mathcal{I}} := \text{Selma},\ Anna^{\mathcal{I}} := \text{Anna},\ Tom^{\mathcal{I}} := \text{Tom},\ Karl^{\mathcal{I}} := \text{Karl}.$$

Obviously the interpretation $\mathcal{I}$ is a common model of $\mathcal{A}$ and $\mathcal{T}$. Because this example will be continued later we give all the important details in a short overview in Figure 2.2. The grey nodes represent elements of the set $human^{\mathcal{I}}$ and the dark grey nodes symbolize membership in the interpretation of the defined concept *happy*.

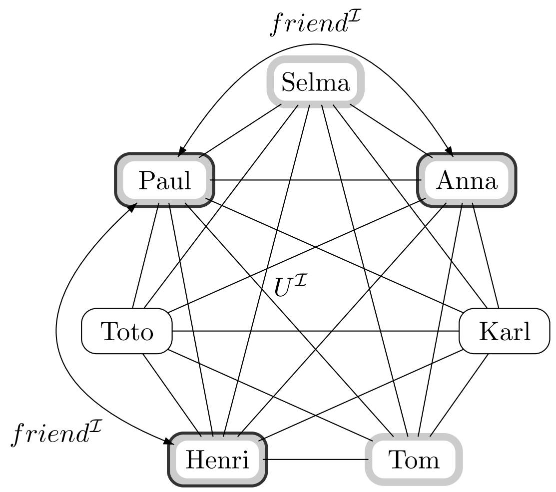

Figure 2.2: Interpretation $\mathcal{I}$

2.1.3 Description Logics Reasoning Tasks

We introduce four important reasoning problems for description logics, which will play an important role in the reduction and the complexity analysis we are going to derive in the next chapter.

Definition 2.11. Let C be a concept, $\mathcal{A}$ an ABox and $\mathcal{T}$ a TBox.

Concept satisfiability: C is *satisfiable* w.r.t. the TBox $\mathcal{T}$ if there exists an interpretation $\mathcal{I}$ that is a model of both C and $\mathcal{T}$.

TBox satisfiability: A TBox $\mathcal{T}$ is satisfiable if there exists a model of $\mathcal{T}$.

ABox consistency: $\mathcal{A}$ is *consistent* w.r.t. the TBox $\mathcal{T}$ if there exists an interpretation $\mathcal{I}$ that is a model of both $\mathcal{A}$ and $\mathcal{T}$.

ABox consequence: An ABox assertion φ is a *consequence* of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$, or $\mathcal{A}, \mathcal{T} \models \varphi$ if every model of $\mathcal{A}$ and $\mathcal{T}$ satisfies φ.

In the next chapter we will gain a deeper understanding about the connection between these problems considered in the different sublanguages of $\mathcal{ALCQIO}^{\mathcal{U}}$.

2.2 Introducing an Action Formalism for Description Logics

The idea of logically specifying dynamical systems has been introduced by John McCarthy more than 50 years ago now, but has been developed further to one of the most popular dynamical system in logics, namely the situation calculus established 1991 by Raymond Reiter [Rei01]. Since then the situation calculus is inspiration and background for many applications, like the logic programming language GOLOG in the field of artificial intelligence. The description logics based action formalism is not only inspired by Reiters work, it can be translated to the situation calculus. This translation is based on two recursive mappings, which carry $\mathcal{ALCQIO}$ concepts over to a formula in one of two free object variables x or y and one free situation variable s, whereat this rests upon standard translation of description logics into first order logics. In contrast to the situation calculus, which is based on full first-order logic with all the accompanying problems, description logics is a decidable fragment of first-order logic. But similarly to the situation calculus the design of a description logics based action theory involves some difficulties. Applying a set of actions within given domain constraints might have wanted or unwanted side effects. Dealing with side effects in an action formalism is called considering the *ramification problem*.

By now the description logics based action formalism has evolved in different ways. First introduced 2005 in the article [BLM+05], actions based on description logics where defined with pre- and post-conditions and supplemented by so-called occlusions. The authors have also proven the decidability of the major inference problem named projection for all sublanguages of $\mathcal{ALCQIO}$ and the corresponding complexity bounds could be given. But the derived results worked only for acyclic TBoxes and very restricted post-conditions with the result, that the ramification problem has been avoided. In such a setting the interpretations of primitive concepts and roles uniquely determine the whole interpretation and therefore no side effects are to be expected.

Later the article [BLL10b] and the corresponding report [BLL10a] could give a formalism, which contains general TBoxes and solves the ramification problem by introducing causal relationships. Then this ideas have been abstracted 2013 in the article [BZ13a] and the accompanying report [BZ13b] where the authors presented a so-called effect function, intended to pick up all possible side effects.

2.2.1 The Definition of the Action Formalism

To develop the theory of actions we merge the definitions given in [BML+05], [BLL10a] and [BZ13b]. For a first understanding we point to the mentioned article [BZ13a], where an overview, that establishes the association of an action formalism and description logics, is given. To form an action formalism based on description logics the following ingredients are necessary:

- The domain constraints given as a TBox $\mathcal{T}$;
- An incomplete description of the initial world given by an ABox $\mathcal{A}$;
- A finite set of action names denoted by Σ;
- A finite set of relevant ABox assertions denoted by $\mathcal{D}$.

An abstract action will be called by letter $\alpha, \beta \in \Sigma$. The set of literals contained in $\mathcal{D}$ is described by the set *Lit*. We further require $\mathcal{A} \subseteq \mathcal{D}$ and the set $\mathcal{D}$ to be closed under negation. When we work within the setting of actions for acyclic TBoxes the set $\mathcal{D}$ plays a minor role, because it collects only assertions explicitly given by the pre- and post-conditions of actions and the ABox. It will be shown in this section, how the set $\mathcal{D}$ can be generated if no creative work to solve the ramification problem is necessary. However, when defining a description logics action theory with regard to general TBoxes the major design process has to be laid in the construction of $\mathcal{D}$.

Let us formulate the straight forward definition of a concrete action without occlusions we will favour in this thesis and take a deeper look how the later to be defined effect function works together with this approach.

Definition 2.12. Let $\mathcal{T}$ be an acyclic TBox $\mathcal{T}$. An *atomic action*

$$\alpha = (\text{pre,post})$$

for $\mathcal{T}$ consists of a finite set pre of ABox assertions, the *pre-conditions*, a finite set post of *post-conditions* of the form φ/ψ where φ is an ABox assertion and ψ is a *primitive literal*, i.e. a literal restricted to primitive concepts. A *composite action* for $\mathcal{T}$ is a finite sequence $\alpha_1, \ldots, \alpha_k$ of atomic actions $\{\alpha_1, \ldots, \alpha_k\} \subseteq \Sigma$ for $\mathcal{T}$.

To define a semantics of actions w.r.t. acyclic TBoxes, let $\mathcal{D}$ be the set of assertions occurring in the initial ABox and in the pre- and post-conditions of action descriptions. Remember the set $\mathcal{D}$ is closed under negation. The literals in $\mathcal{D}$ are supposed to specify how an action changes the actual world and the semantics of actions should reflect how the application of an action changes the world. Altogether the semantics will determine, how the application of an action transforms an interpretation $\mathcal{I}$ into an interpretation $\mathcal{I}'$.

Definition 2.13. Let Σ be the set of action names, $\mathcal{D}$ be the set of relevant assertions with $Lit \subseteq \mathcal{D}$ the set of literals occurring in $\mathcal{D}$, and $\mathcal{T}$ the TBox specifying the domain constraints. The *effect function* $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ w.r.t. Σ, $\mathcal{D}$ and $\mathcal{T}$ is a partial function. If the function $\mathcal{E}$ is defined for a pair $(\alpha, \mathcal{I}) \in \Sigma \times \mathcal{M}(\mathcal{T})$ and for such an $\mathcal{I}$ holds $\mathcal{I} \models$ pre we say α is *applicable to* $\mathcal{I}$. Otherwise we say α is not applicable to $\mathcal{I}$.

Let $\mathcal{D}$ be the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals in $\mathcal{D}$. For every $\alpha \in \Sigma$, the effect function induces a binary relation $\Rightarrow_\alpha^\mathcal{E}$ on $\mathcal{M}(\mathcal{T})$.

Definition 2.14. Let $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ be an effect function w.r.t. Σ, $\mathcal{D}$ and $\mathcal{T}$. Then for $\mathcal{I}, \mathcal{I}' \in \mathcal{M}(\mathcal{T})$ we have $\mathcal{I} \Rightarrow_\alpha^\mathcal{E} \mathcal{I}'$ iff the following conditions are satisfied:

1. $\Delta^\mathcal{I} = \Delta^{\mathcal{I}'}$ and $a^\mathcal{I} = a^{\mathcal{I}'}$ for all $a \in N_I$.
2. $A^{\mathcal{I}'} := \Big(A^\mathcal{I} \cup \{a^\mathcal{I} \mid A(a) \in \mathcal{E}(\alpha, \mathcal{I})\}\Big) \setminus \{a^\mathcal{I} \mid \neg A(a) \in \mathcal{E}(\alpha, \mathcal{I})\}$ for all $A \in N_C$
3. $r^{\mathcal{I}'} := \Big(r^\mathcal{I} \cup \{(a^\mathcal{I}, b^\mathcal{I}) \mid r(a,b) \in \mathcal{E}(\alpha, \mathcal{I})\}\Big) \setminus \{(a^\mathcal{I}, b^\mathcal{I}) \mid \neg r(a,b) \in \mathcal{E}(\alpha, \mathcal{I})\}$
   for all $r \in N_R$.

For $\mathcal{I}, \mathcal{I}' \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I} \Rightarrow_\alpha^\mathcal{E} \mathcal{I}'$ we say α *may transform* the model $\mathcal{I}$ into the model $\mathcal{I}'$.

Let $\alpha_1, \ldots, \alpha_k \in \Sigma$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function w.r.t. Σ, $\mathcal{D}$ and $\mathcal{T}$. We say the composite action $\alpha_1, \ldots, \alpha_k$ may transform $\mathcal{I}$ to $\mathcal{I}'$ and write $\mathcal{I} \Rightarrow_{\alpha_1,\ldots,\alpha_k}^\mathcal{E} \mathcal{I}'$ if there are models $\mathcal{I}_0, \ldots, \mathcal{I}_k \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I} = \mathcal{I}_0$, $\mathcal{I}' = \mathcal{I}_k$ and $I_{i-1} \Rightarrow_{\alpha_i}^\mathcal{E} \mathcal{I}_i$ for $1 \leq i \leq k$.

Let us complete this definitions by some remarks.

Remark 2.15. The effect function is given as a function of an abstract action $\alpha \in \Sigma$ and an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$. If we have a concrete action $\alpha =$ (pre,post) with $\alpha \in \Sigma$, $\mathcal{A}$ ABox, $\mathcal{T}$ acyclic TBox and $\mathcal{I} \in \mathcal{M}(\mathcal{T})$, we can directly define the mapping $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ by setting

$$\mathcal{E}(\alpha, \mathcal{I}) := \{\psi \mid \varphi/\psi \in \text{post} \wedge \mathcal{I} \models \varphi\} \tag{2.1}$$

If such a function $\mathcal{E}$ is defined for a pair α and $\mathcal{I}$ and $\mathcal{I} \models$ pre then α is applicable to $\mathcal{I}$. We obtain for an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$, the concept names $A \in N_C$, the role assertions $r \in N_R$ and $a, b \in N_I$:

$$\begin{aligned}
A(a) \in \mathcal{E}(\alpha, \mathcal{I}) \quad &\Longleftrightarrow \quad \varphi/A(a) \in \text{post} \wedge \mathcal{I} \models \varphi \\
\neg A(a) \in \mathcal{E}(\alpha, \mathcal{I}) \quad &\Longleftrightarrow \quad \varphi/\neg A(a) \in \text{post} \wedge \mathcal{I} \models \varphi \\
r(a,b) \in \mathcal{E}(\alpha, \mathcal{I}) \quad &\Longleftrightarrow \quad \varphi/r(a,b) \in \text{post} \wedge \mathcal{I} \models \varphi \\
\neg r(a,b) \in \mathcal{E}(\alpha, \mathcal{I}) \quad &\Longleftrightarrow \quad \varphi/\neg r(a,b) \in \text{post} \wedge \mathcal{I} \models \varphi
\end{aligned}$$

Obviously, with the effect function given by (2.1), Definition 2.14 states the same transition of an interpretation $\mathcal{I}$ to an interpretation $\mathcal{I}'$ as the definition of the semantics of actions given by Lemma 9 in the report [BML$^+$05].

The next remark considers the effect function for concrete composite actions.

Remark 2.16. Let $\alpha_1, \dots, \alpha_n$ be a concrete composite action, where for $1 \leq i \leq n$ the $\alpha_i = \{\text{pre}_i, \text{post}_i\}$ are concrete actions, $\mathcal{A}$ be an ABox, $\mathcal{T}$ an acyclic TBox and let further $\mathcal{I}_0, \dots, \mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ be interpretations. The set of relevant assertions $\mathcal{D}$ now contains not only the ABox $\mathcal{A}$ but all assertions from $\text{pre}_1, \dots, \text{pre}_n$ to $\text{post}_1, \dots, \text{post}_n$. As supposed by definition, the set $\mathcal{D}$ has to be closed under negation and the set *Lit* contains all literals occurring in $\mathcal{D}$. For a composite action $\alpha_1, \dots, \alpha_n$ we construct only one effect function $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ and therefore the notation $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\dots,\alpha_n} \mathcal{I}_n$ for interpretations $\mathcal{I}, \dots, \mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ is justifiable. This function $\mathcal{E}$ can be defined for every pair $(\alpha_i, \mathcal{I}_{i-1})$ by (2.1).

The last remark concerns both, abstract and concrete actions.

Remark 2.17. If for the composite action $\alpha_1, \dots, \alpha_n$ for a TBox $\mathcal{T}$ and interpretations $\mathcal{I}_0, \dots, \mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\dots,\alpha_n} \mathcal{I}_n$ we know additionally, that for all $1 \leq i \leq n$ the action α_i is applicable to $\mathcal{I}_{i-1}$ we say the composite action $\alpha_1, \dots, \alpha_n$ is applicable to the interpretation $\mathcal{I}_0$.

The authors of [BML+05] state the determinism of the action formalism without occlusions and restricted to acyclic TBoxes and primitive literals. They also give a hint where determinism fails due to semantic or computational problems. For the action formalism given in [BZ13b] these limits could be extended, because in this setting for a given action α and a model $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ there is at most one $\mathcal{I}'$, such that $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha} \mathcal{I}'$. This means the transition of an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ to an interpretation $\mathcal{I}' \in \mathcal{M}(\mathcal{T})$ triggered by an action $\alpha \in \Sigma$ and given by a function $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ is deterministic even for general TBoxes.

2.2.2 The Basic Reasoning Tasks for Actions

First we consider the possibility, that for $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ and an action $\alpha \in \Sigma$ there exists no $\mathcal{I}'$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha} \mathcal{I}'$.

Definition 2.18. Let $\alpha \in \Sigma$, $\mathcal{A}$ an ABox, $\mathcal{D}$ the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function.

The action α is called *consistent w.r.t. the TBox* $\mathcal{T}$ if, for every $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ to which α is applicable, there is an interpretation $\mathcal{I}' \in \mathcal{M}(\mathcal{T})$ such that $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha} \mathcal{I}'$. Otherwise α is called *inconsistent with* $\mathcal{I}$.
If an action α, given for a TBox $\mathcal{T}$, is applicable to an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ and not inconsistent with $\mathcal{I}$ we say α *transforms* the interpretation $\mathcal{I}$ to an interpretation $\mathcal{I}'$.

Note that an action α is inconsistent with an interpretation $\mathcal{I}$ if and only if there exists $\psi \in Lit$ with $\{\psi, \neg\psi\} \subseteq \mathcal{E}(\alpha, \mathcal{I})$. For concrete actions this is the case if and only if there are $\varphi_1/\psi,\ \varphi_2/\neg\psi \in \text{post}$ and $\mathcal{I} \models \varphi_1, \varphi_2$.
Beside the question of consistence of actions, the article [BLM+05] introduces two important reasoning problems for reasoning within knowledge bases. We want to formulate them in terms of actions with effect functions and start with the definition of the inference problem named executability.

Definition 2.19. Let $\mathcal{T}$ be an acyclic TBox, $\alpha_1, \ldots, \alpha_k \in \Sigma$ a composite action, $\mathcal{A}$ an ABox, $\mathcal{D}$ the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function.

The composite action $\alpha_1, \ldots, \alpha_k$ is *executable* in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if the following conditions are true for all $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$:

- α_1 is applicable to $\mathcal{I}$,
- for all i with $1 \leq i < k$ and all interpretations $\mathcal{I}'$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_i} \mathcal{I}'$ the action α_{i+1} is applicable to $\mathcal{I}'$.

Remark 2.20. For concrete composite actions $\alpha_1, \ldots, \alpha_n$, where for $1 \leq i \leq n$ we have $\alpha_i = \{\mathrm{pre}_i, \mathrm{post}_i\}$ and all interpretations $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ this corresponds to the following requirements. The interpretation $\mathcal{I} \models \mathrm{pre}_1$ and for all interpretations $\mathcal{I}'$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_i} \mathcal{I}'$ we have $\mathcal{I}' \models \mathrm{pre}_{i+1}$.
Consider the composite action $\alpha_1, \ldots, \alpha_n$. It is executable to an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$ if and only if for all $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ we know $\alpha_1, \ldots, \alpha_n$ is applicable to $\mathcal{I}$ and $\mathcal{I} \models \mathcal{A}$.

The next definition introduces the major inference problem named projection.

Definition 2.21. Let $\mathcal{T}$ be an acyclic TBox, $\alpha_1, \ldots, \alpha_k \in \Sigma$ a composite action, $\mathcal{A}$ an ABox, $\mathcal{D}$ the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function w.r.t. Σ, $\mathcal{D}$ and $\mathcal{T}$.

The assertion φ is a *consequence* of applying $\alpha_1, \ldots, \alpha_k$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if for all models $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ and all $\mathcal{I}'$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_k} \mathcal{I}'$ we have $\mathcal{I}' \models \varphi$.

The following statement establishes a relationship between the above defined reasoning problems for concrete actions.

Lemma 2.22. *Executability and projection can be reduced in polynomial time to each other.*

We adopt the proof of Lemma 11 in [BML+05] but present it in terms of application of actions given by the effect function, because this proof gives a compact example for understanding actions for TBoxes and inference problems for actions and knowledge bases.

Proof. Let $\alpha_1, \ldots, \alpha_n \in \Sigma$ and for $1 \leq i \leq n$ let $\alpha_i = (\mathrm{pre}_i, \mathrm{post}_i)$ be a concrete composite action for the acyclic TBox $\mathcal{T}$, $\mathcal{D}$ the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function. The composite action $\alpha_1, \ldots, \alpha_n$ is executable in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if and only if

1. α_1 is applicable to every model $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$.
2. For $1 \leq i < n$ all assertions in pre_{i+1} are consequences of applying $\alpha_1, \ldots, \alpha_i$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$.

Condition two obviously is a projection problem and condition one can be seen as a projection problem for the empty action $(\emptyset, \emptyset)$.

Conversely assume we want to know whether φ is a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$. We consider the composite action $\beta_1, \ldots, \beta_n, \beta \in \Sigma$ with $\beta_i := (\emptyset, \mathrm{post}_i)$ for $1 \leq i \leq n$ and $\beta := (\{\varphi\}, \emptyset)$. Then φ is a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if and only if $\beta_1, \ldots, \beta_n, \beta$ is executable. $\square$

Remark 2.23. Recently it has become important in some applications of the description logics based action formalism to consider only action theories for which all actions are consistent. This is hardly a difference to the approach presented in this thesis, because this requirement can be reflected by minimally changing a few definitions. It should be noted in this context, that the authors of the article [BZ13a] consider only consistent actions and we point to comparing the definitions given in this article and in [BLM+05].
The effect function from Definition 2.13 can be defined for $\alpha \in \Sigma$ with $\alpha = \{\mathrm{pre}, \mathrm{post}\}$ and interpretations $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ which yield $\mathcal{I} \models \mathrm{pre}$ only. In this context an action α is applicable to $\mathcal{I}$ if and only if the function $\mathcal{E}$ is defined for α and $\mathcal{I}$.
In Definition 2.14 we then require the action α to be applicable and consistent, which influences the semantics of actions and the transition $\mathcal{I} \Rightarrow_\alpha^\mathcal{E} \mathcal{I}'$. As a result of this changes the inference problem executability becomes a condition for the projection problem. Compare the design of the projection problem from the article [BLM+05] and Definition 2.21, where the satisfiability of the pre-conditions is not of interest. In practice only well constructed, means consistent action theories are considered and therefore the projection problem will mostly be defined under the assumption of executability. Lemma 2.22 yields the polynomial reducibility of executability and projection as given in [BLM+05]. This had originally been motivation to consider the projection problem only. But of course it also justifies the mentioned changes in the definitions and the combination of both inference problems. As a consequence it is a question of taste or necessity, whether we consider the satisfiability of pre-conditions a requirement in the definition of a semantics of actions and the projection problem. In practice one mostly chooses the newer versions, which are provided in [BZ13a]. One of the aims of this thesis is the extension of the results from [BLM+05] and therefore here the earlier definitions are adopted.

In the next chapter we bring all the previous definitions and thoughts together to derive complexity bounds and formulate a reduction theorem. Before we start, let us close this introduction and envision actions and reasoning problems by continuing Example 2.10.

Example 2.24. Let $\mathcal{A}$, $\mathcal{T}$, $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be as defined in Example 2.10.
Let $\alpha_1 = (\mathrm{pre}_1, \mathrm{post}_1), \alpha_2 = (\mathrm{pre}_2, \mathrm{post}_2) \in \Sigma$ with

$$\mathrm{pre}_1 := \mathrm{pre}_2 := \{\mathit{human}(\mathit{Tom})\}$$

$$\begin{aligned}
\text{post}_1 &:= \{\mathit{friend}(\mathit{Paul}, \mathit{Anna})/\mathit{friend}(\mathit{Anna}, \mathit{Tom}),\\
&\quad \mathit{friend}(\mathit{Anna}, \mathit{Paul})/\mathit{friend}(\mathit{Tom}, \mathit{Anna})\}\\
\text{post}_2 &:= \{\neg \mathit{friend}(\mathit{Paul}, \mathit{Karl})/\mathit{human}(\mathit{Karl})\}
\end{aligned}$$

be two actions for $\mathcal{T}$ and $\mathcal{I}$. The set $\mathcal{D}$ contains $\mathcal{A}$, pre_1, pre_2 and the assertion from the post-conditions post_1 and post_2. Additionally $\mathcal{D}$ is closed under negation but nothing else. *Lit* is the set of literals in $\mathcal{D}$. Let further $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$, where

$$\begin{aligned}
\mathcal{E}(\alpha_1, \mathcal{I}) &:= \{\mathit{friends}(\mathit{Anna}, \mathit{Tom}), \mathit{friends}(\mathit{Tom}, \mathit{Anna})\}\\
\mathcal{E}(\alpha_2, \mathcal{I}_1) &:= \{\mathit{human}(\mathit{Karl})\}
\end{aligned}$$

The actions α_1 and α_2 are executable in $\mathcal{A}$ w.r.t $\mathcal{T}$. Clearly $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ and for all $\mathcal{I}_1$, $\mathcal{I}_2$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1} \mathcal{I}_1$, $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\alpha_2} \mathcal{I}_2$ we know $\text{pre}_1 = \text{pre}_2 = \{\mathit{human}(\mathit{Tom})\}$ with $\mathcal{I}_1 \models \{\mathit{human}(\mathit{Tom})\}$ and $\mathcal{I}_2 \models \{\mathit{human}(\mathit{Tom})\}$. Therefore α_1 and α_2 are applicable to $\mathcal{I}$. The interpretation $\mathcal{I}_1$ is illustrated in Figure 2.3. Note that only $\mathcal{I}_1$ meets all requirement from Definition 2.14, namely condition one to three and being a model for $\mathcal{T}$. Interpretation

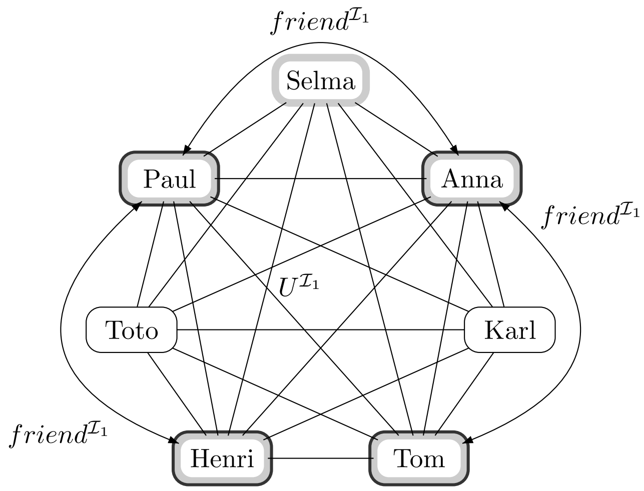

Figure 2.3: Interpretation $\mathcal{I}_1$

$\mathcal{I}_2$ is illustrated in Figure 2.4.
Let us consider $\beta \in \Sigma$ with

$$\beta := (\{\mathit{human}(\mathit{Tom})\}, \{\neg \mathit{friend}(\mathit{Paul}, \mathit{Karl})/\mathit{happy}(\mathit{Karl})\}).$$

The composite action α, β is not applicable to $\mathcal{I}$, because β does not meet the condition, that *happy*(Karl) is a primitive literal. But the composite action α, β gives rise to showing a perspective given by the more abstract approach. What is disturbing the determinism and existence of a successor interpretation $\mathcal{I}_2$ is the fact, that all members of the set $\mathit{happy}^{\mathcal{I}_1}$

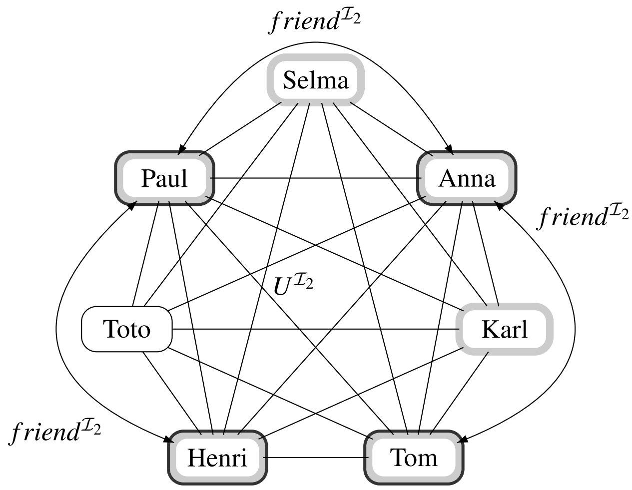

Figure 2.4: Interpretation $\mathcal{I}_2$

need to be member of the set $human^{\mathcal{I}_1}$ and have to be related to an element of the set $human^{\mathcal{I}_1}$ by the role $friend^{\mathcal{I}_1}$. This problems can be solved within the effect function.

$$\mathcal{E}(\beta_2, \mathcal{I}_1) := \{happy(Karl), human(Karl), friend(Karl, Anna)\}$$

This construction would require $human(Karl)$ and $friend(Karl, Anna)$ to be relevant assertions, which is not the case in the setting we assume in this thesis.

