Combination of Constraint Solvers for Free and Quasi-Free Structures*

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Abstract

When combining languages for symbolic constraints, one is typically faced with the problem of how to treat "mixed" constraints. The two main problems are (1) how to define a combined solution structure over which these constraints are to be solved, and (2) how to combine the constraint solving methods for pure constraints into one for mixed constraints. The paper introduces the notion of a "free amalgamated product" as a possible solution to the first problem. We define so-called quasi-free structures (called "strong simply-combinable structures" in a previous publication) as a generalization of free structures. For quasi-free structures over disjoint signatures, we describe a canonical amalgamation construction that yields the free amalgamated product. The combination techniques known from unification theory can be used to combine constraint solvers for quasi-free structures over disjoint signatures into a solver for their free amalgamated product. In addition to term algebras modulo equational theories (i.e., free algebras), the class of quasi-free structures contains many solution structures that are of interest in constraint logic programming, such as the algebra of rational trees, feature structures, and domains consisting of hereditarily finite (wellfounded or non-wellfounded) nested sets and lists.

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1 Introduction

The integration of constraint solvers (i.e., special decision procedures for restricted classes of problems) into general purpose deductive systems aims at a combination of the efficiency of the special method with the universality of the general method. To achieve this goal, different frameworks for deduction and for programming with constraints have been designed (e.g., [11, 25, 23]), and constraint solvers for various constraint languages have been developed (e.g., [14, 27, 2]). Many applications of constraint-based systems, however, require a combination of more than one constraint language, and thus a solver for mixed constraints.

There has been some research on how to combine constraint solvers for specific constraint languages. As early as 1979, Shostak [39] considered the integration of free (i.e., uninterpreted) function symbols into Presburger arithmetic; and in 1981, Stickel [40] described an algorithm for AC-unification with free function symbols, whose termination was finally shown by Fages [22]. In the area of Constraint Logic Programming, mixed constraints are, for example, available in Prolog III [15], where it is possible to formulate conditions on lists of rational trees where some nodes can again be lists etc.; Mukai [31] combines rational trees and record structures, and a domain that integrates rational trees and feature structures has been used in [38]; Rounds [35] introduces set-valued feature structures that interweave ordinary feature structures and non-wellfounded sets, and many other suggestions for integrating sets into logic programming exist [19, 20].

The first more general method for combining decision procedures was proposed by Nelson and Oppen [32]. In their framework, it is possible to combine "decision procedures for two quantifier-free theories into a single decision procedure for their combination." More precisely, Nelson and Oppen consider validity of (implicitly) *universally quantified* formulae in the union of two theories over *disjoint signatures*. The only other restriction on the theories to be combined is that they are *stably-infinite*, i.e., a quantifier-free formula is satisfiable in a model of the theory iff it is satisfiable in an infinite model of the theory [33].

In unification theory, the research on how to combine unification algorithms for equational theories over disjoint signatures has also lead to rather general results [37, 10, 3]. From a logical point of view, one is here interested in validity of *existentially quantified positive* (equational) formulae in all models of the equational theory (or, equivalently, in the free algebra defined by the theory). Thus, the difference to Nelson and Oppen's work is, on the one hand, the existential quantifier prefix, which makes the combination problem more complicated. On the other hand, Nelson and Oppen allow for negation in their formulae, whereas in unification one considers only positive formulae.[1] Another difference between the two situations is that for unification, decidability of the existential positive

[1]Note, however, that the combination result of [3] can be generalized to disunification [4].

theory for the single theories is not sufficient to obtain decidability of the existential positive theory for their combination. In fact, as shown in [3], one needs decidability of unification with *linear constant restrictions* in the single theories. From a logical point of view, this means that the *full* positive theory must be decidable [3].

In the present paper—which simplifies and combines results from [5] and [6]—we generalize the framework for combining decision procedures for unification in equational theories in two directions. As in [5], where we consider free structures instead of just free algebras, the restriction to equational formulae (i.e., a purely functional signature where the only predicate symbol available is equality) is removed. Thus, we allow for relational constraints such as ordering constraints. The more important generalization is, however, that we no longer restrict ourselves to *free* algebras or structures. This is motivated by the fact that many solution domains for symbolic constraints (such as feature algebras and the algebra of rational trees) are not free.

In order to capture such solution domains, we introduce the class of *quasi-free structures* in Section 3.[2] The definition of this class (see Section 3.2) is motivated by an "internal" algebraic characterization of free structures given in Section 3.1. In [6], quasi-free structures have been called strong SC-structures. The new name is motivated by the similarities exhibited in the present paper between free structures and strong SC-structures. The algebra of rational trees [27], feature structures [2, 38], but also domains over hereditarily finite (wellfounded or non-wellfounded) nested sets and lists turn out to be quasi-free structures. The main difference between free structures (treated in [5]) and quasi-free structures is that free structures are generated by a (countably infinite) set of (free) generators, whereas this need not be the case for quasi-free structures (e.g., an infinite rational tree is not generated—in the algebraic sense—by its leaf nodes). Quasi-free structures turn out to have nice algebraic and logical properties, and some useful results in these directions will be collected in Section 3.3 and Section 3.4, respectively.

Given two constraint languages with quasi-free structures as solution domains, it is not a priori clear how to define the *combined solution structure* over which the mixed constraints are to be solved. This is a new problem, which does is not occur in the free case, where the solution structures are defined by logical theories. For example, in unification modulo equational theories E_1, E_2, the single solution structures are the free algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$ for E_1 and E_2. Thus, the obvious candidate for the combined structure is $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$, the free algebra for the union $E_1 \cup E_2$ of the theories.

Section 4 treats, within an algebraic framework, the problem of how to combine quasi-free structures. In the first subsection (Section 4.1), we introduce

[2]It should be noted that the notion of a quasi-free structure is closely related to the concept of a "unification algebra" [36], and to the notion of an "instantiation system" [43].

the abstract notion of a "free amalgamated product" of two *arbitrary* structures. Whenever the free amalgamated product of two given structures $\mathcal{A}_1$ and $\mathcal{A}_2$ exists, it is unique up to isomorphism, and it is the "most general" structure among all structures that are considered as an admissible combination of $\mathcal{A}_1$ and $\mathcal{A}_2$. For the case of free algebras $\mathcal{T}(\Sigma_1, X)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, X)/{=_{E_2}}$, the free amalgamated product yields the combined free algebra $\mathcal{T}(\Sigma_1 \cup \Sigma_2, X)/{=_{E_1 \cup E_2}}$ (see Section 4.2). This indicates that it makes sense to propose the free amalgamated product of two solution structures as an adequate combined solution structure.

In Section 4.3, we introduce an explicit amalgamation construction that, given two quasi-free structures $\mathcal{A}_1$ and $\mathcal{A}_2$ over disjoint signatures Σ_1 and Σ_2, yields a $(\Sigma_1 \cup \Sigma_2)$-structure $\mathcal{A}_1 \otimes \mathcal{A}_2$.[3] We show that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is in fact the free amalgamated product of the two components. As a Σ_i-structure, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is isomorphic to $\mathcal{A}_i$. Consequently, pure Σ_i-constraints are solvable in $\mathcal{A}_i$ iff they are solvable in $\mathcal{A}_1 \otimes \mathcal{A}_2$ $(i = 1, 2)$. Another interesting property of the free amalgamated product $\mathcal{A}_1 \otimes \mathcal{A}_2$ of two quasi-free structures is that it is again a quasi-free structure. For this reason, the amalgamation construction can be used to combine any finite number of quasi-free structures over disjoint signatures. Since free amalgamation of quasi-free structures can be shown to be associative and commutative, the order in which the amalgamation construction is applied is irrelevant.

In Section 5, we show that the decomposition method introduced in [3, 5] can be used to combine constraint solvers for two arbitrary quasi-free structures $\mathcal{A}_1$ and $\mathcal{A}_2$ over disjoint signatures into a solver for $\mathcal{A}_1 \otimes \mathcal{A}_2$. To be more precise, we first show (in Section 5.1) that the scheme reduces the problem of deciding validity of *existential positive sentences* in the combined solution structure to validity of (not necessarily existential) positive sentences in the component structures. Thus, decidability of the *existential* positive theory of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is reduced to decidability of the positive theories of the quasi-free structures $\mathcal{A}_1$ and $\mathcal{A}_2$. It should be noted that the proof of this combination result heavily depends on the explicit construction of the free amalgamated product described in Section 4.3.

In Section 5.2, it is shown that the combination method can also treat *general positive sentences*. Thus, in this subsection, decidability of the *full* positive theory of $\mathcal{A} \otimes \mathcal{B}$ is reduced to decidability of the positive theories of $\mathcal{A}$ and $\mathcal{B}$. The proof of this result depends on the fact that the amalgamation construction is associative. As a consequence of this general combination result, we can deduce that validity of positive sentences is decidable in domains that interweave (finite or rational) trees with feature structures and hereditarily finite (wellfounded or non-wellfounded) sets and lists.

[3]It should be noted that the description of the construction presented here is simpler than the one given in [6], and it allows for much shorter proofs.

2 Formal Preliminaries

A *signature* Σ consists of a set Σ_F of function symbols and a set Σ_P of predicate symbols, each of fixed arity. We assume that equality "=" is a logical constant that does not occur in Σ_P, and which is always interpreted as the identity relation. Σ_F-terms are composed using the function symbols of Σ_F and variables from a countably infinite set V. An atomic Σ-formula is an equation $s = t$ between Σ_F-terms s, t, or a relational formula $p[s_1, \ldots, s_m]$ where p is a predicate symbol in Σ_P of arity m and $s_1, \ldots, s_m$ are Σ_F-terms. A positive Σ-matrix is any Σ-formula obtained from atomic Σ-formulae using conjunction and disjunction only. A *positive* Σ*-formula* is obtained from a positive Σ-matrix by adding an arbitrary quantifier prefix, and an *existential positive* Σ*-formula* is a positive formula where the prefix consists of existential quantifiers only. Sentences are formulae without free variables. The notation $t(v_1, \ldots, v_n)$ (resp. $\varphi(v_1, \ldots, v_n)$) indicates that the set of all (free) variables of the term t (of the formula φ) forms a subset of $\{v_1, \ldots, v_n\}$. Letters $u, v, \ldots$ denote variables, and expressions $\vec{u}, \vec{v}, \ldots$ denote finite, possibly empty sequences of variables.

A Σ-*structure* $\mathcal{A}^\Sigma$ has a non-empty carrier set A, and it interprets each $f \in \Sigma_F$ of arity n as an n-ary (total) function $f_\mathcal{A}$ on A, and each $p \in \Sigma_P$ of arity m as an m-ary relation $p_\mathcal{A}$ on A. Whenever we use a roman letter like A and an expression $\mathcal{A}^\Sigma$ in the same context, the former symbol denotes the carrier set of the Σ-structure denoted by the latter expression. Sometimes we will consider several signatures simultaneously. If Δ is a subset of the signature Σ, then any Σ-structure $\mathcal{A}^\Sigma$ can be considered as a Δ-structure (called the Δ-reduct of $\mathcal{A}^\Sigma$) by just forgetting about the interpretation of the additional symbols. In this situation, $\mathcal{A}^\Delta$ denotes the Δ-reduct of $\mathcal{A}^\Sigma$.

If $\mathcal{A}^\Sigma$ is a Σ-structure, every assignment $\nu : V \to A$ has a unique extension to an evaluation $\widehat{\nu}$ that maps each Σ-term $t = t(v_1, \ldots, v_n)$ to an element $\widehat{\nu}(t) \in A$. An element $a \in A$ is *generated* by the subset A_0 of A if there exists a Σ-term $t = t(v_1, \ldots, v_n)$ and an assignment $\nu : V \to A$ such that $\widehat{\nu}(t) = a$ and $\nu(v_i) \in A_0$ for $i = 1, \ldots, n$. The subset A_1 of A is generated by $A_0 \subseteq A$ if every element $a \in A_1$ is generated by A_0.

We write $\mathcal{A}^\Sigma \models \varphi(a_1, \ldots, a_n)$ to express that the formula $\varphi(v_1, \ldots, v_n)$ is valid in $\mathcal{A}^\Sigma$ under the evaluation that maps v_i to $a_i \in A$ ($1 \leq i \leq n$). Expressions $\vec{a}$ denote finite (possibly empty) sequences $a_1, \ldots, a_k$ of elements of A. In order to simplify notation we will sometimes use $\vec{a}$ also to denote the set $\{a_1, \ldots, a_k\}$.

A Σ-*homomorphism* between two structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ (sometimes called *homomorphic embedding*[4] of $\mathcal{A}^\Sigma$ into $\mathcal{B}^\Sigma$ in the following) is a mapping $h : A \to B$

[4]Note that we allow arbitrary homomorphisms as homomorphic embeddings, i.e., the "embeddings" need not be injective.

such that

$$\begin{aligned} h(f_{\mathcal{A}}(a_1,\ldots,a_n)) &= f_{\mathcal{B}}(h(a_1),\ldots,h(a_n)),\\ p_{\mathcal{A}}[a_1,\ldots,a_n] &\Rightarrow p_{\mathcal{B}}[h(a_1),\ldots,h(a_n)] \end{aligned}$$

for all $f \in \Sigma_F$, $p \in \Sigma_P$, and $a_1,\ldots,a_n \in A$. Letters $h, g, \ldots$ denote homomorphisms, and $Hom^{\Sigma}_{\mathcal{A}-\mathcal{B}}$ denotes the set of all Σ-homomorphisms between $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Sigma}$. In order to increase readability and not to run out of letters, we will often use expressions of the form h^{Σ}_{A-B} to denote an element of $Hom^{\Sigma}_{\mathcal{A}-\mathcal{B}}$. A Σ-endomorphism of $\mathcal{A}^{\Sigma}$ is a homomorphism $h^{\Sigma} : \mathcal{A}^{\Sigma} \to \mathcal{A}^{\Sigma}$. With $End^{\Sigma}_{\mathcal{A}}$ we denote the monoid of all endomorphisms of the Σ-structure $\mathcal{A}^{\Sigma}$, with composition as operation. For a set A, we denote the identity mapping on A by Id_A. If A is the carrier of a Σ-structure $\mathcal{A}^{\Sigma}$, then Id_A is the unit of the monoid $End^{\Sigma}_{\mathcal{A}}$.

A Σ-*isomorphism* is a bijective Σ-homomorphism $h : \mathcal{A}^{\Sigma} \to \mathcal{B}^{\Sigma}$ such that

$$p_{\mathcal{A}}[a_1,\ldots,a_n] \Longleftrightarrow p_{\mathcal{B}}[h(a_1),\ldots,h(a_n)],$$

for all $p \in \Sigma_P$, and all $a_1,\ldots,a_n \in A$. Equivalently, one can require that the inverse mapping h^{-1} is also homomorphic.

Obviously, validity of arbitrary formulae is preserved under isomorphisms. There is a less trivial connection between surjective homomorphisms and positive formulae, which will become important in the proof of correctness of our method for combining constraint solvers (see [29], pp. 143, 144 for a proof).

Lemma 2.1 *Let $h : \mathcal{A}^{\Sigma} \to \mathcal{B}^{\Sigma}$ be a surjective homomorphism between the Σ-structures $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Sigma}$, $\varphi(v_1,\ldots,v_m)$ be a positive Σ-formula, and $a_1,\ldots,a_m$ be elements of A. Then $\mathcal{A}^{\Sigma} \models \varphi(a_1,\ldots,a_m)$ implies $\mathcal{B}^{\Sigma} \models \varphi(h(a_1),\ldots,h(a_m))$.*

Since validity of existential formulae is preserved when going to a superstructure (see, e.g., [28], pp. 131), the following weaker version of Lemma 2.1 holds for arbitrary homomorphisms.

Lemma 2.2 *Let $h : \mathcal{A}^{\Sigma} \to \mathcal{B}^{\Sigma}$ be a homomorphism between the Σ-structures $\mathcal{A}^{\Sigma}$ and $\mathcal{B}^{\Sigma}$, $\varphi(v_1,\ldots,v_m)$ be an existential positive Σ-formula, and $a_1,\ldots,a_m$ be elements of A. Then $\mathcal{A}^{\Sigma} \models \varphi(a_1,\ldots,a_m)$ implies $\mathcal{B}^{\Sigma} \models \varphi(h(a_1),\ldots,h(a_m))$.*

Given a signature Σ, "constraints" are usually introduced as Σ-formulae (of a particular syntactic type) $\varphi(v_1,\ldots,v_n)$ with free variables. The constraint $\varphi(v_1,\ldots,v_n)$ is solvable in the structure $\mathcal{A}^{\Sigma}$ iff there are $a_1,\ldots,a_n \in A$ such that $\mathcal{A}^{\Sigma} \models \varphi(a_1,\ldots,a_n)$. Thus solvability of φ in $\mathcal{A}^{\Sigma}$ and validity of the sentence $\exists v_1 \ldots \exists v_n\ \varphi(v_1,\ldots,v_n)$ in $\mathcal{A}^{\Sigma}$ are equivalent. In this paper we shall always use the second point of view. As constraints we consider existential positive and positive sentences. We are mainly interested in solving "mixed" constraints. This means that we consider two different signatures Σ_1 and Σ_2, with fixed solution structures $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$. A mixed constraint is a positive (or existential positive) $(\Sigma_1 \cup \Sigma_2)$-sentence, which must be solved in an appropriately defined $(\Sigma_1 \cup \Sigma_2)$-structure, the combined solution structure.

If $g : A \to B$ and $h : B \to C$ are mappings, then $g \circ h : A \to C$ denotes their composition. Note that $g \circ h$ means that g is applied first, and then h. Let $g_1 : A \to C$ and $g_2 : B \to D$ be two mappings. We say that g_1 and g_2 coincide on $E \subseteq A \cap B$ iff $g_1(e) = g_2(e)$ for all $e \in E$. The symbol "$\uplus$" denotes disjoint union of sets.

3 Free Structures and Quasi-Free Structures

The algebraic theory of free structures is very similar to the one for free algebras, though considerably less well-known. In the first subsection, we will briefly recall some definitions and results for free structures (see [28, 13, 42] for more information). The usual definition of free structures is *external* in the sense that it refers to a whole class of structures. In the present context (i.e., combination of structures and constraint solvers), a characterization of free structures in terms of their *internal* algebraic structure turns out to be more appropriate. An internal characterization of free structures over countably infinite sets of generators will be used as starting point for the definition of quasi-free structures in the second subsection. In the third and fourth subsection, we derive useful algebraic and logical properties of quasi-free structures.

3.1 Free structures

We start with the usual external characterization of free structures.

Definition 3.1 *Let $\mathcal{K}$ be a class of Σ-structures, let $\mathcal{A}^\Sigma \in \mathcal{K}$ and let X be a subset of A. Then $\mathcal{A}^\Sigma$ is called* free in $\mathcal{K}$ over X *iff $\mathcal{A}^\Sigma$ is generated by X and if every mapping from X into the carrier of a structure $\mathcal{B}^\Sigma \in \mathcal{K}$ can be extended to a Σ-homomorphism of $\mathcal{A}^\Sigma$ into $\mathcal{B}^\Sigma$.*[5]

If $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ are free in the same class $\mathcal{K}$, and if their sets of generators have the same cardinality, then $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ are isomorphic. As shown by the next theorem, it is not really necessary to allow for arbitrary classes of Σ-structures in the definition of free structures. One can restrict the attention to varieties, or to the singleton class consisting of the free structure. As for the case of algebras,

[5]Since $\mathcal{A}^\Sigma$ is generated by X, this homomorphism is unique.

Σ-*varieties* are defined as classes of Σ-structures that are closed under direct products, substructures, and homomorphic images.

Theorem 3.2 *Let $\mathcal{A}^\Sigma$ be a Σ-structure that is generated by X. Then the following conditions are equivalent:*

1. *$\mathcal{A}^\Sigma$ is free over X in some class $\mathcal{K}$ of Σ-structures.*
2. *$\mathcal{A}^\Sigma$ is free over X in some Σ-variety.*
3. *$\mathcal{A}^\Sigma$ is free over X in $\{\mathcal{A}^\Sigma\}$.*

The only non-trivial part of the proof, namely "$1 \to 2$", follows from the fact that an algebra that is free in a class $\mathcal{K}$ is also free in the variety generated by $\mathcal{K}$, i.e., the closure of $\mathcal{K}$ under building direct products, substructures, and homomorphic images (see [28, 13] for details).

The third condition of the theorem gives a characterization of free structures that is independent of any other structure. This motivates the next definition.

Definition 3.3 *A Σ-structure $\mathcal{A}^\Sigma$ is called* free *iff it is free over X in $\{\mathcal{A}^\Sigma\}$ for some subset X of A.*

If X is the chosen set of generators of the free structure $\mathcal{A}^\Sigma$, then we will sometimes indicate this by saying that $(\mathcal{A}^\Sigma, X)$ is free. We can now give the promised internal characterization of free structures over countably infinite sets of generators. It is a simple consequence of known results.

