Combination of Compatible Reduction Orderings that are Total on Ground Terms

– Extended Abstract –

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1 Introduction

Reduction orderings that are *total on ground terms* play an important rôle in many areas of automated deduction. For example, unfailing completion [4]—a variant of Knuth-Bendix completion that avoids failure due to incomparable critical pairs—presupposes such an ordering. In addition, using a reduction ordering that is total on ground terms, one can show that any finite set of ground equations has a decidable word problem [13, 20]. It is very easy to obtain such orderings. Indeed, many of the standard methods for constructing reduction orderings yield orderings that are total on ground terms: both Knuth-Bendix orderings [12] and lexicographic path orderings [10] are total on ground terms if they are based on a *total precedence ordering* on the set of function symbols.

Things become more complex if one is interested in reduction orderings that are *compatible with a given equational theory E*. Such orderings, which are, for example, used in rewriting modulo equational theories [8, 9, 2], can be seen as orderings on E-equivalence classes. E-compatible reduction orderings that are total on (E-equivalence classes of) ground terms can be employed for similar purposes as the usual reduction orderings that are total on ground terms. For example, let AC denote a theory that axiomatizes associativity and commutativity of several binary function symbols, where the signature may contain additional free function symbols. An AC-compatible reduction ordering that is total on ground terms can be used to show that for any finite set G of ground equations, the word problem is decidable for $AC \cup G$ [14, 15]. The first AC-compatible reduction ordering total on ground terms was described in [15]. It is based on a relatively complex polynomial interpretation in which the coefficients of the polynomials are again integer polynomials. Surprisingly, it turned out to be rather hard to construct AC-compatible reduction orderings by appropriately modifying standard orderings such as recursive path orderings [7]. The main idea underlying most proposals in this direction (e.g., [5, 3, 11, 6]) is to apply certain transformations such as flattening to the terms before comparing them with one of the standard path orderings. A major drawback of these approaches is that they impose rather strong restrictions on the precedence orderings on function symbols that may be used. One consequence of these restrictions is that the obtained AC-compatible orderings are not total on ground terms if more than one AC-symbol is present. This problem has finally been overcome in [18, 19], where an AC-compatible reduction ordering total on ground terms is defined that is based on a recursive path ordering (with status). In [17] it was shown that this approach can even be used to construct reduction orderings total on ground terms that are compatible with theories that axiomatize several associative, commutative, associative-commutative, and free symbols.

The present paper proposes a different way of attacking the problem of how to construct E-compatible orderings that are total on ground terms. It was motivated by the observation that it is very easy to define an AC-compatible reduction ordering total on ground terms if there is only one AC-symbol in the signature. Instead of directly defining an AC-compatible ordering total on ground terms for the case of more than one AC-symbol, we try to obtain such an ordering by combining the orderings that exist for the case of one AC-symbol.[1] To be more precise, assume that AC_1 axiomatizes associativity-commutativity of the symbol $+ \in \Sigma_1$ and that AC_2 axiomatizes associativity-commutativity of the symbol $* \in \Sigma_2$, where Σ_1 and Σ_2 are disjoint signatures that may contain additional free function symbols. For $i = 1, 2$, let $\succ_i$ be an AC_i-compatible reduction ordering that is total on the AC_i-equivalence classes of ground terms, i.e., $\succ_i$ can be seen as a total ordering on $\mathcal{T}(\Sigma_i, \emptyset)/_{=_{AC_i}}$. In order to define a reduction ordering that is total on $\mathcal{T}(\Sigma_1 \cup \Sigma_2, \emptyset)/_{=_{AC_1 \cup AC_2}}$ from the given orderings $\succ_1$ and $\succ_2$, we utilize the fact that this combined algebra can be represented as the amalgamated product of the single algebras $\mathcal{T}(\Sigma_i, \emptyset)/_{=_{AC_i}}$. This product was introduced in [1] in the context of combining unification algorithms. The construction of the amalgamated product represents the universe of $\mathcal{T}(\Sigma_1 \cup \Sigma_2, \emptyset)/_{=_{AC_1 \cup AC_2}}$ as a (possibly infinite) tower of layers. In principle, the combined ordering compares elements of the combined algebra first with respect to the layers they are in: elements in higher layers are larger than elements in lower ones. If two elements are in the same layer, then one of the original orderings ($\succ_1$ or $\succ_2$) is used to compare them.