Last but not least we give an example considering the projection problem. Let $Toto \in N_I$ and $\varphi := (\exists U.(\forall friend.\{Selma\}))\,(Toto)$. The question is whether φ is a consequence of applying α_1, α_2 in $\mathcal{A}$ w.r.t. $\mathcal{T}$. By definition this is the case if for all interpretations $\mathcal{J} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ and all $\mathcal{J}' \in \mathcal{M}(\mathcal{T})$ with $\mathcal{J} \Rightarrow^{\mathcal{E}}_{\alpha_1,\alpha_2} \mathcal{J}'$ we have $\mathcal{J}' \models \varphi$. This is the case if

$$Toto^{\mathcal{J}'} \in (\exists U.(\forall friend.\{Selma\}))^{\mathcal{J}'} .$$

Consider $\mathcal{J} := \mathcal{I}$. In Figure 2.4 we observe the set of Selmas friends in $\Delta^{\mathcal{I}_2}$ and therefore also the set $(\exists U.(\forall friend.\{Selma\}))^{\mathcal{I}_2}$ is empty and so $\mathcal{I}_2 \not\models \varphi$ for all individuals in N_I. Therefore φ is not a consequence of applying α_1, α_2 in $\mathcal{A}$ w.r.t. $\mathcal{T}$ because there exists a model, namely $\mathcal{I}$ with $\mathcal{I} \models^{\mathcal{E}}_{\alpha_2,\alpha_2} \mathcal{I}_2$ and $\mathcal{I}_2 \not\models \varphi$.

3 The Action Formalism for DLs Admitting a Universal Role

In this chapter we investigate the influence of the universal role on the description logics based action formalism. Despite the fact that the universal role is provided in one of the most popular description logics, namely $\mathcal{SROIQ}$, universality is not the most current subject of interest in the field of description logics. Syntactically the universal role is introduced by a special role name and its semantics is given by a fixed interpretation. Because this interpretation always links all elements of the domain, the universal role can be considered a kind of top role in analogy to the top concept. Beside this, the universal role is one of the most simple possibilities to make global statements. Where global statements are such declara beeing true for each element of the interpretation domain. Recent research concerns more involved global properties like transitivity, but as we will observe most of the description logics that make global statements contain a universal role. We will see further adding universality to a description logic increases the complexity of reasoning problems in most of the cases.
Our task in this chapter now is to include the universal role in the description logics action formalism and to derive complexity statements for the inference problem projection. To achieve this, the chapter consists of two parts. Where we first look at the general influence of the universal role and its position in the world of description logics we make the above mentioned observation on increasing complexity. The second part then contains the major work. We will reduce the action reasoning problems executability and projection to the results for ABox consequence obtained in the first part. In doing so we orient ourselves on Theorem 14 from the report [BML$^+$05]. All together this chapter contains various complexity results for reasoning within description logics admitting a universal role.

3.1 Complexity Results for Description Logics admitting a Universal Role

The aim of this section is to derive statements about the complexity class containing ABox consequence in all fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$ that include nominals and a universal role. Much is known about description logics containing a universal role U one way ore another, but things are different if we directly postulate the existence of such an U. To obtain complexity results about a special fragment of $\mathcal{ALCQIO}^{\mathcal{U}}$ our argumentation therefore follows a certain pattern. We consider several logics whose complexity is well studied and that contain the description logic of interest, or are contained in it respectively. This way we cannot only touch on various fields of description logics but encircle the complexity of different reasoning tasks. More concrete this means we try to evaluate the complexity of several reasoning problems in fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$ by transferring results from other description logics. To understand the techniques used in doing so we give a brief overview about the complexity theory used in this chapter.

3.1.1 Brief Review of Complexity Theory

Many tasks in computational logics deal with the question whether there exists an algorithm, which solves a given problem. Moreover we ask, how fast this algorithm is at best. The answer to this question is sometimes given by the reduction of one problem to another one for which there exists an algorithm that solves the problem. If this reduction, that takes problem A as input and maps it to problem B, is polynomial in the size of the input than problem A is said to be polynomial reducible to Problem B.
The definition of a complexity class considers several parameters. First there is the model of computation, then we distinguish between deterministic and nondeterministic mode and we also look at the resource wished to be bound, basically time and space. For an introduction to this broad field and the proves of the general statements on complexity theory we refer the to the book [Pap94].
Figure 3.1 shows the hierarchy of the complexity classes used in this thesis. Note that some

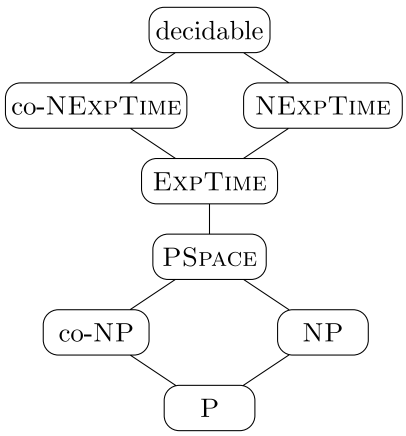

Figure 3.1: Hierarchy of complexity classes.

of these classes are only supposed to be unequal, but so far there exists no proof. What we know by the time hierarchy theorem is that P $\subsetneq$ ExpTime and NP $\subsetneq$ NExpTime, so at least one of the inclusions in Figure 3.1 must be real. What we do not know is one of the unsolved questions of computational complexity theory, namely the question whether P equals NP. It is also unknown for which of the inclusions NP $\subseteq$ PSpace $\subseteq$ ExpTime $\subseteq$ NExpTime equality holds or can be excluded respectively.

The class co-NP, depicted in Figure 3.1, is the class of complements of problems in NP. In computational complexity a decision problem B is the complement of problem A if whenever for a given input the answer is yes for A then the answer for B is no. The complement of a complexity class is the set of all complement problems of a given class. For deterministic complexity classes it is known that they equal their complement class. Whereas equality of the complement classes remains an open problem for the non-deterministic classes NP and NExpTime.

3.1.2 General Results

In the first chapter we introduced several reasoning problem for description logics. These tasks are of course related to each other in various ways and their complexity depends on the description logics they are considered in. We want to start our considerations with a little lemma we quote from [BML+05]. It considers the complexity of ABox consistency and ABox consequence.

Lemma 3.1. *ABox inconsistency and ABox consequence are of the same complexity.*

The proof can be found in detail in [BML+05]. Therein it is shown ABox consequence can be polynomially reduced to ABox consistency with negated role assertions and vice versa. Then ABox consistency with negated role assertions is polynomially reduced to ABox inconsistency without such assertions. Note that ABox inconsistency is the complement problem of ABox consistency.
The next lemma is adopted from Chapter 2 of the book [BCM+03]. It deals with the relationship between concept satisfiability and ABox consequence.

Lemma 3.2. *Within the presence of nominals concept satisfiability and ABox consistency are polynomial reducible to each other.*

In this thesis we only need to consider reasoning problems for description logics containing nominals and therefore the above stated reducibility is of special interest. Without nominals available the lemma is not necessarily true, because ABox consistency may be more difficult than concept satisfiability. This follows from the equivalence of ABox consistency and a problem called instance checking observed in Chapter 2 of the book [BCM+03] and the complexity calculations in [DLNS94]. In presence of nominals the proof can be given as an easy consideration and the quotation of a prominent theorem.

Proof. Clearly concept satisfiability can be reduced to ABox consistency without any further restriction on the underlying DL, because a concept C is satisfiable if and only if the ABox $\{C(a)\}$ is consistent for an arbitrary $a \in N_I$.

Within the presence of nominals in a description logic Theorem 3.7 in [Sch94] shows ABox consistency is polynomial reducible to concept satisfiability. □

3.1.3 Complexity Results with Universal Role

The main question of this section is how the influence of the universal role U will be reflected in the complexity bounds of the inference problem ABox consequence in several description logics. The universal role is not just an arbitrary role but a very powerful tool. When added to $\mathcal{ALC}$ it can be used to define the concept $\forall U.C = (\leq 0\ U\ \neg C)$, which is satisfiable if and only if C is globally satisfiable. Where C is global satisfiable if there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Another well known observation, which is for example provided in the book [BL06], is that the universal role can be used to translate a general TBox $\mathcal{T}$ into a single concept $C_{\mathcal{T}}$. Let

$$C_{\mathcal{T}} := \forall U. \bigsqcap_{D \sqsubseteq E \in \mathcal{T}} \neg D \sqcup E$$

Because the proof of the two implications $\mathcal{I} \models \mathcal{T} \Rightarrow C_{\mathcal{T}}^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $C_{\mathcal{T}}^{\mathcal{I}} \neq \emptyset \Rightarrow \mathcal{I} \models \mathcal{T}$ can be found in [BL06] as well, satisfiability in $\mathcal{ALC}^{\mathcal{U}}$ w.r.t. general TBoxes is reducible to satisfiability in $\mathcal{ALC}^{\mathcal{U}}$ w.r.t. the empty TBox. A consequence of the above observations now is the loss of the possibility to obtain a PSpace-algorithm for reasoning within $\mathcal{ALC}^{\mathcal{U}}$ and beyond. For $\mathcal{ALC}$ ExpTime-hardness of concept satisfiability w.r.t general TBoxes has originally been proven in [Sch91]. Another interesting proof, that uses the above mentioned relation of the universal role and global concept satisfiability, is given by Theorem 3.18 in [Tob01]. We put this fact to record by the following lemma.

Lemma 3.3. *ABox consequence in $\mathcal{ALC}^{\mathcal{U}}$ is ExpTime-hard.*

Proof. The ExpTime-hardness of concept satisfiability w.r.t. general TBoxes for $\mathcal{ALC}$ transfers to ExpTime-hardness of concept satisfiability for $\mathcal{ALC}^{\mathcal{U}}$ w.r.t acyclic TBoxes with the above given internalisation. By the proof of Lemma 3.2 we obtain ExpTime-hardness of ABox consistency and by Lemma 3.1 ExpTime-hardness of ABox consequence in $\mathcal{ALC}^{\mathcal{U}}$.
Note that Lemma 3.3 holds w.r.t. the empty TBox as well as w.r.t. general TBoxes because of the possibility to internalize a TBox in presence of a universal role. □

To transfer complexity results from other DLs we point to the following observation. A universal role U is implicitly contained in many description logics based on $\mathcal{ALC}$. As discussed in the book [BCM+03], the universal role can be simulated by a role U^* that is universal w.r.t. a terminology $\mathcal{T}$. Let for example U^* be the reflexive transitive closure of all roles and possible inverse roles occurring in $\mathcal{T}$, then U^* is universal w.r.t. $\mathcal{T}$. Such an U^* is contained in $\mathcal{ALC}_{reg}$, which is $\mathcal{ALC}$ admitting regular expressions over roles, i.e. union, composition, role identity and the Kleene operator. From the simulation of a universal role by U^* in $\mathcal{ALC}_{reg}$ we can derive several upper bounds for reasoning in fragments $\mathcal{L}^{\mathcal{U}}$ of $\mathcal{ALCQIO}^{\mathcal{U}}$.
Another possibility to construct an U^* which is universal w.r.t. a TBox $\mathcal{T}$, is defining U^* to be transitive and contain all roles and possible inverse roles occurring in $\mathcal{T}$. In this case

upper bounds for complexity results would transfer from $\mathcal{SH}$, the extension of $\mathcal{ALC}$ with transitive roles and role hierarchy.
With this notes on reasoning problems and the universal role we start our considerations on the complexity of ABox consequence in fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$. As the interplay of role constructors and qualified number restriction needs special care we first give the following lemma about the two fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$, that contain nominals and a universal role but not qualified number restriction.

Lemma 3.4. *ABox consequence in $\mathcal{ALCO}^{\mathcal{U}}$ and $\mathcal{ALCIO}^{\mathcal{U}}$ is ExpTime-complete.*

Proof. The lower bounds follow from the ExpTime-hardness of ABox consistence in $\mathcal{ALC}^{\mathcal{U}}$ presented in Lemma 3.3.
In the Description Logics Handbook [BCM+03] Diego Calvanese and Giuseppe De Giacomo argue concept satisfiability in $\mathcal{ALCIO}_{reg}$ is ExpTime-complete. They reduce ExpTime-decidability of concept satisfiability in $\mathcal{ALCIO}_{reg}$ to concept satisfiability in $\mathcal{ALCI}_{reg}$ by referring to the PhD thesis [Gia95] and the article [GL94]. ExpTime-decidability of concept satisfiability in $\mathcal{ALCI}_{reg}$ is then deduced from Theorem 5.18 in the book [BCM+03], which states ExpTime-completeness of concept satisfiability in $\mathcal{ALCQI}_{reg}$. This result goes back to the correspondence between $\mathcal{ALC}_{reg}$ and PDL, first published by Klaus Schild in the report [Sch91], but not only his work, see the notes in Section 5.2.3 of the book [BCM+03].
From Lemma 3.1 and Lemma 3.2 we obtain the polynomial reducibility of concept satisfiability to ABox consequence in presence of nominals which yields the upper bound that proves this lemma. □

Reasoning with qualified number restrictions in combination with special roles needs closer consideration and is a matter of research for its own. In most of the cases the universal role is excluded from qualified number restrictions. For example in any extension of $\mathcal{ALCQ}_{reg}$ and $\mathcal{SHQ}$ complex roles are in general not allowed to occur in qualified number expressions.
With regard to this discussion we split the lemma about the remaining two fragments $\mathcal{ALCQO}^{\mathcal{U}}$ and $\mathcal{ALCQIO}^{\mathcal{U}}$, where we first consider the restriction that U is not occurring in qualified number restriction. Note that parts of this lemma only hold under the assumption of numbers coded in unary.

Lemma 3.5. *Assume the universal role U is not occurring in qualified number restriction. Then ABox consequence in*

1. *$\mathcal{ALCQO}^{\mathcal{U}}$ is ExpTime-complete;*
2. *$\mathcal{ALCQIO}^{\mathcal{U}}$ is co-NExpTime-complete if unary coding of numbers in the input is assumed.*

Proof. The lower bound for the first statement follows from the ExpTime-hardness of ABox consequence in $\mathcal{ALC}^{\mathcal{U}}$ given in Lemma 3.3.

From Corollary 4.13 of the article [Tob00] we know concept satisfiability in $\mathcal{ALCQIO}$ is NExpTime-complete[1], which yields together with Lemma 3.1 and Lemma 3.2 the lower bound for the second statement.

Analogously to concept satisfiability in $\mathcal{ALCIO}_{reg}$ Diego Calvanese and Giuseppe De Giacomo argue in The Description Logic Handbook [BCM+03], that concept satisfiability in $\mathcal{ALCQO}_{reg}$ is polynomial reducible to concept satisfiability in $\mathcal{ALCQ}_{reg}$ by quoting the PhD thesis [Gia95] and the article [GL94]. ExpTime-decidability of concept satisfiability in $\mathcal{ALCQ}_{reg}$ follows again from Theorem 5.18 in the book [BCM+03] which states ExpTime-completeness of concept satisfiability in $\mathcal{ALCQI}_{reg}$. From Lemma 3.1 and Lemma 3.2 we obtain the polynomial reducibility of concept satisfiability to ABox consequence in presence of nominals which yields the upper bound for point one.
From Corollary 6.31 in [Tob01] we know satisfiability of $\mathcal{SHOIQ}$-concepts is NExpTime-hard. The problem is NExpTime–complete if unary coding of numbers in the input is assumed. Together with Lemma 3.1 and Lemma 3.2 this yields the upper bounds for the second item and completes the proof. □

This also completes the collection of results for all fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$ containing nominals and the universal role, but excluding the universal role from qualified number restriction.

In his thesis [Tob01] Stephan Tobies investigated the extension of several fragments of $\mathcal{ALCQIO}$ with boolean roles. A so-called $\mathcal{ALCQIBO}$-role expression is build from normal roles using role intersection, role union and role complement. Obviously the universal role can be simulated in $\mathcal{ALCQIBO}$ by a role $U^* := r \cup \neg r$ for an arbitrary $r \in N_R$. In $\mathcal{ALCQIBO}$ the role U^* is allowed to occur within qualified number restriction. This is not self-evident but the result of a long discussion. Based on the results for $\mathcal{ALCQIBO}$ we start our considerations on $\mathcal{ALCQO}^{\mathcal{U}}$ and $\mathcal{ALCQIO}^{\mathcal{U}}$.

Lemma 3.6. *Assume the universal role U is allowed to occur in qualified number restrictions. Then ABox consequence in $\mathcal{ALCQIO}^{\mathcal{U}}$ is co-NExpTime-complete if unary coding of numbers in the input is assumed.*

Proof. The lower bound follows from point 2 in Lemma 3.5.
From Corollary 5.31 in [Tob01] we know satisfiability of $\mathcal{ALCQIBO}$-concepts is NExpTime-hard. The problem is NExpTime–complete if unary coding of numbers in the input is assumed. Together with Lemma 3.1 and Lemma 3.2 this yields the upper bounds and completes the proof. □

The case $\mathcal{ALCQO}^{\mathcal{U}}$ is more involved.

Lemma 3.7. *Assume the universal role U is allowed to occur in qualified number restrictions. Then ABox consequence in $\mathcal{ALCQO}^{\mathcal{U}}$ is ExpTime-hard and in co-NExpTime if unary coding of numbers in the input is assumed.*

[1] This border also holds under the assumption of binary coding, because $\mathcal{ALCQIO}$ can be translated to C^2, the two variable fragment of first order logic with counting quantifiers, see for example [BML+05]. From [PH05] we know satisfiability in C^2 is in NExpTime even for binary coding of numbers.

Proof. Obviously the lower bound follows from Lemma 3.5 and Lemma 3.6 gives the upper bound. □

An approach to investigate the gap in Lemma 3.7 is the introduction of so-called CBoxes from [Tob01].

Definition 3.8. Let $n \in \mathbb{N}$ and C a concept, then a *cardinality restriction* is of the form $(\leq n\ C)$ or $(\geq n\ C)$. A finite set of cardinality restrictions is called *CBox*.

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ *satisfies a cardinality restriction* $(\leq n\ C)$ or $(\geq n\ C)$ iff $|C^{\mathcal{I}}| \leq n$, or $|C^{\mathcal{I}}| \geq n$ respectively. An interpretation $\mathcal{I}$ *satisfies a CBox* $\mathcal{C}$ if $\mathcal{I}$ satisfies all cardinal restrictions in $\mathcal{C}$.

The notion of cardinality restrictions is motivated by the following lemma, which proves the universal role can be used to simulate cardinality restrictions in presence of qualified number restriction.

Lemma 3.9. *The cardinality restriction $(\leq n\ C)$ is satisfiable if and only if the qualified number restriction $(\leq n\ U\ C)$ is satisfiable, where U is the universal role.*

Proof. Let $\mathcal{I}$ be an interpretation. Then $(\leq n\ C)^{\mathcal{I}}$ is true if and only if $(\leq n\ U\ C)^{\mathcal{I}}$ is true because for $\triangle \in \{\leq, \geq\}$ we know $(\triangle\ n\ U\ C)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid |C^{\mathcal{I}}| \triangle n\}$. □

This enables us to express a cardinality restriction as an $\mathcal{ALCQ}^{\mathcal{U}}$ concept. As an important consequence we can use the universal role to transform a CBox into a single concept.

Lemma 3.10. *CBox satisfiability for $\mathcal{ALCQ}$ is polynomial reducible to concept satisfiability in $\mathcal{ALCQ}^{\mathcal{U}}$.*

The proof is similar to the proof of Lemma 5.32 in [Tob01].

Proof. Let $\mathcal{C}$ be an $\mathcal{ALCQ}$-CBox. We transfer $\mathcal{C}$ into a single $\mathcal{ALCQ}^{\mathcal{U}}$-concept $C_{\mathcal{C}}$ by setting

$$C_{\mathcal{C}} \equiv \bigsqcap_{(\leq\ n\ C) \in \mathcal{C}} (\leq n\ U\ C) \sqcap \bigsqcap_{(\geq\ n\ C) \in \mathcal{C}} (\geq n\ U\ C)$$

Claim 3.11. The concept $C_{\mathcal{C}}$ is satisfiable if and only if the CBox $\mathcal{C}$ is satisfiable.

Let $\mathcal{C}$ be satisfiable with $\mathcal{I}$ interpretation for $\mathcal{C}$. Then by definition the interpretation $\mathcal{I}$ satisfies all $(\leq n\ C) \in \mathcal{C}$ and all $(\geq n\ C) \in \mathcal{C}$ and thus by Lemma 3.9 $\mathcal{I}$ satisfies $C_{\mathcal{C}}$.

Let $\mathcal{I}$ be an interpretation for the concept $C_{\mathcal{C}}$. Then by definition $\mathcal{I}$ satisfies all conjuncts of $C_{\mathcal{C}}$ and thus by Lemma 3.9 the interpretation $\mathcal{I}$ satisfies all $(\leq\ n\ C) \in \mathcal{C}$ and all $(\geq n\ C) \in \mathcal{C}$. □

Because Lemma 3.10 is true for all concepts occurring in cardinality restriction and roles are not part of cardinality restriction, we get the following corollary.

Corollary 3.12. *Let $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCQ}^{\mathcal{U}}, \mathcal{ALCQO}, \mathcal{ALCQO}^{\mathcal{U}}, \mathcal{ALCQI}, \mathcal{ALCQI}^{\mathcal{U}}, \mathcal{ALCQIO}, \mathcal{ALCQIO}^{\mathcal{U}}\}$. Then CBox satisfiability in $\mathcal{L}$ is polynomial reducible to concept satisfiability in $\mathcal{L}^{U}$.*

This is of course especially true in case $\mathcal{L} \in \{\mathcal{ALCQO}, \mathcal{ALCQO}^{\mathcal{U}}\}$. The next lemma combines the complexity results from Corollary 5.8 and Corollary 5.20 in [Tob01]. Together with the previous argumentation this cannot completely fill the gap in Lemma 3.7, but it will be at least a hint how to proceed in this direction.