Theorem 3.4 *A Σ-structure $\mathcal{A}^\Sigma$ is free over the countably infinite set X iff*

1. *$\mathcal{A}^\Sigma$ is generated by X,*
2. *for every finite subset X_0 of X, every mapping $h_0 : X_0 \to A$ can be extended to a* surjective *endomorphism of $\mathcal{A}^\Sigma$.*

Proof. First, assume that $\mathcal{A}^\Sigma$ is free over X in $\{\mathcal{A}^\Sigma\}$. By definition, this implies that X generates $\mathcal{A}^\Sigma$. To show the second condition, assume that $h_0 : X_0 \to A$ is given. Let $h_1 : X \setminus X_0 \to X$ be a bijection (which exists since X is infinite and X_0 is finite), and let $h_0 \dot{\cup} h_1 : X \to A$ be the mapping that coincides with h_0 on X_0 and with h_1 on $X \setminus X_0$. Since $\mathcal{A}^\Sigma$ is free over X in $\{\mathcal{A}^\Sigma\}$, there exists an extension of $h_0 \dot{\cup} h_1$ to an endomorphism h of $\mathcal{A}^\Sigma$. Since $\mathcal{A}^\Sigma$ is generated by X, h is surjective.

Now, assume that $\mathcal{A}^\Sigma$ and X satisfy the two conditions of the right-hand side of the equivalence stated in the theorem. To show that $\mathcal{A}^\Sigma$ is free over X in $\{\mathcal{A}^\Sigma\}$, assume that $h_0 : X \to A$ is given. Let $X_1 \subseteq X_2 \subseteq X_3 \ldots$ be an increasing chain of finite subsets of X such that $X = \bigcup_{i=1}^\infty X_i$. For $i \geq 1$, let h_i be the restriction of h_0 to X_i. By assumption, the mappings h_i can be extended to surjective endomorphisms H_i of $\mathcal{A}^\Sigma$.

Let $\mathcal{A}_i^\Sigma$ denote the substructure of $\mathcal{A}^\Sigma$ generated by X_i. It is easy to see that $i < j$ implies that $\mathcal{A}_i^\Sigma$ is a substructure of $\mathcal{A}_j^\Sigma$, and that H_i and H_j coincide on $\mathcal{A}_i^\Sigma$. In addition, any element a of A is generated by finitely many generators, and thus there exists a least index $i(a)$ such that $a \in A_{i(a)}$.

We define the mapping H_0 from A to A as the "limit" of the homomorphisms H_i; more precisely: $H_0(a) := H_{i(a)}(a)$. It remains to be shown that H_0 is a homomorphism. Thus, let f be an n-ary function symbol, and let $a_1, \ldots, a_n$ be elements of A. For $i := \max\{i(a_1), \ldots, i(a_n)\}$ we have $H_0(a_j) = H_i(a_j)$ for all $j, 1 \leq j \leq n$. In addition, since $f_\mathcal{A}(a_1, \ldots, a_n)$ is also in A_i we have $H_0(f_\mathcal{A}(a_1, \ldots, a_n)) = H_i(f_\mathcal{A}(a_1, \ldots, a_n))$. Since H_i is a homomorphism, we obtain $H_0(f_\mathcal{A}(a_1, \ldots, a_n)) = H_i(f_\mathcal{A}(a_1, \ldots, a_n)) = f_\mathcal{A}(H_i(a_1), \ldots, H_i(a_n)) = f_\mathcal{A}(H_0(a_1), \ldots, H_0(a_n))$. The homomorphism condition for predicates can be proved in the same way. □

If one is interested in the question of how free structures can be constructed, the characterization via varieties is more appropriate. We have seen in Theorem 3.2 that every free structure is free for some variety. Conversely, it can be shown that every non-trivial variety contains free structures with sets of generators of arbitrary cardinality [28]. The well-known Birkhoff Theorem says that a class of Σ_F-algebras is a variety iff it is an equational class, i.e., the class of models of a set of equations. For structures, a similar characterization is possible [28].

Theorem 3.5 *A class $\mathcal{V}$ of Σ-structures is a Σ-variety if, and only if, there exists a set G of atomic Σ-formulae*[6] *such that $\mathcal{V}$ is the class of models of G.*

In this situation, we say that $\mathcal{V}$ is the Σ-variety defined by G, and we write $\mathcal{V} = \mathcal{V}(G)$.

A concrete description of free Σ-structures can be obtained as follows (see [28, 42] for more information). Obviously, the Σ_F-reduct of a free Σ-structure $\mathcal{A}^\Sigma$ is a free Σ_F-algebra, and thus it is (isomorphic to) an E-free Σ_F-algebra $\mathcal{T}(\Sigma_F, X)/{=_E}$ for an equational theory E. In particular, the $=_E$-equivalence classes $[s]$ of Σ_F-terms s constitute the carrier of $\mathcal{A}^\Sigma$. It remains to be shown how the predicate symbols are interpreted on this carrier. Since $\mathcal{A}^\Sigma$ is free over X, any mapping from X into $T(\Sigma_F, X)/{=_E}$ can be extended to a Σ-endomorphism of $\mathcal{A}^\Sigma$. This, together with the definition of homomorphisms of structures, shows

[6]As usual, open formulae are here considered as implicitly universally quantified.

that the interpretation of the predicates must be closed under substitution, i.e., for all $p \in \Sigma_P$, all substitutions σ, and all terms $s_1, \ldots, s_m$, if $p[[s_1], \ldots, [s_m]]$ holds in $\mathcal{A}^\Sigma$ then $p[[s_1\sigma], \ldots, [s_m\sigma]]$ must also hold in $\mathcal{A}^\Sigma$. Conversely, it is easy to see that any extension of the Σ_F-algebra $\mathcal{T}(\Sigma_F, X)/{=_E}$ to a Σ-structure that satisfies this property is a free Σ-structure over X.

Example 3.6 Let Σ_F be an arbitrary set of function symbols, and assume that Σ_P consists of a single binary predicate symbol $\leq$. Consider the (absolutely free) term algebra $\mathcal{T}(\Sigma_F, X)$. We can extend this algebra to a Σ-structure by interpreting $\leq$ as the subterm ordering. Another possibility would be to take a reduction ordering [18] such as the lexicographic path ordering. In both cases, we have closure under substitution, which means that we obtain a free Σ-structure. Constraints involving the subterm ordering or reduction orderings are, for example, important in constrained rewriting [25].

Free structures over countably infinite sets of generators are canonical for the positive theory of their variety in the following sense:

Theorem 3.7 *Let $\mathcal{A}^\Sigma$ be free over the countably infinite set X in the Σ-variety $\mathcal{V}(G)$, and let ϕ be a positive Σ-formula. Then the following are equivalent:*

1. *ϕ is valid in all elements of $\mathcal{V}(G)$, i.e., ϕ is a logical consequence of the set of atomic formulae G.*
2. *ϕ is valid in $\mathcal{A}^\Sigma$.*

This theorem explains why it is appropriate to use free structures over countably infinite sets of generators as solution structures when solving positive constraints. The proof is an easy consequence of Lemma 2.1.

We close this subsection by introducing one more definition. If $(\mathcal{A}^\Sigma, X)$ is free *in* a class of Σ-structures $\mathcal{K}$, then, by definition, $\mathcal{A}^\Sigma \in \mathcal{K}$. Some authors (see e.g., [30]) do not assume $\mathcal{A}^\Sigma \in \mathcal{K}$ when defining the notion "free *for* $\mathcal{K}$." We make use of this less restrictive way of defining "free *for* $\mathcal{K}$" in the following situation:

Definition 3.8 *Let $\mathcal{A}^\Sigma$ and $\mathcal{D}^\Sigma$ be Σ-structures, and assume that $X \subseteq A$ generates $\mathcal{A}^\Sigma$. $(\mathcal{A}^\Sigma, X)$ is called* free for *$\mathcal{D}^\Sigma$ if every mapping $X \to D$ has a unique extension to a homomorphism $h_{A-D} \in Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$.*

3.2 Quasi-free structures

In this section, we generalize the definition of free structures, in order to capture typical domains for constraint-based reasoning such as the algebra of rational trees. As illustrating and motivating example for the abstract definitions, we will use free algebras (i.e., free structures where the relational part Σ_R of the signature is empty). In the sequel, let $\mathcal{T} := \mathcal{T}(\Sigma_F, V)/_{=_E}$ be such an algebra (i.e., $\mathcal{T}$ is free over X in the variety defined by the equational theory E, where X consists of the $=_E$-equivalence classes of variables).

Consider an element $[t]$ of $\mathcal{T}$, i.e., the $=_E$-equivalence class of a term t. Obviously, t contains only finitely many variables $v_1, \ldots, v_n$, which shows that $[t]$ is generated by the finite subset $[v_1], \ldots, [v_n]$ of X. Thus, the image of $[t]$ under an endomorphism of $\mathcal{T}$ is determined by the images of the generators $[v_1], \ldots, [v_n]$. In particular, two endomorphisms of $\mathcal{T}$ that coincide on $[v_1], \ldots, [v_n]$ also coincide on $[t]$.

When looking at non-free structures that are used as solution structures for symbolic constraints, one observes that they satisfy algebraic properties that are very similar to those of free algebras. For example, consider the algebra of rational trees where leaves are labeled by constants or variables. This algebra is not generated by the set of variables (since "generated by" talks about a finite process whereas rational trees may be infinite). Nevertheless, a rational tree t contains only a finite number of variables $v_1, \ldots, v_n$, and two endomorphisms of this algebra that coincide on these variables also coincide on t. This means that the variables occurring in rational trees play a rôle that is similar to the rôle of generators in free algebras, even though they do not generate the algebra. This observation motivates the definition of stable hulls and atom sets given below.

Definition 3.9 *Let A_0, A_1 be subsets of the Σ-structure $\mathcal{A}^\Sigma$. Then A_0* stabilizes *A_1 iff all elements h_1 and h_2 of $\mathrm{End}_{\mathcal{A}}^{\Sigma}$ that coincide on A_0 also coincide on A_1. For $A_0 \subseteq A$ the* stable hull of A_0 *is the set*

$$SH_\Sigma^{\mathcal{A}}(A_0) := \{a \in A \mid A_0 \textit{ stabilizes } \{a\}\}.$$

If $A_0 \subseteq A$ stabilizes a singleton set $\{a\}$, we also say that A_0 stabilizes a. The following two lemmas show that the stable hull of a set A_0 has properties that are similar to those of the subalgebra generated by A_0. Note, however, that the stable hull can be larger than the generated subalgebra (see the example of the algebra of rational trees in 3.17). The proofs of the lemmas are easy, and therefore omitted (see [7] for details).

Lemma 3.10 *Let A_0 be a subset of the carrier A of $\mathcal{A}^\Sigma$ such that $SH_\Sigma^{\mathcal{A}}(A_0)$ is non-empty. Then $SH_\Sigma^{\mathcal{A}}(A_0)$ is the carrier of a Σ-substructure of $\mathcal{A}^\Sigma$, and $A_0 \subseteq SH_\Sigma^{\mathcal{A}}(A_0)$.*

In the sequel, we shall not make a notational distinction between stable hulls and the corresponding Σ-structures.

Lemma 3.11 *Let A_0, A_1 be subsets of the Σ-structure $\mathcal{A}^\Sigma$, and let $h \in End_{\mathcal{A}}^{\Sigma}$. If $h(A_0) \subseteq SH_{\Sigma}^{\mathcal{A}}(A_1)$, then $h(SH_{\Sigma}^{\mathcal{A}}(A_0)) \subseteq SH_{\Sigma}^{\mathcal{A}}(A_1)$.*

Definition 3.12 *The set $X \subseteq A$ is an* atom set for $\mathcal{A}^\Sigma$ *if every mapping $X \to A$ can be extended to an endomorphism of $\mathcal{A}^\Sigma$.*

For the free algebra $\mathcal{T}$ generated by X, the set of generators X obviously is an atom set, and two subalgebras generated by subsets X_0, X_1 of X of the same cardinality are isomorphic. The same holds for atom sets and their stable hulls.

Lemma 3.13 *Let X_0, X_1 be non-empty atom sets of $\mathcal{A}^\Sigma$ of the same cardinality. Then every bijection $h_0 : X_0 \to X_1$ can be extended to an isomorphism between $SH_{\Sigma}^{\mathcal{A}}(X_0)$ and $SH_{\Sigma}^{\mathcal{A}}(X_1)$.*

Proof. Let $h_0 : X_0 \to X_1$ be bijective, and let $h_1 : X_1 \to X_0$ denote the inverse mapping. Since X_0 and X_1 are atom sets, both mappings can be extended to endomorphisms $\widehat{h}_0$ and $\widehat{h}_1$ of $\mathcal{A}^\Sigma$. Now $(\widehat{h}_0 \circ \widehat{h}_1) \in End_{\mathcal{A}}^{\Sigma}$ is an endomorphism that coincides with Id_A on X_0. Therefore, it coincides with Id_A on $SH_{\Sigma}^{\mathcal{A}}(X_0)$. Let g_i denote the restriction of $\widehat{h}_i$ to $SH_{\Sigma}^{\mathcal{A}}(X_i)$ $(i = 0, 1)$. Lemma 3.11 shows that

$$\begin{aligned} g_0 : SH_{\Sigma}^{\mathcal{A}}(X_0) &\to SH_{\Sigma}^{\mathcal{A}}(X_1), \\ g_1 : SH_{\Sigma}^{\mathcal{A}}(X_1) &\to SH_{\Sigma}^{\mathcal{A}}(X_0). \end{aligned}$$

We have $g_0 \circ g_1 = Id_{\mathrm{SH}_{\Sigma}^{\mathcal{A}}(X_0)}$, which implies that g_0 is injective and g_1 is surjective. Symmetrically, we can show that g_0 is surjective and g_1 is injective. Thus, g_0 and g_1 are bijective homomorphisms, and g_i is the inverse of g_{1-i} $(i = 0, 1)$. □

We are now ready to introduce the main concept of this article.

Definition 3.14 *A countably infinite Σ-structure $\mathcal{A}^\Sigma$ is called* quasi-free *iff $\mathcal{A}^\Sigma$ has an infinite atom set X where each $a \in A$ is stabilized by a finite subset of X. We denote this quasi-free structure by $(\mathcal{A}^\Sigma, X)$.*

This definition generalizes the characterization of free structures given in Theorem 3.4. The countably infinite set of generators is replaced by a countably infinite atom set, but we retain some of the properties of generators. In the free case, every element of the structure is generated by a finite set of generators, whereas in the quasi-free case it is stabilized by a finite set of atoms. The following lemma shows that the second condition of Theorem 3.4 is satisfied in the quasi-free case.

Lemma 3.15 *Let X be an infinite atom set of the countably infinite Σ-structure $\mathcal{A}^\Sigma$, and let $X_0 \subseteq X$ be finite. Then every mapping $h_0 : X_0 \to A$ can be extended to a surjective endomorphism of $\mathcal{A}^\Sigma$.*

Proof. Obviously, h_0 can be extended to a surjective mapping $h_1 : X \to A$. Since X is an atom set, h_1 can be extended to an endomorphism $h_2 \in End_{\mathcal{A}}^{\Sigma}$. By construction, h_2 is surjective. □

Remark 3.16 Let A^{Σ} be a Σ-structure and $\mathcal{M}$ be a submonoid of $End_{\mathcal{A}}^{\Sigma}$. We obtain useful variants of the notions of "stabilizer", "stable hull", "atom set", and "quasi-free structure" by always referring to $\mathcal{M}$ instead of $End_{\mathcal{A}}^{\Sigma}$. For example, $X \subseteq A$ is an *atom set for* $\mathcal{A}^{\Sigma}$ *w.r.t.* $\mathcal{M}$ if every mapping $X \to A$ can be extended to an endomorphism in $\mathcal{M}$. We say that $(\mathcal{A}^{\Sigma}, X)$ is *quasi-free with respect to* $\mathcal{M}$ if $(\mathcal{A}^{\Sigma}, X)$ satisfies the corresponding variant of Definition 3.14. In [6], such structures were called simply combinable structures (SC-structures). An example of an SC-structure that is not quasi-free is the domain of feature structures, as described in Examples 3.17 below. Most of the results that we will prove for quasi-free structures can be lifted to the more general class of structures $\mathcal{A}^{\Sigma}$ that are quasi-free with respect to some submonoid $\mathcal{M}$ of $End_{\mathcal{A}}^{\Sigma}$ (see [6] for details).

Examples 3.17 The following examples show that many solution domains for symbolic constraints are indeed quasi-free structures.

Free structures. Obviously, every free structures over a countably infinite set of generators is a quasi-free structure. The atom set is the set of generators of the free structure.

Vector spaces. Let K be a field, let $\Sigma_K := \{+\} \cup \{s_k \mid k \in K\}$. The K-vector space spanned by a countably infinite basis X is a quasi-free structure over the atom set X. Here "+" is interpreted as addition of vectors, and s_k denotes scalar multiplication with $k \in K$.

The algebra of rational trees. Let Σ_F be a finite set of function symbols, and let $\mathcal{R}^{\Sigma_F}$ be the algebra of rational trees [14, 27], where leaves are labelled with constants from Σ_F or with variables from the countably infinite set V. It is easy to see that every mapping $V \to R$ can be extended to a unique endomorphism of $\mathcal{R}^{\Sigma_F}$, and that $(\mathcal{R}^{\Sigma_F}, V)$ is a quasi-free structure. Note, however, that $\mathcal{R}^{\Sigma_F}$ is not generated by V: only the set of *finite* trees is generated by V. In addition, it is easy to see that $\mathcal{R}^{\Sigma_F}$ cannot be a free structure (over any set of generators). Indeed, it is well-known that only trivial equations between Σ_F-terms are valid in $\mathcal{R}^{\Sigma_F}$. Thus, if $\mathcal{R}^{\Sigma_F}$ was free, it would be isomorphic to the absolutely free term algebra, which is not true, however [27].

Hereditarily finite sets. Let $V_{\mathrm{hfs}}(Y)$ be the set of all nested, hereditarily finite (standard, i.e., wellfounded) sets over the countably infinite set of "urelements" Y. Thus, each $M \in V_{\mathrm{hfs}}(Y)$ is finite, and the elements of M are either in Y or in $V_{\mathrm{hfs}}(Y)$, the same holds for elements of elements etc. Wellfounded means that there are no infinite descending membership sequences. Since union is not defined for the urelements $y \in Y$, the urelements will not be treated as sets here. The signature Σ that we want to use contains a binary symbol for union "$\cup$", a unary symbol for set construction $\{\cdot\}$, and a constant ϵ that denotes the empty set.

Let $X := \{\{y\} \mid y \in Y\}$, and $h : X \to V_{\text{hfs}}(Y)$ be an arbitrary mapping. We want to show that there exists a unique extension of h to a Σ-endomorphism $\widehat{h}$ of $V_{\text{hfs}}(Y)$, which is now considered as a Σ-structure. Obviously we have to define $\widehat{h}(\emptyset) := \emptyset$. Each non-empty $M \in V_{\text{hfs}}(Y)$ can uniquely be represented in the form $M = x_1 \cup \ldots \cup x_k \cup \{M_1\} \cup \ldots \cup \{M_l\}$ where $x_i \in X$, for $1 \leq i \leq k$, and where the M_i are the elements of M that belong to $V_{\text{hfs}}(Y)$. By induction (on nesting depth), we may assume that $\widehat{h}(M_i)$ is already defined $(1 \leq i \leq l)$. Obviously $\widehat{h}(M) := h(x_1) \cup \ldots \cup h(x_k) \cup \{\widehat{h}(M_1)\} \cup \ldots \cup \{\widehat{h}(M_l)\}$ is one and the only way of extending $\widehat{h}$ in a homomorphic way to the set M of deeper nesting. For $M = x \in X$ we obtain $\widehat{h}(x) = h(x)$, thus $\widehat{h}$ is an extension of h. Moreover, each mapping $\widehat{h}$ is in fact homomorphic with respect to Σ. In addition, each set $M \in V_{\text{hfs}}(Y)$ involves only finitely many different urelements (induction on the nesting depth). Thus, the Σ-structure $V_{\text{hfs}}(Y)$ is a quasi-free structure with atom set X.

Hereditarily finite non-wellfounded sets. If we use the same signature Σ as above, it can be seen in a similar way that the domain $V_{\text{hfnws}}(Y)$ of hereditarily finite *non-wellfounded* sets[7] over a countably infinite set of urelements Y is a quasi-free structure over the atom set $X = \{\{y\} \mid y \in Y\}$.

Hereditarily finite wellfounded or non-wellfounded lists. The two domains $V_{\text{hfl}}(Y)$ and $V_{\text{hfnwl}}(Y)$ of nested, hereditarily finite (1) wellfounded or (2) non-wellfounded lists over the countably infinite set of urelements Y, under a signature with a binary symbol for concatenation "$\circ$", a (unary) symbol for list construction $\langle\cdot\rangle : l \mapsto \langle l\rangle$, and a constant *nil* for the empty list, are quasi-free structures over the atom set $X := \{\langle y\rangle \mid y \in Y\}$ of all lists with one element $y \in Y$. Formally, these domains can be described as the set of all (1) finite or (2) rational trees where the topmost node has label "$\langle\ \rangle$" (representing a list constructor of varying finite arity), nodes with successors have label "$\langle\ \rangle$", and leaves have labels $y \in Y$ or "$\langle\ \rangle$". A leaf with label "$\langle\ \rangle$" represents the empty list *nil*.

Feature structures. Let *Lab*, *Fea*, and X be mutually disjoint infinite sets of labels, features, and atoms respectively. Following [2], we define a feature tree to be a partial function $t : \mathit{Fea}^* \to \mathit{Lab} \cup X$ whose domain is prefix closed (i.e., if $pq \in \mathit{dom}(t)$ then $p \in \mathit{dom}(t)$ for all words $p, q \in \mathit{Fea}^*$), and in which atoms do not label interior nodes (i.e., if $p(t) = x \in X$ then there is no $f \in \mathit{Fea}$ with $pf \in \mathit{dom}(t)$). As usual, *rational* feature trees are required to have only finitely many subtrees. In addition, they must be finitely branching.