This combination approach is, of course, not restricted to AC-theories. It can be used to combine arbitrary compatible reduction orderings that are to-

[1] This should not be confused with Rubio's approach for combining orderings on disjoint signatures [17]. To obtain his combined ordering, which extends given orderings on terms over the single signatures to an ordering on terms over the union of the signatures, he presupposes the existence of a compatible reduction ordering total on ground terms for the combined signature. In the present paper, the main goal is to show that such an ordering exists.

tal on ground terms, provided that the single theories are over disjoint signatures and satisfy some additional properties that will be introduced below. For example, theories that axiomatize associativity, commutativity, or associativity-commutativity of a binary function symbol satisfy these properties.

2 Compatible reduction orderings

Let Σ be a signature, and let $T(\Sigma, X)$ denote the terms over Σ with variables in X. A *reduction ordering* on $T(\Sigma, X)$ is a strict partial ordering $\succ$ that is Noetherian, stable under Σ-operations (i.e., $s \succ t$ implies $f(\ldots, s, \ldots) \succ f(\ldots, t, \ldots)$ for all $f \in \Sigma$), and stable under substitutions (i.e., $s \succ t$ implies $\sigma(s) \succ \sigma(t)$ for all Σ-substitutions σ). In the following, we will restrict our attention to reduction orderings on ground terms, which means that stability under substitutions can be dispensed with. However, the ground terms that will be considered may contain additional *free constants* from a set of constants C with $C \cap \Sigma = \emptyset$. By a slight abuse of notation, the set of these ground terms will be written as $T(\Sigma, C)$. The only difference between variables and free constants is the fact that constants cannot be replaced by substitutions, and thus it is possible to order them with a reduction ordering.

Let E be a set of identities over Σ, and let $=_E$ denote the equational theory induced by E. A reduction ordering $\succ$ is *E-compatible* iff $s \succ t$, $s =_E s'$, and $t =_E t'$ imply $s' \succ t'$. Thus, an E-compatible reduction ordering induces a well-defined ordering on the set of $=_E$-equivalence classes. For a set of free constants C, the E-free algebra with generators C, i.e., $\mathcal{T}(\Sigma, C)/_{=_E}$, will be denoted by $\langle C \rangle_{\Sigma,E}$. The set of free constants occurring in a term t is denoted by $C(t)$. We call a reduction ordering *total on* $\langle C \rangle_{\Sigma,E}$ (or simply "total on ground terms," if the set of ground terms is clear from the context) iff it induces a total ordering on $\langle C \rangle_{\Sigma,E}$, i.e., iff for all $s, t \in T(\Sigma, C)$ we have $s \succ t$, or $s =_E t$, or $s \prec t$.

If E is a consistent equational theory (i.e., admits models of cardinality greater than 1), then we have $c \neq_E c'$ for every pair of distinct free constants $c, c' \in C$. Thus, an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$ yields a total Noetherian ordering on C. We say that an E-compatible reduction ordering *extends* a total Noetherian ordering $>$ on C iff its restriction to C coincides with $>$. In the following, we consider *only consistent equational theories* (without mentioning it explicitly as a condition).

We close this second by stating some properties of equational theories and reduction orderings compatible with equational theories that will be important for the proof of our combination result:

Lemma 2.1 *1. If there exists a non-empty E-compatible reduction ordering, then E is a regular equational theory. In particular, we have for all terms*

$s, t \in T(\Sigma, C)$ that $s =_E t$ implies $C(s) = C(t)$.