Lemma 3.13. *CBox-satisfiability in $\mathcal{ALCQO}$ is* ExpTime*-complete if unary coding of numbers is assumed. Satisfiability of $\mathcal{ALCQ}$-CBoxes is* NExpTime*-hard if binary coding is used to represent numbers in cardinality restrictions.*

The first fact confirms together with Corollary 3.12 the lower from Lemma 3.7 under the assumption of unary coding. But the second fact shows at least for binary coding, that ABox consequence in $\mathcal{ALCQO}^{\mathcal{U}}$ cannot be decided in ExpTime.
This closes our considerations on the complexity of ABox consequence in presence of a universal role.

3.2 Deciding Executability and Projection with Universal Role

In the first chapter a Description Logics based action theory and one of the most important inference problems for an action formalism, namely the projection problem, have been introduced. One of the most important challenges in working with an action formalism is finding practical solutions for this problem. For the description logics action formalism mainly two different ideas are investigated. On the one side is the so called regression approach. An ABox assertion φ, supposed to be a consequence of applying a composite action to an interpretation, is together with the pre- and post-conditions developed into one possibly huge formula. This formula is then asked to hold w.r.t. the initial ABox and TBox.
The approach this thesis is going to follow, has been introduced 2005 in the article [BLM+05]. Here a so-called reduction ABox and TBox are constructed from the input, i.e. the initial ABox and TBox and the pre- and post-conditions of the composite action. Time stamped copies of all relevant concept and role names of the input are with the intense use of nominals supposed to simulate the application of a composite action to an interpretation. The Abox assertion φ is then asked to hold w.r.t. this modified ABox and TBox.
What the two ideas have in common is the reduction of the projection problem to the well known question of ABox consequence. In practice there exist implementations of this projection procedure for both approaches. For a detailed introduction and comparison of the algorithms we recommend the article [YLL+12].

The centrepiece of the idea to develop the ABox and TBox together with the composite action into a so-called reduction ABox and TBox is given by Theorem 14 in the report [BML+05]. It proves that the complexity of executability and projection for a fragment $\mathcal{L}$ of $\mathcal{ALCQIO}$ coincides with the complexity of ABox consequence in $\mathcal{LO}$, the extension of $\mathcal{L}$ with nominals. In this section we want to improve the constructions this theorem includes. We ask if Theorem 14 from [BML+05] holds for every fragment $\mathcal{L}$ of $\mathcal{ALCQIO}^{\mathcal{U}}$. Before we start to implement the previous thoughts on the reduction from reasoning with actions to ABox consequence we want to prepend the main theorem we are going to prove in the following.

Theorem 3.14. *Let $\mathcal{L}$ be a fragment of $\mathcal{ALCQIO}^{\mathcal{U}}$. Then projection of composite actions formulated in $\mathcal{L}$ can be polynomially reduced to ABox consequence in $\mathcal{LO}$ w.r.t. acyclic TBoxes.*

The inspiring idea from [BML+05] is to define an ABox and a TBox, called $\mathcal{A}_{\text{red}}$ and $\mathcal{T}_{\text{red}}$, such that each single model of them encodes a sequence of interpretations $\mathcal{I}_0, \dots, \mathcal{I}_n$ obtained by applying $\alpha_1, \dots, \alpha_n$ in $\mathcal{A}$. The assertion φ will be encoded in an appropriate assertion φ_{red}. The proof of Theorem 3.14 then directly follows from a technical lemma, which states that φ is a consequence of applying the actions $\alpha_1, \dots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if and only if φ_{red} is a consequence of $\mathcal{A}_{\text{red}}$ w.r.t. $\mathcal{T}_{\text{red}}$.
The design of the reduction ABox and the reduction TBox has to satisfy several requirements. We want the $\mathcal{I}_0$ component of a reduction model $\mathcal{J}$ to be a model of $\mathcal{A}$ and each $\mathcal{I}_i$ should satisfy the post-conditions post_i. The TBox $\mathcal{T}_{\text{red}}$ has to ensure for $0 \le i \le n$ that each $\mathcal{I}_i$ is a model of $\mathcal{T}$. Last but not least we need to describe, that no individual or concept is supposed to change unless this is triggered by an action. We have to transform the dynamic process of applying actions into the static problem of ABox consequence. This will be a central point in the design of the reduction ABox and TBox. We need to ensure by $\mathcal{A}_{\text{red}}$ and $\mathcal{T}_{\text{red}}$ that only the changes triggered by the actions occur in the encoded sequence of the interpretations $\mathcal{I}_1, \dots, \mathcal{I}_n$. The clue here is the distinction between so-called named and unnamed elements. We encode the appliance of actions on the named elements in $\mathcal{A}_{\text{red}}$ and the appliance of actions on unnamed elements in $\mathcal{T}_{\text{red}}$. All this requirements will be reflected in the following definitions.

3.2.1 Preliminaries

For a fragment $\mathcal{LO}^{\mathcal{U}}$ of $\mathcal{ALCQIO}^{\mathcal{U}}$ let $\mathcal{A}$ be an ABox, $\alpha_1, \dots, \alpha_n$ a composite action with $\alpha_i = (\text{pre}_i, \text{post}_i)$ for $1 \le i \le n$, $\mathcal{T}$ an acyclic TBox and φ_0 an assertion.
For a role assertion φ_0 both $r(a, b)$ and $\neg r(a, b)$ can be replaced with $(\exists r.\{b\})(a)$ and $(\forall r.\neg\{b\})(a)$ respectively where $\{a\}$ and $\{b\}$ are nominals. If $\varphi_0 = C(a)$ with C not a concept name, we add a concept definition $A_0 \equiv C$ to the TBox $\mathcal{T}$ and consider $\varphi = A_0(a)$. Therefore we can assume w.l.o.g. that φ_0 is of the form $A_0(a_0)$ for a concept name A_0.
We call $\mathcal{A}$, $\mathcal{T}$, $\alpha_1, \dots, \alpha_n$ and φ_0 the *input*, $\mathcal{A}_{\text{red}}$ the *reduction ABox*, $\mathcal{T}_{\text{red}}$ the *reduction TBox* and φ_{red} the *reduction assertion*.

To define the TBox $\mathcal{T}_{\text{red}}$ we have to make the above mentioned distinction between named elements and unnamed ones. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. An element $x \in \Delta^{\mathcal{I}}$ is called a *named element* of $\mathcal{I}$ if for some $a \in N_I$ used in the input we have $a^{\mathcal{I}} = x$. Otherwise we call $x \in \Delta^{\mathcal{I}}$ an *unnamed element*.

Let N be an auxiliary concept name, denoting the set of individual names used in the input, which are mapped to named elements of interpretations. Let Sub be the smallest set containing all concepts occurring in the input and being closed under taking subconcepts. Finally the labelled concepts and roles are to be introduced. For every $C \in \text{Sub}$ and every $0 \leq i \leq n$ let $T_C^{(i)}$ be a concept name.

For every $0 \leq i \leq n$ let $A^{(i)}$ be an auxiliary concept name for every primitive concept name A used in the input. For $1 \leq i \leq n$ the $A^{(i)}$ represent $A^{\mathcal{I}_i}$ restricted to the named elements. Since concept membership of unnamed elements never changes the unnamed part of $A^{\mathcal{I}_i}$ can be found in $A^{(0)}$.

For every $0 \leq i \leq n$ let $r^{(i)}$ be an auxiliary role name for every role name $r \neq U$ used in the input. For $1 \leq i \leq n$ the $r^{(i)}$ will represent $r^{\mathcal{I}_i}$ restricted to role relationships where both involved domain elements are named. The role relationships of unnamed elements are recorded in $r^{(0)}$.

Let Obj be a set of individual names used in the input and let a_{help} be an auxiliary individual with $a_{\text{help}} \notin \text{Obj}$.

Further for every $0 \leq i \leq n$ let $U^{(i)}$ be an auxiliary role name representing $\Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}$. Note that mapping $U^{(i)}$ to the full relation in every component is disturbing the intention to cover only role relationships between named elements in the components labelled by $1 \leq i \leq n$. But it certainly corresponds to the nature of a universal role.

Now, well equipped with this preparations we can give the definition of $\mathcal{T}_{\text{red}}$ and $\mathcal{A}_{\text{red}}$. We basically follow [BML+05] augmented with concept definitions and assertions reflecting the addition of the universal role and the introduction of the effect function to the definition of an action.

The reduction TBox $\mathcal{T}_{\text{red}}$ consists of several components. The first component states, that every interpretation of a concept N contains exactly the named elements.

$$\mathcal{T}_N := \{N \equiv \bigsqcup_{a \in \text{Obj}} \{a\}\}$$

The TBox component $\mathcal{T}_{Sub}$, given in Figure 3.2, contains one concept definition for every $0 \leq i \leq n$ and every $C \in \text{Sub}$, which is not a defined concept name in $\mathcal{T}$. These concept definitions ensure $T_C^{(i)}$ stands for the interpretation of C after the application of $\alpha_1, \ldots, \alpha_i$. In the construction of $\mathcal{T}_{Sub}$ the concept $A^{(i)}$ only represents the extension of A in the ith interpretation for the named elements. So, to get to the full extension $T_A^{(i)}$, we have to use $A^{(i)}$ for the named elements and $A^{(0)}$ for the unnamed ones. The idea of splitting can also be recognized for role relationships in the lines (3.1) and (3.2) of Figure 3.2 .

Because of the addition of the universal role U the lines (3.3) to (3.6) have been added to $\mathcal{T}_{Sub}$. They reflect the fact, that the interpretation of U never changes and U ranges over the whole domain. We need two different rules for the universal role, depending on

$$
\begin{aligned}
T_A^{(i)} &\equiv (N \sqcap A^{(i)}) \sqcup (\neg N \sqcap A^{(0)}) \qquad \text{if } A \text{ is primitive in } \mathcal{T} \\
T_{\neg C}^{(i)} &\equiv \neg T_C^{(i)} \\
T_{C \sqcap D}^{(i)} &\equiv T_C^{(i)} \sqcap T_D^{(i)} \\
T_{C \sqcup D}^{(i)} &\equiv T_C^{(i)} \sqcup T_D^{(i)} \\
T_{(\geq m\ r\ C)}^{(i)} &\equiv \left(N \sqcap \bigsqcup_{0 \leq j \leq m} (\geq j\ r^{(i)}\ (N \sqcap T_C^{(i)})) \sqcap (\geq (m-j)\ r^{(0)}\ (\neg N \sqcap T_C^{(i)})) \right) \\
&\quad \sqcup \left(\neg N \sqcap (\geq m\ r^{(0)}\ T_C^{(i)}) \right) \qquad (3.1) \\
T_{(\leq m\ r\ C)}^{(i)} &\equiv \left(N \sqcap \bigsqcup_{0 \leq j \leq m} (\leq j\ r^{(i)}\ (N \sqcap T_C^{(i)})) \sqcap (\leq (m-j)\ r^{(0)}\ (\neg N \sqcap T_C^{(i)})) \right) \\
&\quad \sqcup \left(\neg N \sqcap (\leq m\ r^{(0)}\ T_C^{(i)}) \right) \qquad (3.2) \\
T_{(\geq m\ U\ C)}^{(i)} &\equiv \left(\geq m\ U^{(i)}\ T_C^{(i)} \right) \qquad (3.3) \\
T_{(\leq m\ U\ C)}^{(i)} &\equiv \left(\leq m\ U^{(i)}\ T_C^{(i)} \right) \qquad (3.4) \\
T_{\forall U.C}^{(i)} &\equiv \left(\forall U^{(i)}.T_C^{(i)} \right) \qquad (3.5) \\
T_{\exists U.C}^{(i)} &\equiv \left(\exists U^{(i)}.T_C^{(i)} \right) \qquad (3.6)
\end{aligned}
$$

Figure 3.2: The concept definitions in $\mathcal{T}_{Sub}$.

whether we allow U to occur in number restriction, which will subsume existential and value restriction, or not. In the first case the lines (3.3) and (3.4) of Figure 3.2 will be part of $\mathcal{T}_{Sub}$ and in the other case we add the lines (3.5) and (3.6) to $\mathcal{T}_{Sub}$.

Now we can join $\mathcal{T}_{\text{red}}$:

$$\mathcal{T}_{\text{red}} := T_{Sub} \cup T_N \cup \{T_A^{(i)} \equiv T_E^{(i)} \mid A \equiv E \in \mathcal{T}, 0 \leq i \leq n\}$$

The reduction ABox $\mathcal{A}_{\text{red}}$ also consists of several components. We first introduce abbreviations for $0 \leq i \leq n$ to simplify $\mathcal{A}_{\text{red}}$. Note that this version differs from the construction given in [BML+05], because we can benefit from the universal role.

$$\begin{array}{rcl} p_i(C(a)) & := & \exists U.(\{a\} \sqcap T_C^{(i)}) \\ p_i(r(a,b)) & := & \exists U.(\{a\} \sqcap \exists r^{(i)}.\{b\}) \\ p_i(\neg r(a,b)) & := & \exists U.(\{a\} \sqcap \forall r^{(i)}.\neg\{b\}) \end{array}$$

The first component of $\mathcal{A}_{\text{red}}$ formalizes the satisfaction of the post-conditions. For $1 \leq i \leq n$ let

$$\mathcal{A}_{\text{post}}^{(i)} = \{a_{\text{help}} : (p_{i-1}(\varphi) \rightarrow p_i(\psi)) \mid \varphi/\psi \in \text{post}_i)\}.$$

The next component of $\mathcal{A}_{\text{red}}$ formalizes the minimisation of changes on named elements. For $1 \leq i \leq n$ the ABox $\mathcal{A}_{\text{min}}^{(i)}$ contains the following assertions:

- for every $a \in \text{Obj}$ and every primitive concept name A

$$a : \left((A^{(i-1)} \sqcap \bigsqcap_{\varphi/\neg A(a) \in \text{post}_i} \neg p_{(i-1)}(\varphi)) \rightarrow A^{(i)} \right)$$

$$a : \left((\neg A^{(i-1)} \sqcap \bigsqcap_{\varphi/A(a) \in \text{post}_i} \neg p_{(i-1)}(\varphi)) \rightarrow \neg A^{(i)} \right)$$

- for all $a, b \in \text{Obj}$ and every role name $r \in N_R$

$$a : \left((\exists r^{(i-1)}.\{b\} \sqcap \bigsqcap_{\varphi/\neg r(a,b) \in \text{post}_i} \neg p_{(i-1)}(\varphi)) \rightarrow \exists r^{(i)}.\{b\} \right)$$

$$a : \left((\forall r^{(i-1)}.\neg\{b\} \sqcap \bigsqcap_{\varphi/r(a,b) \in \text{post}_i} \neg p_{(i-1)}(\varphi)) \rightarrow \forall r^{(i)}.\neg\{b\} \right)$$

The following component of $\mathcal{A}_{\text{red}}$ ensures the first interpretation $\mathcal{I}_0$ of the encoded sequence is a model of $\mathcal{A}$.

$$\mathcal{A}_{\text{ini}} := \{T_C^{(0)} \mid C(a) \in \mathcal{A}\} \cup \{r^{(0)}(a,b) \mid r(a,b) \in \mathcal{A}\} \cup \{\neg r^{(0)}(a,b) \mid \neg r(a,b) \in \mathcal{A}\}$$

We can now assemble $\mathcal{A}_{\text{red}}$:

$$\mathcal{A}_{\text{red}} := \mathcal{A}_{\text{ini}} \cup \mathcal{A}_{\text{post}}^{(1)} \cup \ldots \cup \mathcal{A}_{\text{post}}^{(n)} \cup \mathcal{A}_{\text{min}}^{(1)} \cup \ldots \cup \mathcal{A}_{\text{min}}^{(n)}$$

Finally we define the reduction assertion:

$$\varphi_{\text{red}} := T_{A_0}^{(n)}(a_0)$$

This completes the preliminaries we need to state the reduction lemma proving Theorem 3.14. But before we discuss the influence of the universal role on our action formalism we want to continue Example 2.10 to support an understanding of the construction of time stamped concepts evolving the named elements only.

Example 3.15. Recall in Example 2.10 we defined a circle of friends, the interpretation $\mathcal{I}$ and the two actions

$$\begin{aligned}
\alpha_1 &:= (\{human(Tom)\}, \{friend(Paul, Anna)/friend(Anna, Tom),\\
&\qquad\qquad friend(Anna, Paul)/friend(Tom, Anna)\})\\
\alpha_2 &:= (\{human(Tom)\}, \{\neg friend(Paul, Karl)/human(Karl)\}).
\end{aligned}$$

In this context Paul, Anna, Selma, Tom, Karl $\in \Delta^{\mathcal{I}}$ are the named elements of $\mathcal{I}$ and the unnamed elements of $\mathcal{I}$ are Henri, Lisa and Toto from $\Delta^{\mathcal{I}}$. Therefore everything that concerns Henri can always be found in the zero-part of the concept and role names. We can compare the evolution of these elements in Figures. 2.2, 2.3 and 2.4 as well.
The concept name N is used to denote the set of individuals that are mapped to named elements by $\mathcal{I}$:

$$N \equiv \{Paul\} \sqcup \{Anna\} \sqcup \{Selma\} \sqcup \{Tom\} \sqcup \{Karl\}$$

The concept names $human^{(0)}, human^{(1)}$ and $human^{(2)}$ evolve from the concept name $human$ by applying α_1 and α_2.

$$\begin{aligned}
human^{(0)} &\equiv \{Paul\} \sqcup \{Anna\} \sqcup \{Selma\} \sqcup \{Tom\} \sqcup \{Henri\}\\
human^{(1)} &\equiv \{Paul\} \sqcup \{Anna\} \sqcup \{Selma\} \sqcup \{Tom\}\\
human^{(2)} &\equiv \{Paul\} \sqcup \{Anna\} \sqcup \{Selma\} \sqcup \{Tom\} \sqcup \{Karl\}
\end{aligned}$$

3.2.2 Considerations on $\mathcal{A}_{\text{red}}$

In the report [BML+05] abbreviations named p_i have been defined by the use of auxiliary roles r_a and an auxiliary individual a_{help} with $a_{\text{help}} \notin \text{Obj}$. Where for each individual name $a \in N_I$ there is an auxiliary role r_a added to N_R, which connects each individual name with a and only with a:

$$\mathcal{A}_{aux} := \{(\exists r_b.\{b\} \sqcap \forall r_b.\{b\})(a) \mid a \in \text{Obj} \cup \{a_{\text{help}}\}, b \in \text{Obj}\}$$

$$p_{i1}(C(a)) := \forall r_a.T_C^{(i)}$$

The auxiliary role names can be replaced by the universal role U, because by definition U connects all individuals with each other.

$$p_i(C(a)) := \exists U.(\{a\} \sqcap T_C^{(i)})$$

Note that with the use of nominals the assertion $r(a,b)$ can be replaced by $(\exists r.\{b\})(a)$ and the assertion $\neg r(a,b)$ by $(\forall r.\neg\{b\})(a)$ with $\{a\}$, $\{b\}$ nominals. So the definition of p_i and p_{i1} covers role assertions and negated role assertions respectively.
The abbreviation p_i is intended to indicate concept or role membership in the ith interpretation, i.e. after the appliance of $\alpha_1, \ldots, \alpha_i$ to a given interpretation $\mathcal{I}$. This is motivated by the following calculations.
Recall that $\mathrm{Obj} \cup \{a_{\mathrm{help}}\} \subseteq N_I$ and therefore $(\mathrm{Obj} \cup \{a_{\mathrm{help}}\})^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$. Further we know $\Delta^{\mathcal{I}} \neq \emptyset$ by definition and $a_{\mathrm{help}} \in \mathrm{Obj} \cup \{a_{\mathrm{help}}\}$, so $\mathrm{Obj} \cup \{a_{\mathrm{help}}\} \neq \emptyset$.

$$\begin{aligned}
p_{i1}(C(a))^{\mathcal{I}} &= \left(\exists U.(\{a\} \sqcap T_C^{(i)})\right)^{\mathcal{I}} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \exists e.(d,e) \in U^{\mathcal{I}} \wedge e \in (\{a\} \sqcap T_C^{(i)})^{\mathcal{I}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \exists e.(d,e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \wedge e = a^{\mathcal{I}} \wedge a^{\mathcal{I}} \in (T_C^{(i)})^{\mathcal{I}})\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid (d, a^{\mathcal{I}}) \in \Delta^{\mathcal{I}} \times \{a\}^{\mathcal{I}} \wedge a^{\mathcal{I}} \in (T_C^{(i)})^{\mathcal{I}})\},
\end{aligned}$$

$$\begin{aligned}
p_i(C(a))^{\mathcal{I}} &= \left(\forall r_a.T_C^{(i)}\right)^{\mathcal{I}} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \forall e.(d,e) \in r_a^{\mathcal{I}} \to e \in (T_C^{(i)})^{\mathcal{I}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \forall e.(d,e) \in (\mathrm{Obj} \cup \{a_{\mathrm{help}}\})^{\mathcal{I}} \times \{a\}^{\mathcal{I}} \to e \in (T_C^{(i)})^{\mathcal{I}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid (d, a^{\mathcal{I}}) \in (\mathrm{Obj} \cup \{a_{\mathrm{help}}\})^{\mathcal{I}} \times \{a\}^{\mathcal{I}} \to a^{\mathcal{I}} \in (T_C^{(i)})^{\mathcal{I}}\}
\end{aligned}$$

This shows p_{i1} and p_i equally indicate membership in the ith interpretation. Let us make note of an equivalence that is a direct consequence of the above calculations.

$$p_{i1}(C(a))^{\mathcal{I}} \neq \emptyset \iff p_i(C(a))^{\mathcal{I}} \neq \emptyset \iff a^{\mathcal{I}} \in (T_C^{(i)})^{\mathcal{I}} \tag{3.7}$$

3.2.3 Reducing Projection with Universal Role to ABox Consequence

The reduction stated in Theorem 3.14 directly results from the construction of the ABox $\mathcal{A}_{\mathrm{red}}$, the TBox $\mathcal{T}_{\mathrm{red}}$ and the following lemma.

Lemma 3.16. *Let $\alpha_1, \ldots, \alpha_n \in \Sigma$, $\mathcal{T}$ TBox, $\mathcal{A}$ ABox and φ an ABox assertion. Further let $\mathcal{A}_{red}$, $\mathcal{T}_{red}$, φ_{red} be constructed from $\mathcal{A}$, $\mathcal{T}$ and φ as given in the previous section. Then the following holds:*

φ is a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ iff $\mathcal{A}_{red}, \mathcal{T}_{red} \models \varphi_{red}$.

We will outline the ideas of the indirect proof of Lemma 15 from [BML+05] and show the reduction holds for all fragments $\mathcal{L}^{\mathcal{U}}$ of $\mathcal{ALCQIO}^{\mathcal{U}}$

Proof. We start giving some abbreviations used in the proof. Let Con denote the set of concept names occurring in the input, Prim the concept names from the input which are primitive in $\mathcal{T}$, Rol the set of role names occurring in the input, $\mathrm{Obj}^{\mathcal{I}}$ for $\mathcal{I}$ interpretation the set $\{a^{\mathcal{I}} \mid a \in \mathrm{Obj}\}$ and Assert the set of assertions occurring in the input.
Further let $\mathcal{D}$ be the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ the effect function w.r.t. Σ, $\mathcal{D}$ and $\mathcal{T}$.

$\Rightarrow$:

Assume $\mathcal{A}_{\mathrm{red}}, \mathcal{T}_{\mathrm{red}} \not\models \varphi_{\mathrm{red}}$. So there exists an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{A}_{\mathrm{red}}$, $\mathcal{J} \models \mathcal{T}_{\mathrm{red}}$ but $\mathcal{J} \not\models \varphi_{\mathrm{red}} = T^{(n)}_{A_0}(a_0)$. It is to show under this assumption the assertion $\varphi = A_0(a_0)$ is not a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$. This is achieved by giving interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ such that for $1 \leq i \leq n$ we have $\mathcal{I}_{i-1} \Rightarrow^{\mathcal{E}}_{\alpha_i} \mathcal{I}_i$ and $\mathcal{I}_n \not\models A_0(a_0)$.