We use the set R of all rational feature trees as carrier set of a structure $\mathcal{R}^\Sigma$

[7]Non-wellfounded sets, sometimes called hypersets, became prominent through [1]. They can have infinite descending membership sequences. The hereditarily finite non-wellfounded sets are those having a "finite picture," see [1] for details.

whose signature contains a unary predicate L for every label $L \in Lab$, and a binary predicate f for every $f \in Fea$. The interpretation $L_{\mathcal{R}}$ of L in $\mathcal{R}$ is the set of all rational feature trees having root label L. The interpretation $f_{\mathcal{R}}$ of f consists of all pairs $(t_1, t_2) \in R \times R$ such that $t_1(f)$ is defined and t_2 is the subtree of t_1 at f. The structure $\mathcal{R}^{\Sigma}$ defined this way can be seen as a non-ground version of the solution domain used in [2]. We will call $\mathcal{R}^{\Sigma}$ the non-ground structure of rational feature trees. We will show that the set of feature trees that consist of a single leaf node that is labeled by an element of X is an atom set of $\mathcal{R}^{\Sigma}$ w.r.t. a certain monoid $\mathcal{M}$ (see Remark 3.16). We identify this set in the obvious way with X.

Each mapping $h : X \to R$ has a unique extension to an endomorphism of $\mathcal{R}^{\Sigma}$ that acts like a substitution, replacing each leaf with label $x \in X$ by the feature tree $h(x)$. With composition, the set of these substitution-like endomorphisms yields a monoid $\mathcal{M}$. It is not difficult to see that $(\mathcal{R}^{\Sigma}, X)$ is quasi-free with respect to $\mathcal{M}$. However, $\mathcal{R}^{\Sigma}$ has endomorphisms (not belonging to $\mathcal{M}$) that modify non-leaf nodes (e.g., by introducing new feature-edges for such internal nodes). Since these modifications of non-leaf nodes are independent of the images of elements of X, the set X is not an atom set w.r.t. all endomorphisms, and thus $(\mathcal{R}^{\Sigma}, X)$ is not quasi-free.

Now suppose that we introduce, following [38], additional arity predicates F for every finite set $F \subseteq Fea$. The interpretation $F_{\mathcal{R}}$ of F consists of all feature trees t where the root of t has a label $L \in Lab$ and where F is (exactly) the set of all features departing from the root of t. Let Δ be the extended signature. Then $(\mathcal{R}^{\Delta}, X)$ is a quasi-free structure. We shall call it the non-ground structure of rational feature trees with arity.

As can be seen from the previous examples, there is often an interesting ground variant of a given quasi-free structure. The following definition formalizes this relationship.

Definition 3.18 *Let $(\mathcal{A}^{\Sigma}, X)$ be a quasi-free structure such that $SH_{\Sigma}^{\mathcal{A}}(\emptyset)$ is non-empty. Then $\mathcal{A}_G^{\Sigma} := SH_{\Sigma}^{\mathcal{A}}(\emptyset)$ is called the* ground substructure *of $(\mathcal{A}^{\Sigma}, X)$.*

3.3 Algebraic properties of quasi-free structures

Before we can turn to the combination of quasi-free structures, we must establish some useful properties of these structures.

Lemma 3.19 *Let $(\mathcal{A}^{\Sigma}, X)$ be a quasi-free structure.*

1. *$\mathcal{A}^{\Sigma} = SH_{\Sigma}^{\mathcal{A}}(X)$ and every mapping $X \to A$ has a* unique *extension to an endomorphism of $\mathcal{A}^{\Sigma}$.*

2. *Let* $X_0 \subseteq X$*. Then we have* $SH^{\mathcal{A}}_{\Sigma}(X_0) \cap X = X_0$*.*

3. *For all finite sets* $\{a_1, \dots, a_n\} \subseteq A$ *there exists a* unique minimal *finite subset* Y *of* X *such that* $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(Y)$*.*

Proof. (1) Since every element of A is stabilized by a finite subset of X, the atom set X stabilizes the whole structure A, which means that $\mathcal{A}^{\Sigma} = SH^{\mathcal{A}}_{\Sigma}(X)$. Existence of the extension follows from the fact that X is an atom set, and uniqueness is an immediate consequence of $\mathcal{A}^{\Sigma} = SH^{\mathcal{A}}_{\Sigma}(X)$.

(2) The inclusion $X_0 \subseteq SH^{\mathcal{A}}_{\Sigma}(X_0)$ follows from Lemma 3.10. For the other direction, assume that an atom $x \in X$ is in $SH^{\mathcal{A}}_{\Sigma}(X_0) \setminus X_0$. Let $h_1, h_2 : X \to A$ be mappings that coincide on X_0, but differ on x. Because X is an atom set, there are endomorphisms $\widehat{h}_1, \widehat{h}_2$ extending h_1, h_2. Since $\widehat{h}_1$ and $\widehat{h}_2$ coincide on X_0, they coincide on $x \in SH^{\mathcal{A}}_{\Sigma}(X_0)$. This is a contradiction to our assumption that h_1 and h_2 differ on x.

(3) Since $(\mathcal{A}^{\Sigma}, X)$ is quasi-free, every finite set $\{a_1, \dots, a_n\} \subseteq A$ is stabilized by a finite subset of X. Let X_0, X_1 be two finite subsets of X such that $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(X_i)$ for $i = 0, 1$. We claim that $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(X_0 \cap X_1)$. In fact, let h_0, h_1 be two endomorphisms that coincide on $X_0 \cap X_1$. We may choose an endomorphism $h_{0,1} \in End^{\Sigma}_{\mathcal{A}}$ that coincides with h_0 on X_0 and with h_1 on X_1. Such an endomorphism exists since $(\mathcal{A}^{\Sigma}, X)$ is quasi-free. Now h_0 and $h_{0,1}$ coincide on $\{a_1, \dots, a_n\}$, and h_1 and $h_{0,1}$ coincide on $\{a_1, \dots, a_n\}$. This shows that h_0 and h_1 coincide on $\{a_1, \dots, a_n\}$, and thus we have proved $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(X_0 \cap X_1)$. Obviously, this implies that there exists a unique minimal finite subset Y of X such that $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(Y)$. □

The third statement of the lemma shows that the notion "is stabilized by" behaves better than the notion "is generated by." In fact, minimal sets of generators need not be unique, as demonstrated by the next example.

Example 3.20 We consider the quotient term algebra $\mathcal{T}(\Sigma_F, V)/{=_E}$, where Σ_F consists of one unary function symbol f, V is countably infinite, and $E = \{f(x) = f(y)\}$. Obviously, the carrier of $\mathcal{T}(\Sigma_F, V)/{=_E}$ consists of the $=_E$-classes $\{x_i\}$ for $x_i \in V$ and one additional class $[f(\cdot)] := \{f(t) \mid t \in T(\Sigma_F, V)\}$. It is easy to see that for all $x_i \in V$, the element $[f(\cdot)]$ of $\mathcal{T}(\Sigma_F, V)/{=_E}$ is generated by $\{x_i\}$. However, $[f(\cdot)]$ is not generated by $\emptyset$. Thus, there are infinitely many minimal sets of generators of $[f(\cdot)]$.

Definition 3.21 *Let* $(\mathcal{A}^{\Sigma}, X)$ *be a quasi-free structure, and let* $\{a_1, \dots, a_n\} \subseteq A$*. The* stabilizer $Stab^{\mathcal{A}}_{\Sigma}(a_1, \dots, a_n)$ *of* $\{a_1, \dots, a_n\}$ *is the (unique) minimal finite subset* Y *of* X *such that* $\{a_1, \dots, a_n\} \subseteq SH^{\mathcal{A}}_{\Sigma}(Y)$*.*

For the case of term algebras (i.e., absolutely free algebras), the stabilizer of a term is the set of variables (i.e., generators) occurring in this term. In the more general case of arbitrary quasi-free structures, using this as an intuition will help to understand the definitions and proofs. Note, however, that the notion of a stabilizer is still well-defined (and turns out to be extremely useful) in contexts where "*the* minimal set of generators occurring in an element" is no longer unique. The next lemma is an immediate consequence of Definition 3.21 and of the definition of the stable hull.

Lemma 3.22 *Let $(\mathcal{A}^\Sigma, X)$ be a quasi-free structure, and let Y be a subset of X. Then $SH^{\mathcal{A}}_\Sigma(Y) = \{a \in A \mid Stab^{\mathcal{A}}_\Sigma(a) \subseteq Y\}$.*

Lemma 3.23 *Let $(\mathcal{A}^\Sigma, X)$ be a quasi-free structure, let $h \in End^\Sigma_{\mathcal{A}}$ and $a \in A$. Then $Stab^{\mathcal{A}}_\Sigma(h(a)) \subseteq h(Stab^{\mathcal{A}}_\Sigma(a))$.*

Proof. If $m_1, m_2 \in End^\Sigma_{\mathcal{A}}$ coincide on $h(Stab^{\mathcal{A}}_\Sigma(a))$, then $h \circ m_1$ and $h \circ m_2$ coincide on $Stab^{\mathcal{A}}_\Sigma(a)$. But then $h \circ m_1(a) = h \circ m_2(a)$ and $m_1(h(a)) = m_2(h(a))$. □

The stabilizing effect of $Stab^{\mathcal{A}}_\Sigma(a)$ for a is not restricted to $End^\Sigma_{\mathcal{A}}$. Under suitable conditions on the Σ-structure $\mathcal{D}^\Sigma$, $Stab^{\mathcal{A}}_\Sigma(a)$ stabilizes a with respect to $Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$. Before we can formulate this in a more precise way, we must generalize Definition 3.8 to the quasi-free case.

Definition 3.24 *Let $\mathcal{A}^\Sigma, \mathcal{D}^\Sigma$ be Σ-structures, and let $X \subseteq A$. Then $(\mathcal{A}^\Sigma, X)$ is called* quasi-free for $\mathcal{D}^\Sigma$ *if every mapping $X \to D$ has a unique extension to a homomorphism $h_{A-D} \in Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$.*

Note that every quasi-free structure is quasi-free for itself.

Lemma 3.25 *Let $(\mathcal{A}^\Sigma, X)$ be quasi-free, and assume that $(\mathcal{A}^\Sigma, X)$ is quasi-free for $\mathcal{D}^\Sigma$. Let $h_1, h_2 \in Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$, $a \in A$ and $Y \subseteq X$.*

1. *If h_1 and h_2 coincide on $Stab^{\mathcal{A}}_\Sigma(a)$, then $h_1(a) = h_2(a)$.*
2. *If h_1 and h_2 coincide on Y, then h_1 and h_2 coincide on $SH^{\mathcal{A}}_\Sigma(Y)$.*

Proof. To prove the first part, suppose that $h_1, h_2 \in Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$ coincide on $Stab^{\mathcal{A}}_\Sigma(a)$. The joint image $D_0 := \{h_1(b) \mid b \in A\} \cup \{h_2(b) \mid b \in A\}$ of h_1 and h_2 is at most countably infinite. Let $h_{X-D_0} : X \to D_0$ be a surjective mapping such that h_{X-D_0}, h_1 and h_2 coincide on $Stab^{\mathcal{A}}_\Sigma(a)$. Let h_{A-D} denote the unique extension of h_{X-D_0} to an element of $Hom^\Sigma_{\mathcal{A}-\mathcal{D}}$. Let $g_0 : X \to A$ be a mapping such that (1) $g_0(x) = x$ for all $x \in Stab^{\mathcal{A}}_\Sigma(a)$, and (2) $g_0(y)$ is an element of the set $h^{-1}_{A-D}(h_1(y))$, for all $y \in X \setminus Stab^{\mathcal{A}}_\Sigma(a)$. Let g be the unique extension of g_0 to an endomorphism of $\mathcal{A}$. From (1) we know that $g(a) = a$. Now $g \circ h_{A-D} \in Hom^{\Sigma}_{\mathcal{A}-\mathcal{D}}$ coincides on X with h_1. Since $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{D}^{\Sigma}$, we know that $g \circ h_{A-D} = h_1$. This yields $h_{A-D}(a) = h_1(a)$. Symmetrically it follows that $h_{A-D}(a) = h_2(a)$. Thus in fact $h_1(a) = h_2(a)$. By Lemma 3.22, the second part of the lemma is a trivial consequence of the first one. □

In Section 4.3, where we introduce a construction that combines quasi-free structures over disjoint signatures, we need to embed a given quasi-free structure into an isomorphic superstructure. Here, the usual notion of isomorphism between structures is not sufficient, however, since the atom sets must also be taken into account.

Definition 3.26 *Let $(\mathcal{A}^{\Sigma}, X)$ and $(\mathcal{B}^{\Sigma}, Y)$ be quasi-free. A* qf-isomorphism *between $(\mathcal{A}^{\Sigma}, X)$ and $(\mathcal{B}^{\Sigma}, Y)$ is an isomorphism $h : \mathcal{A}^{\Sigma} \to \mathcal{B}^{\Sigma}$ that maps X onto Y.*

The next lemma shows that qf-isomorphic structures are quasi-free for the same class of structures (in the sense introduced in Definition 3.24).

Lemma 3.27 *Let $(\mathcal{A}^{\Sigma}, X)$ and $(\mathcal{B}^{\Sigma}, Y)$ be qf-isomorphic quasi-free structures, and let $\mathcal{D}^{\Sigma}$ be a Σ-structure. If $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{D}^{\Sigma}$, then also $(\mathcal{B}^{\Sigma}, Y)$ is quasi-free for $\mathcal{D}^{\Sigma}$. In particular, since any quasi-free structure is quasi-free for itself, $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{B}^{\Sigma}$ and $(\mathcal{B}^{\Sigma}, Y)$ is quasi-free for $\mathcal{A}^{\Sigma}$.*

Proof. Let $h_{A-B} : \mathcal{A}^{\Sigma} \to \mathcal{B}^{\Sigma}$ be a Σ-isomorphism that maps X onto Y. Suppose that $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{D}^{\Sigma}$. Let $h_{Y-D} : Y \to D$ be a mapping. Let h_{X-Y} be the restriction of h_{A-B} to X. Since $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{D}^{\Sigma}$ there exists a unique extension of $h_{X-D} := h_{X-Y} \circ h_{Y-D}$ to a homomorphism $h_{A-D} : \mathcal{A}^{\Sigma} \to \mathcal{D}^{\Sigma}$. Then $h_{B-D} := h^{-1}_{A-B} \circ h_{A-D} : \mathcal{B}^{\Sigma} \to \mathcal{D}^{\Sigma}$ is a Σ-homomorphism extending h_{Y-D}.

It remains to be shown that this homomorphism is unique. First, note that $(*)$ $h_{A-D} = h_{A-B} \circ h_{B-D}$. Let $g_{B-D} : \mathcal{B}^{\Sigma} \to \mathcal{D}^{\Sigma}$ be any Σ-homomorphism extending h_{Y-D}. Then $(**)$ $g_{A-D} := h_{A-B} \circ g_{B-D}$ is a Σ-homomorphism extending h_{X-D}. Since $(\mathcal{A}^{\Sigma}, X)$ is quasi-free for $\mathcal{D}^{\Sigma}$ we have $g_{A-D} = h_{A-D}$. Now, $(*)$ and $(**)$ imply $h_{B-D} = g_{B-D}$. □

The following two results show that one can always find qf-isomorphic substructures and superstructures of a given quasi-free structure. For free structures, showing these results is almost trivial. For quasi-free structures it requires rather long and tedious technical proofs, which are therefore deferred to an Appendix.

Lemma 3.28 *Let $(\mathcal{B}^{\Sigma}, Y)$ be a quasi-free structure. Let Z be an infinite subset of Y, and let $\mathcal{C}^{\Sigma} := SH^{\mathcal{B}}_{\Sigma}(Z)$. Then the following holds:*

1. $(\mathcal{C}^\Sigma, Z)$ *is quasi-free, and* $(\mathcal{B}^\Sigma, Y)$ *and* $(\mathcal{C}^\Sigma, Z)$ *are qf-isomorphic.*

2. *For each* $c \in C$*, we have* $Stab^{\mathcal{B}}_\Sigma(c) = Stab^{\mathcal{C}}_\Sigma(c)$.

3. *For each* $U \subseteq Z$, $SH^{\mathcal{B}}_\Sigma(U) = SH^{\mathcal{C}}_\Sigma(U)$.

Theorem 3.29 *Let* $(\mathcal{A}^\Sigma, X)$ *be a quasi-free structure. Then there exists a quasi-free superstructure* $(\mathcal{B}^\Sigma, Y)$ *with the following properties:*

1. $Y \setminus X$ *is infinite.*

2. $X \subseteq Y$*, and* $\mathcal{A}^\Sigma = SH^{\mathcal{B}}_\Sigma(X)$.

3. $(\mathcal{A}^\Sigma, X)$ *and* $(\mathcal{B}^\Sigma, Y)$ *are qf-isomorphic.*

4. *If* $X \subseteq Z \subseteq Y$*, and if* $\mathcal{C}^\Sigma = SH^{\mathcal{B}}_\Sigma(Z)$*, then* $\mathcal{A}^\Sigma = SH^{\mathcal{C}}_\Sigma(X)$*, and* $(\mathcal{A}^\Sigma, X)$ *and* $(\mathcal{C}^\Sigma, Z)$ *are qf-isomorphic.*

3.4 Logical properties of quasi-free structures

Using the notion of stabilizers, the validity of positive formulae in quasi-free structures can be characterized in an algebraic way. This characterization is essential for proving correctness of our method of combining constraint solvers for quasi-free structures.

Lemma 3.30 *Let* $(\mathcal{A}^\Sigma, X)$ *be a quasi-free structure, and let*

$$\gamma = \forall \vec{u}_1 \exists \vec{v}_1 \dots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k)$$

be a positive Σ*-sentence, where* φ *is a positive (not necessarily quantifier-free) formula. Then the following conditions are equivalent:*

1. $\mathcal{A}^\Sigma \models \forall \vec{u}_1 \exists \vec{v}_1 \dots \forall \vec{u}_k \exists \vec{v}_k \ \varphi(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k)$.

2. *There exist* $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \dots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}$ *such that*
   (a) $\mathcal{A}^\Sigma \models \varphi(\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k)$,
   (b) *all atoms in the sequences* $\vec{x}_1, \dots, \vec{x}_k$ *are distinct,*
   (c) *for all* $j, 1 \leq j \leq k$*, the components of* $\vec{x}_j$ *are not contained in* $Stab^{\mathcal{A}}_\Sigma(\vec{e}_1) \cup \dots \cup Stab^{\mathcal{A}}_\Sigma(\vec{e}_{j-1})$.

Proof. The proof is by induction on the number of quantifier alternations k. For $k = 0$, there is nothing to show since in this case (1) and (2.a) coincide. For the induction step, assume that

$$\gamma = \forall \vec{u}_1 \exists \vec{v}_1 \dots \forall \vec{u}_k \exists \vec{v}_k \forall \vec{u}_{k+1} \exists \vec{v}_{k+1}\ \varphi(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k, \vec{u}_{k+1}, \vec{v}_{k+1})$$

is a positive Σ-sentence. Let

$$\varphi'(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k) := \forall \vec{u}_{k+1} \exists \vec{v}_{k+1}\ \varphi(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k, \vec{u}_{k+1}).$$

"$1 \Rightarrow 2$:" Assume that

$$\mathcal{A}^\Sigma \models \forall \vec{u}_1 \exists \vec{v}_1 \dots \forall \vec{u}_k \exists \vec{v}_k \forall \vec{u}_{k+1} \exists \vec{v}_{k+1}\ \varphi(\vec{u}_1, \vec{v}_1, \dots, \vec{u}_k, \vec{v}_k, \vec{u}_{k+1}, \vec{v}_{k+1}).$$

By induction hypothesis, there exist $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \dots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}$ such that $\mathcal{A}^\Sigma \models \varphi'(\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k)$, all atoms in the sequences $\vec{x}_1, \dots, \vec{x}_k$ are distinct, and for all $j, 1 \leq j \leq k$, the components of $\vec{x}_j$ are not contained in $Stab_\Sigma^\mathcal{A}(\vec{e}_1) \cup \dots \cup Stab_\Sigma^\mathcal{A}(\vec{e}_{j-1})$. Because φ' starts with a block of universal quantifiers, we may substitute the variables $\vec{u}_{k+1}$ by an arbitrary sequence $\vec{x}_{k+1} \in \vec{X}$ of distinct atoms that are "new" in the sense that none of them occurs in the finite set $Stab_\Sigma^\mathcal{A}(\vec{e}_1) \cup \dots \cup Stab_\Sigma^\mathcal{A}(\vec{e}_k) \cup \vec{x}_1 \cup \dots \cup \vec{x}_k$.[8] Thus, we obtain $\mathcal{A}^\Sigma \models \exists \vec{v}_{k+1} \varphi(\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k, \vec{x}_{k+1}, \vec{v}_{k+1})$. Because of the existential quantifier on $\vec{v}_{k+1}$, we can deduce that there exist elements $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \dots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}, \vec{x}_{k+1} \in \vec{X}, \vec{e}_{k+1} \in \vec{A}$ satisfying all the properties (a), (b), and (c) from above. Note that (b) and (c) place no restriction on the elements of the sequence $\vec{e}_{k+1}$.