2. *If there exists a non-empty E-compatible reduction ordering, then for any free constant $c \in C$ and term $t \in T(\Sigma, C)$ we can have $c =_E t$ only if c occurs exactly once in t.*

3. *If $\succ$ is an E-compatible reduction ordering total on $\langle C\rangle_{\Sigma,E}$, then $c \in C(t)$ for a free constant $c \in C$ and a term $t \neq_E c$ implies $t \succ c$.*

4. *Let $\succ$ be an E-compatible reduction ordering total on $\langle C\rangle_{\Sigma,E}$, and assume that $0 \in \Sigma$ is a signature constant and $c \in C$ is a free constant. If there exists a term s* containing 0 *such that $s =_E c$, then 0 is the smallest element of $\langle C\rangle_{\Sigma,E}$ with respect to $\succ$.*

3 Combination of compatible reduction orderings

In principle, we want to solve the following combination problem: Let Σ_1, Σ_2 be disjoint signatures and E_1, E_2 be equational theories over the respective signature. Assume that, for $i = 1, 2$ and any set C of free constants, there exists an E_i-compatible reduction ordering $\succ_i$ that is total on $\langle C\rangle_{\Sigma_i,E_i}$. Can the orderings $\succ_1, \succ_2$ be used to construct an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$?

The next example demonstrates that this is not always possible.

Example 3.1 Let $\Sigma_1 := \{+, 0\}$, $\Sigma_2 := \{*, 1\}$, $E_1 := \{x + 0 = x\}$, and $E_2 := \{x*1 = x\}$. It is easy to see that there exist E_i-compatible reduction orderings $\succ_i$ that are total on $\langle C\rangle_{\Sigma_i,E_i}$. In fact, since any term in $T(\Sigma_1, C)$ is $=_{E_1}$-equivalent to a term in $T(\{+\}, C)$, and since $=_{E_1}$ is the syntactic equality on $T(\{+\}, C)$, one can simply take a lexicographic path ordering that is induced by a well-ordering of C. The same argument applies to E_2.

However, assume that $\succ$ is an $(E_1 \cup E_2)$-compatible reduction ordering total on $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$. Obviously, we have $c+0 =_{E_1\cup E_2} c$ and $c*1 =_{E_1\cup E_2} c$. By Property 4 of Lemma 2.1, both 0 and 1 must be the smallest element in $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$, which is a contradiction since $0 \neq_{E_1\cup E_2} 1$.

In our general combination result, this kind of problem is avoided by restricting the attention to theories whose signatures do not contain constant symbols, i.e., the only constants that may occur are free constants.[2]

There is a second restriction that must hold for our method to apply. The orderings $\succ_1, \succ_2$ must satisfy the following constant dominance condition:

[2] Actually, it would be sufficient to apply this restriction to one of the two theories to be combined.

Definition 3.2 Let $\succ$ be an E-compatible reduction ordering total on $\langle C\rangle_{\Sigma,E}$. Then $\succ$ satisfies the *constant dominance condition (CDC)* iff for all $t \in T(\Sigma, C)$ and $c \in C$ such that $c \succ c'$ for all $c' \in C(t)$, we have $c \succ t$.

Intuitively, this means that large constants dominate terms containing only small constants. An arbitrary E-compatible reduction ordering total on ground terms need not satisfy this property. For certain equational theories, however, the existence of an arbitrary E-reduction ordering total on ground terms implies the existence of such an ordering that also satisfies the CDC. Let C be a countably infinite set of free constants. For a term $t \in T(\Sigma, C)$ and a free constant $c \in C$, let $|t|_c$ denote the number of occurrences of c in t. We say that the equational theory E is *strongly regular* iff $s =_E t$ implies $|s|_c = |t|_c$ for all terms $s, t \in T(\Sigma, C)$ and free constants c.

Lemma 3.3 *Let E be strongly regular. If there exists an E-compatible reduction ordering total on $\langle C\rangle_{\Sigma,E}$, then there also exists such an ordering that additionally satisfies the CDC.*

For example, theories axiomatizing commutativity, associativity, or associativity-commutativity of a binary function symbol are obviously strongly regular.

Our method for combining compatible reduction orderings depends on the representation of $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$ as the free amalgamated product of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$, as introduced in [1].[3]

The free amalgamated product

The free amalgamated product of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$ is defined using two ascending towers of the following form: We consider disjoint sets of free constants $C_\infty = \bigcup_{i=0}^{\infty} C_i$ and $D_\infty = \bigcup_{i=0}^{\infty} D_i$ such that $C_0 = C$. In addition, for $n \geq 0$, let A_n be the carrier set of $\langle \bigcup_{i=0}^{n} C_i\rangle_{\Sigma_1,E_1}$, and let B_{n+1} be the carrier set of $\langle \bigcup_{i=0}^{n} D_i\rangle_{\Sigma_2,E_2}$. The partitioning of C_∞ and D_∞ into the sets C_i and D_i is such that sets on corresponding floors of the double tower shown in Figure 1 have the same cardinality.