For $0 \leq i \leq n$ the interpretations $\mathcal{I}_i$ are constructed from $\mathcal{J}$:

$$
\begin{aligned}
\Delta^{\mathcal{I}_i} &:= \Delta^{\mathcal{J}}\\
a^{\mathcal{I}_i} &:= a^{\mathcal{J}} \text{ for } a \in \mathrm{Obj}\\
A^{\mathcal{I}_i} &:= (T^{(i)}_A)^{\mathcal{J}} \text{ for } A \in \mathrm{Con}\\
r^{\mathcal{I}_i} &:= \Big((r^{(i)})^{\mathcal{J}} \cap (N^{\mathcal{J}} \times N^{\mathcal{J}})\Big) \cup \Big((r^{(0)})^{\mathcal{J}} \cap (\Delta^{\mathcal{J}} \times (\neg N)^{\mathcal{J}} \cup (\neg N)^{\mathcal{J}} \times \Delta^{\mathcal{J}})\Big)\\
&\quad\ \text{for } r \in \mathrm{Rol}\backslash U\\
U^{\mathcal{I}_i} &:= (U^{(i)})^{\mathcal{J}} = \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} = \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i}
\end{aligned}
$$

According to Definition 2.14 applying an action α transforms a model $\mathcal{I}$ into a model $\mathcal{I}'$ and both models share the same domain. This is important here, because the universal role U is defined to be always the full relation.

The next claim can be found in the proof of Lemma 15, given as Claim 1 in the report [BML+05] It collects some technical equivalences where, based on this claim, it can be shown for all $0 \leq i \leq n$ that $\mathcal{I}_i \in \mathcal{M}(\mathcal{T})$, $\mathcal{I}_0 \models \mathcal{A}$ and $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}_n$ holds for the above defined interpretations.

Claim 3.17. For $0 \leq i \leq n$ the following holds:

1. If $a \in \mathrm{Obj}$, then $a^{\mathcal{I}_i} \in A^{\mathcal{I}_i} \iff a^{\mathcal{J}} \in (A^{(i)})^{\mathcal{J}}$, for all $A \in \mathrm{Prim}$.
   If $x \notin \mathrm{Obj}^{\mathcal{J}}$, then $x \in A^{\mathcal{I}_i} \iff x \in (A^{(0)})^{\mathcal{J}}$, for all $A \in \mathrm{Prim}$.

2. a) If $a, b \in \mathrm{Obj}$ then, for all $r \in \mathrm{Rol}\backslash U$: $(a^{\mathcal{I}_i}, b^{\mathcal{I}_i}) \in r^{\mathcal{I}_i} \iff (a^{\mathcal{J}}, b^{\mathcal{J}}) \in (r^{(i)})^{\mathcal{J}}$
   If $x \notin \mathrm{Obj}^{\mathcal{J}}$ or $y \notin \mathrm{Obj}^{\mathcal{J}}$ then, for all $r \in \mathrm{Rol}\backslash U$:
   $(x, y) \in r^{\mathcal{I}_i} \iff (x, y) \in (r^{(0)})^{\mathcal{J}}$

   b) If $a, b \in \mathrm{Obj}$ then: $(a^{\mathcal{I}_i}, b^{\mathcal{I}_i}) \in U^{\mathcal{I}_i} \iff (a^{\mathcal{J}}, b^{\mathcal{J}}) \in (U^{(i)})^{\mathcal{J}}$
   If $x \notin \mathrm{Obj}^{\mathcal{J}}$ or $y \notin \mathrm{Obj}^{\mathcal{J}}$ then: $(x, y) \in U^{\mathcal{I}_i} \iff (x, y) \in (U^{(0)})^{\mathcal{J}}$

3. $E^{\mathcal{I}_i} = (T_E^{(i)})^{\mathcal{J}}$ for every $E \in \text{Sub}$

4. $\mathcal{I}_i \models \varphi \iff \mathcal{J} \models (p_i(\varphi))(a)$ for all $\varphi \in \text{Assert}$ and $a \in \text{Obj} \cup \{a_{\text{help}}\}$

Proof. We only proof the facts added due to the use of the universal role. For completeness we refer to the report [BML+05]. Note that in comparison to the mentioned Claim 1 the second point is split in two cases. This follows from the necessity to show the interpretations for $r = U$ fall into place.

2.b) Let $a, b \in \text{Obj}$ and $x \notin \text{Obj}^{\mathcal{J}}$ or $y \notin \text{Obj}^{\mathcal{J}}$. Then for $0 \le i \le n$:

$$\begin{array}{rcl} (a^{\mathcal{I}_i}, b^{\mathcal{I}_i}) \in U^{\mathcal{I}_i} = \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} & \iff & (a^{\mathcal{J}}, b^{\mathcal{J}}) \in (U^{(i)})^{\mathcal{J}} = \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \\ (x, y) \in U^{\mathcal{I}_i} = \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} & \iff & (x, y) \in (U^{(0)})^{\mathcal{J}} = \Delta^{\mathcal{I}_0} \times \Delta^{\mathcal{I}_0} \end{array}$$

3. The equality is shown by structural induction on E. So the components (3.3) and (3.4) or (3.5) and (3.6) respectively of T_{Sub} are missing cases.

   First let $E = (\ge m\ U\ C)$. Then for $0 \le i \le n$ we can evaluate:

$$\begin{array}{rcl} (\ge m\ U\ C)^{\mathcal{I}_i} & = & \{a \in \Delta^{\mathcal{I}_i} \mid |\{b \in \Delta^{\mathcal{I}_i} \mid (a, b) \in U^{\mathcal{I}_i} \wedge b \in C^{\mathcal{I}_i}\}| \ge m\} \\ & = & \{a \in \Delta^{\mathcal{J}} \mid |\{b \in \Delta^{\mathcal{J}} \mid (a, b) \in (U^{(i)})^{\mathcal{J}} \wedge b \in (T_C^{(i)})^{\mathcal{J}}\}| \ge m\} \\ & = & \left(\ge m\ U^{(i)}\ T_C^{(i)}\right)^{\mathcal{J}} = \left(T^{(i)}_{(\ge m\ U\ C)}\right)^{\mathcal{J}} \end{array}$$

   This holds, because for all $0 \le i \le n$ we can apply the definition of $U^{(i)}$ and induction on $C^{\mathcal{I}_i}$. Further we know $\Delta^{\mathcal{I}_i} = \Delta^{\mathcal{J}}$ by definition. The last line reflects the definition of an interpretation of the at least number restriction and the last step follows from line (3.3) in the definition of $\mathcal{T}_{Sub}$.

   $E = (\le m\ U\ C)$ can be shown similar to the previous case.

   Let $E = (\forall U.C)$. With the same argumentation as above and line (3.5) we have for $0 \le i \le n$

$$\begin{array}{rcl} (\forall U.C)^{\mathcal{I}_i} & = & \{d \in \Delta^{\mathcal{I}_i} \mid \forall e.(d, e) \in U^{\mathcal{I}_i} \to e \in C^{\mathcal{I}_i}\} \\ & = & \{d \in \Delta^{\mathcal{J}} \mid \forall e.(d, e) \in (U^{(i)})^{\mathcal{J}} \to e \in (T_C^{(i)})^{\mathcal{J}}\} \\ & = & \left(\forall U^{(i)}.T_C^{(i)}\right)^{\mathcal{J}} = \left(T^{(i)}_{\forall U.C}\right)^{\mathcal{J}} \end{array}$$

   $E = (\exists U.C)$ can be shown similar to the previous case.

4. For $\varphi \in \text{Assert}$ and $a \in \text{Obj} \cup \{a_{\text{help}}\}$ we know by definition

$$\mathcal{J} \models (p_i(\varphi))(a) \iff a^{\mathcal{J}} \in (p_i(\varphi))^{\mathcal{J}}.$$

Without loss of generality let $\varphi = C(b)$ for $C \in \mathrm{Con}$ and $b \in \mathrm{Obj} \cup \{a_{\mathrm{help}}\}$, see in Section 3.2.1 the comment on φ_0, then for $0 \leq i \leq n$

$$a^{\mathcal{J}} \in (p_i(C(b)))^{\mathcal{J}} = \begin{cases} \Delta^{\mathcal{J}} & \iff b^{\mathcal{J}} \in (T_C^{(i)})^{\mathcal{J}} \\ \emptyset & \text{else.} \end{cases}$$

This follows from the definition of p_i and the equivalence in (3.7). For $0 \leq i \leq n$ we know $b^{\mathcal{I}_i} = b^{\mathcal{J}} \in (T_C^{(i)})^{\mathcal{J}}$ from the definition of $\mathcal{I}_i$ and $(T_C^{(i)})^{\mathcal{J}} = C^{\mathcal{I}_i}$ by the third point of this claim. Summing up we obtain

$$\mathcal{J} \models (p_i(\varphi))(a) \iff b^{\mathcal{I}_i} \in C^{\mathcal{I}_i} \iff \mathcal{I}_i \models C(b) = \varphi.$$

This completes the proof of the claim. □

With Claim 3.17 we can show the above defined interpretations work as intended. The proof outlines the ideas from the report [BML+05] where also missing details can be found. In contrast to the original version we are using the notion of the effect function. We will pay a special attention to the resulting differences.

$\mathcal{I}_0 \models \mathcal{A}$ follows from $\mathcal{J} \models \mathcal{A}_{\mathrm{ini}}$ and point two and three of Claim 3.17. For $0 \leq i \leq n$ we have $\mathcal{I}_i \in \mathcal{M}(\mathcal{T})$ because $\mathcal{J} \models \{T_A^{(i)} \equiv T_E^{(i)} \mid A \equiv E \in \mathcal{T}, 0 \leq i \leq n\}$ and point three of Claim 3.17.

Let $\alpha_1, \ldots, \alpha_n$ be given for $1 \leq i \leq n$ by $\alpha_i = (\mathrm{post}_i, \mathrm{pre}_i)$. Then for all $1 \leq i \leq n$ and all $\varphi/\psi \in \mathrm{post}_i$ Equation (2.1) and Remark 2.16 give the effect function such that

$$\mathcal{I}_{i-1} \models \varphi \Rightarrow \psi \in \mathcal{E}(\alpha_i, \mathcal{I}_{i-1}).$$

The implication $\mathcal{I}_{i-1} \models \varphi \Rightarrow \mathcal{I}_i \models \psi$ now holds because of point four of Claim 3.17 and $\mathcal{J} \models \mathcal{A}_{\mathrm{post}}$. To prove the necessary minimization of changes in the transition from $\mathcal{I}_{i-1}$ to $\mathcal{I}_i$, let for all $1 \leq i \leq n$ be $\varphi/\psi \in \mathrm{post}_i$ with $\psi \in \{A(a), \neg A(a), r(a,b), \neg r(a,b)\}$ primitive literal and $\mathcal{I}_{i-1} \models \neg\psi$. From $\mathcal{J} \models \mathcal{A}_{\mathrm{min}}^{(i)}$ and Claim 3.17 we get $\mathcal{I}_{i-1} \not\models \varphi \Rightarrow \mathcal{I}_i \models \neg\psi$.
From the atomic construction of the semantics of $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ follows for all $1 \leq i \leq n$ that $\mathcal{I}_{i-1} \Rightarrow_{\alpha_i}^{\mathcal{E}} \mathcal{I}_i$. The $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are unique and so $\mathcal{I}_0 \Rightarrow_{\alpha_1,\ldots,\alpha_n}^{\mathcal{E}} \mathcal{I}_n$.

With the fourth point of Claim 3.17 in mind, the implications

$$\mathcal{J} \not\models T_{A_0}^{(n)}(a_0) \Rightarrow \mathcal{J} \not\models \exists U.\left(\{a_0\} \sqcap T_{A_0}^{(n)}\right) \Rightarrow \mathcal{J} \not\models (p_n(A_0))\,(a_0) \Rightarrow \mathcal{I}_n \not\models A_0(a_0)$$

show, that for an ABox assertion φ, if φ_{red} is not a consequence of $\mathcal{J}$, the constructed interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$ witness φ is not a consequence of applying the composite action $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$.
Together with the previously obtained fact $\mathcal{I}_0 \Rightarrow_{\alpha_1,\ldots,\alpha_n}^{\mathcal{E}} \mathcal{I}_n$ this proves the implication.

$\Leftarrow$:

To proof the other implication let us assume $\varphi = A_0(a_0)$ is not a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$. So there exist interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$ such that $\mathcal{I}_0 \models \mathcal{A}$, $\mathcal{I}_{i-1} \Rightarrow^{\mathcal{E}}_{\alpha_i} \mathcal{I}_i$ for all $1 \leq i \leq n$ and $\mathcal{I}_n \not\models A_0(a_0)$. We have to show $T^{(n)}_{A_0}(a_0)$ is not a consequence of $\mathcal{A}_{\text{red}}$ w.r.t. $\mathcal{T}_{\text{red}}$ by construction of a model $\mathcal{J} \models \mathcal{A}_{\text{red}}, \mathcal{T}_{\text{red}}$ with $\mathcal{J} \not\models \varphi_{\text{red}}$. The proof of this implication starts with a claim about the evolution of the interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$. Thereby a special task will be to reflect the membership of the unnamed elements. A pendant to this proof can be found in the proof of Lemma 15 in the report [BML+05] under the name Claim 2.

Claim 3.18. Let $1 \leq i \leq n$.

1. For all $a \in \text{Obj}$ and $A \in \text{Prim}$ the following holds:

$$\mathcal{I}_{i-1} \models A(a) \wedge \forall \varphi/\neg A(a) \in \text{post}_i,\ \mathcal{I}_{i-1} \not\models \varphi \quad \Rightarrow \mathcal{I}_i \models A(a)$$
$$\mathcal{I}_{i-1} \models \neg A(a) \wedge \forall \varphi/A(a) \in \text{post}_i,\ \mathcal{I}_{i-1} \not\models \varphi \quad \Rightarrow \mathcal{I}_i \models \neg A(a)$$

   For all $a, b \in \text{Obj}$ and $r \in \text{Rol}\backslash U$ the following holds:

$$\mathcal{I}_{i-1} \models r(a,b) \wedge \forall \varphi/\neg r(a,b) \in \text{post}_i,\ \mathcal{I}_{i-1} \not\models \varphi \quad \Rightarrow \mathcal{I}_i \models r(a,b)$$
$$\mathcal{I}_{i-1} \models \neg r(a,b) \wedge \forall \varphi/r(a,b) \in \text{post}_i,\ \mathcal{I}_{i-1} \not\models \varphi \quad \Rightarrow \mathcal{I}_i \models \neg r(a,b)$$

2. $x \notin \text{Obj}^{\mathcal{I}_0} \Rightarrow (x \in A^{\mathcal{I}_i} \iff x \in A^{\mathcal{I}_0})$ for all $A \in \text{Prim}$.
3. a) $x \notin \text{Obj}^{\mathcal{I}_0} \vee y \notin \text{Obj}^{\mathcal{I}_0} \Rightarrow ((x,y) \in r^{\mathcal{I}_i} \iff (x,y) \in r^{\mathcal{I}_0})$ for all $r \in \text{Rol}\backslash U$.
   b) $x \notin \text{Obj}^{\mathcal{I}_0} \vee y \notin \text{Obj}^{\mathcal{I}_0} \Rightarrow ((x,y) \in U^{\mathcal{I}_i} \iff (x,y) \in U^{\mathcal{I}_0})$

Proof. Because ψ is a primitive literal the universal role U is not contained in ψ and therefore we only need to complement the proof from [BML+05] by the case 3.b). So let $x \notin \text{Obj}^{\mathcal{I}_0}$ or $y \notin \text{Obj}^{\mathcal{I}_0}$. Then $(x,y) \in U^{\mathcal{I}_i} \iff (x,y) \in U^{\mathcal{I}_0}$. But this is just

$$(x,y) \in \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \iff (x,y) \in \Delta^{\mathcal{I}_0} \times \Delta^{\mathcal{I}_0}$$

because $\Delta^{\mathcal{I}_i} = \Delta^{\mathcal{I}_0}$ by definition, which completes the proof. □

We construct an interpretation $\mathcal{J}$ from the given interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$ as follows:

$$\begin{array}{rcl}
\Delta^{\mathcal{J}} & := & \Delta^{\mathcal{I}_0}\ (= \Delta^{\mathcal{I}_1} = \ldots = \Delta^{\mathcal{I}_n}) \\
a^{\mathcal{J}} & := & a^{\mathcal{I}_0}\ (= a^{\mathcal{I}_1} = \ldots = a^{\mathcal{I}_n}) \text{ for } a \in \text{Obj} \\
a^{\mathcal{J}}_{\text{help}} & := & d \text{ for an arbitrary but unused } d \in \Delta^{\mathcal{J}} \\
N^{\mathcal{J}} & := & \{a^{\mathcal{J}} \mid a \in \text{Obj}\} \\
(U^{(i)})^{\mathcal{J}} & := & U^{\mathcal{I}_i} = \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \\
(A^{(i)})^{\mathcal{J}} & := & A^{\mathcal{I}_i} \text{ for } A \in \text{Con and } 0 \leq i \leq n \\
(r^{(i)})^{\mathcal{J}} & := & r^{\mathcal{I}_i} \text{ for } r \in \text{Rol and } 0 \leq i \leq n \\
(T^{(i)}_C)^{\mathcal{J}} & := & C^{\mathcal{I}_i} \text{ for all } C \in \text{Sub and } 0 \leq i \leq n
\end{array}$$

For this interpretation $\mathcal{J}$ we are going to proof $\mathcal{J} \models \mathcal{A}_{\text{red}}, \mathcal{T}_{\text{red}}$ and $\mathcal{J} \not\models \varphi_{\text{red}}$.

We start proving an important equality that holds for $\mathcal{J}$ for all $0 \le i \le n$, $\varphi \in$ Assert and $a \in N_I$:

$$\mathcal{I}_i \models \varphi \iff \mathcal{J} \models (p_i(\varphi))(a) \qquad (*)$$

By the definition of the interpretation of assertions we know for all $0 \le i \le n$ that $\mathcal{J} \models (p_i(\varphi))(a) \iff a^{\mathcal{J}} \in (p_i(\varphi))^{\mathcal{J}}$. Without loss of generality let $\varphi = C(b)$ with $C \in$ Con and $b \in \text{Obj} \cup \{a_{\text{help}}\}$ arbitrary. From (3.7) follows for all $0 \le i \le n$

$$a^{\mathcal{J}} \in (p_i(C(b)))^{\mathcal{J}} = \begin{cases} \Delta^{\mathcal{J}} & \iff b^{\mathcal{J}} \in (T_C^{(i)})^{\mathcal{J}} \\ \emptyset & \text{else} \end{cases}$$

By definition we have $b^{\mathcal{I}_i} = b^{\mathcal{J}} \in (T_C^{(i)})^{\mathcal{J}}$ and $(T_C^{(i)})^{\mathcal{J}} = C^{\mathcal{I}_i}$.
This together proofs equality in $(*)$, because:

$$\mathcal{J} \models (p_i(\varphi))(a) \iff b^{\mathcal{J}} \in (T_C^{(i)})^{\mathcal{J}} \iff b^{\mathcal{I}_i} \in C^{\mathcal{I}_i} \iff \mathcal{I}_i \models C(b) = \varphi$$

The proof of $\mathcal{J} \models \mathcal{A}_{\text{red}}$ is split into the three components of $\mathcal{A}_{\text{red}}$.

1. $\mathcal{J} \models \mathcal{A}_{\text{ini}}$ follows from $\mathcal{I}_0 \models \mathcal{A}$ and the definition of $\mathcal{J}$.
2. $\mathcal{J} \models \mathcal{A}_{\text{post}}^{(i)}$: Let $1 \le i \le n$ and $\alpha_i = \{\text{pre}_i, \text{post}_i\}$. For $\mathcal{I}_0, \dots, \mathcal{I}_n$ and all $\varphi/\psi \in \text{post}_i$ we know $\mathcal{I}_{i-1} \models \varphi \Rightarrow \mathcal{I}_i \models \psi$.
   From $(*)$ follows $\mathcal{J} \models (p_{i-1}(\varphi) \rightarrow p_i(\psi))(a_{\text{help}})$.
3. $\mathcal{J} \models \mathcal{A}_{\text{min}}^{(i)}$: For $1 \le i \le n$ the first point of of Claim 3.18 gives the minimization of changes in the transition from $\mathcal{I}_{i-1}$ to $\mathcal{I}_i$. Everything else follows from $(*)$ and the definition of $\mathcal{J}$.

Showing $\mathcal{J} \models \mathcal{T}_{\text{red}}$ is split into the three components of $\mathcal{T}_{\text{red}}$.

1. $\mathcal{J} \models T_N$ from the definition of $\mathcal{J}$.
2. For all $0 \le i \le n$ we have $\mathcal{J} \models \{T_A^{(i)} \equiv T_E^{(i)} \mid A \equiv E \in \mathcal{T}\}$ from the definition of $\mathcal{J}$.
3. $\mathcal{J} \models T_{Sub}$ is shown in [BML+05] by structural induction on $E \in Sub$ and the second and third point of Claim 3.18. We will use the parts proven in [BML+05] and add four cases. First let $E = (\ge m\ U\ C)$. We show for $0 \le i \le n$ the interpretation $\mathcal{J}$ satisfies the concept definitions $T^{(i)}_{(\ge\ m\ U\ C)} \equiv \left(\ge m\ U^{(i)}\ T_C^{(i)}\right)$.

$$\begin{aligned} \left(T^{(i)}_{(\ge\ m\ U\ C)}\right)^{\mathcal{J}} &= (\ge m\ U\ C)^{\mathcal{I}_i} \\ &= \{a \in \Delta^{\mathcal{I}_i} \mid |\{b \in \Delta^{\mathcal{I}_i} \mid (a,b) \in U^{\mathcal{I}_i} \wedge b \in C^{\mathcal{I}_i}\}| \ge m\} \\ &= \{a \in \Delta^{\mathcal{J}} \mid |\{b \in \Delta^{\mathcal{J}} \mid (a,b) \in (U^{(i)})^{\mathcal{J}} \wedge b \in (T_C^{(i)})^{\mathcal{J}}\}| \ge m\} \\ &= \left(\ge m\ U^{(i)}\ T_C^{(i)}\right)^{\mathcal{J}} \end{aligned}$$

Which holds by the definition of $\mathcal{J}$ and induction on $C^{\mathcal{I}_i} = (T_C^{(i)})^{\mathcal{J}}$. The case $E = (\leq m\ U\ C)$ can be shown by similar arguments.

Let $E = (\exists U.C)$. It is to show for $0 \leq i \leq n$ the interpretation $\mathcal{J}$ satisfies the concept definitions $T^{(i)}_{(\exists U.C)} \equiv \left(\exists U^{(i)}.T_C^{(i)}\right)$.

$$\begin{aligned}
\left(T^{(i)}_{(\exists U.C)}\right) &= (\exists U.C)^{\mathcal{I}_i} \\
&= \{d \in \Delta^{\mathcal{I}_i} \mid \exists e.(d,e) \in U^{\mathcal{I}_i} \wedge e \in C^{\mathcal{I}_i}\} \\
&= \{d \in \Delta^{\mathcal{J}} \mid \exists e.(d,e) \in (U^{(i)})^{\mathcal{J}} \wedge e \in (T_C^{(i)})^{\mathcal{J}}\} \\
&= \left(\exists U^{(i)}.T_C^{(i)}\right)^{\mathcal{J}}
\end{aligned}$$

This holds by the definition of $\mathcal{J}$ and induction on $C^{\mathcal{I}_i} = (T_C^{(i)})^{\mathcal{J}}$. Again we can show the case $E = (\forall U.C)$ by similar arguments.

For $\varphi = A_0(a_0)$ we obtain $(A_0)^{\mathcal{I}_n} = (T_{A_0}^{(n)})^{\mathcal{J}}$ from the definition of $\mathcal{J}$. Together with $(*)$ we get for an arbitrary $a \in N_I$:

$$\mathcal{I}_n \not\models A_0(a_0) \Rightarrow \mathcal{J} \not\models (p_i(A_0(a_0)))\,(a) \Rightarrow a_0^{\mathcal{J}} \notin T_{A_0}^{(n)} \Rightarrow \mathcal{J} \not\models T_{A_0}^{(n)}(a_0)$$

for $a \in N_I$. This completes the proof of this implication and finally yields Lemma 3.16. $\square$

If we assume unary coding of numbers in qualified number restriction the size of $\mathcal{A}_{\text{red}}$, $\mathcal{T}_{\text{red}}$ and φ_{red} is clearly polynomial in the size of the input. So Lemma 3.16 immediately gives Theorem 3.14.

Remark 3.19. If we assume binary coding the resulting upper bounds will increase by one exponential, so the size of the knowledge base $(\mathcal{A}_{\text{red}}, \mathcal{T}_{\text{red}})$ is not anymore polynomial in the size of the input. Note that a concept $(\leq\ m\ r\ C)$ occuring in a concept definition in $\mathcal{T}$ is in unary coding assumed to be of the size $(|m| + 1 + |C|)$ and in binary coding of the size $(\log(m) + 1 + |C|)$. Therefore $\mathcal{T}_{\text{red}}$ is polynomial in the size of $\mathcal{T}$ if numbers are coded in unary but increases by one exponential in case of binary coding.