"$2 \Rightarrow 1$:" For the converse direction, assume that there exist elements $\vec{x}_1 \in \vec{X}, \vec{e}_1 \in \vec{A}, \dots, \vec{x}_k \in \vec{X}, \vec{e}_k \in \vec{A}, \vec{x}_{k+1} \in \vec{X}, \vec{e}_{k+1} \in \vec{A}$ such that

(a) $\mathcal{A}^\Sigma \models \varphi(\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k, \vec{x}_{k+1}, \vec{e}_{k+1})$,

(b) all atoms in the sequences $\vec{x}_1, \dots, \vec{x}_k, \vec{x}_{k+1}$ are distinct,

(c) for all $j, 1 \leq j \leq k+1$, the components of $\vec{x}_j$ are not contained in $Stab_\Sigma^\mathcal{A}(\vec{e}_1) \cup \dots \cup Stab_\Sigma^\mathcal{A}(\vec{e}_{j-1})$.

Let $\vec{a}_{k+1}$ denote a sequence of arbitrary elements of A, such that $\vec{x}_{k+1}$ and $\vec{a}_{k+1}$ have the same length. By Lemma 3.15, the mapping h_0 that fixes all elements in $Stab_\Sigma^\mathcal{A}(\vec{e}_1) \cup \dots \cup Stab_\Sigma^\mathcal{A}(\vec{e}_k) \cup \vec{x}_1 \cup \dots \cup \vec{x}_k$ and maps each component of $\vec{x}_{k+1}$ to the corresponding component of $\vec{a}_{k+1}$ can be extended to a surjective endomorphism h of $\mathcal{A}^\Sigma$. By Lemma 2.1 we obtain $\mathcal{A}^\Sigma \models \varphi(\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k, \vec{a}_{k+1}, h(\vec{e}_{k+1}))$, since h keeps all elements in $\vec{x}_1, \vec{e}_1, \dots, \vec{x}_k, \vec{e}_k$ fixed. The arbitrary choice of $\vec{a}_{k+1}$ shows

[8] Here we use $\vec{x}_i$ also to denote the *set* of elements of the sequence.

that $\mathcal{A}^\Sigma \models \varphi'(\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_k, \vec{e}_k)$. The induction hypothesis applied to φ' and the definition of φ' yield

$$\mathcal{A}^\Sigma \models \forall \vec{u}_1 \exists \vec{v}_1 \ldots \forall \vec{u}_k \exists \vec{v}_k \forall \vec{u}_{k+1} \exists \vec{v}_{k+1} \varphi(\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_k, \vec{v}_k, \vec{u}_{k+1}, \vec{v}_{k+1}),$$

which concludes the proof of the lemma. □

Readers that are familiar with our work on combining unification algorithms should note the close relationship between the second condition of the lemma and the notion of a *linear constant restriction* (cf. [3]). In fact, both conditions play a very similar rôle. To see this, consider a prefix $\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_{i-1}, \vec{e}_{i-1}, \vec{x}_i$ of the sequence in Condition 2. Condition (c) makes sure that the atoms in $\vec{x}_i$ do not occur in the stabilizers of the elements $\vec{e}_1, \ldots, \vec{e}_{i-1}$ preceding $\vec{x}_i$ in the order of the enumeration. In a solution σ of a unification problem with linear constant restrictions, a constant c (corresponding to an atom x above) must not occur (corresponding to "is not in the stabilizer" above) in the image $v\sigma$ (corresponding to an element e above) if v comes before c in the linear order of the restriction.

4 Combination of Quasi-Free Structures

This section is concerned with the problem of how to combine two quasi-free structures over disjoint signatures into a new structure over the union of both signatures. First, we will introduce an algebraic framework for combining structures, which is not restricted to quasi-free structures or disjoint signatures.[9] This framework tries to formalize our intuition of what to expect from a canonical combination of two structures. Second, we show that for the case of free structures, this framework really yields the canonical combined structure. In the third subsection, we describe an explicit construction for combining two quasi-free structures over disjoint signatures, and in the fourth subsection we show that the result of this construction coincides with what our abstract framework proposes as canonical combined structure.

4.1 Combination of structures

Let $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ be two structures. What conditions should a $(\Sigma_1 \cup \Sigma_2)$-structure $\mathcal{C}^{\Sigma_1 \cup \Sigma_2}$ satisfy to be called a "canonical combination" of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$? The central notion of this subsection will be obtained after three steps, each introducing a restriction that is motivated by the example of the combination of free algebras, i.e., term algebras modulo equational theories. The structures $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ will be called the *components* in the sequel.

[9] Even if we later restrict considerations to the case of disjoint signatures, the following general definitions might turn out to be a good starting point for an investigation of non-disjoint combination problems.

Restriction 1: *Homomorphisms that embed the components into the combined structure must exist. If the components share a common substructure, then the embedding homomorphisms must agree on this substructure.*

In fact, a minimal requirement seems to be that both structures must in some sense be embedded in their combination. It would, however, be too restrictive to demand that the components are substructures of the combined structure. For the case of non-trivial equational theories E_1, E_2 over disjoint signatures Σ_1, Σ_2, there exist 1–1-embeddings of $\mathcal{T}(\Sigma_1, V)/{=_{E_1}}$ and $\mathcal{T}(\Sigma_2, V)/{=_{E_2}}$ into $\mathcal{T}(\Sigma_1 \cup \Sigma_2, V)/{=_{E_1 \cup E_2}}$. For non-disjoint signatures, however, these "embeddings" need no longer be 1–1. Note that even for disjoint signatures Σ_1 and Σ_2 there is a common part, namely the trivial structure represented by the set V of variables. A reasonable requirement is that elements of the common part are mapped to the same element of the combined structure by the homomorphic embeddings. To be as general as possible, we do not assume that the "common part" is really a substructure of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$. Instead, we assume that it is just homomorphically embedded in both structures. These considerations motivate the following formalization of Restriction 1.

Definition 4.1 *Let Σ_1 and Σ_2 be signatures, and let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2$. A triple $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ with given homomorphic embeddings*

$$h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Gamma \quad \textit{and} \quad h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Gamma$$

is called an amalgamation base*. The structure $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$* closes *the amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ iff there are homomorphisms*

$$h^{\Sigma_1}_{B_1-D} : \mathcal{B}_1^{\Sigma_1} \to \mathcal{D}^{\Sigma_1} \quad \textit{and} \quad h^{\Sigma_2}_{B_2-D} : \mathcal{B}_2^{\Sigma_2} \to \mathcal{D}^{\Sigma_2}$$

such that $h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-D} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-D}$. We call $(\mathcal{D}^{\Sigma_1 \cup \Sigma_2}, h^{\Sigma_1}_{B_1-D}, h^{\Sigma_2}_{B_2-D})$ an amalgamated product *of $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$.*

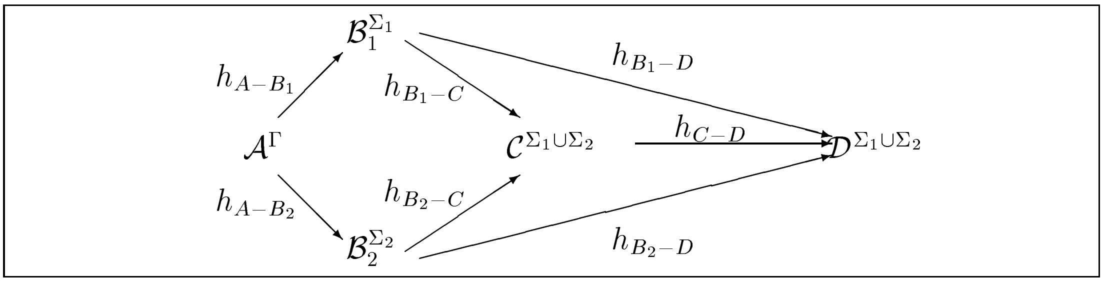
If the embedding homomorphisms are irrelevant or clear from the context, we will also call the structure $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$ alone an amalgamated product of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ over $\mathcal{A}^\Gamma$. For a given amalgamation base, there usually exist various structures that can be used to close this base. Which one should be seen as a canonical closure? Motivated by the example of free structures, where the canonical combined structure is again free, we are interested in "most general" amalgamated products.

Restriction 2: *We are interested in structures closing the amalgamation base that are as general as possible.*

In principle, we consider a structure $\mathcal{C}$ to be more general than a structure $\mathcal{D}$ iff there is a homomorphism of $\mathcal{C}$ into $\mathcal{D}$. Thus, a possible formalization of Restriction 2 seems to be to ask for an amalgamated product

$$(\mathcal{C}^{\Sigma_1 \cup \Sigma_2}, h^{\Sigma_1}_{B_1-C}, h^{\Sigma_2}_{B_2-C})$$

such that for each amalgamated product $(\mathcal{D}^{\Sigma_1 \cup \Sigma_2}, h^{\Sigma_1}_{B_1-D}, h^{\Sigma_2}_{B_2-D})$ of the amalgamation base there exists a unique $(\Sigma_1 \cup \Sigma_2)$-homomorphism h_{C-D} such that $h_{B_i-D} = h_{B_i-C} \circ h_{C-D}$, for $i = 1, 2$. This situation is illustrated in the following figure.



It turns out, however, that requiring a most general element among *all* possible amalgamated products is too strong. Informally, the reason is that not all amalgamated products of a given amalgamation base share "relevant" structural properties with the component structures of the base. To be more precise, we consider the example of free algebras $\mathcal{B}_1^{\Sigma_1} := \mathcal{T}(\Sigma_1, V)/_{=_{E_1}}$ and $\mathcal{B}_2^{\Sigma_2} := \mathcal{T}(\Sigma_2, V)/_{=_{E_2}}$, with common "substructure" $\mathcal{A}^{\Gamma} := \mathcal{T}(\Sigma_1 \cap \Sigma_2, V)$. The canonical combined algebra is the free algebra $\mathcal{T}(\Sigma_1 \cup \Sigma_2, V)/_{=_{E_1 \cup E_2}}$, which is in fact most general (in the sense introduced above) among all amalgamated products that satisfy $E_1 \cup E_2$, i.e., all elements of $\mathcal{V}(E_1 \cup E_2)$. An arbitrary product $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$ of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ may, however, invalidate some axioms of $E_1 \cup E_2$. In this case, it may not be possible to find an appropriate homomorphism from $\mathcal{T}(\Sigma_1 \cup \Sigma_2, V)/_{=_{E_1 \cup E_2}}$ to $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$. For this reason, we allow for the possibility of restricting the attention to a certain subclass of all amalgamated products.

> **Restriction 3:** *Only admissible combinations of the two components are considered. The class of admissible structures should share relevant structural properties with these components.*

For the case of free algebras, the obvious candidate for the class of admissible structures is the variety defined by the union of the component theories, i.e., $Adm(\mathcal{T}(\Sigma_1, V)/_{=_{E_1}}, \mathcal{T}(\Sigma_2, V)/_{=_{E_2}}) = \mathcal{V}(E_1 \cup E_2)$. In Section 4.2, we will give an algebraic reformulation of the definition of this class (for the case of free structures instead of only free algebras). An appropriate class of admissible structures for the quasi-free case will be obtained as an obvious generalization of this reformulation. In the remainder of this subsection, however, we make no assumption on the specific form of the class of admissible structures. We just assume that such a class is given. An amalgamated product is called admissible iff it belongs to the class of admissible structures.

Definition 4.2 *Let* $(\mathcal{A}^{\Gamma}, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ *be an amalgamation base, and assume that a class* $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ *of admissible* $(\Sigma_1 \cup \Sigma_2)$*-structures is fixed. The admissible amalgamated product* $(\mathcal{C}^{\Sigma_1\cup\Sigma_2}, h^{\Sigma_1}_{B_1-C}, h^{\Sigma_2}_{B_2-C})$ *of* $\mathcal{B}_1^{\Sigma_1}$ *and* $\mathcal{B}_2^{\Sigma_2}$ *over* $\mathcal{A}^{\Gamma}$ *is called a* free amalgamated product with respect to $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ *iff for every admissible amalgamated product* $(\mathcal{D}^{\Sigma_1\cup\Sigma_2}, h^{\Sigma_1}_{B_1-D}, h^{\Sigma_2}_{B_2-D})$ *of* $\mathcal{B}_1^{\Sigma_1}$ *and* $\mathcal{B}_2^{\Sigma_2}$ *over* $\mathcal{A}^{\Gamma}$ *there exists a* unique *homomorphism* $h^{\Sigma_1\cup\Sigma_2}_{C-D} : \mathcal{C}^{\Sigma_1\cup\Sigma_2} \to \mathcal{D}^{\Sigma_1\cup\Sigma_2}$ *such that*

$$h^{\Sigma_1}_{B_1-D} = h^{\Sigma_1}_{B_1-C} \circ h^{\Sigma_1\cup\Sigma_2}_{C-D} \quad \text{and} \quad h^{\Sigma_2}_{B_2-D} = h^{\Sigma_2}_{B_2-C} \circ h^{\Sigma_1\cup\Sigma_2}_{C-D}.$$

Free amalgamated products need not exist, but if they exist they are unique up to isomorphism.

Theorem 4.3 *Let* $(\mathcal{A}^{\Gamma}, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ *be an amalgamation base with fixed homomorphic embeddings* $h^{\Gamma}_{A-B_1} : \mathcal{A}^{\Gamma} \to \mathcal{B}_1^{\Gamma}$ *and* $h^{\Gamma}_{A-B_2} : \mathcal{A}^{\Gamma} \to \mathcal{B}_2^{\Gamma}$*. The free amalgamated product of* $\mathcal{B}_1^{\Sigma_1}$ *and* $\mathcal{B}_2^{\Sigma_2}$ *over* $\mathcal{A}^{\Gamma}$ *with respect to a given class* $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ *is unique up to* $(\Sigma_1 \cup \Sigma_2)$*-isomorphism.*

Proof. Let $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ and $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ be free amalgamated products of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ over $\mathcal{A}^{\Gamma}$ with respect to $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$. It follows that both structures belong to the class of admissible structures $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$. Since $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ is an admissible amalgamated product, there exist homomorphisms $h^{\Sigma_1}_{B_1-C} : \mathcal{B}_1^{\Sigma_1} \to \mathcal{C}^{\Sigma_1}$ and $h^{\Sigma_2}_{B_2-C} : \mathcal{B}_2^{\Sigma_2} \to \mathcal{C}^{\Sigma_2}$ such that $h^{\Gamma}_{A-B_1} \circ h^{\Sigma_1}_{B_1-C} = h^{\Gamma}_{A-B_2} \circ h^{\Sigma_2}_{B_2-C}$. Similarly there exist homomorphisms $h^{\Sigma_1}_{B_1-D} : \mathcal{B}_1^{\Sigma_1} \to \mathcal{D}^{\Sigma_1}$ and $h^{\Sigma_2}_{B_2-D} : \mathcal{B}_2^{\Sigma_2} \to \mathcal{D}^{\Sigma_2}$ such that $h^{\Gamma}_{A-B_1} \circ h^{\Sigma_1}_{B_1-D} = h^{\Gamma}_{A-B_2} \circ h^{\Sigma_2}_{B_2-D}$.

Since $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ is a free amalgamated product, there exists a unique homomorphism $f^{\Sigma_1\cup\Sigma_2}_{C-D} : \mathcal{C}^{\Sigma_1\cup\Sigma_2} \to \mathcal{D}^{\Sigma_1\cup\Sigma_2}$ such that

$$h^{\Sigma_1}_{B_1-D} = h^{\Sigma_1}_{B_1-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D} \quad \text{and} \quad h^{\Sigma_2}_{B_2-D} = h^{\Sigma_2}_{B_2-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D}.$$

Similarly, there exists a unique homomorphism $f^{\Sigma_1\cup\Sigma_2}_{D-C} : \mathcal{D}^{\Sigma_1\cup\Sigma_2} \to \mathcal{C}^{\Sigma_1\cup\Sigma_2}$ such that

$$h^{\Sigma_1}_{B_1-C} = h^{\Sigma_1}_{B_1-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C} \quad \text{and} \quad h^{\Sigma_2}_{B_2-C} = h^{\Sigma_2}_{B_2-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C}.$$

This implies $h^{\Sigma_1}_{B_1-C} = h^{\Sigma_1}_{B_1-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C} = h^{\Sigma_1}_{B_1-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C}$, and similarly we obtain $h^{\Sigma_2}_{B_2-C} = h^{\Sigma_2}_{B_2-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C}$.

Since $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ is a *free* amalgamated product, $\mathcal{C}^{\Sigma_1\cup\Sigma_2} \in Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ implies that there exists a unique $(\Sigma_1 \cup \Sigma_2)$-endomorphism $h^{\Sigma_1\cup\Sigma_2}$ of $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ such that

$$\begin{aligned} h^{\Sigma_1}_{B_1-C} &= h^{\Sigma_1}_{B_1-C} \circ h^{\Sigma_1\cup\Sigma_2} \\ h^{\Sigma_2}_{B_2-C} &= h^{\Sigma_2}_{B_2-C} \circ h^{\Sigma_1\cup\Sigma_2}. \end{aligned}$$

We have just seen that $f^{\Sigma_1\cup\Sigma_2}_{C-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C}$ satisfies these properties, and obviously, Id_C satisfies them as well. This shows that $f^{\Sigma_1\cup\Sigma_2}_{C-D} \circ f^{\Sigma_1\cup\Sigma_2}_{D-C} = Id_C$. Symmetrically, one can also show $f^{\Sigma_1\cup\Sigma_2}_{D-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D} = Id_D$.

To sum up, we have shown that $f^{\Sigma_1\cup\Sigma_2}_{C-D}$ and $f^{\Sigma_1\cup\Sigma_2}_{D-C}$ are isomorphisms that are inverse to each other. □

The theorem justifies to speak about *the* free amalgamated product of two structures (provided that the embedding homomorphisms and the class of admissible structures are fixed). In this situation, we will sometimes denote the free amalgamated product of $\mathcal{B}_1$ and $\mathcal{B}_2$ by $\mathcal{B}_1 \odot \mathcal{B}_2$. The product operation is obviously commutative, if the definition of the class of admissible structures satisfies $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}) = Adm(\mathcal{B}_2^{\Sigma_2}, \mathcal{B}_1^{\Sigma_1})$. In order to obtain associativity as well, we need some additional conditions on the class of admissible structures.

Before formulating these restrictions, we extend the definition of an amalgamation base and of the free amalgamated product to the case of three structures.[10] Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$. A quadruple $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ with given homomorphic embeddings

$$h^\Gamma_{A-B_i} : \mathcal{A}^\Gamma \to \mathcal{B}_i^\Gamma \quad (i = 1, 2, 3)$$

is called a *simultaneous amalgamation base*. The structure $\mathcal{D}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}$ *closes* the simultaneous amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ iff, for $i = 1, 2, 3$, there are homomorphisms $h^{\Sigma_i}_{B_i-D} : \mathcal{B}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ such that

$$h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-D} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-D} = h^\Gamma_{A-B_3} \circ h^{\Sigma_3}_{B_3-D}.$$

In this case, $(\mathcal{D}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}, h^{\Sigma_1}_{B_1-D}, h^{\Sigma_2}_{B_2-D}, h^{\Sigma_3}_{B_3-D})$ is a *simultaneous amalgamated product* of $\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ over $\mathcal{A}^\Gamma$.

Now, assume that a class of admissible structures $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ is fixed. The admissible simultaneous amalgamated product

$$(\mathcal{C}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}, h^{\Sigma_1}_{B_1-C}, h^{\Sigma_2}_{B_2-C}, h^{\Sigma_3}_{B_3-C})$$

of $\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ over $\mathcal{A}^\Gamma$ is called a *free simultaneous amalgamated product with respect to* $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ iff for every admissible simultaneous amalgamated product $(\mathcal{D}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}, h^{\Sigma_1}_{B_1-D}, h^{\Sigma_2}_{B_2-D}, h^{\Sigma_3}_{B_3-D})$ there exists a *unique* homomorphism

$$f^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}_{C-D} : \mathcal{C}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3} \to \mathcal{D}^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}$$

such that $g^{\Sigma_i}_{B_i-D} = h^{\Sigma_i}_{B_i-C} \circ f^{\Sigma_1\cup\Sigma_2\cup\Sigma_3}_{C-D}$, for $i = 1, 2, 3$. As for the binary free amalgamated product, one can show that the free simultaneous amalgamated product is unique up to isomorphism, provided that it exists.

[10]The extension to an arbitrary number $n \geq 2$ of structures should then be obvious.

Theorem 4.4 (Associativity of free amalgamation)
Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, and let $\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ be structures with fixed homomorphic embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Gamma$, $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Gamma$, and $h^\Gamma_{A-B_3} : \mathcal{A}^\Gamma \to \mathcal{B}_3^\Gamma$. Assume that the free amalgamated products $\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}$, $\mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$, $\mathcal{B}_1^{\Sigma_1} \odot B_2^{\Sigma_2}$, and $(\mathcal{B}_1^{\Sigma_1} \odot B_2^{\Sigma_2}) \odot \mathcal{B}_3^{\Sigma_3}$ exist, and that the classes of admissible structures satisfy

$$\begin{array}{rcl} \mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}) & \in & Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}), \\ (\mathcal{B}_1^{\Sigma_1} \odot \mathcal{B}_2^{\Sigma_2}) \odot \mathcal{B}_3^{\Sigma_3} & \in & Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}), \ \textit{and} \\ Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) & \subseteq & Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}) \cap Adm(\mathcal{B}_1^{\Sigma_1} \odot \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \cap \\ & & Adm(\mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \cap Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}). \end{array}$$

Then we have $(\mathcal{B}_1^{\Sigma_1} \odot \mathcal{B}_2^{\Sigma_2}) \odot \mathcal{B}_3^{\Sigma_3} \simeq \mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$, and this structure is the free simultaneous amalgamated product of $\mathcal{B}_1^{\Sigma_1}$, $\mathcal{B}_2^{\Sigma_2}$, and $\mathcal{B}_3^{\Sigma_3}$ over $\mathcal{A}^\Gamma$.