Thus, there are bijections $h_0 : A_0 \to D_0$, $g_1 : B_1 \setminus D_0 \to C_1$, and for all $n \geq 1$, bijections $h_n : A_n \setminus (A_{n-1} \cup C_n) \to D_n$ and $g_{n+1} : B_{n+1} \setminus (B_n \cup D_n) \to C_{n+1}$.

Let A_∞ be the carrier set of $\langle C_\infty\rangle_{\Sigma_1,E_1}$, i.e., the union of all set in the left tower, and let B_∞ be the carrier set of $\langle D_\infty\rangle_{\Sigma_2,E_2}$, i.e., the union of all set in the right tower. The above bijections can be used in the obvious way to define

[3]It should be noted, however, that we use a slightly modified construction, which is not as symmetric as the original one, but more easy to adapt to our purposes.

$\dots$	$\dots$
$A_{n+1} \setminus (A_n \cup C_{n+1})$	D_{n+1}
C_{n+1}	$B_{n+1} \setminus (B_n \cup D_n)$
$A_n \setminus (A_{n-1} \cup C_n)$	D_n
C_n	$B_n \setminus (B_{n-1} \cup D_{n-1})$
$\dots$	$\dots$
C_2	$B_2 \setminus (B_1 \cup D_1)$
$A_1 \setminus (A_0 \cup C_1)$	D_1
C_1	$B_1 \setminus D_0$
A_0	D_0

Figure 1: The double tower of the amalgamation construction.

bijections

$$h_\infty := \bigcup_{i=0}^{\infty} h_i \cup g_{i+1}^{-1} : A_\infty \to B_\infty \quad \text{and} \quad g_\infty := \bigcup_{i=0}^{\infty} h_i^{-1} \cup g_{i+1} : B_\infty \to A_\infty.$$

By definition, A_∞ is equipped with a Σ_1-structure, and the bijections h_∞ and g_∞ can be used to carry the Σ_2-structure on B_∞ to A_∞ (see [1] for details). As shown in [1], the $(\Sigma_1 \cup \Sigma_2)$-algebra $\mathcal{A}_\infty$ with carrier set A_∞ that is obtained this way is isomorphic to $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$.

An ordering on the free amalgamated product

As mentioned above, we assume that the signatures Σ_1 and Σ_2 do not contain constant symbols, i.e., the only constants are free constants. In addition, assume that, for $i = 1, 2$, there is a mechanism for constructing E_i-compatible reduction orderings that satisfies the following properties:

1. For any finite or countably infinite set of free constants C and any total Noetherian ordering $>$ on C, the mechanism yields an E_i-compatible reduction ordering $\succ_{C,>}^{(i)}$ that extends $>$, is total on $\langle C \rangle_{\Sigma_i, E_i}$, and satisfies the CDC.
2. The mechanism is monotone in the following sense: Let $C_1 \subseteq C_2$, let $>_1$ be a total Noetherian ordering on C_1, and let $>_2$ be a total Noetherian

ordering on C_2 such that $>_1 \subseteq >_2$. Then $\succ^{(i)}_{C_1,>_1} \subseteq \succ^{(i)}_{C_2,>_2}$.

3. The mechanism is invariant under monotone renaming of free constants. To be more precise, let $>_1$ be a total Noetherian ordering on C_1, $>_2$ be a total Noetherian ordering on C_2, and let $\pi : C_1 \to C_2$ be an order isomorphism. Then $s \succ^{(i)}_{C_1,>_1} t$ implies $\pi(s) \succ^{(i)}_{C_2,>_2} \pi(t)$, where the terms $\pi(s), \pi(t)$ are obtained from s, t by replacing the free constants in these terms by their π-images.