3.3 Complexity Results for Executability and Projection

After discussing the complexity of ABox consequence in presence of a universal role in the first part of this chapter and proving the reduction theorem in the second part we can bring this thoughts together to obtain the following complexity bounds.

Corollary 3.20. *Projection and executability of actions w.r.t. acyclic TBoxes are*

1. *ExpTime-complete for $\mathcal{ALC}^{\mathcal{U}}$, $\mathcal{ALCO}^{\mathcal{U}}$, $\mathcal{ALCI}^{\mathcal{U}}$ and $\mathcal{ALCIO}^{\mathcal{U}}$;*
2. *ExpTime-complete for $\mathcal{ALCQ}^{\mathcal{U}}$ and $\mathcal{ALCQO}^{\mathcal{U}}$ if the universal role U is not occurring in qualified number restriction;*

3. *in co-NExpTime for $\mathcal{ALCQ}^{\mathcal{U}}$ and $\mathcal{ALCQO}^{\mathcal{U}}$ if the universal role U is allowed in qualified number restriction;*

4. *in co-NExpTime for $\mathcal{ALCQI}^{\mathcal{U}}$;*

5. *co-NExpTime-complete for $\mathcal{ALCQIO}^{\mathcal{U}}$.*

where 2,3,4 and 5 presuppose that numbers in number restrictions are coded in unary.

Proof. This complexity results follow from Theorem 3.14, which gives the reduction from projection to ABox consequence and the three lemmata Lemma 3.4, Lemma 3.5 and Lemma 3.6, which prove the complexity of ABox consequence in the different description logics. □

Corollary 3.20 brings together all thoughts, presented in this chapter. It is therefore a good last point for our considerations on description logics admitting a universal role.

3.4 Spin-off: Observations on Actions with Effect Function

When writing the prove of Lemma 3.16 we came to the conclusion one of the most important next steps now is to develop the process of abstraction further. Through this chapter we integrated the most simple global property, namely the universal role to the description logics action formalism. But we also had to note that the more important and more involved global role statement is transitivity. Or on the side of concepts to investigate the embedding of abstract TBoxes. In their report [BML+05] the authors argue, adding even transitivity exceeds the possibilities of the action formalism so far. Therefore we want to look again at Theorem 3.14 and the literature we used to introduce abstract actions. The there discussed approach goes beyond the ideas we used in the proof of Theorem 3.14, but the introduction of an effect function assures the determinism we need to cross the limits given by simply avoiding the ramification problem. We presented the action formalism by introducing a so-called effect function. Recall that this function presented in Definition 2.13 maps tuples of action names and interpretations satisfying the TBox to a set of literals. It has its origin in the article [BZ13a] where it is introduced to construct a more abstract action calculus. The there developed ideas, based also on the article [BLL10b], point the way towards a more powerful action formalism, but in our proof of Theorem 3.14 the effect function only played a minor role. It is used to form a new language but we are still talking about the same circumstances as [BLM+05] and we used only half the possibilities contained in this approach.
The authors of [BZ13a] start from the assumption that actions are given by abstract names. Their effect, which has previously been defined by the pre- and post-conditions, is then completely determined by a function. Such a function of course has to fulfil several requirements we will take a look at later. The reduction ABox can be indexed by subsets

of *Lit* which are the literals contained in the set of relevant assertions. This is done by regarding so-called updated interpretations, which illustrate the interplay between actions and their interpretation by a function mapping tuples of actions and interpretations to a non-contradictory subset of *Lit*.

In the following we take a closer look at this abstraction and proof some small details we did not find explicitly in the literature. First we introduce some notions and give the definition of two kinds of types of interpretations from [BZ13a]. Then we show some properties of these types, which motivate their influence on the set of interpretations, that hold after applying an action. Recall that the authors of [BZ13a] consider only consistent actions. To be concurrent at this point, we refer to Remark 2.23 and adopt this assumption.

Definition 3.21. Let $E \subseteq Lit$ a non-contradictory set of literals. From the set E the *updated interpretation* $\mathcal{I}^E$ can be constructed:

$$
\begin{aligned}
A^{\mathcal{I}^E} &:= \Big(A^{\mathcal{I}} \cup \{a^{\mathcal{I}} \mid A(a) \in E\}\Big) \setminus \{a^{\mathcal{I}} \mid \neg A(a) \in E\} \text{ for all A } \in N_C \\
r^{\mathcal{I}^E} &:= \Big(r^{\mathcal{I}} \cup \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid r(a,b) \in E\}\Big) \setminus \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid \neg r(a,b) \in E\} \text{ for all } r \in N_R
\end{aligned}
$$

Let $\neg E := \{\neg L \mid L \in E\}$ where double negation is eliminated in the sense of Remark 2.9.

Defining abstract actions unravels the satisfiability of the pre- and post-conditions into the construction of the effect function. Now by updated interpretation this idea is again abstracted by looking at the behaviour of arbitrary but non-contradictory subsets $E \subseteq Lit$. As observed in the article [BZ13a] this gives rise to another simplification. By regarding updated interpretations we can merge composite actions into a single action.

Definition 3.22. Let $\mathcal{T}$, $\mathcal{A}$ TBox and ABox and $\alpha_1, \ldots, \alpha_n \in \Sigma$. For $\mathcal{I}_0 \in \mathcal{M}(\mathcal{T})$ and $1 \leq i \leq n$ let the interpretation $\mathcal{I}_i$ be defined by $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_i} \mathcal{I}_i$. Then for $1 < i \leq n$

$$
\begin{aligned}
E_1 &:= \mathcal{E}(\alpha_1, \mathcal{I}_0) \\
E_i &:= E_{i-1} \setminus \neg\mathcal{E}(\alpha_i, \mathcal{I}_{i-1}) \cup \mathcal{E}(\alpha_i, \mathcal{I}_{i-1})
\end{aligned}
$$

Remark 3.23. Note that we have for all $1 \leq i \leq n$ the identity $\mathcal{I}_0^{E_i} = \mathcal{I}_i$ and especially $\mathcal{I}_0^{E_n} = \mathcal{I}_n$. Because from page 16 of the report [BZ13b] we know the following identity for updated interpretations:

$$
(\mathcal{I}^E)^{E'} = \mathcal{I}^{E \setminus \neg E' \cup E'}
$$

Where the set E_n is not the result of applying α_n to $\mathcal{I}_{n-1}$ but concentrates the process of applying a sequence of actions in one subset of *Lit*.

For a given $\alpha \in \Sigma$, we are actually interested in the non-contradictory sets $E \subseteq Lit$ where $E = \mathcal{E}(\alpha, \mathcal{I})$ for an $\mathcal{I} \in \mathcal{M}(\mathcal{T})$. In the report [BZ13b] all non-contradictory subsets of *Lit* are considered, regardless of whether the updated interpretations $\mathcal{I}^E$ are models of the TBox. Then these sets are sorted, where the sets for which there does not exists an $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ such that $E = (\alpha, \mathcal{I})$ are to be sorted out. How this can be done lies in the answer to the following question: If we fix an $\alpha \in \Sigma$, how does the set $\{\mathcal{E}(\alpha, \mathcal{I}) \mid \mathcal{I} \in \mathcal{M}(\mathcal{T})\}$

look like, and how can we wisely limit the set of interpretations we have to consider? In their slides from the talk coming with the paper [BZ13a] Franz Baader Benjamin and Zarrieß mention that the introduced types give rise to an equivalence relation on the set of models for a TBox. For interpretations $\mathcal{I}, \mathcal{I}' \in \mathcal{M}(\mathcal{T})$ we need a relation, which yields $\mathcal{I} \cong \mathcal{I}'$ if and only if $\mathcal{I}$ and $\mathcal{I}'$ have the same to defined type.
An action theory is given by an ABox and TBox $\mathcal{A}$, $\mathcal{T}$, the set of relevant assertions $\mathcal{D}$, a finite set of actions Σ and the effect function $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$. On the set of interpretations for a given TBox the following definition presents two kinds of types for an interpretation.

Definition 3.24. Let $\mathcal{T}$, $\mathcal{A}$ be an TBox and ABox, $\mathcal{I} \in \mathcal{M}(\mathcal{T}) \cap \mathcal{M}(\mathcal{A})$ an interpretation, $\mathcal{D}$ be the set of relevant assertions and $Lit \subseteq \mathcal{D}$ the set of literals occurring in $\mathcal{D}$. Then the *static type* and the *dynamic type* are defined as follows:

$$\begin{aligned} s - type(\mathcal{I}) &:= \{\varphi \in \mathcal{D} \mid \mathcal{I} \models \varphi\} \\ d - type(\mathcal{I}) &:= \{(\varphi, E) \mid \varphi \in \mathcal{D}, E \subseteq Lit \text{ non-contradictory}, \mathcal{I}^E \models \varphi\} \end{aligned}$$

where $\mathcal{I}^E$ is the updated interpretation from Definition 3.21.

At first we want to consider the relation on the set of interpretations induced by the static type.

Definition 3.25. Let the relation $\cong_s \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ be defined as follows:

$$\mathcal{I}_1 \cong_s \mathcal{I}_2 :\iff s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$$

Lemma 3.26. *The relation $\cong_s \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ is an equivalence relation.*

Proof. Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \mathcal{M}(\mathcal{T})$. Then $\cong_s$ is reflexive ($\mathcal{I} \cong_s \mathcal{I}$) because $s - type(\mathcal{I})$ is a fixed set, symmetric ($\mathcal{I}_1 \cong_s \mathcal{I}_2 \Rightarrow \mathcal{I}_2 \cong_s \mathcal{I}_1$) and transitive ($\mathcal{I}_1 \cong_s \mathcal{I}_2$, $\mathcal{I}_2 \cong_s \mathcal{I}_3 \Rightarrow \mathcal{I}_1 \cong_s \mathcal{I}_3$) because $\cong_s$ is defined by equality of the static types. □

As mentioned in the previous chapter, developing an action formalism can also start from the construction of the sets $\mathcal{D}$ and Lit, the function $\mathcal{E}$ and abstracting actions. Using the notion of static types the authors of the report [BZ13b] formulate some requirements the effect function has to meet to generate a solid action formalism.

Definition 3.27. A DL based action formalism given by $\mathcal{A}, \mathcal{T}, \Sigma, \mathcal{D}, \mathcal{E}$ is called *admissible*, if all actions contained in Σ are consistent and the following conditions hold.

A1 If $s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$, then $\alpha \in \Sigma$ is applicable to $\mathcal{I}_1$ if and only if α is applicable to $\mathcal{I}_2$.

A2 If $s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$ and $\alpha \in \Sigma$ is applicable to $\mathcal{I}_1$ then $\mathcal{E}(\alpha, \mathcal{I}_1) = \mathcal{E}(\alpha, \mathcal{I}_2)$.

If the conditions A1 and A2 are satisfied for our action formalism then for $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ arbitrary and $t := s - type(\mathcal{I})$ a static type the set $\mathcal{E}(\alpha, t) := \mathcal{E}(\alpha, \mathcal{I})$ is well defined.

A3 For a given static type $t := d - type(\mathcal{I})$ it is decidable whether $\mathcal{E}(\alpha, t)$ is defined and can be effectively computed.

We show these requirements are reasonable by proving the action formalism introduced in Section 2.2 is admissible.

Lemma 3.28. *Let $\mathcal{T}$ be an acyclic TBox, $\mathcal{A}$ ABox, Σ a finite set of consistent actions where all $\alpha \in \Sigma$ are concrete actions, i.e. given by a set of pre- and post-conditions. Further let the set $\mathcal{D}$ be closed under negation and contain $\mathcal{A}$, the set of ABox assertions in the pre-conditions and all assertions contained in the post-conditions. Let the effect function $\mathcal{E} : \Sigma \times \mathcal{M}(\mathcal{T}) \to 2^{Lit}$ be defined by 2.1. Then the action formalism given by $\mathcal{A}, \mathcal{T}, \Sigma, \mathcal{D}, \mathcal{E}$ is admissible.*

Proof. Let $\alpha = (\text{pre}, \text{post}) \in \Sigma$ be a concrete consistent action and $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2 \in \mathcal{M}(\mathcal{T})$.

A1: Let $s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$. By definition of the static type we know that $\mathcal{I}_1 \in \mathcal{M}(\mathcal{A}) \iff \mathcal{I}_2 \in \mathcal{M}(\mathcal{A})$ and $\mathcal{I}_1 \models \text{pre} \iff \mathcal{I}_2 \models \text{pre}$. Further we know by definition the effect function $\mathcal{E}$ is defined for $(\alpha, \mathcal{I}_1)$ and $(\alpha, \mathcal{I}_2)$ if and only if $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{M}(\mathcal{A})$. Summing up we conclude the action α is applicable to $\mathcal{I}_1$ if and only if the effect function $\mathcal{E}$ is defined for $(\alpha, \mathcal{I}_1)$ and $\mathcal{I}_1 \models \text{pre}$. This is the case if and only if the function $\mathcal{E}$ is defined for $(\alpha, \mathcal{I}_2)$ and $\mathcal{I}_2 \models \text{pre}$, which means α is applicable to $\mathcal{I}_2$.

A2: Let $s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$ and α be applicable to $\mathcal{I}_1$. For all $\varphi \in \mathcal{D}$ we know by the equality of the static types $\mathcal{I}_1 \models \varphi \iff \mathcal{I}_2 \models \varphi$. We conclude from the construction of the set $\mathcal{D}$:

$$\begin{aligned} \mathcal{E}(\alpha, \mathcal{I}_1) &= \{\psi \mid \varphi/\psi \in \text{post} \wedge \mathcal{I}_1 \models \varphi\} \\ &= \{\psi \mid \varphi/\psi \in \text{post} \wedge \mathcal{I}_2 \models \varphi\} \\ &= \mathcal{E}(\alpha, \mathcal{I}_2) \end{aligned}$$

A3: The ABox $\mathcal{A}$ is finite, the set Σ is finite and every $\alpha \in \Sigma$ is given by a finite set of pre- and post-conditions. So the set $\mathcal{D}$ is finite and every non-contrary $t \subseteq \mathcal{D}$ that contains $\mathcal{A}$ and the set of assertions pre is a possible static type for an $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$. For all such t, we can decide whether an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ with $\forall_{\varphi \in t} \mathcal{I} \models \varphi$ exists. If this is the case we can compute such an interpretation and the set $\mathcal{E}(\alpha, t)$. $\square$

Because for an admissible action formalism the effect function does not distinguish between interpretations of the same static type, we can give a first answer to the question about the properties of the set of effects of an action.

$$\{\mathcal{E}(\alpha, \mathcal{I}) \mid \mathcal{I} \in \mathcal{M}(\mathcal{T})\} = \{\mathcal{E}(\alpha, t) \mid t = s - type(\mathcal{I}) \text{ for an } \mathcal{I} \in \mathcal{M}(\mathcal{T})\} \tag{3.8}$$

Can we just look at the different static types we can construct over a well formed set $\mathcal{D}$? No because as Franz Baader and Benjamin Zarrieß observe we need to distinguish between interpretations that are mapped by an action to different types of interpretations. Look at Example 20 from [BZ13b]. It gives an admissible action formalism and two interpretations of the same static type. But the interpretations they are transformed to, are of different static type. What we really need is a relation that satisfies all the properties of $\cong_s$ and that is further compatible with an action.

Remark 3.29. The ordered pair $\langle \mathcal{M}(\mathcal{T}), \alpha \rangle$ with $\alpha : \mathcal{M}(\mathcal{T}) \to \mathcal{M}(\mathcal{T})$ is a unary algebra.

Note that we are only slightly double using notation. On the one hand α is nothing more that an action name and the other α is a function taking one interpretation to another. For $\alpha \in \Sigma$ the assignment $\alpha : \mathcal{M}(\mathcal{T}) \to \mathcal{M}(\mathcal{T})$, where $\alpha : \mathcal{I} \to \mathcal{I}' \iff \mathcal{I} \Rightarrow_\alpha^\mathcal{E} \mathcal{I}'$ is defined for all $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ and further for all $\mathcal{I} \in \mathcal{M}(\mathcal{T})$ there is a unique $\mathcal{I}' \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I} \Rightarrow_\alpha^\mathcal{E} \mathcal{I}'$ if and only if we assume all actions to be consistent.

Due to the simplicity of $\langle \mathcal{M}(\mathcal{T}), \alpha \rangle$ one could argue seeing an algebra is overdone. But only with this perspective we can properly refine $\cong_s$ to an equivalence relation on $\mathcal{M}(\mathcal{T})$ that is compatible with an action.

Definition 3.30. Let $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{M}(\mathcal{T})$. The relation $\cong_d \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ is defined as follows:

$$\mathcal{I}_1 \cong_d \mathcal{I}_2 :\iff d - type(\mathcal{I}_1) = d - type(\mathcal{I}_2)$$

Lemma 3.31. *The relation $\cong_d \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ is a congruence relation on $\langle \mathcal{M}(\mathcal{T}), \alpha \rangle$.*

Proof. Let $\mathcal{I}_0, \mathcal{I}_1, \mathcal{J}_0, \mathcal{J}_1 \in \mathcal{M}(\mathcal{T})$, $\alpha \in \Sigma$. The proof $\cong_d$ is an equivalence relation on the set $\mathcal{M}(\mathcal{T})$ is analogous to the proof of Lemma 3.26.

From Lemma 22 in the report [BZ13b] we can conclude, that $\cong_d \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ is also a congruence relation. Because this Lemma states in case $\mathcal{I}_0 \Rightarrow_\alpha^\mathcal{E} \mathcal{I}_1$ and $\mathcal{J}_0 \Rightarrow_\alpha^\mathcal{E} \mathcal{J}_1$ the implication $d - type(\mathcal{I}_0) = d - type(\mathcal{J}_0) \Rightarrow d - type(\mathcal{I}_1) = d - type(\mathcal{J}_1)$ holds. ☐

The next lemma proves that the congruence relation $\cong_d$ really is a refinement of the relation $\cong_s$ as recommended.

Lemma 3.32. *Let $\cong_d, \cong_s \in \mathcal{M}(\mathcal{T}) \times \mathcal{M}(\mathcal{T})$ be the equivalence relations induced by the static and dynamic types. Then the following holds:*

$$\cong_d \subseteq \cong_s \tag{3.9}$$

Proof. The authors of the report [BZ13b] conclude from the equivalence

$$(\varphi, \emptyset) \in d - type(\mathcal{I}) \iff \mathcal{I} \models \varphi \iff \varphi \in s - type(\mathcal{I})$$

that

$$d - type(\mathcal{I}_1) = d - type(\mathcal{I}_2) \Rightarrow s - type(\mathcal{I}_1) = s - type(\mathcal{I}_2)$$

We conclude $\mathcal{I}_1 \cong_d \mathcal{I}_2 \Rightarrow \mathcal{I}_1 \cong_s \mathcal{I}_2$ and therefore $\mathcal{I}_1 \times \mathcal{I}_2 \in \cong_d \Rightarrow \mathcal{I}_1 \times \mathcal{I}_2 \in \cong_s$. ☐

Remark 3.33. From (3.8) and (3.9) we obtain for an admissible action formalism

$$\{\mathcal{E}(\alpha, \mathcal{I}) \mid \mathcal{I} \in \mathcal{M}(\mathcal{T})\} = \{\mathcal{E}(\alpha, \mathcal{I}) \mid [\mathcal{I}]_{\cong_d} \in (\mathcal{M}(\mathcal{I}) / \cong_d)\}$$

where every $[\mathcal{I}]_{\cong_d} \in (\mathcal{M}(\mathcal{I})/\cong_d)$ represents a different dynamic type of an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$. Lemma 23 from the report [BZ13b] implies, that we can compute effectively for every pair (φ, E) with $\varphi \in \mathcal{D}$ an assertion and $E \subset \mathit{Lit}$ a non-contradictory set of literals, whether it is a dynamic type for an interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T})$. Thus for finite $\mathcal{D}$ the quotient algebra $(\langle \mathcal{M}(\mathcal{T}), \alpha \rangle / \cong_d)$ and therefore also the set $\{\mathcal{E}(\alpha, \mathcal{I}) \mid \mathcal{I} \in \mathcal{M}(\mathcal{T})\}$ is finite and computable.

With this background we can now start to investigate for description logics with various properties what is needed to form an admissible action formalism. For a given assertion φ, an interesting knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and a composite action α let $\varphi^{(E)}$, $\mathcal{A}_{\text{red}}^{(E)} := A_{\text{eff}}^{(E)}$ be as defined in the report [BZ13b] and $\mathcal{T}_{\text{red}}^{(E)} = \mathcal{T}_{\text{red}} \cup \{T_C^{(E)} \sqsubseteq T_D^{(E)} \mid E \sqsubseteq D \in \mathcal{T}\}$ where we understand $\mathcal{T}_{\text{red}}$ as introduced in [BZ13b]. The next step will be to prove, based on the previously obtained knowledge on types and updated interpretations, if the following holds for the knowledge base $\mathcal{K}$: The assertion φ is a consequence of applying $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if and only if for all $E \in \{\mathcal{E}(\alpha, \mathcal{I}) \mid [\mathcal{I}]_{\cong_d} \in (\mathcal{M}(\mathcal{I})/\cong_d)\}$ it is true for all $\mathcal{J} \in \mathcal{M}\left(\mathcal{T}_{\text{red}}^{(E)}\right) \cap \mathcal{M}\left(\mathcal{A}_{\text{red}}^{(E)}\right)$ that $\mathcal{J} \models \varphi^{(E)}$.
We would like to let these reflections rest for the time being and close the observations on the effect function at this point.

Let us summarize this chapter. At the beginning we pursued a line of thoughts how a universal role and description logics work together. We investigated the complexity of ABox consequence for the fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$ in presence of nominals and the universal role. Furthermore we discussed the pros and cons of allowing the universal role in qualified number restriction and investigated the correspondence between the universal role and CBoxes in this context.
In the second part of this chapter we deduced complexity results for the action inference problem projection in the fragments of $\mathcal{ALCQIO}^{\mathcal{U}}$ which contain a universal role. This was based on the previously obtained bounds for ABox consequence and the construction of a reduction ABox and a reduction TBox, mainly inspired by [BML+05]. As an idea that appeared out of the thoughts on this section we presented some remarks about the direction the research of the description logics based action formalism is going at the moment. Some observations on this powerful new ideas, that might give a whole new drive to the action formalism closed this chapter.

4 Deciding Projection with Boolean Conjunctive Queries

Through the last chapter we have been dealing with a simple ABox assertion φ as a consequence of an ABox or as a consequence of applying actions $\alpha_1, \ldots, \alpha_n$ in an ABox w.r.t. a TBox respectively. If we want to consider more complex assertions, enriched by restricted existential quantification and the use of variables this leads to queries. This idea is not only of theoretic interest but comes from database querying. Formerly a database has usually been a relational structure where much effort has been put in a constructive design. This closed system could then be considered to entail a query. Lately, mainly triggered by growing use and range of the internet, it has turned into focus to consider also possibly distributed unsorted data source. This data storage can be managed by the so-called Ontology-Based Data Access, or in short OBDA. This approach enables us to query a data source w.r.t. an ontology. A good introduction how this works with a description logics TBox as ontology is given in [Cal12]. OBDA generalizes query answering in relational databases and Figure 4.1 gives an impression of how description logics work as an ontology. The ABox is understood to organize the data and the TBox to link it with

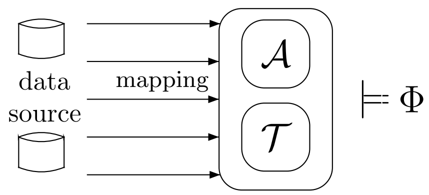

Figure 4.1: Ontology-based data access.

each other.

In the previous chapter the projection problem for ABox assertions has been reduced to a well studied problem for description logics, namely ABox consequence. Considering the description logics action formalism and reasoning problems related to query answering an attempt to reduce the projection problem to query entailment is self-evident. In this chapter we want to adopt the previously obtained results and the idea of the construction of a reduction ABox and TBox. Based on the techniques developed in the report [BML+05] we want to decide whether a query holds in an ABox w.r.t. a TBox after applying a sequence of actions.