The proof of this theorem, which is again deferred to the Appendix, can be given on a rather abstract level (manipulation of arrows, i.e., homomorphisms). Note, however, that proving in a particular situation that the prerequisites of the theorem are satisfied is usually not possible on this abstract external level; it may require deep knowledge about the internal structure of the involved structures.

Notions of "amalgamated product," similar to the one given above, can be found in universal algebra, model theory, and in category theory (see, e.g., [29, 12, 21]). There are, however, certain differences between our situation and the typical situations in which amalgamation occurs in other areas. In algebra or model theory, amalgamation has been introduced for *particular classes of algebraic structures* such as groups, fields, skew fields etc. Amalgamation is studied for such a fixed class of structures over the same signature, and it is assumed that these structures all satisfy the same set of axioms (e.g., those for groups, fields, skew fields, etc.). In our case, algebras over different signatures are amalgamated, and these algebras satisfy different types of axioms (or are not defined by axioms at all).

4.2 The free amalgamated product of free structures

Let $\mathcal{B}_1^{\Sigma_1}$ be free over V in the variety $\mathcal{V}(G_1)$, and let $\mathcal{B}_2^{\Sigma_2}$ be free over V in the variety $\mathcal{V}(G_2)$, for atomic theories G_1 and G_2 over the signatures Σ_1 and Σ_2 respectively, where $G_1 \cup G_2$ is non-trivial.[11] We will show that the free amalgamated product of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ is free over V in the variety $\mathcal{V}(G_1 \cup G_2)$.

First, we must fix the "common part" of the amalgamation base, the embedding homomorphisms, and the class of admissible structures. Since we want to

[11]As for the purely equational case, we call a theory defined by relational and equational atomic formulae non-trivial iff it has models of cardinality greater than 1.

show that the notion of a free amalgamated product is natural even in the non-disjoint case, we do not assume that Σ_1 and Σ_2 are disjoint. We take as common part $\mathcal{A}^\Gamma$ the absolutely free structure over V with signature $\Gamma = \Sigma_1 \cap \Sigma_2$, i.e., the free structure over V in the class of all Γ-structures. The embedding homomorphisms $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}^\Gamma_1$ and $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}^\Gamma_2$ are the unique extensions of Id_V to Γ-homomorphisms between these structures. As motivated in the previous section (for the case of free algebras), we use

$$Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}) := \mathcal{V}(G_1 \cup G_2)$$

as class of admissible structures.

Proposition 4.5 *Let $\Gamma = \Sigma_1 \cap \Sigma_2$, let $\mathcal{A}^\Gamma$ be the absolutely free structure over V, let G_1 and G_2 be sets of atomic formulae over the signatures Σ_1 and Σ_2 respectively, where $G_1 \cup G_2$ is non-trivial, and let $\mathcal{B}_i^{\Sigma_i}$ be free over V in the variety $\mathcal{V}(G_i)$ $(i = 1, 2)$. The free amalgamated product with respect to $\mathcal{V}(G_1 \cup G_2)$ of the amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2})$ introduced above is isomorphic to the free structure over V in the variety $\mathcal{V}(G_1 \cup G_2)$.*

Proof. Let $\mathcal{C}^{\Sigma_1 \cup \Sigma_2}$ be the free structure over V in the variety $\mathcal{V}(G_1 \cup G_2)$, which exists since $G_1 \cup G_2$ is assumed to be non-trivial. Since this structure is in $\mathcal{V}(G_1 \cup G_2)$, it is an admissible structure. The Σ_1-reduct $\mathcal{C}^{\Sigma_1}$ of $\mathcal{C}^{\Sigma_1 \cup \Sigma_2}$ satisfies G_1, and the Σ_2-reduct $\mathcal{C}^{\Sigma_2}$ satisfies G_2. Since $\mathcal{B}_i^{\Sigma_i}$ is free over V for the class of all models of G_i, there exists a unique Σ_i-homomorphism $h^{\Sigma_i}_{B_i-C} : \mathcal{B}_i^{\Sigma_i} \to \mathcal{C}^{\Sigma_i}$ that extends Id_V (for $i = 1, 2$).

Let $h^\Gamma_{A-B_1}$ and $h^\Gamma_{A-B_2}$ be as above. It follows that

$$h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-C} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-C},$$

since both homomorphisms represent the unique extension of Id_V to a Γ-homomorphism $\mathcal{A}^\Gamma \to \mathcal{C}^\Gamma$. Thus, we have shown that the free structure $\mathcal{C}^{\Sigma_1 \cup \Sigma_2}$ is in fact an admissible amalgamated product of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$ over $\mathcal{A}^\Gamma$ with respect to $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}) = \mathcal{V}(G_1 \cup G_2)$.

In order to show that it is the free product, assume that $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$ is an admissible structure in $\mathcal{V}(G_1 \cup G_2)$, and that homomorphisms $h^{\Sigma_i}_{B_i-D} : \mathcal{B}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ $(i = 1, 2)$ satisfying

$$h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-D} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-D}$$

are given. Let $f_0 : V \to D$ be the restriction of $h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-D} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-D}$ to V. Since $\mathcal{D}^{\Sigma_1 \cup \Sigma_2}$ is an admissible structure, it is an element of $\mathcal{V}(G_1 \cup G_2)$, and since $\mathcal{C}^{\Sigma_1 \cup \Sigma_2}$ is free over V in the class $\mathcal{V}(G_1 \cup G_2)$, the mapping $f_0 : V \to D$ has a unique extension to a homomorphism $f^{\Sigma_1 \cup \Sigma_2}_{C-D} : \mathcal{C}^{\Sigma_1 \cup \Sigma_2} \to \mathcal{D}^{\Sigma_1 \cup \Sigma_2}$.

Since $h^{\Sigma_1}_{B_1-C}$ and $h^\Gamma_{A-B_1}$ coincide with Id_V on V, $h^{\Sigma_1}_{B_1-C} \circ f^{\Sigma_1 \cup \Sigma_2}_{C-D}$ and $h^{\Sigma_1}_{B_1-D}$ are two Σ_1-homomorphisms $\mathcal{B}_1^{\Sigma_1} \to \mathcal{D}^{\Sigma_1}$ that coincide on V. Thus $h^{\Sigma_1}_{B_1-C} \circ f^{\Sigma_1 \cup \Sigma_2}_{C-D} =$

$h^{\Sigma_1}_{B_1-D}$, since $\mathcal{B}_1^{\Sigma_1}$ is free over V in $\mathcal{V}(G_1)$, and the Σ_1-reduct $\mathcal{D}^{\Sigma_1}$ of $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ satisfies G_1. Similarly, one can prove that $h^{\Sigma_2}_{B_2-C} \circ f^{\Sigma_1\cup\Sigma_2}_{C-D} = h^{\Sigma_2}_{B_2-D}$.

It remains to be shown that $f^{\Sigma_1\cup\Sigma_2}_{C-D}$ is unique with this property. Since $h^{\Sigma_1}_{B_1-C}$ coincides with Id_V on V, any $(\Sigma_1 \cup \Sigma_2)$-homomorphism $f : \mathcal{C}^{\Sigma_1\cup\Sigma_2} \to \mathcal{D}^{\Sigma_1\cup\Sigma_2}$ satisfying $h^{\Sigma_1}_{B_1-C} \circ f = h^{\Sigma_1}_{B_1-D}$ coincides with $h^{\Sigma_1}_{B_1-D}$ on V. Since $\mathcal{C}^{\Sigma_1\cup\Sigma_2}$ is free, there can be only one such homomorphism. □

In its given form, it is not clear how to generalize the definition of the class of admissible structures from free structures to arbitrary quasi-free structures. The following reformulation will turn out to be more appropriate for this purpose (see Section 4.4).

Proposition 4.6 *Let X be a countably infinite set, let G_1 and G_2 be sets of (relational and equational) atomic formulae over the signatures Σ_1 and Σ_2, where $G_1 \cup G_2$ is non-trivial, and let $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2}$, respectively, be free over X in the varieties $\mathcal{V}(G_1)$ and $\mathcal{V}(G_2)$. Then $\mathcal{V}(G_1 \cup G_2)$ is the class of all structures $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ such that $(\mathcal{B}_i^{\Sigma_i}, X)$ is free for $\mathcal{D}^{\Sigma_i}$, for $i = 1, 2$.*

Proof. Let $\mathcal{D}^{\Sigma_1\cup\Sigma_2} \in \mathcal{V}(G_1 \cup G_2)$. Since $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ satisfies $G_1 \cup G_2$, its Σ_1-reduct $\mathcal{D}^{\Sigma_1}$ satisfies G_1 and its Σ_2-reduct $\mathcal{D}^{\Sigma_2}$ satisfies G_2. Since $\mathcal{B}_i^{\Sigma_i}$ is free over X in the class of all models of G_i, it is in particular free for $\mathcal{D}^{\Sigma_i}$ $(i = 1, 2)$.

Conversely, let $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ be a $(\Sigma_1 \cup \Sigma_2)$-structure such that $(\mathcal{B}_i^{\Sigma_i}, X)$ is free for $\mathcal{D}^{\Sigma_i}$, for $i = 1, 2$. We must show that $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ satisfies the atomic formulae in $G_1 \cup G_2$. We restrict our attention to equations; relational atomic formulae can be treated analogously. Let $s = t$ be an equation in $G_1 \cup G_2$, and let $v_1, \ldots, v_n$ be the set of variables occurring in $s = t$. Now, assume that $\mathcal{D}^{\Sigma_1\cup\Sigma_2}$ does not satisfy $s = t$. Without loss of generality, we assume that $s = t$ is in G_1. Thus, there exist elements $c_1, \ldots, c_n$ of D such that

$$\mathcal{D}^{\Sigma_1} \not\models s(c_1, \ldots, c_n) = t(c_1, \ldots, c_n).$$

Since $\mathcal{B}_1^{\Sigma_1}$ is a model of G_1, we know that for arbitrary generators $x_1, \ldots, x_n \in X$ we have

$$\mathcal{B}_1^{\Sigma_1} \models s(x_1, \ldots, x_n) = t(x_1, \ldots, x_n).$$

Let $f : X \to D$ be a mapping that extends $\{x_1 \mapsto c_1, \ldots, x_n \mapsto c_n\}$. By assumption, f can be extended to a homomorphism $\phi : \mathcal{B}_1^{\Sigma_1} \to \mathcal{D}^{\Sigma_1}$. By Lemma 2.2 this implies that $\mathcal{D}^{\Sigma_1} \models s(c_1, \ldots, c_n) = t(c_1, \ldots, c_n)$, which is a contradiction. □

4.3 An amalgamation construction for quasi-free structures

We describe an explicit construction for closing any amalgamation base where the two components are quasi-free structures over disjoint signatures. In Section 4.4 we will prove that the constructed amalgamated product is in fact the free amalgamated product. Having such an explicit construction rather than just an abstract algebraic characterization of the free amalgamated product will become important in the proof of correctness of our method for combining constraint solvers. The description of the construction given below is considerably different from the one presented in [5, 6]. The main advantage of this new description is that it allows for shorter and simpler proofs.

Let $(\mathcal{A}_1^{\Sigma_1}, X)$ and $(\mathcal{A}_2^{\Sigma_2}, X)$ be quasi-free structures over disjoint signatures Σ_1 and Σ_2 such that $A_1 \cap A_2 = X$. We consider the amalgamation base $(X, \mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$, where the common part is just the set of atoms X. For $i = 1, 2$, the embedding "homomorphisms" $h_{X-A_i} : X \to \mathcal{A}_i^{\Sigma_i}$ are given by Id_X. In order to close this amalgamation base, we first embed each component $(\mathcal{A}_i^{\Sigma_i}, X)$ into an isomorphic superstructure $(\mathcal{B}_i^{\Sigma_i}, Y_i)$ satisfying Conditions 1–4 of Theorem 3.29 ($i = 1, 2$). In addition, we assume without loss of generality that $B_1 \cap B_2 = X$. Our goal is to construct (for $i = 1, 2$) a Σ_i-structure $\mathcal{C}_i^{\Sigma_i}$, which is a superstructure of $\mathcal{A}_i^{\Sigma_i}$ and a substructure of $\mathcal{B}_i^{\Sigma_i}$. The construction will provide us with a bijection between C_2 and C_1 satisfying certain properties. This bijection can be used to carry the Σ_2-structure of $\mathcal{C}_2^{\Sigma_2}$ over to C_1. The $(\Sigma_1 \cup \Sigma_2)$-structure obtained in this way is the result of the construction. The properties of the bijection will guarantee that this result is in fact the free amalgamated product of the component structures. For defining the required bijection, the notion of a fibre will be important.

Definition 4.7 *Let B_1, B_2, X, Y_1, Y_2 as above.* Fibres *are either* 1*-fibres or* 2*-fibres. A* 1-fibre *is of the form $F = \{x\}$ for $x \in X$, and a* 2-fibre *is of the form $F = \{y, b\}$ where $y \in Y_i \setminus X$ and $b \in B_j \setminus Y_j$ for $\{i, j\} = \{1, 2\}$. For a fibre F and $i = 1, 2$, let $F(i)$ be the unique element of F in B_i. The* index *of a 2-fibre F is j iff $F(j)$ is the non-atom element of F.*

The fibring construction

Let $b_1, b_2, b_3, \ldots$ be an enumeration of $B_{1,2} := B_1 \cup B_2$. Using this enumeration, we construct an ascending tower of sets $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ where each $\mathcal{F}_i$ is a set of mutually disjoint fibres. In addition, each set $\mathcal{F}_i$ contains only finitely many 2-fibres. We start with $\mathcal{F}_0 := \{\{x\} \mid x \in X\}$, i.e., $\mathcal{F}_0$ is the set of all 1-fibres. Now, assume that $\mathcal{F}_k$ has already been defined, and that all fibres of $\mathcal{F}_k$ are mutually disjoint. When defining $\mathcal{F}_{k+1}$, we distinguish two situations.

Case 1: If there exists an element b of $B_{1,2}$, say in B_i, such that

1. each element of the stabilizer $Stab_{\Sigma_i}^{\mathcal{B}_i}(b)$ belongs to a fibre $F \in \mathcal{F}_k$, but

2. b itself does not belong to a fibre $F \in \mathcal{F}_k$,

then we proceed as follows: Let b_{min} be the first element of $B_{1,2}$ (in the enumeration $b_1, b_2, b_3, \ldots$) satisfying the two Properties 1 and 2, and let i be such that $b_{min} \in B_i$. For the other index $j \neq i$, we select an atom $z \in Y_j$ that does not belong to any fibre of $\mathcal{F}_k$. Such an atom exists since $\mathcal{B}_j^{\Sigma_1}$ satisfies Condition 1 of Theorem 3.29, and $\mathcal{F}_k$ is assumed to contain only finitely many 2-fibres. We define $F_{k+1} := \{b_{min}, z\}$, and $\mathcal{F}_{k+1} := \mathcal{F}_k \cup \{F_{k+1}\}$. Note that F_{k+1} is indeed a 2-fibre since b_{min} cannot be an atom. In fact, it is easy to see that any atom x has the singleton set $\{x\}$ as its stabilizer. Thus, an atom cannot satisfy the Conditions 1 and 2 simultaneously.

Case 2: Otherwise, we define $\mathcal{F}_{k+1} := \mathcal{F}_k$.

By definition, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$, and at each level k, all fibres of $\mathcal{F}_k$ are mutually disjoint.

The definition of the amalgamated structure

Let $\mathcal{F} := \bigcup_{k \geq 0} \mathcal{F}_k$ be the set of all fibres introduced by the construction. We say that an element of $B_{1,2}$ is fibred iff it belongs to a fibre of $\mathcal{F}$. $C_{1,2} := \bigcup_{F \in \mathcal{F}} F$ is the set of all fibred elements of $B_{1,2}$, in particular $C_{1,2} \subseteq B_{1,2}$. Let $C_i := C_{1,2} \cap B_i$ denote the set of fibred elements of B_i, and let $Z_i := C_{1,2} \cap Y_i$ $(i = 1, 2)$.

Lemma 4.8 *$C_i = SH_{\Sigma_i}^{\mathcal{B}_i}(Z_i)$, and thus it is the carrier of a substructure $\mathcal{C}_i^{\Sigma_i}$ of $\mathcal{B}_i^{\Sigma_i}$.*

Proof. By Lemma 3.10, it is sufficient to show the first part of the lemma. First, we show $SH_{\Sigma_i}^{\mathcal{B}_i}(Z_i) \subseteq C_i$. Assume that $b \in SH_{\Sigma_i}^{\mathcal{B}_i}(Z_i)$. If b is an atom, then $b \in Z_i \subseteq C_i$ (by Lemma 3.19). Thus, assume that b is not an atom. By Lemma 3.22, $Stab_{\Sigma_i}^{\mathcal{B}_i}(b)$ is a finite subset of Z_i. Since all elements of Z_i are fibred, there is a minimal $k_b \geq 0$ such that each element of $Stab_{\Sigma_i}^{\mathcal{B}_i}(b)$ is contained in a fibre of $\mathcal{F}_{k_b}$. Thus, b satisfies Condition 1 of the construction for all $k \geq k_b$. As long as b is not included in a fibre of $\mathcal{F}_k$, Condition 2 is satisfied as well. Since only finitely many elements in the enumeration can precede b, it will at some stage of the construction be the minimal element satisfying both conditions, and will thus be included in a fibre of $\mathcal{F}$.

Conversely, assume that $b \in C_i$. If b is an atom, then $b \in Z_i \subseteq SH_{\Sigma_i}^{\mathcal{B}_i}(Z_i)$. A non-atom element of $B_{1,2}$ is only fibred after all elements of its stabilizer are fibred. Thus, we know that $Stab_{\Sigma_i}^{\mathcal{B}_i}(b) \subseteq Z_i$, which implies $b \in SH_{\Sigma_i}^{\mathcal{B}_i}(Z_i)$. □

Now, we define appropriate bijections between C_1 and C_2. Each element $c \in C_{1,2}$ belongs to a unique fibre F_c of $\mathcal{F}$. We define the bijections $h_{i,j} : C_i \to C_j$ by mapping each $c \in C_i$ to $F_c(j)$, the unique element of F_c belonging to C_j ($\{i, j\} = \{1, 2\}$). Obviously this implies $h_{i,j} = {h_{j,i}}^{-1}$. Note that any element x of X belongs to a 1-fibre, and thus

$$h_{i,j}(x) = x \text{ for all } x \in X. \tag{4.9}$$

The bijections $h_{1,2}$ and $h_{2,1}$ are now used to carry the Σ_2-structure of $\mathcal{C}_2^{\Sigma_2}$ to C_1: Let f be an n-ary function symbol of Σ_2, let p be an n-ary predicate symbol of Σ_2, and let $a_1, \ldots, a_n \in C_1$. We define

$$\begin{array}{rcl} f_{\mathcal{C}_1}(a_1, \ldots, a_n) & := & h_{2,1}(f_{\mathcal{C}_2}(h_{1,2}(a_1), \ldots, h_{1,2}(a_n))) \\ p_{\mathcal{C}_1}[a_1, \ldots, a_n] & :\Longleftrightarrow & p_{\mathcal{C}_2}[h_{1,2}(a_1), \ldots, h_{1,2}(a_n)]. \end{array}$$

In the same way, we impose the Σ_1-structure of $\mathcal{C}^{\Sigma_1}$ on C_2. Thus, both $\mathcal{C}_1$ and $\mathcal{C}_2$ can be seen as $(\Sigma_1 \cup \Sigma_2)$-structures. Let $\Sigma := \Sigma_1 \cup \Sigma_2$. By construction, the mappings

$$h_{1,2} \text{ and } h_{2,1} \text{ are inverse } \Sigma\text{-isomorphisms between } \mathcal{C}_1^{\Sigma} \text{ and } \mathcal{C}_2^{\Sigma}. \tag{4.10}$$

For this reason, it is irrelevant which of these two structures is taken as the result of the construction. In the following, we use $\mathcal{C}_1^{\Sigma}$ as the amalgamated structure obtained by the construction, and we will sometimes denote this structure by $\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$.