Theorem 3.4 *Assume that Σ_1 and Σ_2 are disjoint signatures that do not contain constant symbols, and that, for $i = 1, 2$, there exist mechanisms for constructing E_i-compatible reduction orderings total on ground terms satisfying the three conditions from above.*

1. *Then there exists an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$.*

2. *If the word problem for E_i and the orderings $\succ^{(i)}_{C,>}$ are decidable for $i = 1, 2$, then the combined ordering is also decidable.*

Instead of giving a formal proof of the first part of the theorem (which would violate the page limit), we give an intuitive description of how this ordering looks like. Its definition depends on the representation of $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$ as the free amalgamated product $\mathcal{A}_\infty$ of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$. Going from bottom to top, one simultaneously defines an ordering on A_∞ and B_∞ by induction. Elements that belong to different levels of one of the towers are compared according to their height in the tower. Elements in a level $A_n \setminus (A_{n-1} \cup C_n)$ are compared with respect to the E_1-compatible ordering on A_n obtained by the mechanism (assuming that the precedence ordering on $\bigcup_{i=0}^n C_i$ is already defined). Elements in a level C_n are ordered using the bijection $g_n : B_n \setminus (B_{n-1} \cup D_{n-1}) \to D_n$ (assuming that the ordering on $B_n \setminus (B_{n-1} \cup D_{n-1})$ is already defined). The right tower is treated analogously.

In this construction, the induction base is given by an arbitrary total Noetherian ordering on C. The combined ordering obtained this way depends on the set C and on the ordering on C used for starting the inductive construction. Thus, we again obtain a construction mechanism that transforms a given total Noetherian ordering on a set of free constants C into an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$. The combined ordering does not satisfy the CDC. However, if E_1 and E_2 are strongly regular, then so is $E_1 \cup E_2$. Thus, Lemma 3.3 can be used to modify the combined ordering into one satisfying the CDC. It can be shown that the mechanism satisfies the other properties required in Theorem 3.4. Consequently, the construction can be applied iteratedly, provided that the involved theories are strongly regular.

The decision procedure for the combined ordering depends on a method that is similar to the approach used to show that the word problem for $E_1 \cup E_2$ is decidable, provided that the word problems for the single theories E_1, E_2 are decidable (see, e.g., [16]).

4 Conclusion

The aim of this work was to develop a general approach for combining compatible orderings that are total on ground terms. The main motivation was that it is often relatively easy to design such orderings for "small" signatures and theories, whereas it is rather involved to give a direct definition of an appropriate ordering in the case of signatures that contain several symbols axiomatized by equational theories over disjoint subsets of the signature. As an example, we have mentioned the case of signatures containing free symbols and more than one AC-symbol.

The main restrictions that must hold for this combination approach to apply are

1. The signatures of the single theories must not contain constant symbols, i.e., the only available constants are free constants.
2. Both theories must admit compatible orderings total on ground terms that satisfy the constant dominance condition (CDC).

These restrictions seem to be not overly severe. In fact, we have shown by an example that a violation of the first condition may lead to cases where a compatible ordering total on ground terms does not exist for the combined theory. In addition, for strongly regular theories (such as associativity, commutativity, or associativity-commutativity of a binary function symbol), the existence of a compatible orderings total on ground terms implies the existence such an ordering that also satisfies the CDC.

A major drawback of the presented combination approach is that until now it does not yield a non-trivial ordering for terms with variables. Indeed, we have defined an ordering on $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$, where the elements of C are treated as free *constants*. For an ordering on terms with variables, one must also have stability under substitution. For some application (e.g., the decision problem for ground equations modulo AC), having an ordering on ground terms is sufficient. For other applications where one works with terms containing variables (such as unfailing completion), this is not quite satisfactory. For example, for unfailing completion, using an ordering where all terms with variables are incomparable would mean that none of the identities can be oriented into a rule, and thus all of them must be used in both directions to compute critical pairs. Thus, an important open problem is to extend the combined ordering in a non-trivial way to an ordering on terms with variables. It might be that this makes additional restrictions on the theories necessary (such as requiring them to be collapse-free).