4.1 Introducing Queries over Description Logics Knowledge Bases

The concept of a query is very powerful because queries contain variables in a precisely defined syntax. They enable us to express the situation that there exists an element which is connected to an individual $a \in N_I$ by a relation $r \in N_R$ without naming it. This not further specified element can be represented by a variable.
The chapter starts with the definition of the syntactic and semantic concept of queries in terms of description logics. To distinguish queries from assertions we will denote them by capital Greek letters. In the following the idea of a homomorphism from a query to an interpretation and how it is used to fully characterize the semantics of queries will be given. Well prepared with this formal introduction we bring together the description logics based action formalism and queries. The question if a query is true after applying a finite sequence of actions will be called projection problem. Finally our considerations will converge in a version of Theorem 3.14 that concerns queries.
Bringing together description logics and queries of course doesn't appear out of nowhere. The following definitions and results are based on the paper [CDGL+09] where database access, by use of a popular family of description logics called $DL - Lite$, is introduced. Especially the idea of a homomorphism is presented in Definition 2.5, Theorem 2.6 and Example 2.7 at great length. A short but precise introduction into conjunctive queries, the semantics given by a match for an interpretation and a query and also the query entailment problem can be found in the article [GLHS08]. Where this article treats conjunctive query answering for the description logic $\mathcal{SHIQ}$. Last but not least the article [BBL13] about temporal query answering in $DL-Lite$ has influenced this thesis. The detailed introduction and examples have been of great benefit to understand Boolean conjunctive queries in context of description logics.

4.1.1 Syntax and Semantics of Conjunctive Queries

Under various conditions query answering becomes undecidable. In this thesis therefore we want to consider a common restriction on queries, namely Boolean conjunctive queries. Generally speaking a query for description logics is a first order formula $q = \varphi(v_1, \ldots, v_k)$ built from variables and individual names. The unary and binary predicates come from N_C and N_R respectively and the free variables of the first order formula φ are among the variables $v_1, \ldots, v_k \subseteq N_V$ which are called answer variables. The formal definition of a Boolean conjunctive query will be presented in the following definition.

Definition 4.1. Let N_V be a countably infinite set of variables, N_I, N_R and N_C be countably infinite sets of individual names, role names and concept names. The sets N_V, N_I, N_R and N_C are supposed to be disjoint.

A *conjunctive query* is a restriction of a first order query $\Phi = \exists u_1, \ldots, u_n.\psi$, where $u_1, \ldots, u_n \subseteq N_V$ are distinct variables and ψ is a possibly empty conjunction of atoms of the form

- $A(t)$ with $A \in N_C$ and $t \in N_V \cup N_I$ a concept atom or
- $r(s,t)$ with $r \in N_R$ and $s, t \in N_V \cup N_I$ a role atom.

The free variables of a conjunctive query are among the answer variables. The number of free variables is called *arity*. Conjunctive queries of arity 0, i.e. conjunctive queries without free occurring variables, are called *Boolean conjunctive queries* and abbreviated by BCQ.

We denote the set of variables occurring in a query Φ by $Var(\Phi)$ and the set of individual names occurring in Φ by $Ind(\Phi)$. By $\alpha \in \Phi$ we denote the occurrence of the concept or role atom α in Φ.

The next definition gives the semantics of conjunctive queries. This is done by introducing a partial function from the individuals and variables of a query to a non-empty set which will be called a match for an interpretation. Some authors give the semantics of queries using the notion of a homomorphism. We present the technical approach first and introduce homomorphisms as an equivalent characterisation later.

Definition 4.2. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation and $\pi : N_V \cup N_I \to \Delta^{\mathcal{I}}$ a partial function such that $\pi(a) = a^{\mathcal{I}}$ for all $a \in dom(\pi) \cap N_I$. The relation $\mathcal{I} \models^{\pi} \psi$ for a quantifier free first-order formula ψ with $Var(\psi) \subseteq dom(\pi)$ is inductively defined:

$$\begin{aligned}
\mathcal{I} \models^{\pi} A(t) &\iff \pi(t) \in A^{\mathcal{I}} \\
\mathcal{I} \models^{\pi} r(t_1, t_2) &\iff (\pi(t_1), \pi(t_2)) \in r^{\mathcal{I}} \\
\mathcal{I} \models^{\pi} \psi_1 \wedge \psi_2 &\iff \mathcal{I} \models^{\pi} \psi_1 \wedge \mathcal{I} \models^{\pi} \psi_2
\end{aligned}$$

Let $\Phi = \exists u_1, \ldots, u_n.\psi$ be a conjunctive query. A *match* for an interpretation $\mathcal{I}$ and Φ is a mapping $\pi : Var(\Phi) \cup Ind(\Phi) \to \Delta^{\mathcal{I}}$ such that $\pi(a) = a^{\mathcal{I}}$ for all $a \in Ind(\Phi)$ and $\mathcal{I} \models^{\pi} \psi$. Let $\{v_1, \ldots, v_k\} \subseteq N_V$ be the set of all free variables occurring in Φ. If we have $\pi(v_i) = a_i^{\mathcal{I}}$ for $1 \leq i \leq k$, then π is called an $(a_1, \ldots, a_k)$-match for $\mathcal{I}$ and Φ.
If a match for $\mathcal{I}$ and Φ exists, we say $\mathcal{I}$ satisfies Φ and write $\mathcal{I} \models \Phi$, if we want to give a specific match we write $\mathcal{I} \models \Phi[a_1, \ldots, a_k]$.

A *certain answer* for a k-ary conjunctive query $\Phi(v_1, \ldots, v_k)$ and a knowledge base $\mathcal{K}$ is a tuple $(a_1, \ldots, a_k) \subseteq N_I$ such that $a_1, \ldots, a_k$ occur in $\mathcal{K}$ and $\mathcal{I} \models \Phi[a_1, \ldots, a_k]$. The answer to a Boolean query is either the empty tuple $(,)$ considered as *true* or the empty set $\emptyset$ considered as $false$.

The role constructor U has been introduced to talk about unnamed elements. Boolean conjunctive queries now enlarge our possibilities to get general information about unnamed elements by introducing variables.

Remark 4.3. As observed in the article [BBL13] Boolean conjunctive queries extend what can be expressed by an $\mathcal{ALCQIO}$ assertion. An arbitrary ABox assertion can without loss of generality be given by $\varphi = A_0(a)$ with $A_0 \in N_C$. Such an assertion φ can be expressed by the Boolean conjunctive query $\exists u.A_0(u)$. On the other hand the Boolean conjunctive query

$\Phi = \exists y.r(y, y)$, which says there exists a loop in the model without naming the individual which has the loop, can neither be expressed in $\mathcal{ALCQIO}$ nor with the additional use of a universal role.

4.1.2 A Homomorphism of a Query to an Interpretation

The semantics of queries can also be defined using the notion of a homomorphism. The identity of the two approaches is a remarkable result we will give following the definition of a homomorphism from a Boolean conjunctive query to an interpretation. This definition is based upon the one given in [BBL13], whereas an approach to extend the definition of a homomorphism to general conjunctive queries can be found in [CDGL+09].

Definition 4.4. Let $\mathcal{I} = (\Delta^\mathcal{I}, .^\mathcal{I})$ be an interpretation and $\Phi = \exists u_1, \ldots, u_n.\psi(u_1, \ldots, u_n)$ a Boolean conjunctive query. A mapping $\mu : Var(\Phi) \cup Ind(\Phi) \to \Delta^\mathcal{I}$ is a homomorphism from Φ to $\mathcal{I}$ if

$$\begin{aligned}
&\mu(a) = a^\mathcal{I} \text{ for all } a \in Ind(\Phi)\\
&\mu(s) \in A^\mathcal{I} \text{ for all } A \in N_C \text{ and } A(s) \in \Phi\\
&(\mu(s), \mu(t)) \in r^\mathcal{I} \text{ for all } r \in N_R \text{ and } r(s,t) \in \Phi
\end{aligned}$$

The characterisation of the semantics of queries by a homomorphism has originally been stated by Ashok K. Chandra and Philip M. Merlin in [CM77]. Because this observation is of such importance it is also known as the Chandra and Merlin Theorem. It basically states an interpretation $\mathcal{I}$ is a model of Φ if and only if there exists a homomorphism $\mu : Var(\Phi) \cup Ind(\Phi) \to \Delta^\mathcal{I}$. The context of Boolean conjunctive queries makes the proof easy. Nevertheless the proof also teaches us how to construct a homomorphism from an interpretation and backwards and this will enable us to switch between the constructive and abstract level.

Theorem 4.5. *Let $\Phi = \exists u_1, \ldots, u_n.\psi(u_1, \ldots, u_n)$ be a Boolean conjunctive query and $\mathcal{I} = (\Delta^\mathcal{I}, .^\mathcal{I})$ an interpretation. Then $\mathcal{I} \models \Phi$ if and only if there is a homomorphism from Φ to $\mathcal{I}$.*

The proof is done by technical comparing the definitions. Due to our specific interest the result is formulated for Boolean conjunctive queries only, but it can easily be extended to conjunctive queries. Given a Boolean conjunctive query Φ and an interpretation $\mathcal{I}$ we will show that, if we have a match π for $\mathcal{I}$ and Φ, we can construct a homomorphism from Φ to $\mathcal{I}$ and vice versa.

Proof. Recall $\mathcal{I}$ is model of a Boolean conjunctive query Φ if there is a match

$$\pi : Var(\Phi) \cup Ind(\Phi) \to \Delta^\mathcal{I}$$

for $\mathcal{I}$ and Φ. Whereas a homomorphism is a mapping

$$\mu : Var(\Phi) \cup Ind(\Phi) \to \Delta^\mathcal{I}$$

from Φ to $\mathcal{I}$. By setting $\pi = \mu$ the equivalence of the two approaches is obvious. We prove the simultaneous satisfiability of the conditions for μ being a homomorphism and π being a match that determines an interpretation from the atomic items:

- For all $a \in Ind(\Phi)$ we know $\pi(a) = \mu(a) = a^{\mathcal{I}}$.
- Let $\alpha \in \Phi$ be a concept atom where $\alpha = A(s)$ with $A \in N_C$. We know for every $s \in Var(\Phi) \cup Ind(\Phi)$ with $\mu(s) = \pi(s)$ that $\mu(s) \in A^{\mathcal{I}}$ if and only if $\pi(s) \in A^{\mathcal{I}}$ if and only if $\mathcal{I} \models^{\pi} A(s)$.
- Let $\alpha \in \Phi$ be a role atom $\alpha = r(s,t)$ with $r \in N_C$ and $s,t \in Var(\Phi) \cup Ind(\Phi)$. We know from $\mu(s) = \pi(s)$ and $\mu(t) = \pi(t)$ that $(\mu(s), \mu(t)) \in r^{\mathcal{I}}$ if and only if $(\pi(s), \pi(t)) \in r^{\mathcal{I}}$ if and only if $\mathcal{I} \models^{\pi} r(s,t)$.
- Let $\psi = \alpha_1 \wedge \ldots \wedge \alpha_k$ with $\alpha_a, \ldots, \alpha_k$ concept atoms. From the above items we gather $\mathcal{I} \models^{\pi} \alpha_1 \wedge \ldots \wedge \alpha_k$ if and only if for all $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$ it is true that $\mathcal{I} \models^{\pi} \alpha$. Therefore π is a match for every atom $\alpha_1, \ldots, \alpha_k$ if and only if the homomorphism conditions are true for all concept atoms $\alpha \in \Phi$.

This completes the proof. $\square$

4.1.3 The Query Entailment Problem

This section introduces one of the most important problems for Boolean conjunctive queries, namely query answering over a description logics ontology. For a given knowledge base $\mathcal{K}$ and a Boolean conjunctive query Φ, it is the question whether every model of $\mathcal{K}$ entails Φ.

Definition 4.6. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be an knowledge base and Φ a Boolean conjunctive query. If for every $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ the interpretation $\mathcal{I}$ is a model of Φ then Φ is called a *consequence of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$* and we write $\mathcal{K} \models \Phi$.

The question, whether a Boolean conjunctive query Φ is a consequence of an ABox $\mathcal{A}$ w.r.t. a TBox $\mathcal{T}$ is called the *query entailment problem.*

The next theorem includes statements about decidability of the query entailment problem for description logics knowledge bases, that include nominals, inverses and qualified number restriction. It is a direct consequence of a theorem from Birte Glimm and Sebastian Rudolph, namely Theorem 2 presented in the joint article [GR09].

Theorem 4.7. *Let $\mathcal{K}$ be an $\mathcal{ALCQIO}$ knowledge base and Φ a Boolean conjunctive query. Then $\mathcal{K} \models \Phi$ is decidable.*

The introduction into Boolean conjunctive queries closes with an example.

Example 4.8. Let the sets N_I, N_C and N_R, the knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and the interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be defined as in Example 2.10 and illustrated in Figure 2.2 . Let N_V be a set of variables with $u_1, u_2 \in N_V$. Then

$$\Phi := \exists u_1, u_2(friend(u_1, Anna)) \wedge friend(u_2, Tom))$$

is a Boolean conjunctive query.

The interpretation $\mathcal{I}$ is not a model of Φ. There cannot exist a homomorphism from Φ to $\mathcal{I}$ because by definition of $\mathcal{I}$ we cannot get $(\mu(u_2), \mu(Tom)) \in friend^\mathcal{I}$ with $\mu(Tom) = Tom^\mathcal{I}$. There does not exist $a \in \Delta^\mathcal{I}$ with $(a, Tom) \in friend^\mathcal{I}$. Therefore $\mathcal{I} \not\models friend(u_2, Tom)$, and so $\mathcal{I} \not\models \Phi$. Because $\mathcal{I}$ is a common model of $\mathcal{A}$ and $\mathcal{T}$, it follows Φ is not a consequence of $\mathcal{A}$ w.r.t. $\mathcal{T}$ and so $\mathcal{K}$ does not entail Φ.

4.1.4 The Action Formalism for Description Logics and Boolean Conjunctive Queries

In this section we bring together query answering for Boolean conjunctive queries and the description logics based action formalism. We start with the introduction of a Boolean conjunctive query as a consequence of applying actions to a knowledge base. Then we define the projection problem within this setting and ask if we can obtain a result similar to Theorem 3.14 that states the reducibility of projection for Boolean conjunctive queries to query entailment. This reduction will again be based on the definition of a reduction ABox and TBox presented in the report [BML+05]. We therefore have to require the description logic on which we base the action formalism to contain nominals.
The first definition introduces the projection problem for Boolean conjunctive queries with respect to a knowledge base and composite actions.

Definition 4.9. Let Φ be a Boolean conjunctive query. Let further $\mathcal{T}$ be an acyclic TBox, $\mathcal{A}$ an ABox, $\alpha_1, \ldots, \alpha_k \in \Sigma$ a composite action for $\mathcal{T}$, $\mathcal{D}$ the set of relevant assertions, $Lit \subseteq \mathcal{D}$ the set of literals contained in $\mathcal{D}$ and $\mathcal{E} : \Sigma \times \mathcal{T} \to 2^{Lit}$ the effect function w.r.t. $\Sigma, \mathcal{D}$ and $\mathcal{T}$.

Projection: The Boolean conjunctive query Φ is a *consequence of applying* $\alpha_1, \ldots, \alpha_k$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if for all models $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ and all $\mathcal{I}'$ with $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}'$ we have $\mathcal{I}' \models \Phi$.

Remark 4.10. Executability of actions, as defined in Definition 2.19, and projection with queries cannot be proven to be reducible to each other the same way as shown in Lemma 2.22 because an action $\alpha \in \Sigma$ is not defined to contain Boolean conjunctive queries in the pre-conditions. But of course we can consider Boolean conjunctive queries as a consequence of consistent actions as well and Lemma 2.22 would yield the justification, see Remark 2.23.

Example 4.11. Recall from Example 2.10 the sets N_I, N_C and N_R, the ABox $\mathcal{A}$, the TBox $\mathcal{T}$ and the interpretation $\mathcal{I}$ given in Figure 2.2 and further from Example 2.24 the actions $\alpha_1, \alpha_2 \in \Sigma$, the effect function $\mathcal{E} : \Sigma \times \mathcal{T} \to 2^{Lit}$ and the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ given in Figures. 2.3 and 2.4. Let us look again at the Boolean conjunctive query

$$\Phi := \exists u_1, u_2(friend(u_1, Anna)) \wedge friend(u_2, Tom))$$

From Example 2.24 we know $\mathcal{I} \Rightarrow^{\mathcal{E}}_{\alpha_1,\alpha_2} \mathcal{I}_2$. If for $\mu : Var(\Phi) \cup Ind(\Phi) \to \Delta^{\mathcal{I}_2}$ we set $\mu(Tom) := Tom^{\mathcal{I}}$, $\mu(Anna) := Anna^{\mathcal{I}}$, $\mu(u_1) := Tom^{\mathcal{I}_2}$, $\mu(u_2) := Anna^{\mathcal{I}_2}$ then $(\mu(Anna), \mu(Tom)) \in friend^{\mathcal{I}_2}$ and $(\mu(Tom), \mu(Anna)) \in friend^{\mathcal{I}_2}$. This proofs μ is a homomorphism from Φ to $\mathcal{I}_2$ and therefore $\mathcal{I}_2 \models \Phi$.

4.2 Reducing Projection with Boolean Conjunctive Queries to Query Entailment

We are now well prepared to state the reduction theorem which we would like to have for Boolean conjunctive queries. We will look how far we can go, if we walk on the lines of the ideas from Theorem 14 and Lemma 15 in [BML$^+$05]. Recall that these two results show with use of nominals that projection of composite actions can be polynomially reduced to ABox consequence. In the next theorem we will state this result for Boolean conjunctive queries, that will now take the place of the ABox assertion φ in Lemma 3.16.

Theorem 4.12. *Let $\mathcal{L}$ be a fragment of $\mathcal{ALCQIO}$. Then projection of composite actions with Boolean conjunctive queries formulated in $\mathcal{L}$ can be exponentially reduced to query entailment in $\mathcal{LO}$ w.r.t. acyclic TBoxes.*

The proof of this theorem will occupy us for the remaining part of this chapter. It will be the consequence of a lemma, showing that the Boolean conjunctive query Φ is a consequence of applying composite action α to a knowledge base if and only if we can construct a reduction query Ψ from Φ and a reduction knowledge base $\mathcal{K}_{\text{red}}$ from $\mathcal{K}$ and $\mathcal{K}$ entails Ψ.

4.2.1 Considering Boolean Conjunctive Queries

Let us illustrate in the following discussion and examples what we have to regard in the construction of reduction query, which can be considered a consequence of $\mathcal{A}_{\text{red}}$ w.r.t. $\mathcal{T}_{\text{red}}$. Thereby we understand the reduction ABox $\mathcal{A}_{\text{red}}$ and the reduction TBox $\mathcal{T}_{\text{red}}$ to be as constructed in the proof of Theorem 3.14.

Discussion The unary and binary predicates of a query Φ come from concepts and roles occurring in the knowledge base. If we want to define a Boolean conjunctive query Φ_{red} for which we can ask whether it is a consequence of $\mathcal{A}_{\text{red}}$ w.r.t. $\mathcal{T}_{\text{red}}$, we have to replace the predicates in Φ with their labelled copies and pendants.

Let us start with the example $\Phi = \exists a.A(a)$ where $A \in N_C$ and $a \in N_V$. We have to formulate this simple query as a consequence of an ABox $\mathcal{A}_{\text{red}}$ w.r.t. a TBox $\mathcal{T}_{\text{red}}$, but we do not know whether the variable $x \in N_V$ will be mapped to a named or unnamed element by an interpretation $\mathcal{I}$.

$$\begin{aligned}\Phi_{\text{red}} &:= \exists x.(A^{(n)}(x) \wedge N(x)) \vee (A^{(0)}(x) \wedge \neg N(x)) \\ &= \exists x.((A^{(n)} \sqcap N)) \sqcup (A^{(0)} \sqcap \neg N))(x)\end{aligned}$$

By adding a concept definition $A^{(*)} \equiv (A^{(n)} \sqcap N)) \sqcup (A^{(0)} \sqcap \neg N)$ to $\mathcal{T}_{\text{red}}$ this would yield a conjunctive query.

The other atomic example considers the Boolean conjunctive query $\Phi = \exists a, b.r(a, b)$ with $r \in N_R$ and $a \in N_I \cup N_V$, $b \in N_V$. We can express the same content for $\mathcal{T}_{\text{red}}$ and $\mathcal{A}_{\text{red}}$, where we split the named and unnamed elements:

$$\Phi_{\text{red}} = \exists x, y. \left(r^{(n)}(x, y) \wedge N(x) \wedge N(y)\right) \vee \left(r^{(0)}(x, y) \wedge (\neg N(x) \vee \neg N(y))\right)$$

We cannot obtain a concept definition for $\mathcal{T}_{\text{red}}$ similar to the above example for concept atoms. In the construction of $\mathcal{T}_{\text{red}}$, $\mathcal{A}_{\text{red}}$ and φ_{red} we have used nominals to transform role assertions into concept assertions. But the process of forming nominals from variables is subject of current research. It is presented in the article [KMKH11], where it is called nominal schemas and its semantics is defined by interpreting variables as place holders for named individuals. This approach does not cover our whole problem and because it has an own influence on complexity we will not make use of the there developed technique in this thesis. Any approach to formulate a query analogous to the first example is either not a conjunctive query or not a query over $\mathcal{ALCQIO}$.

Let us make a last observation, before we present a solution approach. It shows, that there is no abbreviation to obtain a reduction query, because we need to know for every variable in the input and every interpretation, whether it is mapped to a named or unnamed element by this interpretation. We have talked about the splitting of named and unnamed elements before, but using variables brings up a problem in the construction of $\mathcal{A}_{\text{red}}$, we have to be aware of. Consider the following example:

Example 4.13. Let $A, B \in N_C$, $x \in N_V$, $a \in N_I$. Further let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a knowledge base with the TBox $\mathcal{T} = \emptyset$ and the ABox $\mathcal{A} = \{A(a)\}$, let $\Phi = \exists x.A(x)$ be a Boolean conjunctive query and $\alpha = (\emptyset, \{A(a)/\neg B(a), A(a)/\neg A(a)\})$ an action for $\mathcal{T}$. For the interpretations $\mathcal{I}_0 \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ and $\mathcal{I}_1 \in \mathcal{M}(\mathcal{T})$ given in Figure 4.2 with $\mathcal{I}_0 \Rightarrow_\alpha^\mathcal{E} \mathcal{I}_1$ we observe $\mathcal{I}_1 \not\models \Phi$ and therefore $\mathcal{K} \not\models \Phi$.

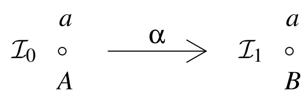

Figure 4.2: Interpretations $\mathcal{I}_0$ and $\mathcal{I}_1$ for $\mathcal{K}$ and α.

We know about the labelled concepts $A^{(0)}, B^{(0)}, A^{(1)}, B^{(1)} \in N_C$ and about the reduction ABox, the membership $A^{(0)}(a) \in \mathcal{A}_{\text{red}}$, $(p_0(A(a)) \rightarrow p_1(B(a)))(a_{help}) \in \mathcal{A}_{\text{red}}$ and further $(p_0(A(a)) \rightarrow p_1(\neg A(a)))(a_{help}) \in \mathcal{A}_{\text{red}}$. For an interpretation $\mathcal{J} \in \mathcal{M}(\mathcal{A}_{\text{red}}) \cap \mathcal{M}(\mathcal{T}_{\text{red}})$, given in Figure 4.3, we observe $\mathcal{T}_{\text{red}}, \mathcal{A}_{\text{red}} \models \exists x.A^{(0)}(x)$ and $\mathcal{T}_{\text{red}}, \mathcal{A}_{\text{red}} \not\models \exists x.A^{(1)}(x)$.

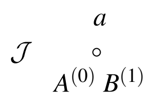

Figure 4.3: Interpretation $\mathcal{J}$ for $\mathcal{A}_{\text{red}}$ and $\mathcal{T}_{\text{red}}$.

This example demonstrates the following problem. We collect the knowledge about the unnamed elements in the zero labelled concepts, but these concepts also indicate membership in the initial ABox. We have to make sure for the variables in the reduction query, that they only stand for the unnamed elements of the zero labelled part. We will implement these preliminaries by requiring a concept membership for every input element of the reduction query, either in the named elements or in the unnamed elements.

4.2.2 Considering Unions of Conjunctive Queries

In their article [GR10] and the accompanying report [RG10] on conjunctive query answering in presence of nominals, inverses and qualified number restriction Birte Glimm and Sebastian Rudolph show decidability of entailment for $\mathcal{ALCQIO}$ knowledge bases for so-called unions of conjunctive queries. Because these queries include a moment of choosing they turn into our focus to overcome the problem of deciding for the right Boolean conjunctive query for every interpretation $\mathcal{J} \in \mathcal{M}(\mathcal{A}_{\text{red}}) \cap \mathcal{M}(\mathcal{T}_{\text{red}})$.
First we introduce unions of conjunctive queries and their semantics and give the extension of the query entailment problem for unions of conjunctive queries. The following discussion is then already the preparation for the lemma, which proves Theorem 4.12 by reducing projection with Boolean conjunctive queries to query entailment with unions of conjunctive queries.