Properties of the amalgamation construction

Before we show that the construction really yields the free amalgamated product, let us list some useful properties:

$$(\mathcal{C}_i^{\Sigma_i}, Z_i) \text{ and } (\mathcal{A}_i^{\Sigma_i}, X) \text{ are qf-isomorphic (for } i = 1, 2). \tag{4.11}$$

(4.11) follows from the fact that (for $i = 1, 2$) $\mathcal{B}_i^{\Sigma_i}$ satisfies Condition 4 of Theorem 3.29. In addition, by Lemma 3.28 we have

$$\forall d \in C_i : Stab_{\Sigma_i}^{\mathcal{C}_i}(d) = Stab_{\Sigma_i}^{\mathcal{B}_i}(d) \text{ and } \forall U \subseteq Z_i : SH_{\Sigma_i}^{\mathcal{C}_i}(U) = SH_{\Sigma_i}^{\mathcal{B}_i}(U) \tag{4.12}$$

For $i = 1, 2$, each set of fibres $\mathcal{F}_k$ determines a set $\mathcal{F}_k^i := \{F(i) \mid F \in \mathcal{F}_k\} \subseteq C_i$. Now, (4.12) and the definition of the fibring construction imply:

$$\text{If } c \in C_i \setminus Z_i \text{ is in } \mathcal{F}_{k+1}^i, \text{ then } Stab_{\Sigma_i}^{\mathcal{C}_i}(c) \subseteq \mathcal{F}_k^i \text{ (for } i = 1, 2). \tag{4.13}$$

In order to show that $\mathcal{C}_1^{\Sigma}$ closes the amalgamation base $(X, \mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$, we define

$$h_{A_1-C_1} := Id_{A_1} \text{ and } h_{A_2-C_1} := h_{2,1}|_{A_2}. \tag{4.14}$$

By definition of $h_{A_i-C_1}$ and (4.9) we know that

$$h_{A_i-C_1}|_X = Id_X \text{ (for } i = 1, 2). \tag{4.15}$$

Thus, $h_{X-A_1} \circ h_{A_1-C_1} = Id_X = h_{X-A_2} \circ h_{A_2-C_1}$, which shows:

Lemma 4.16 *The amalgamated structure $\mathcal{C}_1^{\Sigma}$ obtained by the construction is an amalgamated product of $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$.*

Definition 4.17 *The enumeration $b_1, b_2, b_3, \ldots$ defines a strict linear ordering $\prec_X$ on X. In addition, a strict linear ordering $\prec_i$ on the complements $C_i \setminus X$ is given by the order in which the elements of $C_i \setminus X$ are fibred: We define $c \prec_i d$ iff, for some k, $c \in \mathcal{F}_k^i$ and $d \notin \mathcal{F}_k^i$. With $<_i$ we denote the unique strict linear ordering on C_i that extends both $\prec_X$ and $\prec_i$, and makes each element of X smaller than each element of $C_i \setminus X$ $(i = 1, 2)$.*

As an easy consequence of this definition, we obtain

$$\forall c, d \in C_i\colon c <_i d \quad \text{iff} \quad h_{i,j}(c) <_j h_{i,j}(d) \quad (\{i,j\} = \{1,2\}), \tag{4.18}$$

$$\forall c, d \in C_i\colon c <_i d \quad \text{implies} \quad d \notin Stab_{\Sigma_i}^{\mathcal{C}_i}(c) \quad (i \in \{1,2\}). \tag{4.19}$$

Note that (4.18) is trivial, and that (4.19) follows from (4.13).

4.4 The free amalgamated product of quasi-free structures

In this subsection, we will show that the amalgamation construction presented above really yields the free amalgamated product of the quasi-free component structures. In the sequel, $(\mathcal{A}_1^{\Sigma_1}, X)$ and $(\mathcal{A}_2^{\Sigma_2}, X)$ denote quasi-free structures over disjoint signatures, which are used as the input components of the amalgamation construction. As before, we consider the amalgamation base $(X, \mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$ where the embedding homomorphisms h_{X-A_i} are given by Id_X. We also refer to other entities introduced in the amalgamation construction, such as $\mathcal{C}_i^{\Sigma}, \mathcal{B}_i^{\Sigma_i}, h_{i,j}, \mathcal{F}_k$, etc. Recall that $\Sigma := \Sigma_1 \cup \Sigma_2$. First, we must fix the class of admissible structures with respect to which the free product is to be built. Proposition 4.6 motivates the following definition.

Definition 4.20 *Let $(X, \mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$ be the amalgamation base introduced above. Then we choose*

$$Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}) := \{\mathcal{D}^{\Sigma_1 \cup \Sigma_2} \mid (\mathcal{A}_i^{\Sigma_i}, X) \text{ is quasi-free for } \mathcal{D}^{\Sigma_i}, \text{ for } i = 1, 2\}$$

as the class of admissible structures (cf. Definition 3.24).

We obtain

Theorem 4.21 *$\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ is the free amalgamated product of the quasi-free structures $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ with respect to the amalgamation base and the class $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$ of admissible structures defined above.*

Proof. Lemma 4.16 states that $\mathcal{C}_1^{\Sigma}$ is an amalgamated product of $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$. It remains to be shown that this is the *free* amalgamated product w.r.t. $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$.

First, we prove that $\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ is in the chosen class $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$ of admissible structures. By Lemma 3.27 and (4.11), $(\mathcal{A}_1^{\Sigma_1}, X)$ is quasi-free for $\mathcal{C}_1^{\Sigma_1}$ and $(\mathcal{A}_2^{\Sigma_2}, X)$ is quasi-free for $\mathcal{C}_2^{\Sigma_2}$. Since $\mathcal{C}_1^{\Sigma_2}$ and $\mathcal{C}_2^{\Sigma_2}$ are isomorphic, $(\mathcal{A}_2^{\Sigma_2}, X)$ is also quasi-free for $\mathcal{C}_1^{\Sigma_2}$. This shows $\mathcal{C}_1^{\Sigma} \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$.

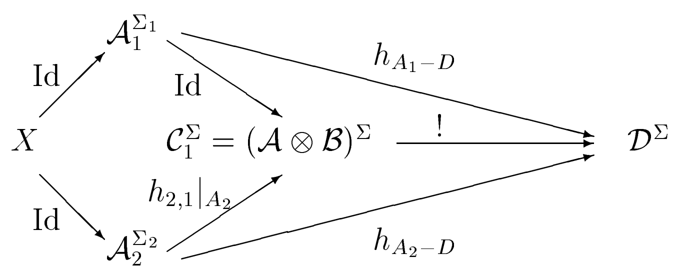
In order to show that $\mathcal{C}_1^{\Sigma}$ is the most general admissible structure closing the amalgamation base, assume that $\mathcal{D}^{\Sigma}$ is another admissible amalgamated product of $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$, i.e.,

$$(\mathcal{A}_i^{\Sigma_i}, X) \text{ is quasi-free for } \mathcal{D}^{\Sigma_i}, \text{ for } i = 1, 2, \tag{4.22}$$

and there are Σ_i-homomorphism $h_{A_i-D} : \mathcal{A}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ such that h_{A_1-D} and h_{A_2-D} coincide on X. (Recall that Id_X is the "homomorphism" embedding X into the components $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ of the amalgamation base.) We must show that there exists a unique Σ-homomorphism $h_{C_1-D} : \mathcal{C}_1^{\Sigma} \to \mathcal{D}^{\Sigma}$ that satisfies

$$(*) \quad h_{A_i-D} = h_{A_i-C_1} \circ h_{C_1-D}, \text{ for } i = 1, 2.$$

Recall that $h_{A_1-C_1} = Id_{A_1}$ and $h_{A_2-C_1} = h_{1,2}|_{A_2}$, by (4.14). This situation is illustrated in the following figure.



Definition of the homomorphism: Let us start with a simple remark. A given mapping $h_k : \mathcal{F}_k \to D$ induces two mappings $h_k^i := \{\langle F(i), d\rangle \mid \langle F, d\rangle \in h_k\} : \mathcal{F}_k^i \to D$. By definition of the bijections $h_{i,j}$ we have, for all i, j such that $\{i, j\} = \{1, 2\}$,

$$h_k^i \text{ and } h_{i,j} \circ h_k^j \text{ coincide on } \mathcal{F}_k^i. \tag{4.23}$$

We define *an ascending tower of mappings* $h_0 \subseteq h_1 \subseteq h_2 \subseteq \ldots$, where $h_k : \mathcal{F}_k \to D$ $(k = 0, 1, 2, \ldots)$. Thus, we have $h_0^i \subseteq h_1^i \subseteq h_2^i \subseteq \ldots$, for $i = 1, 2$. At each step k of the construction of this tower, we will show that, for $i = 1, 2$,

$$h_k^i \text{ can be extended to a } \Sigma_i\text{-homomorphism } g_{C_i-D} : \mathcal{C}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}. \tag{4.24}$$

For the case $k = 0$, recall that $\mathcal{F}_0 = \{\{x\} \mid x \in X\}$. We define

$$h_0 : \mathcal{F}_0 \to D : \{x\} \mapsto h_{A_1-D}(x) = h_{A_2-D}(x). \tag{4.25}$$

By (4.11), (4.22), and Lemma 3.27,

$$(\mathcal{C}_i^{\Sigma_i}, Z_i) \text{ is quasi-free for } \mathcal{D}^{\Sigma_i}, \text{ for } i = 1, 2. \tag{4.26}$$

Since the induced mappings h_0^i are functions from $X \subseteq Z_i$ to D, property (4.24) for h_0^i follows directly from (4.26) (for $i = 1, 2$).

For the *induction step*, suppose that h_k is already defined, and that we must define h_{k+1}. In the amalgamation construction, we have distinguished two cases. If $\mathcal{F}_{k+1} = \mathcal{F}_k$, then we define $h_{k+1} := h_k$. In this case, property (4.24) is satisfied by induction hypothesis for the mappings $h_{k+1}^i = h_k^i$ (for $i = 1, 2$). Thus, assume that $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{F\}$, where F is a 2-fibre, say, with index 1. Thus F is of the form $\{c, z\}$, where $c \in C_1 \setminus Z_1$ and $z \in Z_2$, and these two elements do not occur in a fibre of $\mathcal{F}_k$. By (4.13), $Stab_{\Sigma_1}^{\mathcal{C}_1}(c) \subseteq \mathcal{F}_k^1$, and thus h_k^1 is defined on $Stab_{\Sigma_1}^{\mathcal{C}_1}(c)$. By induction, property (4.24) holds for k, i.e., there exists at least one extension of h_k^1 to a Σ_1-homomorphism $g_{C_1-D} : \mathcal{C}_1^{\Sigma_1} \to \mathcal{D}^{\Sigma_1}$. By Lemma 3.25 and (4.26), all such extensions yield the same value—say, $d \in D$—for c. We define $h_{k+1} := h_k \cup \{\langle\{c, z\}, d\rangle\}$.

We must show that condition (4.24) holds for $k + 1$. By choice of d, this is trivial for $i = 1$. Let $U^k = Z_2 \cap \mathcal{F}_k^2$ denote the set of all atoms in Z_2 that are fibred in $\mathcal{F}_k$, and let h_{U^k-D} denote the restriction of h_k^2 to U^k. By (4.26), the mapping $h_{U_k-D} \cup \{\langle z, d\rangle\}$ can be extended to a Σ_2-homomorphism $g_{C_2-D} : \mathcal{C}_2^{\Sigma_2} \to \mathcal{D}^{\Sigma_2}$. In order to show (4.24) for $i = 2$ it suffices to prove that g_{C_2-D} extends h_k^2. By induction, h_k^2 has an extension to a Σ_2-homomorphism $g'_{C_2-D} : \mathcal{C}_2^{\Sigma_2} \to \mathcal{D}^{\Sigma_2}$. Since g'_{C_2-D} and g_{C_2-D} coincide on U_k, these homomorphisms coincide on $SH_{\Sigma_2}^{\mathcal{C}_2}(U_k)$, by Lemma 3.25. Recall that a non-atom, say b, is only fibred if its stabilizer is already fibred, and thus b is in the stable hull of the already fibred atoms. For this reason, $\mathcal{F}_k^2$—which constitutes the domain of h_k^2—is a subset of $SH_{\Sigma_2}^{\mathcal{B}_2}(U_k)$. By (4.12), $SH_{\Sigma_2}^{\mathcal{B}_2}(U_k) = SH_{\Sigma_2}^{\mathcal{C}_2}(U_k)$, and thus g_{C_2-D} extends h_k^2. This completes the inductive definition of the ascending tower of mappings $h_0 \subseteq h_1 \subseteq h_2 \subseteq \ldots$, and the proof that these mappings satisfy (4.24).

We use this tower to define $H := \bigcup_{k \geq 0} h_k$. Let $h_{C_i-D} := \bigcup_{k \geq 0} h_k^i : C_i \to D$ be the mappings induced by H ($i = 1, 2$). We claim that, for $i = 1, 2$, h_{C_i-D} is a Σ_i-homomorphism. In fact, let $c_1, \ldots, c_n \in C_i$, and let f be a function symbol in Σ_i. Choose a level k such that $c_1, \ldots, c_n$ and $f^{\mathcal{C}_i}(c_1, \ldots, c_n)$ are fibred in $\mathcal{F}_k$. Note that $f^{\mathcal{C}_i}(c_1, \ldots, c_n)$ is eventually fibred if $c_1, \ldots, c_n$ are fibred (see Lemma 4.8). Since h_{C_i-D} extends h_k^i, and since the latter mapping can be extended to a Σ_i-homomorphism g_{C_i-D}, by (4.24), it follows that

$$\begin{aligned} h_{C_i-D}(f^{\mathcal{C}_i}(c_1, \ldots, c_n)) &= g_{C_i-D}(f^{\mathcal{C}_i}(c_1, \ldots, c_n)) \\ &= f^{\mathcal{D}}(g_{C_i-D}(c_1), \ldots, g_{C_i-D}(c_n)) \\ &= f^{\mathcal{D}}(h_{C_i-D}(c_1), \ldots, h_{C_i-D}(c_n)). \end{aligned}$$

This shows that h_{C_i-D} is a homomorphism with respect to the function symbols in Σ_i. Similarly, it can be shown that h_{C_i-D} is a homomorphism with respect to the relational symbols in Σ_i. It remains to be shown that h_{C_1-D} is even a Σ-homomorphism and that it satisfies $(*)$.

Property (4.23) shows that $h_{C_1-D} = h_{1,2} \circ h_{C_2-D}$ and $h_{C_2-D} = h_{2,1} \circ h_{C_1-D}$. Since $h_{1,2}$ is a Σ-isomorphism (4.10) and h_{C_2-D} is a Σ_2-homomorphism, the first identity implies that h_{C_1-D} is also a Σ_2-homomorphism. Since we already know that it is a Σ_1-homomorphism, this shows that h_{C_1-D} is a Σ-homomorphism. Now, (4.15) and (4.25) imply that the Σ_i-homomorphisms h_{A_i-D} and $h_{A_i-C_1} \circ h_{C_1-D}$ coincide on X. From (4.22) it follows that $h_{A_i-D} = h_{A_i-C_1} \circ h_{C_1-D}$ (for $i = 1, 2$), i.e., $(*)$ holds.

Uniqueness of the homomorphism: Assume that $g^*_{C_1-D} : \mathcal{C}_1^\Sigma \to \mathcal{D}^\Sigma$ is a Σ-homomorphism such that

$$h_{A_i-D} = h_{A_i-C_1} \circ g^*_{C_1-D}, \text{ for } i = 1, 2. \tag{4.27}$$

Let $G = \{\langle\{c, h_{1,2}(c)\}, d\rangle \mid \langle c, d\rangle \in g^*_{C_1-D}\}$. By definition of $h_{1,2}$, $\{c, h_{1,2}(c)\}$ is a fibre of $\mathcal{F}$, for every pair $\langle\{c, h_{1,2}(c)\}, d\rangle \in G$. The definition of H and (4.27) imply that H and G coincide on $\mathcal{F}_0$. Now, suppose that H and G coincide on $\mathcal{F}_k$. Obviously, if $\mathcal{F}_k = \mathcal{F}_{k+1}$, then H and G coincide on $\mathcal{F}_{k+1}$. Thus, assume that $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{\{c, z\}\}$, where we assume without loss of generality that the 2-fibre $\{c, z\}$ has index 1. As we have seen above, all endomorphisms extending h_k^1 coincide on c. It follows that H and G coincide on $\{c, z\}$, and thus H and G coincide on $\mathcal{F}_{k+1}$. Thus, we have shown by induction that H and G coincide. Hence $g^*_{C_1-D}$ and h_{C_1-D} coincide. □

The following corollary, which is an easy consequence of the above proof, will become important in the next subsection.

Corollary 4.28 *Let $\mathcal{D}^\Sigma \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$. Then every mapping $h_{X-D} : X \to D$ has a unique extension to a Σ-homomorphism $h_{C_1-D} : \mathcal{C}_1^\Sigma = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2} \to \mathcal{D}^\Sigma$.*

Proof. Let $h_{X-D} : X \to D$ be a mapping. Since $(\mathcal{A}_i^{\Sigma_i}, X)$ is quasi-free for $\mathcal{D}^{\Sigma_i}$, there exists a unique extension of h_{X-D} to a Σ_i-homomorphism $h_{A_i-D} : \mathcal{A}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ (for $i = 1, 2$). Given the homomorphisms h_{A_1-D} and h_{A_2-D}, which coincide on X, there exists a unique Σ-homomorphism $h_{C_1-D} : \mathcal{C}_1^\Sigma \to \mathcal{D}^\Sigma$ such that $h_{A_i-D} = h_{A_i-C_1} \circ h_{C_1-D}$ (for $i = 1, 2$), as we have shown in the previous proof. It follows from (4.15) that h_{C_1-D} extends h_{X-D}, which shows the existence of an extension of h_{X-D}.

To show uniqueness, assume that g_{C_1-D} is another extension of h_{X-D}. Because of (4.14) and (4.15), this implies that $h_{A_i-C_1} \circ g_{C_1-D}$ extends h_{X-D} to a Σ_i-homomorphism $\mathcal{A}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ (for $i = 1, 2$). Uniqueness of these extensions implies $h_{A_i-C_1} \circ g_{C_1-D} = h_{A_i-D}$. But we know that h_{C_1-D} is unique with this property. □

4.5 Multiple and iterated amalgamation

The explicit amalgamation construction introduced above can easily be generalized to a construction that combines an arbitrary number $n \geq 2$ of quasi-free structures over disjoint signatures.[12] The proof given in the above subsection can also be generalized to show that the extended construction yields the n-fold simultaneous free amalgamated product, provided that the following obvious generalization of the class of admissible structures is used:

$$Adm(\mathcal{A}_1^{\Sigma_1}, \ldots, \mathcal{A}_n^{\Sigma_n}) = \{\mathcal{D}^{\Sigma_1 \cup \ldots \cup \Sigma_n} \mid \mathcal{A}_i^{\Sigma_i} \text{ is quasi-free for } \mathcal{D}^{\Sigma_i}, \text{ for } 1 \leq i \leq n\}. \quad (4.29)$$

In this subsection, we show that it is not really necessary to introduce the explicit amalgamation construction for the case $n > 2$ since the free amalgamated product can also be obtained by iterated application of the construction to two structures. Obviously, iterated application is only possible if the structure obtained by the construction is again quasi-free. The following proposition shows that this prerequisite is satisfied.

Proposition 4.30 *The free amalgamated product of two quasi-free structures with common atom set X is a quasi-free structure with atom set X.*

Proof. We show that $(\mathcal{C}_1^{\Sigma}, X) = (\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}, X)$ is a quasi-free structure. In the proof of Theorem 4.21 we have seen that $\mathcal{C}_1^{\Sigma}$ is an admissible structure. Thus, if we choose $\mathcal{D}^{\Sigma} := \mathcal{C}_1^{\Sigma}$ in Corollary 4.28, we obtain that every mapping $h_{X-C_1} : X \to C_1$ can be extended to an endomorphism of $\mathcal{C}_1^{\Sigma}$. Thus X is an atom set for $\mathcal{C}_1^{\Sigma}$. It remains to be shown that every element $a \in C_1$ is stabilized—with respect to $\mathcal{C}_1^{\Sigma}$—by a finite subset of X.

To this purpose, we show by *induction on k* ($k \geq 0$) that each element of $\mathcal{F}_k^i$ is stabilized—with respect to $\mathcal{C}_i^{\Sigma}$—by a finite subset of X (for $i = 1, 2$).

For $k = 0$, we have $\mathcal{F}_0^i = X$ $(i = 1, 2)$, and thus the claim is trivially satisfied since $x \in X$ is stabilized by itself.

$k \to k+1$: For $\mathcal{F}_{k+1} = \mathcal{F}_k$ there is nothing to be shown. Otherwise, we have $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{F\}$, where F is a 2-fibre. We assume without loss of generality that F has index 1, i.e., $F = \{c, z\}$ where $c \in C_1 \setminus Z_1$ and $z \in Z_2$, and both elements of F do not occur in a fibre of $\mathcal{F}_k$. By (4.13), we know that $Stab_{\Sigma_1}^{\mathcal{C}_1}(c) \subseteq \mathcal{F}_k^1$. Hence, by induction hypothesis, the elements of $Stab_{\Sigma_1}^{\mathcal{C}_1}(c)$ are stabilized—with respect to $\mathcal{C}_1^{\Sigma}$—by a finite subset U of X. We show that U stabilizes c and z.

Let us first consider c. Take two Σ-endomorphisms h_1 and h_2 of $\mathcal{C}_1^{\Sigma}$ that coincide on U. By choice of U, we have $h_1(y) = h_2(y)$ for all $y \in Stab_{\Sigma_1}^{\mathcal{C}_1}(c)$.

[12] It is even possible to amalgamate a countably infinite number of quasi-free structures in this way.

Since h_1 and h_2 can also be seen as Σ_1-endomorphisms of $\mathcal{C}_1^{\Sigma_1}$, we know that $h_1(c) = h_2(c)$. Hence U stabilizes c with respect to $\mathcal{C}_1^{\Sigma}$.