References

[1] F. Baader and K.U. Schulz. Combination of constraint solving techniques: An algebraic point of view. In *Proceedings of the 6th International Conference on Rewriting Techniques and Applications, Kaiserslautern (Germany)*, volume 914 of *Springer LNCS*, pages 352–366, Berlin, 1995.

[2] L. Bachmair. *Canonical Equational Proofs.* Birkhäuser, Boston, Basel, Berlin, 1991.

[3] L. Bachmair. Associative-commutative reduction orderings. *Information Processing Letters*, 43:21–27, 1992.

[4] L. Bachmair, N. Dershowitz, and D.A. Plaisted. Completion without failure. In H. Aït-Kaci and M. Nivat, editors, *Resolution of Equations in Algebraic Structures*, volume 2: Rewriting Techniques, chapter 1, pages 1–30. Academic Press, New York, 1989.

[5] L. Bachmair and D.A. Plaisted. Associative path orderings. In J. P. Jouannaud, editor, *Proceedings of the International Conference on Rewriting Techniques and Applications*, volume 202 of *Springer LNCS*, pages 241–254, Berlin-Heidelberg-New York, 1986.

[6] C. Delor and L. Puel. Extension of the associative path ordering to a chain of associative commutative symbols. In C. Kirchner, editor, *Proceedings of the Fifth International Conference on Rewriting Techniques and Applications (Montreal, Canada)*, volume 690 of *Springer LNCS*, pages 389–404, Berlin, 1993.

[7] N. Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.

[8] G. Peterson and M.E. Stickel. Complete sets of reductions for some equational theories. *Journal of the ACM*, 28:223–264, 1981.

[9] J.-P. Jouannaud and H. Kirchner. Completion of a set of rules modulo a set of equations. *SIAM Journal on Computing*, 15:1155–1196, 1984.

[10] S. Kamin and J.-J. Levy. Two generalizations of the recursive path ordering. Univ. of Illinois at Urbana-Champaign. Unpublished manuscript, 1980.

[11] D. Kapur, G. Sivakumar, and H. Zhang. A new method for proving termination of AC-rewrite systems. In *Proceedings of the Tenth International Conference of Foundations of Software Technology and Theoretical Computer Science*, volume 472 of *Springer LNCS*, pages 133–148, Berlin, 1990.

[12] D.E. Knuth and P.B. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–267. Pergamon Press, Oxford, 1970.

[13] D. S. Lankford. Canonical inference. Memo ATP-32, Automatic Theorem Proving Project, University of Texas, Austin, TX, December 1975.

[14] C. Marche. On ground AC-completion. In R. Book, editor, *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications (Como, Italy)*, volume 488 of *Springer LNCS*, pages 411–422, Berlin, 1991.

[15] P. Narendran and M. Rusinowitch. Any ground associative-commutative theory has a finite canonical system. In R. Book, editor, *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications (Como, Italy)*, volume 488 of *Springer LNCS*, pages 423–433, Berlin, 1991.

[16] T. Nipkow. Combining matching algorithms: The regular case. In N. Dershowitz, editor, *Proceedings of the Third International Conference on Rewriting Techniques and Applications (Chapel Hill, NC)*, volume 355 of *Springer LNCS*, pages 343–358, Berlin, 1989.

[17] A. Rubio. *Automated Deduction with Constrained Clauses.* PhD Thesis, Universita Politècnica de Catalunya, Barcelona, Spain, 1994.

[18] A. Rubio and R. Nieuwenhuis. A precedence-based total AC-compatible ordering. In C. Kirchner, editor, *Proceedings of the Fifth International Conference on Rewriting Techniques and Applications (Montreal, Canada)*, volume 690 of *Springer LNCS*, pages 374–388, Berlin, 1993.

[19] A. Rubio and R. Nieuwenhuis. A total AC-compatible ordering based on RPO. *Theoretical Computer Science*, 142, 1995.

[20] W. Snyder. Efficient ground completion: An $O(nlogn)$ algorithm for generating reduced sets of ground rewrite rules equivalent to a set of ground equations E. In N. Dershowitz, editor, *Proceedings of the Third International Conference on Rewriting Techniques and Applications (Chapel Hill, NC)*, volume 355 of *Springer LNCS*, pages 419–433, Berlin, 1989.