Definition 4.14. A union of conjunctive queries, abbreviated by UCQ, is the disjunction of a finite set of conjunctive queries. For $\Psi := \Phi_1 \vee \ldots \vee \Phi_n$ with $\Phi_1, \ldots, \Phi_n$ conjunctive queries $Var(\Psi) = \bigcup_{\Phi \in \{\Phi_1, \ldots, \Phi_n\}} Var(\Phi)$ and $Ind(\Psi) = \bigcup_{\Phi \in \{\Phi_1, \ldots, \Phi_n\}} Ind(\Phi)$.
By $\alpha \in \Psi$ and $\Psi := \Phi_1 \vee \ldots \vee \Phi_n$ where $\Phi_1 \vee \ldots \vee \Phi_n$ conjunctive queries and α concept or role atom we denote the existence of a $\Phi \in \{\Phi_1, \ldots, \Phi_n\}$ with $\alpha \in \Phi$.

The semantics of unions of conjunctive queries is based on the semantics of its conjuncts. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and Ψ a union of conjunctive queries. For $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ with $\Phi_1, \ldots, \Phi_n$ conjunctive queries we set $\mathcal{I} \models \Psi :\iff \exists_{\Phi \in \{\Phi_1, \ldots, \Phi_n\}} \mathcal{I} \models \Phi$.

The above definition gives the notion of a union of conjunctive queries for a disjunction of general conjunctive queries. Note that in this thesis we are only talking about unions of Boolean conjunctive queries. With these reservations, we can extend our notion of a homomorphism to unions of conjunctive queries.

Definition 4.15. Let $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ be a union of conjunctive queries where $\Phi_1, \ldots, \Phi_n$ are Boolean conjunctive queries. A mapping $\mu : Var(\Psi) \cup Ind(\Psi) \rightarrow \Delta^{\mathcal{I}}$ is a *homomorphism from* Ψ *to* $\mathcal{I}$ if and only if there is at least one disjunct $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ such that μ is a homomorphism from Φ_i to $\mathcal{I}$.

The interpretation for a union of conjunctive queries and a homomorphism of a union of conjunctive queries to an interpretation is defined to be compatible.

Lemma 4.16. *Let $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ be a union of Boolean conjunctive queries and $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ be an interpretation. Then $\mathcal{I} \models \Psi$ if and only if there is a homomorphism from Ψ to $\mathcal{I}$.*

Proof. This in an obvious consequence of Definition 4.14, where the semantics of unions of conjunctive queries is introduced, Definition 4.15, where a homomorphism from a union of conjunctive queries to an interpretation is given and Theorem 4.5, stating the characterisation of semantics of Boolean conjunctive queries by homomorphisms. □

Definition 4.17. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a knowledge base and $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ a union of conjunctive queries. We say $\mathcal{K}$ *entails* Ψ if for every interpretation $\mathcal{I} \in \mathcal{M}(\mathcal{T}) \cap \mathcal{M}(\mathcal{A})$ there exists an $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ such that $\mathcal{I} \models \Phi_i$.

The next lemma is a result from the already mentioned work of Birte Glimm and Sebastian Rudolph.

Theorem 4.18. *Let $\mathcal{K}$ be an $\mathcal{ALCQIO}$ knowledge base and $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ a union of conjunctive queries. The question whether $\mathcal{K} \models \Psi$ is decidable.*

Proof. This is a direct consequence of Theorem 43 in the report [RG10], where the decidability of query entailment is stated for the description logic $\mathcal{ALCOIF}b$. In Section 3.2.1. Birte Glimm and Sebastian Rudolph derive, why this result also holds for an $\mathcal{ALCHOIQ}b$ knowledge base, which is an extension of $\mathcal{ALCQIO}$. □

The preliminaries for proving the reduction theorem will be to form a union of conjunctive queries Ψ from a given Boolean conjunctive query Φ and the construction of the reduction ABox and TBox. Based on this we show the equivalence we need to prove Theorem 4.12. Thereby we do not want to change the construction of the reduction knowledge base, given by $\mathcal{T}_{\text{red}}$ and $\mathcal{A}_{\text{red}}$ provided by the report [BML+05], because we are going to exploit the proof of Theorem 3.14. We start with discussing examples of simple Boolean conjunctive queries and derive requirements for the resulting union of conjunctive queries.

Discussion Consider first $\Phi_1 = \exists x.A(x)$. We form

$$\Psi_1 = \Big(\exists x.A^{(n)}(x) \wedge N(x)\Big) \vee \Big(\exists x.A^{(0)}(x) \wedge M(x)\Big)$$

which is a union of Boolean conjunctive queries, providing the right information if we add $M \in N_C$ and the concept definition $M \equiv \neg N$ to $\mathcal{T}_{\text{red}}$.

Consider $\Phi_2 = \exists y, z.r(y, z)$. We form

$$\Psi_2 = \Big(\exists y, z.r^{(n)}(y, z) \wedge N(y) \wedge N(z)\Big) \vee \Big(\exists y, z.r^{(0)}(y, z) \wedge M(y)\Big) \vee \Big(\exists y, z.r^{(0)}(y, z) \wedge M(z)\Big)$$

which is a union of Boolean conjunctive queries.

To transform the BCQ $\Phi = \exists x, y, z.A(x) \wedge r(y, z)$ we need to keep in mind that two variables can be mapped to the same element of $\Delta^{\mathcal{I}}$ by an interpretation $\mathcal{I}$. So the union of conjunctive queries Ψ, that caries the same information as Φ, needs to be a union of all possible combinations of named and unnamed elements:

$$\begin{aligned}
\Psi \quad = \quad & \exists x, y, z.A^{(n)}(x) \wedge N(x) \wedge r^{(n)}(y, z) \wedge N(y) \wedge N(z) \\
\vee\ & \exists x, y, z.A^{(n)}(x) \wedge N(x) \wedge r^{(0)}(y, z) \wedge M(y) \\
\vee\ & \exists x, y, z.A^{(n)}(x) \wedge N(x) \wedge r^{(0)}(y, z) \wedge M(z) \\
\vee\ & \exists x, y, z.A^{(0)}(x) \wedge M(x) \wedge r^{(n)}(y, z) \wedge N(y) \wedge N(z) \\
\vee\ & \exists x, y, z.A^{(0)}(x) \wedge M(x) \wedge r^{(0)}(y, z) \wedge M(y) \\
\vee\ & \exists x, y, z.A^{(0)}(x) \wedge M(x) \wedge r^{(0)}(y, z) \wedge M(z)
\end{aligned}$$

This is a very small example and already a huge resulting union of conjunctive queries. But from the above discussion we know every disjunct in Ψ can be necessary.

We observe, especially for $x \in Var(\Phi)$, the mark $M(x)$, always combined with the concept $A^{(0)}(x)$ assures, that this disjunct is never unintentional satisfied by any interpretation.

With this examples in mind we start to formulate the construction of a union of Boolean conjunctive queries Ψ from a Boolean conjunctive query Φ.

Preliminaries Let $\mathcal{A}_{\text{red}}$, $\mathcal{T}_{\text{red05}}$ be the reduction ABox and TBox defined in the report [BML+05] and set $\mathcal{T}_{\text{red}} := \mathcal{T}_{\text{red05}} \cup \{M \equiv \neg N\}$.
Let Φ be a Boolean conjunctive query. Step by step we construct the union of conjunctive queries Ψ.

1. $Var(\Psi) = Var(\Phi)$ and $Ind(\Psi) = Ind(\Phi)$

2. For $\Phi = \exists u_1, \ldots, u_m.\varphi(u_1, \ldots, u_m)$ we start with $\Psi := \exists u_1, \ldots, u_m.\emptyset$.

3. For each occurrence $A(s) \in \Phi$ we need to copy all the already existing conjuncts of Ψ. To one half we add the atoms $A^{(n)}(s)$ and $N(s)$ and to the other half we add the atoms $A^{(0)}(s)$ and $M(s)$.

4. For each occurrence $r(s, t) \in \Phi$ we triplicate all the already existing conjuncts of Ψ. To the first group we add the atoms $r^{(n)}(s, t)$ and $N(s)$ and $N(t)$. To the second group we add the atoms $r^{(0)}(s, t)$ and $M(s)$ and to the last group we add the atoms $r^{(0)}(s, t)$ and $M(t)$.

Remark 4.19. Consider the Boolean conjunctive query Φ and the union of Boolean conjunctive queries $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$, obtained from the above instructions. For individuals $a \in Ind(\Phi)$ every disjunct $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ that contains the term $M(a)$ is always false, because all individuals used in the input are by definition mapped to named elements. In an implementation one could drop this terms. Because they do no harm in a union of queries we leave them here. Another observation is every disjunct, that contains for an $u \in Var(\Phi) \cup Ind(\Phi)$ both terms $M(u)$ and $N(u)$, is in itself contradictory and therefore not satisfiable. They can also be cancelled in an implementation. There are several arguments how to reduce the conjuncts of this query. Ignoring these practical issues is just laziness to do the more involved definition, not to think of a later distinction of cases. Note that there are example where every conjunct is necessary. So this distinction has no influence on the worst case complexity, but it shows in best case the union of conjunctive queries can contain a single conjunct.

Let us illustrate the construction of the union of conjunctive queries Ψ from the Boolean conjunctive query Φ by Figures 4.4. From this construction we obtain a UCQ Ψ, which contains a disjunct for all possible combinations where the variables are mapped to named and unnamed elements and we respect the unique name assumption. Obviously this is

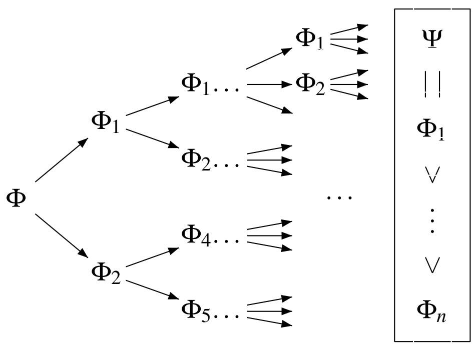

Figure 4.4: Construction of the union of conjunctive queries Ψ from the Boolean conjunctive query Φ.

always possible and the above instruction gives the desired result.

The following lemma states the reduction we need to proof the main result of this chapter, namely Theorem 4.12.

Lemma 4.20. *Let $\alpha_1, \ldots, \alpha_n \in \Sigma$, $\mathcal{T}$ TBox, $\mathcal{A}$ ABox and Φ a Boolean conjunctive query. Further let $\mathcal{A}_{red}$, $\mathcal{T}_{red}$ and Ψ be constructed from $\mathcal{A}$, $\mathcal{T}$ and Φ as given in the preliminaries.*

The Boolean conjunctive query Φ is a consequence of applying the composite action $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$ if and only if the union of conjunctive queries Ψ is a consequence of $\mathcal{A}_{red}$ w.r.t. $\mathcal{T}_{red}$.

We show the following: Every time there exists for every interpretation $\mathcal{I}_0 \in \mathcal{M}(T) \cap \mathcal{M}(\mathcal{A})$ and $\mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}_n$ a homomorphism from Φ to $\mathcal{I}_n$ we can construct for every interpretation $\mathcal{J} \in \mathcal{M}(\mathcal{T}_{\text{red}}) \cap \mathcal{M}(\mathcal{A}_{\text{red}})$ a homomorphism from Ψ to $\mathcal{J}$ and backwards.

Proof. $\Rightarrow$:

Let $\mathcal{J} \in \mathcal{M}(\mathcal{A}_{\text{red}}) \cap \mathcal{M}(\mathcal{T}_{\text{red}})$ arbitrary and assume Φ is a consequence of applying the composite action $\alpha_1, \ldots, \alpha_n$ in $\mathcal{A}$ w.r.t. $\mathcal{T}$. Therefore we know for all interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n \in \mathcal{M}(\mathcal{T})$ with $\mathcal{I}_0 \models \mathcal{A}$ and $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}_n$ that there exists a homomorphism from Φ to $\mathcal{I}_n$. We are going to show by construction the existence of a homomorphism μ_2 from Ψ to $\mathcal{J}$.

Let $\mathcal{I}_0, \ldots, \mathcal{I}_n$ be constructed from $\mathcal{J}$, where for $0 \le i \le n$

$$\begin{aligned}
\Delta^{\mathcal{I}_i} &:= \Delta^{\mathcal{J}} \\
a^{\mathcal{I}_i} &:= a^{\mathcal{J}} \text{ for } a \in \text{Obj} \\
A^{\mathcal{I}_i} &:= (T_A^{(i)})^{\mathcal{J}} \text{ for } A \in \text{Con} \\
r^{\mathcal{I}_i} &:= \Big((r^{(i)})^{\mathcal{J}} \cap (N^{\mathcal{J}} \times N^{\mathcal{J}})\Big) \cup \Big((r^{(0)})^{\mathcal{J}} \cap (\Delta^{\mathcal{J}} \times (M)^{\mathcal{J}} \cup (M)^{\mathcal{J}} \times \Delta^{\mathcal{J}})\Big) \text{ for } r \in \text{Rol}
\end{aligned}$$

For $0 \le i \le n$ we know by the proof of Lemma 15 presented in the report [BML$^+$05] (see the similar Lemma 3.16 with Claim 3.17 in this thesis, which works in presence of a universal role) that $\mathcal{I}_i \in \mathcal{M}(\mathcal{T})$, $\mathcal{I}_0 \models \mathcal{A}$ and $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}_n$.

By assumption $\mathcal{I}_n \models \Phi$, so for all $x \in Var(\Phi) \cup Ind(\Phi)$ there exists a homomorphism $\mu_1 : Var(\Phi) \cup Ind(\Phi) \to \Delta^{\mathcal{I}_n}$ from Φ to $\mathcal{I}_n$. Because Ψ is a union of Boolean conjunctive queries, let $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ with $\Phi_1, \ldots, \Phi_n$ Boolean conjunctive queries and further $\mu_2 : Var(\Psi) \cup Ind(\Psi) \to \Delta^{\mathcal{J}}$ is a function with

$$\begin{aligned}
\mu_2(a) &:= \mu_1(a) \quad \text{for all } a \in Ind(\Psi) \\
\mu_2(u) &:= \mu_1(u) \quad \text{for all } u \in Var(\Psi)
\end{aligned}$$

The function $\mu_2 : Var(\Psi) \cup Ind(\Psi) \to \Delta^{\mathcal{J}}$ is well defined, because $Ind(\Psi) = Ind(\Phi)$, $Var(\Psi) = Var(\Phi)$ and $\Delta^{\mathcal{I}_n} = \Delta^{\mathcal{J}}$ by definition. The proof that μ_2 from Ψ to $\mathcal{J}$ is a homomorphism from a union of conjunctive queries to an interpretation checks Definition 4.15 and the depending Definition 4.4. It is therefore split in three steps.

1. By definition $\mu_2(a) = \mu_1(a) = a^{\mathcal{I}_n} = a^{\mathcal{J}}$ for all $a \in Ind(\Psi)$.

2. Every disjunct of the union of conjunctive queries Ψ contains either $A^{(n)}(x) \wedge N(x)$ or $A^{(0)}(x) \wedge M(x)$ where $x \in Var(\Psi) \cup Ind(\Psi)$. In both cases we know by construction of Ψ from Φ, that $A(x) \in \Phi$ with $A \in N_C$ and $M, N \in N_C$. By the definition of μ_2 and the fact, that μ_1 is a homomorphism from Φ to $\mathcal{I}_n$, it follows $\mu_2(x) = \mu_1(x) \in A^{\mathcal{I}_n}$ where
$$A^{\mathcal{I}_n} = \left(T_A^{(n)}\right)^{\mathcal{J}} = \left(N \sqcap (A^{(n)})\right)^{\mathcal{J}} \cup \left((\neg N \sqcap (A^{(0)})\right)^{\mathcal{J}}.$$

Note that the two sets are disjoint and so $\mu_2(x)$ is contained in exactly one of them. For the concept atoms $N(x) \in \Psi$, $M(x) \in \Psi$ we know by the definition of $\mathcal{J}$ that $N^{\mathcal{J}} = \bigcup_{a \in Obj}\{a\}^{\mathcal{J}} = \bigcup_{a \in Obj}\{a\}^{\mathcal{I}_n}$ and $M^{\mathcal{J}} = (\neg N)^{\mathcal{J}} = \Delta^{\mathcal{J}} \backslash N^{\mathcal{J}}$.

Let $\mu_2(x) \in \left(N \sqcap (A^{(n)})\right)^{\mathcal{J}}$. Then $\mu_2(x) \in N^{\mathcal{J}}$ and $\mu_2(x) \in (A^{(n)})^{\mathcal{J}}$.

Let $\mu_2(x) \in \left(\neg N \sqcap (A^{(0)})\right)^{\mathcal{J}}$. Then $\mu_2(x) \in M^{\mathcal{J}}$ and $\mu_2(x) \in (A^{(0)})^{\mathcal{J}}$.

3. Every disjunct of Ψ contains either $r^{(n)}(y,z) \wedge N(y) \wedge N(z)$ or $r^{(0)}(y,z) \wedge M(y)$ or $r^{(0)}(y,z) \wedge M(z)$ where $y, z \in Var(\Psi) \cup Ind(\Psi)$. In both cases we know by construction of Ψ from Φ, that $r(y,z) \in \Phi$ with $r \in N_R$ and $M, N \in N_C$. By the definition of μ_2 and the fact, that μ_1 is a homomorphism from Φ to $\mathcal{I}_n$ we know $(\mu_2(y), \mu_2(z)) = (\mu_1(y), \mu_1(z)) \in r^{\mathcal{I}_n}$ where

$$\begin{aligned} r^{\mathcal{I}_n} &= \left((r^{(n)})^{\mathcal{J}} \cap (N^{\mathcal{J}} \times N^{\mathcal{J}})\right) \cup \left((r^{(0)})^{\mathcal{J}} \cap (\Delta^{\mathcal{J}} \times M^{\mathcal{J}} \cup M^{\mathcal{J}} \times \Delta^{\mathcal{J}})\right) \\ &= \left((r^{(n)})^{\mathcal{J}} \cap (N^{\mathcal{J}} \times N^{\mathcal{J}})\right) \cup \left((r^{(0)})^{\mathcal{J}} \cap \Delta^{\mathcal{J}} \times M^{\mathcal{J}}\right) \cup \left((r^{(0)})^{\mathcal{J}} \cap M^{\mathcal{J}} \times \Delta^{\mathcal{J}})\right). \end{aligned}$$

Clearly now $(\mu_2(y), \mu_2(z))$ is either contained in the first or in the last two sets.

Let $(\mu_2(y), \mu_2(z)) \in \left((r^{(n)})^{\mathcal{J}} \cap (N^{\mathcal{J}} \times N^{\mathcal{J}})\right)$. Then we know for the role and concept atoms $(\mu_2(y), \mu_2(z)) \in (r^{(n)})^{\mathcal{J}}$, $\mu_1(y) \in N^{\mathcal{J}}$, $\mu_2(z) \in N^{\mathcal{J}}$.

Let $(\mu_2(y), \mu_2(z)) \in \left((r^{(0)})^{\mathcal{J}} \cap \Delta^{\mathcal{J}} \times M^{\mathcal{J}}\right)$ or $(\mu_2(y), \mu_2(z)) \in \left((r^{(0)})^{\mathcal{J}} \cap M^{\mathcal{J}} \times \Delta^{\mathcal{J}})\right)$. Then $(\mu_2(y), \mu_2(z)) \in (r^{(0)})^{\mathcal{J}}$ and $\mu_1(y) \in M^{\mathcal{J}}$ or $\mu_2(z) \in M^{\mathcal{J}}$.

By the construction of Ψ from all possible combinations of the discussed atoms, μ_2 is a homomorphism from at least one of the disjuncts Φ_i of Ψ to $\mathcal{J}$ and therefore μ_2 is a homomorphism from Ψ to $\mathcal{J}$.

$\Leftarrow$:

Let there be arbitrary interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$ with $\mathcal{I}_0 \Rightarrow^{\mathcal{E}}_{\alpha_1,\ldots,\alpha_n} \mathcal{I}_n$ and let for all interpretations $\mathcal{J} \in \mathcal{M}(\mathcal{A}_{\text{red}}) \cap \mathcal{M}(\mathcal{T}_{\text{red}})$ be true that $\mathcal{J} \models \Psi$ with $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ where every $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ is a Boolean conjunctive queries.

Consider the interpretation $\mathcal{J}$ be constructed from $\mathcal{I}_0, \ldots, \mathcal{I}_n$, where for $0 \leq i \leq n$

$$\begin{aligned} \Delta^{\mathcal{J}} &:= \Delta^{\mathcal{I}_n} \; (= \Delta^{\mathcal{I}_0} = \ldots = \Delta^{\mathcal{I}_{n-1}}) \\ a^{\mathcal{J}} &:= a^{\mathcal{I}_n} \; (= a^{\mathcal{I}_0} = \ldots = a^{\mathcal{I}_{n-1}}) \text{ for } a \in \text{Obj} \\ N^{\mathcal{J}} &:= \{a^{\mathcal{J}} \mid a \in \text{Obj}\} \\ M^{\mathcal{J}} &:= \Delta^{\mathcal{J}} \backslash N^{\mathcal{J}} \\ (A^{(i)})^{\mathcal{J}} &:= A^{\mathcal{I}_i} \text{ for } A \in \text{Con} \\ (r^{(i)})^{\mathcal{J}} &:= r^{\mathcal{I}_i} \text{ for } r \in \text{Rol} \\ (T_C^{(i)})^{\mathcal{J}} &:= C^{\mathcal{I}_i} \text{ for all } C \in \text{Sub} \end{aligned}$$

From the obvious fact, that $\mathcal{J} \models M \equiv \neg N$ and the proof of Lemma 15 presented in the report [BML+05] (see also Lemma 3.16 in this thesis) we know $\mathcal{J} \models \mathcal{A}_{\text{red}}$ and $\mathcal{J} \models \mathcal{T}_{\text{red}}$. By assumption there exists a homomorphism $\mu_2 : Var(\Psi) \cup Ind(\Psi) \rightarrow \Delta^{\mathcal{J}}$ from Ψ to $\mathcal{J}$. We can construct μ_1 from Φ to $\mathcal{I}_n$ composed of μ_2. For $\mu_1 : Var(\Phi) \cup Ind(\Phi) \rightarrow \Delta^{\mathcal{I}_n}$ let

$$\begin{aligned} \mu_1(a) &:= \mu_2(a) \quad \text{for all } a \in Ind(\Phi) \\ \mu_1(u) &:= \mu_2(u) \quad \text{for all } u \in Var(\Phi) \end{aligned}$$

The function $\mu_1 : Var(\Phi) \cup Ind(\Phi) \rightarrow \Delta^{\mathcal{I}_n}$ is well defined, because $Ind(\Phi) = Ind(\Psi)$, $Var(\Phi) = Var(\Psi)$ by definition of Φ_f and $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}_n}$ by definition of $\mathcal{J}$. The proof that μ_1 from Φ to $\mathcal{I}_n$ is a homomorphism from a union of conjunctive queries checks Definition 4.15 and the depending Definition 4.4. It will be split in three steps.

1. By definition $\mu_1(a) = \mu_2(a) = a^{\mathcal{J}} = a^{\mathcal{I}_n}$ for all $a \in Ind(\Phi)$.

2. Consider $A(s) \in \Phi$ arbitrary with $A \in N_C$ and $s \in Var(\Phi) \cup Ind(\Phi)$. From the construction of the union of conjunctive queries Ψ from Φ we know every disjunct of Ψ contains either $A^{(n)}(s) \wedge N(s)$ or $A^{(0)}(s) \wedge M(s)$. Further μ_2 is a homomorphism from $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ to the interpretation $\mathcal{J}$ and by definition there exists at least one disjunct $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ such that μ_2 is a homomorphism from Φ_i to $\mathcal{J}$.

   Let $A^{(n)}(s) \in \Phi_i$ and $N(s) \in \Phi_i$. Then we know, because by definition μ_2 is a homomorphism from the Boolean conjunctive query Φ_i to the interpretation $\mathcal{J}$, that $\mu_1(s) = \mu_2(s) \in (A^{(n)})^{\mathcal{J}}$, where $(A^{(n)})^{\mathcal{J}} = A^{\mathcal{I}_n}$ and $\mu_1(s) = \mu_2(s) \in N^{\mathcal{J}}$, where by definition of $\mathcal{J}$ we know $N^{\mathcal{J}} = \{a^{\mathcal{J}} \mid a \in \text{Obj}\}$. In conclusion $\mu_1(s) \in A^{\mathcal{I}_n}$ and $\mu_1(s)$ is a named element.