Next we consider z. Let g_1 and g_2 be two Σ-endomorphisms of $\mathcal{C}_2^{\Sigma}$ that coincide on U. Since $h_{1,2}$ and $h_{2,1}$ leave elements of X fixed, $h_1 := h_{1,2} \circ g_1 \circ h_{2,1}$ and $h_2 := h_{1,2} \circ g_2 \circ h_{2,1}$ are Σ-endomorphisms of $\mathcal{C}_1^{\Sigma}$ that coincide on U. This implies, as we have just seen, that $h_1(c) = h_2(c)$. But then $g_1(z) = (h_{2,1} \circ h_1 \circ h_{1,2})(z) = h_{1,2}(h_1(h_{2,1}(z))) = h_{1,2}(h_1(c)) = h_{1,2}(h_2(c)) = h_{1,2}(h_2(h_{2,1}(z))) = (h_{2,1} \circ h_1 \circ h_{1,2})(z) = g_2(z)$. To see that these equalities hold, recall that, by definition, $h_{1,2}$ maps $z \in C_1$ to the element of C_2 in its fibre, i.e., to c. Thus, we have shown that U stabilizes z with respect to $\mathcal{C}_2^{\Sigma}$, which completes the proof of the proposition. □

Corollary 4.31 *$(\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}, X)$ is a quasi-free structure that is quasi-free for each $\mathcal{D}^{\Sigma} \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2})$.*

Proof. This follows immediately from Corollary 4.28 and Proposition 4.30. □

Obviously, the set of admissible structures, as introduced in Definition 4.20, satisfies $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}) = Adm(\mathcal{A}_2^{\Sigma_2}, \mathcal{A}_1^{\Sigma_1})$. Thus, free amalgamation of quasi-free structures is commutative. Since the amalgamation construction can be iterated, the question arises whether the construction is associative as well. In order to answer this question in the affirmative, we must show that the assumptions of Theorem 4.4 are satisfied. In this case, the theorem also shows that simultaneous free amalgamation and iterated free amalgamation yield the same result.

As an obvious consequence of the definition of the class of admissible structures for $n \geq 2$ (see (4.29)), we obtain $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3}) \subseteq Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}) \cap Adm(\mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3})$.

Lemma 4.32
$Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3}) \subseteq Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}) \cap Adm(\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_2})$.

Proof. Let $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Assume that $\mathcal{D}^{\Sigma} \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3})$. This means that $(\mathcal{A}_i^{\Sigma_i}, X)$ is quasi-free for $\mathcal{D}^{\Sigma_i}$, for $i = 1, 2, 3$. Hence $\mathcal{D}^{\Sigma_2 \cup \Sigma_3} \in Adm(\mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3})$. But then Corollary 4.31 implies that $(\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}, X)$ is quasi-free for $\mathcal{D}^{\Sigma_2 \cup \Sigma_3}$. Therefore $\mathcal{D}^{\Sigma} \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3})$. This shows that $Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3}) \subseteq Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3})$. The other inclusion follows by symmetry. □

Lemma 4.33
$\{\mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}), (\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}) \otimes \mathcal{A}_3^{\Sigma_3}\} \subseteq Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3})$.

Proof. We show $\mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}) \in Adm(\mathcal{A}_1^{\Sigma_1}, \mathcal{A}_2^{\Sigma_2}, \mathcal{A}_3^{\Sigma_3})$. (The other inclusion follows by symmetry.) Obviously, the quasi-free structure $(\mathcal{A}_1^{\Sigma_1}, X)$, is quasi-free for itself, and thus quasi-free for the Σ_1-isomorphic structure $\mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3})$. For the same reasons, $(\mathcal{A}_i^{\Sigma_i}, X)$ is quasi-free for $\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}$, for $i = 2, 3$. Since $\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{A}_3^{\Sigma_3}$ and $\mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2 \otimes \mathcal{A}_3)^{\Sigma_2 \cup \Sigma_3}$ are isomorphic $(\Sigma_2 \cup \Sigma_3)$-structures, $(\mathcal{A}_i^{\Sigma_i}, X)$ is quasi-free for $(\mathcal{A}_1 \otimes (\mathcal{A}_2 \otimes \mathcal{A}_3))^{\Sigma_i}$, for $i = 2, 3$. □

To sum up, we have shown that Theorem 4.4 can be applied, which yields:

Theorem 4.34 *Modulo isomorphism, free amalgamation of quasi-free structures with disjoint signatures over the same atom set is associative, and free simultaneous amalgamation coincides with iterated free amalgamation.*

5 Combining Constraint Solvers for Quasi-Free Structures

Let $(\mathcal{A}_1^{\Sigma_1}, X)$ and $(\mathcal{A}_2^{\Sigma_2}, X)$ be quasi-free structures over disjoint signatures Σ_1 and Σ_2, and let $\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ denote their free amalgamated product, as constructed in the previous section, where $\Sigma = \Sigma_1 \cup \Sigma_2$. This section is devoted to proving the following combination result for constraint solvers over quasi-free structures.

Theorem 5.1 *The positive theory of $\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ is decidable, provided that the positive theories of the quasi-free structures $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ are decidable.*

First, we show how constraint solvers for the positive theories of $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ can be combined to a constraint solver for the *existential* positive theory of $\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$. In a second subsection, it is shown that this result can be lifted to the full positive theory of $\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$.

5.1 The Existential Positive Case

In this subsection, we prove a restricted version of Theorem 5.1.

Theorem 5.2 *The existential positive theory of $\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ is decidable, provided that the positive theories of the quasi-free structures $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ are decidable.*

The same theorem can be proved for the simultaneous free amalgamated product of $n \geq 2$ quasi-free components over disjoint signatures. To keep the proof simpler, we restrict our attention to the case $n = 2$.

The decomposition algorithm described below decomposes an existential positive Σ-sentence φ_0 into a finite set of pairs (α, β), where α is a positive Σ_1-sentence and β is a positive Σ_2-sentence. This algorithm coincides with the one described in [5], where it has been used in the restricted context of combination problems for free structures. Steps similar to Step 1, 3, and the labelling in Step 4 are present in most methods for combining unification algorithms. Nelson and Oppen's combination method for universal theories [32] explicitly uses Step 1, and implicitly, Step 3 is also present.

Before we can describe the algorithm, we must introduce some notation. In the following, V denotes an infinite set of variables used by the first-order languages under consideration. Let t be a Σ-term. This term is called *pure* iff it is either a Σ_1-term or a Σ_2-term. An equation is pure iff it is an equation between pure terms of the same signature. A relational formula $p[s_1, \ldots, s_m]$ is pure iff $s_1, \ldots, s_m$ are pure terms of the signature of p. Now assume that t is a non-pure term whose topmost function symbol is in Σ_1. A subterm s of t is called *alien subterm* of t iff its topmost function symbol belongs to Σ_2 and every proper superterm of s in t has its top symbol in Σ_1. Alien subterms of terms with top symbol in Σ_2 are defined analogously. For a relational formula $p[s_1, \ldots, s_m]$, alien subterms are defined as follows: if s_i has a top symbol whose signature is different from the signature of p, then s_i itself is an alien subterm; otherwise, any alien subterm of s_i is an alien subterm of $p[s_1, \ldots, s_m]$.

The decomposition algorithm

Let φ_0 be an existential positive Σ-sentence. Without loss of generality, we may assume that φ_0 has the form $\exists \vec{u}_0\ \gamma_0$, where γ_0 is a conjunction of atomic formulae. Indeed, since existential quantifiers distribute over disjunction, a sentence $\exists \vec{u}_0\ (\gamma_1 \vee \gamma_2)$ is valid iff $\exists \vec{u}_0\ \gamma_1$ or $\exists \vec{u}_0\ \gamma_2$ is valid.

Step 1: Transform non-pure atomic formulae.

(1) Equations $s = t$ of γ_0 where s and t have topmost function symbols belonging to different signatures are replaced by (the conjunction of) two new equations $u = s, u = t$, where u is a new variable. The quantifier prefix is extended by adding an existential quantification for u.
(2) As a result, we may assign a unique label Σ_1 or Σ_2 to each atomic formula that is not an equation between variables. The label of an equation $s = t$ is the signature of the topmost function symbols of s and/or t. The label of a relational formula $p[s_1, \ldots, s_m]$ is the signature of p.
(3) Now alien subterms occurring in atomic formulae are successively replaced by new variables. For example, assume that $s = t$ is an equation in the current formula, and that s contains the alien subterm s_1. Let u be a variable not occurring in the current formula, and let s' be the term obtained from s by replacing s_1 by u. Then the original equation is replaced

by (the conjunction of) the two equations $s' = t$ and $u = s_1$. The quantifier prefix is extended by adding an existential quantification for u. The equation $s' = t$ keeps the label of $s = t$, and the label of $u = s_1$ is the signature of the top symbol of s_1. Relational atomic formulae with alien subterms are treated analogously. This process is iterated until all atomic formulae occurring in the conjunctive matrix are pure. It is easy to see that this is achieved after finitely many iterations.

Step 2: Remove atomic formulae without label.
Equations between variables occurring in the conjunctive matrix are removed as follows: If $u = v$ is such an equation then one removes $\exists u$ from the quantifier prefix and $u = v$ from the matrix. In addition, every occurrence of u in the remaining matrix is replaced by v. This step is iterated until the matrix contains no equations between variables.

Let φ_1 be the new sentence obtained this way. The matrix of φ_1 can be written as a conjunction $\gamma_{1,\Sigma_1} \wedge \gamma_{1,\Sigma_2}$, where γ_{1,Σ_1} is a conjunction of all atomic formulae from φ_1 with label Σ_1, and γ_{1,Σ_2} is a conjunction of all atomic formulae from φ_1 with label Σ_2. There are three different types of variables occurring in φ_1: *shared variables* occur both in γ_{1,Σ_1} and in γ_{1,Σ_2}; Σ_1-*variables* occur only in γ_{1,Σ_1}; and Σ_2-*variables* occur only in γ_{1,Σ_2}. Let $\vec{u}_{1,\Sigma_1}$ be the tuple of all Σ_1-variables, $\vec{u}_{1,\Sigma_2}$ be the tuple of all Σ_2-variables, and $\vec{u}_1$ be the tuple of all shared variables.[13] Obviously, φ_1 is equivalent to the sentence

$$\exists \vec{u}_1 \, (\exists \vec{u}_{1,\Sigma_1} \; \gamma_{1,\Sigma_1} \wedge \exists \vec{u}_{1,\Sigma_2} \; \gamma_{1,\Sigma_2}) \, .$$

The next two steps of the algorithm are nondeterministic, i.e., a given sentence is transformed into finitely many new sentences. Here the idea is that the original sentence is valid iff at least one of the new sentences is valid.

Step 3: Variable identification.
Consider all possible partitions of the set of all shared variables. Each of these partitions yields one of the new sentences as follows. The variables in each class of the partition are "identified" with each other by choosing an element of the class as representative, and replacing in the sentence all occurrences of variables of the class by this representative. Quantifiers for replaced variables are removed.

Let $\exists \vec{u}_2 \, (\exists \vec{u}_{1,\Sigma_1} \; \gamma_{2,\Sigma_1} \wedge \exists \vec{u}_{1,\Sigma_2} \; \gamma_{2,\Sigma_2})$ denote one of the sentences obtained by Step 3, where $\vec{u}_2$ denotes the sequence of all representatives of shared variables.

[13]The order in these tuples can be chosen arbitrarily.

Step 4: Choose signature labels and ordering.
We choose a label Σ_1 or Σ_2 for every (shared) variable in $\vec{u}_2$, and a linear ordering $<$ on these variables.

For each of the choices made in Step 3 and 4, the algorithm yields a pair (α, β) of sentences as output.

Step 5: Generate output sentences.
The sentence $\exists\vec{u}_2(\exists\vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1} \wedge \exists\vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2})$ is split into two sentences

$$\alpha = \forall\vec{v}_1\exists\vec{w}_1 \ldots \forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1}$$

and

$$\beta = \exists\vec{v}_1\forall\vec{w}_1 \ldots \exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2}.$$

Here $\vec{v}_1\vec{w}_1 \ldots \vec{v}_k\vec{w}_k$ is the unique re-ordering of $\vec{u}_2$ along $<$. The variables $\vec{v}_i$ ($\vec{w}_i$) are the variables with label Σ_2 (label Σ_1).

Thus, the overall output of the algorithm is a finite set of pairs of sentences. Note that the sentences α and β are positive formulae, but they need no longer be existential positive formulae.

Obviously, Theorem 5.2 follows immediately as soon as we have shown that the decomposition algorithm is sound and complete.

Correctness of the decomposition algorithm

First, we show soundness of the algorithm, i.e., if one of the output pairs is valid then the original sentence was valid.

Lemma 5.3 *If $\mathcal{A}_1^{\Sigma_1} \models \alpha$ and $\mathcal{A}_2^{\Sigma_2} \models \beta$ for some output pair (α, β), then $\mathcal{C}_1^{\Sigma} \models \varphi_0$.*

Proof. Since $\mathcal{C}_1^{\Sigma_1}$ and $\mathcal{A}_1^{\Sigma_1}$ are Σ_1-isomorphic structures (4.11), we know that $\mathcal{C}_1^{\Sigma_1} \models \alpha$. Accordingly, we also have $\mathcal{C}_2^{\Sigma_2} \models \beta$. Moreover, since $\mathcal{C}_1^{\Sigma_2}$ and $\mathcal{C}_2^{\Sigma_2}$ are isomorphic, we know that $\mathcal{C}_1^{\Sigma_2} \models \beta$, i.e., the Σ_2-reduct of the Σ-structure $\mathcal{C}_1^{\Sigma}$ satisfies β. This means

$$\mathcal{C}_1^{\Sigma_1} \models \forall\vec{v}_1\exists\vec{w}_1 \ldots \forall\vec{v}_k\exists\vec{w}_k\exists\vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1}(\vec{v}_1, \vec{w}_1, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_1}), \quad (5.4)$$

$$\mathcal{C}_1^{\Sigma_2} \models \exists\vec{v}_1\forall\vec{w}_1 \ldots \exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2}(\vec{v}_1, \vec{w}_1, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_2}). \quad (5.5)$$

Because of the existential quantification over $\vec{v}_1$ in (5.5), there exist elements $\vec{a}_1 \in \vec{C}_1$ such that

$$\mathcal{C}_1^{\Sigma_2} \models \forall\vec{w}_1 \ldots \exists\vec{v}_k\forall\vec{w}_k\exists\vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2}(\vec{a}_1, \vec{w}_1, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_2}). \quad (5.6)$$

Because of the universal quantification over $\vec{v}_1$ in (5.4) we have

$$\mathcal{C}_1^{\Sigma_1} \models \exists \vec{w}_1 \ldots \forall \vec{v}_k \exists \vec{w}_k \exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1}(\vec{a}_1, \vec{w}_1, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_1}).$$

Because of the existential quantification over $\vec{w}_1$ in this formula there exist elements $\vec{c}_1 \in \vec{C}_1$ such that

$$\mathcal{C}_1^{\Sigma_1} \models \forall \vec{v}_2 \exists \vec{w}_2 \ldots \forall \vec{v}_k \exists \vec{w}_k \exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1}(\vec{a}_1, \vec{c}_1, \vec{v}_2, \vec{w}_2, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_1}).$$

Because of the universal quantification over $\vec{w}_1$ in (5.6) we have

$$\mathcal{C}_1^{\Sigma_2} \models \exists \vec{v}_2 \forall \vec{w}_2 \ldots \exists \vec{v}_k \forall \vec{w}_k \exists \vec{u}_{1,\Sigma_2} \ \gamma_{2,\Sigma_2}(\vec{a}_1, \vec{c}_1, \vec{v}_2, \vec{w}_2, \ldots, \vec{v}_k, \vec{w}_k, \vec{u}_{1,\Sigma_2}).$$

Iterating this argument, we thus obtain

$$\begin{array}{rcl} \mathcal{C}_1^{\Sigma_1} & \models & \exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1}(\vec{a}_1, \vec{c}_1, \ldots, \vec{a}_k, \vec{c}_k, \vec{u}_{1,\Sigma_1}), \\ \mathcal{C}_1^{\Sigma_2} & \models & \exists \vec{u}_{1,\Sigma_2} \ \gamma_{2,\Sigma_2}(\vec{a}_1, \vec{c}_1, \ldots, \vec{a}_k, \vec{c}_k, \vec{u}_{1,\Sigma_2}). \end{array}$$

It follows that

$$\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1}(\vec{a}_1, \vec{c}_1, \ldots, \vec{a}_k, \vec{c}_k, \vec{u}_{1,\Sigma_1}) \wedge \exists \vec{u}_{1,\Sigma_2} \ \gamma_{2,\Sigma_2}(\vec{a}_1, \vec{c}_1, \ldots, \vec{a}_k, \vec{c}_k, \vec{u}_{1,\Sigma_2}).$$

Obviously, this implies that

$$\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_2 \left(\exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1} \wedge \exists \vec{u}_{1,\Sigma_2} \ \gamma_{2,\Sigma_2} \right),$$

i.e., one of the sentences obtained after Step 3 of the algorithm holds in $\mathcal{C}_1^{\Sigma}$. It is easy to see that this implies that $\mathcal{C}_1^{\Sigma} \models \varphi_0$. □

Next, we show completeness of the decomposition algorithm, i.e., if the input sentence was valid then there exists a valid output pair.

Lemma 5.7 *If $\mathcal{C}_1^{\Sigma} \models \varphi_0$ then $\mathcal{A}_1^{\Sigma_1} \models \alpha$ and $\mathcal{A}_2^{\Sigma_2} \models \beta$ for some output pair (α, β).*

Proof. Assume that $\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_0 \gamma_0$. Obviously, this implies that

$$\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_1 \left(\exists \vec{u}_{1,\Sigma_1} \ \gamma_{1,\Sigma_1}(\vec{u}_1, \vec{u}_{1,\Sigma_1}) \wedge \exists \vec{u}_{1,\Sigma_2} \ \gamma_{1,\Sigma_2}(\vec{u}_1, \vec{u}_{1,\Sigma_2}) \right),$$

i.e., $\mathcal{C}_1^{\Sigma}$ satisfies the sentence that is obtained after Step 2 of the decomposition algorithm. Thus there exists an assignment $\nu : V \to C_1$ such that

$$\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_{1,\Sigma_1} \ \gamma_{1,\Sigma_1}(\nu(\vec{u}_1), \vec{u}_{1,\Sigma_1}) \wedge \exists \vec{u}_{1,\Sigma_2} \ \gamma_{1,\Sigma_2}(\nu(\vec{u}_1), \vec{u}_{1,\Sigma_2}).$$

In *Step 3* of the decomposition algorithm, we identify two shared variables u and u' of $\vec{u}_1$ if, and only if, $\nu(u) = \nu(u')$. With this choice,

$$\mathcal{C}_1^{\Sigma} \models \exists \vec{u}_{1,\Sigma_1} \ \gamma_{2,\Sigma_1}(\nu(\vec{u}_2), \vec{u}_{1,\Sigma_1}) \wedge \exists \vec{u}_{1,\Sigma_2} \ \gamma_{2,\Sigma_2}(\nu(\vec{u}_2), \vec{u}_{1,\Sigma_2}). \tag{5.8}$$

Here, as in the algorithm, $\vec{u}_2$ denotes the set of shared variables that are used as representatives after variable identification. Accordingly, if $\mu := \nu \circ h_{1,2} : V \to C_2$ denotes the corresponding assignment of elements of C_2 to variables, then we have

$$\mathcal{C}_2^{\Sigma} \models \exists \vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1}(\mu(\vec{u}_2), \vec{u}_{1,\Sigma_1}) \wedge \exists \vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2}(\mu(\vec{u}_2), \vec{u}_{1,\Sigma_2}), \tag{5.9}$$

since $h_{1,2}$ is an isomorphism. Moreover, because of the variable identification chosen above,

$$\text{all components of } \nu(\vec{u}_2) \text{ and of } \mu(\vec{u}_2) \text{ are distinct.} \tag{5.10}$$

In *Step 4*, a variable u in $\vec{u}_2$ is labeled with Σ_1 if $\nu(u)$ belongs to a 1-fibre, or to a 2-fibre with index 1. Equivalently we could demand that $\nu(u) \in X \cup (C_1 \setminus Z_1)$. Accordingly, a variable u in $\vec{u}_2$ is labeled with Σ_2, if $\nu(u)$ belongs to a 2-fibre with index 2, which means that $\nu(u) \in Z_1 \setminus X$. Note that this implies

$$\nu(u) \in Z_1 \setminus X \text{ for all shared variables } u \text{ with label } \Sigma_2, \tag{5.11}$$

$$\mu(u) \in Z_2 \text{ for all shared variables } u \text{ with label } \Sigma_1. \tag{5.12}$$

Property (5.12) holds by definition of 2-fibres and of the isomorphism $h_{1,2}$.

For two shared variables v and v' we define $v < v'$ iff $\nu(v) <_1 \nu(v')$ where $<_1$ is the ordering on C_1 introduced in Definition 4.17. Note that, by (4.18), $\nu(v) <_1 \nu(v')$ is equivalent to $\mu(v) <_2 \mu(v')$. Now, let

$$\begin{aligned} \alpha &= \forall \vec{v}_1 \exists \vec{w}_1 \ldots \forall \vec{v}_k \exists \vec{w}_k \exists \vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1} \\ \beta &= \exists \vec{v}_1 \forall \vec{w}_1 \ldots \exists \vec{v}_k \forall \vec{w}_k \exists \vec{u}_{1,\Sigma_2}\ \gamma_{2,\Sigma_2} \end{aligned}$$

be the output pair that is obtained by these choices. Thus, in $\vec{v}_1\vec{w}_1 \ldots \vec{v}_k\vec{w}_k$ variables of $\vec{u}_2$ that are smaller with respect to $<$ always precede larger variables, and the tuples $\vec{v}_i$ (resp. $\vec{w}_i$) represent the blocks of variables of type Σ_2 (resp. Σ_1).