   Let $A^{(0)}(s) \in \Phi_i$ and $M(s) \in \Phi_i$. Then we know, because $\mu_2 : Var(\Psi) \cup Ind(\Psi) \rightarrow \Delta^{\mathcal{J}}$ is a homomorphism from the BCQ Φ_i to $\mathcal{J}$, that $\mu_1(s) = \mu_2(s) \in (A^{(0)})^{\mathcal{J}}$ and $\mu_1(s) = \mu_2(s) \in M^{\mathcal{J}}$, where $(A^{(0)})^{\mathcal{J}} = A^{\mathcal{I}_0}$ and $M^{\mathcal{J}} = \Delta^{\mathcal{J}} \backslash N^{\mathcal{J}} = \{a^{\mathcal{J}} \mid a \notin \text{Obj}\}$. We know the equivalence $\mu_1(s) \notin \text{Obj}^{\mathcal{I}_0} \iff \mu_2(s) \notin N^{\mathcal{J}}$ by the definition of $\mathcal{J}$. By Claim 2 from the proof of Lemma 15 in [BML+05] (see the similar Claim 3.18 in this thesis) we know for $\mu_1(s) \notin \text{Obj}^{\mathcal{I}_0}$ that $\mu_1(s) \in A^{\mathcal{I}_0} \iff \mu_1(s) \in A^{\mathcal{I}_n}$.

3. Consider $r(s,t) \in \Phi$ arbitrary with $r \in N_R$ and $s, t \in Var(\Phi) \cup Ind(\Phi)$. From the definition of the UCQ $\Psi = \Phi_1 \vee \ldots \vee \Phi_n$ we know every disjunct of Ψ contains either $r^{(n)}(s,t) \wedge N(s) \wedge N(t)$ or $r^{(0)}(s,t) \wedge M(s)$ or $r^{(0)}(s,t) \wedge M(t)$. By definition and assumption there is at least one disjunct $\Phi_i \in \{\Phi_1, \ldots, \Phi_n\}$ such that μ_2 is a homomorphism from Φ_i to $\mathcal{J}$.

   Let $r^{(n)}(s,t) \in \Phi_i$, $N(s) \in \Phi_i$ and $N(t) \in \Phi_i$. Then we know from considerations, similar to the previous point, $(\mu_1(s), \mu_1(t)) = (\mu_2(s), \mu_2(t)) \in (r^{(n)})^{\mathcal{J}} = r^{\mathcal{I}_n}$ and $\mu_1(s)$ and $\mu_1(t)$ are named elements.

   Let $r^{(0)}(s,t) \in \Phi_i$ and $M(s) \in \Phi_i$ or $r^{(0)}(s,t) \in \Phi_i$ and $M(t) \in \Phi_i$ respectively. Then $(\mu_1(s), \mu_1(t)) = (\mu_2(s), \mu_2(t)) \in (r^{(0)})^{\mathcal{J}} = r^{\mathcal{I}_0}$ and $\mu_1(s) \notin \text{Obj}^{\mathcal{I}_0}$ or $\mu_2(s) \notin \text{Obj}^{\mathcal{I}_0}$.

Therefore by Claim 2 from [BML+05] (or again the similar Claim 3.18 in this thesis) we get $(\mu_1(s), \mu_1(t)) \in r^{\mathcal{I}_0} \iff (\mu_1(s), \mu_1(t)) \in r^{\mathcal{I}_n}$.

By assumption there is at least one disjunct Φ_i of Ψ such that μ_2 is a homomorphism from Φ_i to $\mathcal{J}$. By construction of the union of conjunctive queries Ψ from the input query Φ we know from the above argumentation μ_1 is a homomorphism from Φ to $\mathcal{I}_n$. □

The union of conjunctive queries generated in the preliminaries of Lemma 4.20 is exponential in the size of the Boolean conjunctive query from the input. For $\mathcal{T}_{\text{red}}$ and $\mathcal{A}_{\text{red}}$ we know from the report [BML+05] that they are polynomial in the size of the initial ABox and TBox if we assume unary coding. If we assume numbers in number restrictions are coded in binary, the size of the TBox $\mathcal{T}_{\text{red}}$ given in [BML+05] is exponential in size of the input. Hence, this lemma gives the ExpTime reduction from projection with Boolean conjunctive queries to entailment with unions of conjunctive queries and therefore Theorem 4.12 is a direct consequence. This gives rise to the following results about projection.

Corollary 4.21. *Let $\mathcal{K}$ be an $\mathcal{ALCQIO}$ knowledge base. Projection with Boolean conjunctive queries is decidable.*

Proof. This is a consequence of Theorem 4.12 and Lemma 4.20 which yield exponential reducibility of projection with Boolean conjunctive queries to query entailment with unions of Boolean conjunctive queries. The decidability of query entailment for unions of conjunctive queries is given in Theorem 4.18. □

Corollary 4.22. *Projection with Boolean conjunctive queries is in 3-ExpTime for $\mathcal{ALC}$, $\mathcal{ALCO}$, $\mathcal{ALCQ}$ and $\mathcal{ALCQO}$.*

Proof. This is a consequence of Theorem 4.12 and Lemma 4.20. Further Birte Glimm, Ulrike Sattler and Ian Horrocks provide in the proof of Theorem 9 and Lemma 10 in their article [GHS08] an algorithm that solves the query entailment problem for unions of conjunctive queries in $\mathcal{SHOQ}$ in deterministic time single exponential in the size of the knowledge base and double exponential in the size of the query. Together with the above calculations for $\mathcal{A}_{\text{red}}$, $\mathcal{T}_{\text{red}}$ and the union of conjunctive queries Ψ this yields an algorithm that solves projection with Boolean conjunctive queries in time double exponential in the size of the knowledge base and three times exponential in size of the query. □

4.2.3 Improvements to Reduce Complexity in the Reduction

We constructed the union of conjunctive queries Ψ from the Boolean conjunctive query Φ by a recursive rule. This is easy to handle in a proof, but not very efficient, when it comes to complexity considerations and implementations. Let us construct Ψ from Φ with the help of an explicit mapping rather than by recursion.

Definition 4.23. Let $\Phi = \exists u_1, \ldots, u_m.\phi(u_1, \ldots, u_m)$ be a boolean conjunctive query and $f : Var(\Phi) \cup Ind(\Phi) \to \{0, 1\}$ a function. Then $\Phi_f := \exists u_1, \ldots, u_m.\varphi_f(u_1, \ldots, u_m)$, where

$$\begin{aligned} A(t) \in \Phi \wedge f(t) = 1 &\iff A^{(n)}(t) \in \Phi_f \wedge N(t) \in \Phi_f \\ A(t) \in \Phi \wedge f(t) = 0 &\iff A^{(0)}(t) \in \Phi_f \wedge M(t) \in \Phi_f \\ r(s,t) \in \Phi \wedge f(s) = 1 \wedge f(t) = 1 &\iff r^{(n)}(s,t) \in \Phi_f \wedge N(s) \in \Phi_f \wedge N(t) \in \Phi_f \\ r(s,t) \in \Phi \wedge f(s) = 0 &\iff r^{(0)}(s,t) \in \Phi_f \wedge M(s) \in \Phi_f \\ r(s,t) \in \Phi \wedge f(t) = 0 &\iff r^{(0)}(s,t) \in \Phi_f \wedge M(t) \in \Phi_f \end{aligned}$$

is a Boolean conjunctive query with $Var(\Phi_f) = Var(\Phi)$ and $Ind(\Phi_f) = Ind(\Phi)$.
It is well known, that there exist for $n = |Var(\Phi) \cup Ind(\Phi)|$ a total of 2^n different functions $f : (Var(\Phi) \cup Ind(\Phi)) \to \{0, 1\}$.

From the different Boolean conjunctive queries we construct the union of conjunctive queries:

$$\Psi := \bigvee_{f:Var(\Phi)\cup Ind(\Phi)\to\{0,1\}} \Phi_f$$

with $Var(\Psi) = Var(\Phi)$ and $Ind(\Psi) = Ind(\Phi)$

Let us prove for a given Boolean conjunctive query Φ and a knowledge base $\mathcal{K}$, that entailment with a union of conjunctive queries obtained by the recursive approach is equivalent to entailment with a union of conjunctive queries obtained by construction from the functions $f : (Var(\Phi) \cup Ind(\Phi)) \to \{0, 1\}$.

Lemma 4.24. *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a knowledge base, Φ a Boolean conjunctive query. $\Psi_1 = \Phi_1 \vee \ldots \vee \Phi_n$ is the union of conjunctive queries obtained from Φ by the recursive rule and $\Psi_2 = \Theta_1 \vee \ldots \vee \Theta_m$ is the union of conjunctive queries explicitly obtained from Φ and the functions $f : (Var(\Phi) \cup Ind(\Phi)) \to \{0, 1\}$. Then the following is true:*

$$\mathcal{K} \models \Psi_1 \iff \mathcal{K} \models \Psi_2$$

Proof. First of all we know by construction of the unions of conjunctive queries from Φ, that $Var(\Psi_1) = Var(\Psi_2) = Var(\Phi)$ and $Ind(\Psi_1) = Ind(\Psi_2) = Ind(\Phi)$

Assume $\mathcal{K} \models \Psi_1$.
For every disjunct $\Phi_i \in \Psi_1$ we can either construct $f : (Var(\Phi) \cup Ind(\Phi)) \to \{0, 1\}$ with $f(x) = 1 \iff N(x) \in \Phi_i$, $f(x) = 0 \iff M(x) \in \Phi_i$, or the disjunct Φ_i is contradictory, because for an $x \in Var(\Psi_1) \cup Ind(\Psi_1)$ it contains both $M(x)$ and $N(x)$.
Consider an arbitrary $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ with $\mathcal{I} \models \Psi_1$, where for $\Phi_i \in \Psi_1$ we have $\mathcal{I} \models \Phi_i$. Then Φ_i is not contradictory. Thus there exists a function $f : Var((\Phi) \cup Ind(\Phi)) \to \{0, 1\}$ and we know by construction for all $s, t \in Var(\Psi_1) \cup Ind(\Psi_1)$ and $j \in \{0, n\}$:

$$A^{(j)}(s,t) \in \Phi_f \iff A^{(j)}(s,t) \in \Phi_i, \quad r^{(j)}(s,t) \in \Phi_f \iff r^{(j)}(s,t) \in \Phi_i$$

$$N(t) \in \Phi_f \iff N(t) \in \Phi_i, \quad M(t) \in \Phi_f \iff M(t) \in \Phi_i$$

Therefore $\mathcal{I} \models \Phi_f \iff \mathcal{I} \models \Phi_i$ and also $\mathcal{K} \models \Psi_2$.

Assume $\mathcal{K} \models \Psi_2$.
For every function $f : (Var((\Psi_2) \cup Ind(\Phi)) \to \{0,1\}$ we can find a disjunct Φ_i in Ψ_1 with $f(x) = 1 \iff N(x) \in \Phi_i$, $f(x) = 0 \iff M(x) \in \Phi_i$ and for all $x \in Var(\Phi) \cup Ind(\Phi)$ there are not both concept atoms $M(x)$ and $N(x)$ conjuncts in Φ_i, because by construction Ψ_1 contains all suitable combinations of labelled concept and role atoms.
Consider an arbitrary $\mathcal{I} \in \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{T})$ with $\mathcal{I} \models \Psi_2$, where for $\Phi_f \in \Psi_2$ we have $\mathcal{I} \models \Phi_f$. Then for the corresponding function $f : (Var(\Phi) \cup Ind(\Phi)) \to \{0,1\}$ we choose the above discussed disjunct $\Phi_i \in \Psi_1$ with $\mathcal{I} \models \Phi_f \iff \mathcal{I} \models \Phi_i$ and also $\mathcal{K} \models \Psi_1$. $\square$

The definition of a knowledge base $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ entailing a union conjunctive queries yields the following:

$$\mathcal{K} \models \bigvee_{f:Var(\Phi)\cup Ind(\Phi)\to\{0,1\}} \Phi_f \iff \forall_{\mathcal{J}\in\mathcal{M}(\mathcal{A})\cap\mathcal{M}(\mathcal{T})} \exists_{f:Var(\Phi)\cup Ind(\Phi)\to\{0,1\}} \mathcal{J} \models \Phi_f$$

Remark 4.25. We can exclude for all $a \in N_I$ the concepts $M(a)$, because by definition the individuals used in the input are always mapped to named elements. But we need the concepts M and N as labels in the proof, where they are used to mark every element from the input. In practice, when forming this union of conjunctive queries, we can exclude all functions mapping $f(a) = 0$ for $a \in N_I$.

Let us summarize what we have done in this chapter. We started with the introduction of Boolean conjunctive queries, their semantics and the notion of a homomorphism from a query to an interpretation. Some examples and considerations about the reduction ABox and TBox and a suitable query where then leading to unions of conjunctive queries. These queries crossed our way because they enable us to choose the right Boolean conjunctive query for every interpretation of the reduction ABox and TBox. Then we have proven a lemma, which yielded the exponential reducibility from the inference problem projection with Boolean conjunctive queries to entailment with unions of conjunctive queries. From this reduction we obtained a decision result for $\mathcal{ALCQIO}$ and a reference for an ExpTime-algorithm that solves the projection problem with Boolean conjunctive queries in $\mathcal{ALCQO}$.

5 Summary and Future Work

This thesis contains, beside an introduction to description logics, two chapters where action problems are reduced to reasoning problems. This reduction basically follows the ideas and constructions first introduced by Franz Baader, Carsten Lutz, Maja Miličić, Ulrike Sattler and Frank Wolter in the article [BLM⁺05] and the accompanying report [BML⁺05].
At this point what remains is to review the results obtained in this thesis and to point to open problems and thoughts.

In the second chapter we introduced actions, given by an effect function, where we have used the terms of a concrete and an abstract action. It is left to show what are the requirements for the abstract approach to subsume the different attempts to give a concrete description logics action formalism.

In the third chapter we extended the reduction from executability and projection of composite actions to ABox consequence by the universal role. To obtain complexity results from this reduction we needed to investigate ABox consequence in the several sublanguages of $\mathcal{ALCQIO}^{\mathcal{U}}$ that contain nominals and a universal role. Unfortunately there is a gap in Lemma 3.7. We do not have a completeness result for ABox consequence in $\mathcal{ALCQO}^{\mathcal{U}}$, where the universal role is allowed to occur in qualified number restriction. I suspect for unary coding of numbers NExpTime-hardness can be proven by a reduction from a bounded domino problem.
In more than one case we had to assume unary coding of numbers in number restriction. The upper bound for concept satisfiability in $\mathcal{ALCQIO}^{\mathcal{U}}$, where numbers are coded in binary remains an open problem whose solution would derestrict all limitations we had to make in this direction.
In the proof of the reduction theorem also several thoughts remain open, for example on nominals. Every $r(a,b)$ occurring in the ABox is split into a concept definition using nominals $A_{r_b} \equiv \exists r.\{b\}$ and a concept assertion $A_{r_b}(a)$. Up to now we do not know whether this is really necessary. Tobies tells us in [Tob01] that a set of cardinality restrictions contained in a so-called CBox can be used to replace nominals. It could be interesting to examine, whether this approach leads to satisfying observations for the description logics action formalism. But as CBoxes also play a minor role in research on description logics we do not know how a CBox could find its place in the reduction. However the correspondence between cardinality restrictions and nominals, observed in Lemma 5.5. by [Tob01] can be interesting, particularly together with the observed connection between the universal role and CBoxes.
Finally we took a closer look at the development of the description logics based action formalism and the new introduced effect function. Whereas we know now that the most

simple global statement is covered by the approach developed in the report [BML+05], with regard to transitivity the limits of semantic expressiveness have been reached. But it is not unlikely that the abstraction of applying actions by the introduction of an effect function yields a solution to the semantic problems arising from transitive roles in the knowledge base or even as a part of an action. As observed before transitive roles play an important role as they are contained in the most powerful, but still decidable description logics. Therefore it can be a next step to investigate which conditions an admissible action formalism has to meet to cover transitivity.

In the fourth chapter we considered queries as a consequence of actions and we thought about projection with Boolean conjunctive queries. Our considerations about reduction queries were leading to unions of conjunctive queries. We could prove reducibility of projection with Boolean conjunctive queries to entailment with unions of conjunctive queries, where this reduction is based on the reduction for projection with assertions and the construction of the reduction ABox and TBox introduced in the report [BML+05]. The discussion about an extension of this results will also lead to better results and greater knowledge of solving the projection problem with Boolean conjunctive queries. An important question here again is under which circumstances this reduction can be extended to description logics that allow for transitive roles. Because we reduce projection with Boolean conjunctive queries to entailment with unions of conjunctive queries it would be important to have more complexity results for this problem. Obviously we can only obtain strong complexity bounds for projection with Boolean conjunctive queries if we know more about query entailment for knowledge bases that contain nominals. At the end of the fourth chapter we gave an explicit construction of the union of conjunctive queries from the Boolean conjunctive query, considered in the input. Clearly there are exponentially many functions $f : Var(\Phi) \cup Ind(\Phi) \rightarrow \{0, 1\}$, but each of them determines a conjunct, that is only polynomial in the input. The last discussion about this thesis was to use this alternative approach, to push the 3-ExpTime result from Corollary 4.22 to a 2-ExpTime border. Because of a lack of time, this issue remains open for further exploration.

Bibliography

[BBL13] Franz Baader, Stefan Borgwardt, and Marcel Lippmann. On the complexity of temporal query answering. LTCS-Report 13-01, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2013.

[BCM+03] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications.* Cambridge University Press, 2003.

[BL06] F. Baader and C. Lutz. Description logic. In Patrick Blackburn, Johan van Benthem, and Frank Wolter, editors, *The Handbook of Modal Logic*, pages 757–820. Elsevier, 2006.

[BLL10a] Franz Baader, Marcel Lippmann, and Hongkai Liu. Adding causal relationships to DL-based action formalisms. LTCS-Report 10-01, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany, 2010.

[BLL10b] Franz Baader, Marcel Lippmann, and Hongkai Liu. Using causal relationships to deal with the ramification problem in action formalisms based on description logics. In Christian G. Fermüller and Andrei Voronkov, editors, *Proceedings of the 17th International Conference on Logic for Programming, Artifical Intelligence, and Reasoning (LPAR-17)*, volume 6397 of *Lecture Notes in Computer Science (subline Advanced Research in Computing and Software Science)*, pages 82–96, Yogyakarta, Indonesia, October 2010. Springer-Verlag.

[BLM+05] F. Baader, C. Lutz, M. Milicic, U. Sattler, and F. Wolter. Integrating description logics and action formalisms: First results. In *Proceedings of the Twentieth National Conference on Artificial Intelligence (AAAI-05)*, Pittsburgh, PA, USA, 2005.

[BML+05] F. Baader, M. Milicic, C. Lutz, U. Sattler, and F. Wolter. Integrating description logics and action formalisms for reasoning about web services. LTCS-Report LTCS-05-02, Chair for Automata Theory, Institute for Theoretical Computer Science, Dresden University of Technology, Germany, 2005.

[BZ13a] Franz Baader and Benjamin Zarrieß. Verification of golog programs over description logic actions. In Pascal Fontaine, Christophe Ringeissen, and Renate A. Schmidt, editors, *Proceedings of the 9th International Symposium on Frontiers of Combining Systems (FroCoS 2013)*, Lecture Notes in Computer Science, Nancy, France, September 2013. Springer-Verlag.

[BZ13b] Franz Baader and Benjamin Zarrieß. Verification of golog programs over description logic actions. LTCS-Report 13-08, Chair of Automata Theory, TU Dresden, Dresden, Germany, 2013.

[Cal12] Diego Calvanese. Description logics for accessing data. Lecture at the EPCL Basic Training Camp, 2012.

[CDGL+09] Diego Calvanese, Giuseppe De Giacomo, Domenico Lembo, Maurizio Lenzerini, Antonella Poggi, Mariano Rodriguez-Muro, and Riccardo Rosati. Ontologies and databases: The dl-lite approach. In Sergio Tessaris and Enrico Franconi, editors, *Semantic Technologies for Informations Systems - 5th Int. Reasoning Web Summer School (RW 2009)*, volume 5689 of *Lecture Notes in Computer Science*, pages 255–356. Springer, 2009.

[CM77] Ashok K. Chandra and Philip M. Merlin. Optimal implementation of conjunctive queries in relational data bases. In *Proceedings of the ninth annual ACM symposium on Theory of computing*, STOC '77, pages 77–90, New York, NY, USA, 1977. ACM.

[DLNS94] Francesco M. Donini, Maurizio Lenzerini, Daniele Nardi, and Andrea Schaerf. Deduction in concept languages: From subsumption to instance checking. *J. Log. Comput.*, 4(4):423–452, 1994.

[GHS08] Birte Glimm, Ian Horrocks, and Ulrike Sattler. Unions of conjunctive queries in shoq. In *KR*, pages 252–262, 2008.

[Gia95] Giuseppe De Giacomo. *Decidability of class-based knowledge representation formalisms*. PhD thesis, Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", 1995.

[GL94] Giuseppe De Giacomo and Maurizio Lenzerini. Boosting the correspondence between description logics and propositional dynamic logics. In Barbara Hayes-Roth and Richard E. Korf, editors, *Proc. of the 12th Nat. Conf. on Artificial Intelligence (AAAI 1994)*, volume 1, pages 205–212. AAAI Press, The MIT Press, 1994.

[GLHS08] Birte Glimm, Carsten Lutz, Ian Horrocks, and Ulrike Sattler. Conjunctive query answering for the description logic shiq. *J. Artif. Intell. Res. (JAIR)*, 31:157–204, 2008.

[GR09] Birte Glimm and Sebastian Rudolph. Conjunctive query entailment: Decidable in spite of o, i, and q. In *Description Logics*, 2009.

[GR10] Birte Glimm and Sebastian Rudolph. Status qio: Conjunctive query entailment is decidable. In *KR*, 2010.

[KMKH11] Markus Krötzsch, Frederick Maier, Adila Alfa Krisnadhi, and Pascal Hitzler. Nominal schemas for integrating rules and description logics. In Riccardo Rosati, Sebastian Rudolph, and Michael Zakharyaschev, editors, *Description Logics*, volume 745 of *CEUR Workshop Proceedings*. CEUR-WS.org, 2011.

[Neb91] B. Nebel. Terminological cycles: Semantics and computational properties. In J. F. Sowa, editor, *Principles of Semantic Networks: Explorations in the Representation of Knowledge*, pages 331–361. Kaufmann, San Mateo, 1991.

[Pap94] Christos M. Papadimitriou. *Computational complexity*. Addison-Wesley, Reading, Massachusetts, 1994.

[PH05] Ian Pratt-Hartmann. Complexity of the two-variable fragment with counting quantifiers. *Journal of Logic, Language and Information*, 14(3):369–395, 2005.

[Rei01] Raymond Reiter. *Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems*. The MIT Press, Massachusetts, MA, 2001.

[RG10] Sebastian Rudolph and Birte Glimm. Nominals, inverses, counting, and conjunctive queries or: Why infinity is your friend! *J. Artif. Intell. Res. (JAIR)*, 39:429–481, 2010.

[Sch91] Klaus Schild. A correspondence theory for terminological logics: Preliminary report. In John Mylopoulos and Raymond Reiter, editors, *Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI 1991)*, pages 466–471. Morgan Kaufmann, 1991.

[Sch94] Andrea Schaerf. Reasoning with individuals in concept languages. *Data and Knowledge Engineering*, 13(2):141–176, 1994.

[Tob00] Stephan Tobies. The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. *Journal of Artificial Intelligence Research*, 12:199–217, 2000.

[Tob01] Stephan Tobies. *Complexity Results and Practical Algorithms*. PhD thesis, LuFG Theoretical Computer Science, RWTH-Aachen, Germany, 2001.

[vHvHLP07] Frank van Harmelen, Frank van Harmelen, Vladimir Lifschitz, and Bruce Porter. *Handbook of Knowledge Representation*. Elsevier Science, San Diego, USA, 2007.

[YLL+12] Wael Yehia, Hongkai Liu, Marcel Lippmann, Franz Baader, and Mikhail Soutchanski. Experimental results on solving the projection problem in action formalisms based on description logics. In Yevgeny Kazakov, Domenico Lembo, and Frank Wolter, editors, *Proceedings of the 25th International Workshop on Description Logics (DL-2012)*, volume 846 of *CEUR Workshop Proceedings*, Rome, Italy, June 2012. CEUR-WS.org.

List of Figures

ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Investigations on an Action Formalism based on the Description Logic $\mathcal{ALCQIO}$" unter Betreuung von Herrn Marcel Lippmann, Herrn Prof. Franz Baader und Herrn Prof. Bernhard Ganter selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum | Unterschrift