Let $\vec{x}_i := \nu(\vec{v}_i)$ and $\vec{e}_i := \nu(\vec{w}_i)$ $(1 \le i \le k)$. Note that the elements in the sequence $\vec{x}_i$ are atoms in Z_1, by (5.11). We claim that the sequence $\vec{x}_1, \vec{e}_1, \ldots, \vec{x}_k, \vec{e}_k$ satisfies Condition 2 of Lemma 3.30 for $\varphi = \exists \vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1}$ and $\mathcal{C}_1^{\Sigma_1}$. Part (a) of this condition is satisfied since (5.8) implies $\mathcal{C}_1^{\Sigma_1} \models \exists \vec{u}_{1,\Sigma_1}\ \gamma_{2,\Sigma_1}(\nu(\vec{u}_2), \vec{u}_{1,\Sigma_1})$. As an immediate consequence of (5.10) we obtain that Part (b) of the condition is satisfied as well. By (4.19) and the choice of $<$, Part (c) of the condition also holds. This shows that we can apply Lemma 3.30, which yields $\mathcal{C}_1^{\Sigma_1} \models \alpha$. Since $\mathcal{A}_1^{\Sigma_1}$ is isomorphic to $\mathcal{C}_1^{\Sigma_1}$, this implies $\mathcal{A}_1^{\Sigma_1} \models \alpha$. Symmetrically it follows that $\mathcal{A}_2^{\Sigma_2} \models \beta$. □

5.2 The general positive case

The goal of this subsection is to show that the decomposition method introduced above can be extended such that it becomes possible to decide validity of *general* positive sentences in the free amalgamated product $\mathcal{C}_1^{\Sigma} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$. The main idea is to transform positive sentences (with arbitrary quantifier prefix) into existential positive sentences by Skolemizing the universally quantified variables. At this step, all universal quantifiers of a given sentence φ in prenex form are removed. For each variable u that is universally quantified in φ, a Skolem term $f_u(\vec{v})$ is introduced, where f_u is a new uninterpreted function symbol and where $\vec{v}$ denotes the sequence of existential variables of φ that precede u in the quantifier prefix of φ. In the matrix of the new formula, all occurrences of u are replaced by $f_u(\vec{v})$. We are Skolemizing *universally* quantified variables since we are interested in validity of the sentence and not in satisfiability.

In principle, the decomposition algorithm for positive sentences is now applied twice to decompose the input sentence into three positive sentences α, β, ρ, whose validity must respectively be decided in $\mathcal{A}_1^{\Sigma_1}$, $\mathcal{A}_2^{\Sigma_2}$, and the absolutely free term algebra over the Skolem functions.

The extended decomposition algorithm

The input is a positive sentence φ_1 in the mixed signature $\Sigma_1 \cup \Sigma_2$. We assume that φ_1 is in prenex normalform, and that the matrix of φ_1 is in disjunctive normalform. The algorithm proceeds in two phases.

Phase 1: Via Skolemization of universally quantified variables, φ_1 is transformed into an existential sentence φ_1' over the signature $\Sigma_1 \cup \Sigma_2 \cup \Gamma_1$. Here Γ_1 is the signature consisting of all the new Skolem function symbols that have been introduced.

Suppose that φ_1' is of the form $\exists \vec{u}_1 (\bigvee \gamma_{1,i})$, where the $\gamma_{1,i}$ are conjunctions of atomic formulae. Obviously, φ_1' is equivalent to $\bigvee(\exists \vec{u}_1 \ \gamma_{1,i})$, and thus it is sufficient to decide validity of the sentences $\exists \vec{u}_1 \ \gamma_{1,i}$. Each of these sentences is used as input for the decomposition algorithm.

The atomic formulae in $\gamma_{1,i}$ may contain symbols from the two (disjoint) signatures Σ_1 and $\Sigma_2 \cup \Gamma_1$. In Phase 1 we treat the sentences $\exists \vec{u}_1 \gamma_{1,i}$ by means of Steps 1–5 of the decomposition algorithm, finally splitting them into positive Σ_1-sentences α and positive $(\Sigma_2 \cup \Gamma_1)$-sentences φ_2. Thus, the output of Phase 1 is a finite set of pairs (α, φ_2).

Phase 2: In the second phase, φ_2 is treated exactly as φ_1 was treated before, applying Skolemization to universally quantified variables and Steps 1–5 of the decomposition algorithm a second time. Now we consider the two (disjoint) signatures Σ_2 and $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_2 contains the Skolem functions that are introduced by the Skolemization step of Phase 2. We obtain output pairs of the form (β, ρ), where β is a positive sentence over the signature Σ_2 and ρ is a positive sentence over the signature Γ. Together with the corresponding sentence α (over the signature Σ_1) we thus obtain triples (α, β, ρ) as output.

For each of these triples, the sentence α is now tested for validity in $\mathcal{A}_1^{\Sigma_1}$, β is tested for validity in $\mathcal{A}_2^{\Sigma_2}$, and ρ is tested for validity in the absolutely free term algebra $\mathcal{T}(\Gamma, X)$ with countably many generators X, i.e., the free algebra over X for the class of all Γ-algebras.[14] We have seen that this structure is a quasi-free structure with atom set X (Examples 3.17 (3)).

Correctness of the extended decomposition algorithm

We must show that the original sentence φ_1 is valid iff for one of the output triples, all three components are valid in the respective structures. The proof depends on the following lemma, which exhibits an interesting connection between Skolemization and free amalgamation with an absolutely free algebra.

Lemma 5.13 *Let $\mathcal{A}_1^{\Sigma}$ be a quasi-free structure with atom set X, and let γ be a positive Σ-sentence. Suppose that the existential positive sentence γ' is obtained from γ via Skolemization of the universally quantified variables in γ, introducing the set of Skolem function symbols Γ. Let $\mathcal{A}_2^{\Gamma} := \mathcal{T}(\Gamma, X)$ be the absolutely free term algebra over Γ with generators X, and let $\mathcal{C}_1^{\Sigma\cup\Gamma}$ be the free amalgamated product of $\mathcal{A}_1^{\Sigma}$ and $\mathcal{A}_2^{\Gamma}$. Then $\mathcal{A}_1^{\Sigma} \models \gamma$ if, and only if, $\mathcal{C}_1^{\Sigma\cup\Gamma} \models \gamma'$.*

Proof. In order to avoid notational overhead, we assume without loss of generality that existential and universal quantifiers alternate in γ,[15] i.e., $\gamma = \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k)$. Skolemization yields the existential formula $\gamma' = \exists v_1 \ldots \exists v_k\ \varphi(f_1, v_1, f_2(v_1), v_2, \ldots, f_k(v_1, \ldots, v_{k-1}), v_k)$. Thus, Γ consists of k distinct new Skolem functions $f_1, f_2, \ldots, f_k$ having the arities $0, 1, \ldots, k-1$, respectively.

First, assume that $\mathcal{A}_1^{\Sigma} \models \gamma$. The structures $\mathcal{A}_1^{\Sigma}$ and $\mathcal{C}_1^{\Sigma}$ are isomorphic by (4.11), and thus

$$\mathcal{C}_1^{\Sigma} \models \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k). \tag{5.14}$$

Suppose that the Skolem symbols $f_1, f_2, \ldots, f_k$ are interpreted by the functions $f_1^{\mathcal{C}_1}, \ldots, f_k^{\mathcal{C}_1}$ on the carrier C_1 of $\mathcal{C}_1^{\Sigma\cup\Gamma}$. Because of (5.14) there exists $a_1 \in C_1$ such that $\mathcal{C}_1^{\Sigma\cup\Gamma} \models \forall u_2 \exists v_2 \ldots \forall u_k \exists v_k\ \varphi(f_1^{\mathcal{C}_1}, a_1, u_2, v_2, \ldots, u_k, v_k)$. Iterating this argument, we obtain $a_1, \ldots, a_k \in C_1$ such that

$$\mathcal{C}_1^{\Sigma\cup\Gamma} \models \varphi(f_1^{\mathcal{C}_1}, a_1, f_2^{\mathcal{C}_1}(a_1), a_2, \ldots, f_k^{\mathcal{C}_1}(a_1, \ldots, a_{k-1}), a_k).$$

[14] Note that Γ contains no predicate symbols.

[15] Obviously one can introduce additional quantifiers over variables not occurring in γ to generate an equivalent formula of this form.

This yields

$$\mathcal{C}_1^{\Sigma\cup\Gamma} \models \exists v_1 \dots \exists v_k \ \varphi(f_1, v_1, f_2(v_1), v_2, \dots, f_k(v_1, \dots, v_{k-1}), v_k),$$

i.e., $\mathcal{C}_1^{\Sigma\cup\Gamma} \models \gamma'$.

For the converse direction, assume that

$$\mathcal{C}_1^{\Sigma\cup\Gamma} \models \exists v_1 \dots \exists v_k \ \varphi(f_1, v_1, f_2(v_1), v_2, \dots, f_k(v_1, \dots, v_{k-1}), v_k).$$

There exist $a_1, \dots, a_k \in C_1$ such that

$$\mathcal{C}_1^{\Sigma\cup\Gamma} \models \varphi(f_1^{\mathcal{C}_1}, a_1, f_2^{\mathcal{C}_1}(a_1), a_2, \dots, f_k^{\mathcal{C}_1}(a_1, \dots, a_{k-1}), a_k), \tag{5.15}$$

where $f_1^{\mathcal{C}_1}, \dots, f_k^{\mathcal{C}_1}$ again denote the functions on C_1 that interpret the symbols $f_1, \dots, f_k$.

Our goal is to apply Lemma 3.30. Property (5.15) shows that the sequence

$$f_1^{\mathcal{C}_1}, a_1, f_2^{\mathcal{C}_1}(a_1), a_2, \dots, f_k^{\mathcal{C}_1}(a_1, \dots, a_{k-1}), a_k$$

satisfies part (a) of Condition 2 of Lemma 3.30. In order to show that part (b) is satisfied as well, we apply the isomorphism $h_{1,2}$ to the elements $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1})$ of the sequence. Since $\mathcal{C}_2^{\Gamma}$ is the absolutely free term algebra, its carrier is the set of Γ-terms over the set (of variables) Z_2, and thus the symbols f_i interpret themselves. Consequently, we have $h_{1,2}(f_1^{\mathcal{C}_1}) = f_1$, $h_{1,2}(f_2^{\mathcal{C}_1}(a_1)) = f_2(h_{1,2}(a_1))$, ..., $h_{1,2}(f_k^{\mathcal{C}_1}(a_1, \dots, a_{k-1})) = f_k(h_{1,2}(a_1), \dots, h_{1,2}(a_{k-1}))$, which implies that these are distinct non-atom elements of $\mathcal{C}_2^{\Gamma}$. By the definition of $h_{1,2}$, they belong to fibres with index 2, which are of the form $\{f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1}), f_i(h_{1,2}(a_1), \dots, h_{1,2}(a_{i-1}))\}$. Since $h_{1,2}$ is a bijection, and by the definition of fibres, we know that the elements $f_1^{\mathcal{C}_1}, f_2^{\mathcal{C}_1}(a_1), \dots, f_k^{\mathcal{C}_1}(a_1, \dots, a_{k-1})$ are distinct atoms of $\mathcal{C}_1$. Thus, part (b) of Condition 2 of Lemma 3.30 holds.

Consider a $\mathcal{C}_1$-atom $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1})$ and an element a_j, where $1 \leq j \leq i-1$. In order to prove the remaining part (c) of Condition 2 of Lemma 3.30, we must show that $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1}) \notin Stab_{\Sigma}^{\mathcal{C}_1}(a_j)$. First, let us consider the situation where $a_j \in Z_1$ is an atom. Obviously $h_{1,2}(a_j)$ and $h_{1,2}(f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1})) = f_i(h_{1,2}(a_1), \dots, h_{1,2}(a_{i-1}))$ are distinct elements of C_2. Hence, by (4.10), a_j and $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1})$ are distinct atoms, which yields $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1}) \notin Stab_{\Sigma}^{\mathcal{C}_1}(a_j) = \{a_j\}$. Second, assume that $a_j \in C_1 \setminus Z_1$ is non-atomic. Then $h_{1,2}(a_j) \in Z_2$ and $h_{1,2}(a_j) \in Stab_{\Gamma}^{C_2}(f_i(h_{1,2}(a_1), \dots, h_{1,2}(a_{i-1})))$. Hence $h_{1,2}(a_j) <_2 f_i(h_{1,2}(a_1), \dots, h_{1,2}(a_{i-1}))$, by (4.19), and $a_j <_1 f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1})$, by (4.18). But then $f_i^{\mathcal{C}_1}(a_1, \dots, a_{i-1}) \notin Stab_{\Sigma}^{\mathcal{C}_1}(a_j)$, by (4.19). This completes the proof that Condition 2 of Lemma 3.30 is satisfied.

Applying the lemma, we obtain

$$\mathcal{C}_1^{\Sigma\cup\Gamma} \models \forall u_1 \exists v_1 \dots \forall u_k \exists v_k \ \varphi(u_1, v_1, \dots, u_k, v_k).$$

Since $\gamma = \forall u_1 \exists v_1 \ldots \forall u_k \exists v_k\ \varphi(u_1, v_1, \ldots, u_k, v_k)$ is a pure Σ-formula, and since $\mathcal{A}_1^{\Sigma}$ and $\mathcal{C}_1^{\Sigma}$ are isomorphic, this shows $\mathcal{A}_1^{\Sigma} \models \gamma$. □

Correctness of the extended decomposition algorithm is an easy consequence of this lemma.

Proposition 5.16 *$\mathcal{C}_1^{\Sigma_1 \cup \Sigma_2} \models \varphi_1$ if, and only if, there exists an output triple (α, β, ρ) such that $\mathcal{A}_1^{\Sigma_1} \models \alpha$, $\mathcal{A}_2^{\Sigma_2} \models \beta$, and $\mathcal{T}(\Gamma, X) \models \rho$, where Γ consists of the Skolem functions introduced in Phase 1 and 2 of the algorithm.*

Proof. As before, let "$\otimes$" denote the free amalgamated product of two quasi-free structures, as constructed in Section 4.3. Assume that $\mathcal{C}_1^{\Sigma_1 \cup \Sigma_2} = \mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2} \models \varphi_1$. By Lemma 5.13 and Theorem 4.34, this implies that $(\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}) \otimes \mathcal{T}(\Gamma_1, X) \simeq \mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{T}(\Gamma_1, X)) \models \varphi_1'$, where φ_1' is the formula obtained from φ_1 by Skolemization. Let $\exists \vec{u}_1 \gamma_1$ be one of the disjuncts in φ_1' satisfied by $\mathcal{A}_1^{\Sigma_1} \otimes (\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{T}(\Gamma_1, X))$. Since the decomposition algorithm is correct, one of the output pairs (α, φ_2) generated by applying the decomposition algorithm to $\exists \vec{u}_1 \gamma_1$ satisfies $\mathcal{A}_1^{\Sigma_1} \models \alpha$ and $\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{T}(\Gamma_1, X) \models \varphi_2$.

Proposition 4.5 (applied for the case of two empty equational theories) implies that $\mathcal{T}(\Gamma_1, X) \otimes \mathcal{T}(\Gamma_2, X) \simeq \mathcal{T}(\Gamma_1 \cup \Gamma_2, X)$. Applying Lemma 5.13 and Theorem 4.34 a second time, we obtain $(\mathcal{A}_2^{\Sigma_2} \otimes \mathcal{T}(\Gamma_1, X)) \otimes \mathcal{T}(\Gamma_2, X) \simeq \mathcal{A}_2^{\Sigma_2} \otimes \mathcal{T}(\Gamma_1 \cup \Gamma_2, X) \models \varphi_2'$, where φ_2' is the positive existential sentence that is obtained from φ_2 via Skolemization. The decomposition algorithm, applied to φ_2', thus yields an output pair (β, ρ) at the end of Phase 2 such that $\mathcal{A}_2^{\Sigma_2} \models \beta$ and $\mathcal{T}(\Gamma_1 \cup \Gamma_2, X) \models \rho$.

It is easy to see that all arguments used during this proof also apply in the other direction. □

The proposition shows that decidability of the positive theory of the free amalgamated product $\mathcal{A}_1^{\Sigma_1} \otimes \mathcal{A}_2^{\Sigma_2}$ can be reduced to decidability of the positive theories of $\mathcal{A}_1^{\Sigma_1}$, $\mathcal{A}_2^{\Sigma_2}$, and of an absolutely free term algebra $\mathcal{T}(\Gamma, X)$. It is well-known that the whole first-order theory of absolutely free term algebras is decidable [28, 27, 16]. Thus, Theorem 5.1 follows immediately. In connection with the Theorems 4.34 and 4.30, the following generalization is obtained.

Theorem 5.17 *If $(\mathcal{A}_1^{\Sigma_1}, X), \ldots, (\mathcal{A}_n^{\Sigma_n}, X)$ are quasi-free structures over disjoint signatures, then the full positive theory of the free simultaneous amalgamated product $\mathcal{A}_1^{\Sigma_1} \otimes \cdots \otimes \mathcal{A}_n^{\Sigma_n}$ is decidable, provided that the positive theories of all structures $\mathcal{A}_i^{\Sigma_i}$ are decidable ($1 \leq i \leq n$).*

6 Applicability of the combination method

In addition to deriving some specific decidability results for combined structures, we will discuss the conditions under which our method is applicable.

Free structures

In Section 3.1, we have seen that a free structure is always free for some variety. In addition, the free structure in countably many generators is canonical for the positive theory of its variety. This yields the following specialization of Theorem 5.1.

Theorem 6.1 *Let $\mathcal{V}(G_1)$ be a Σ_1-variety and $\mathcal{V}(G_2)$ be a Σ_2-variety for disjoint signatures Σ_1 and Σ_2. The positive theory of the $(\Sigma_1 \cup \Sigma_2)$-variety $\mathcal{V}(G_1 \cup G_2)$ is decidable, provided that the positive theories of $\mathcal{V}(G_1)$ and of $\mathcal{V}(G_2)$ are decidable.*

Simple examples of free structures with a non-trivial relational part are (absolutely free) term algebras that are equipped with an ordering that is invariant under substitution, such as the lexicographic path ordering or the subterm ordering. For our combination result to apply, however, the positive theory of these structures must be decidable. For a total lexicographic path ordering, this is not the case. For the subterm ordering, the existential theory is decidable, but the full first-order theory is undecidable [17]. Decidability of the positive theory is still an open problem. For partial lexicographic path orderings, even decidability of the existential theory is unknown. It should be noted, however, that these decidability and undecidability results refer to ground term algebras (i.e., absolutely free algebras with an empty set of generators). Since we are interested in free structures in countably many generators (see the discussion below), it is not quite clear how relevant the cited results are in our context.

Deciding the full positive theory of quasi-free structures

The prerequisite for combining constraint solvers with the help of our decomposition algorithms is that validity of arbitrary positive sentences is decidable in both components (see Theorem 5.1 and Theorem 5.2). If we leave the realm of free structures, not many results are known that show that the positive theory of a particular quasi-free structure is decidable. For two of the quasi-free structures introduced in Examples 3.17, however, even the full first-order theory is known to be decidable:

- The first-order theory of the algebra of rational trees—like the theory of the algebra of finite trees—is decidable [27]. Maher considers ground tree algebras, but over possibly infinite signatures, which shows that his result can be lifted to the non-ground case by treating variables as constants.

- The first-order theory of the structure of rational feature trees with arity (as introduced in Examples 3.17) is decidable. The decidability result has been obtained for the ground structure by giving a complete axiomatization [8]. It is, however, easy to see that all axioms hold in the non-ground structure as well. Thus, the ground and the non-ground variant are elementary equivalent, which implies that the first-order theory of the non-ground structure is also decidable.

In general, the problem of deciding validity of existential positive sentences and the problem of deciding validity of arbitrary positive sentences in a given structure can be quite different. For the case of quasi-free structures, however, the following variant of Lemma 3.30 shows that the difference is not drastic. The proof can be found in [7].

Lemma 6.2 *Let $(\mathcal{A}^\Sigma, X)$ be a quasi-free structure with non-empty ground substructure $\mathcal{A}^\Sigma_G$ (cf. Def. 3.18), let*

$$\forall\vec{u}_1\exists\vec{v}_1\ldots\forall\vec{u}_k\exists\vec{v}_k\ \varphi(\vec{u}_1,\vec{v}_1,\ldots,\vec{u}_k,\vec{v}_k)$$

be a positive Σ-sentence, and let, for all $i, 1\le i\le k$, $\vec{x}_i$ be an arbitrary (but fixed) sequence of length $|\vec{u}_i|$ of distinct atoms such that distinct sequences $\vec{x}_i$ and $\vec{x}_j$ do not have common elements. Let $X_{1,i}$ denote the set of all atoms occurring in the sequences $\vec{x}_1,\ldots,\vec{x}_i$ $(i=1,\ldots,k)$. Then the following conditions are equivalent:

1. $\mathcal{A}^\Sigma \models \forall\vec{u}_1\exists\vec{v}_1\ldots\forall\vec{u}_k\exists\vec{v}_k\ \varphi(\vec{u}_1,\vec{v}_1,\ldots,\vec{u}_k,\vec{v}_k)$,
2. *there exist $\vec{e}_1\in SH^{\mathcal{A}}_\Sigma(X_{1,1}),\ldots,\vec{e}_k\in SH^{\mathcal{A}}_\Sigma(X_{1,k})$ such that* $\mathcal{A}^\Sigma\models\varphi(\vec{x}_1,\vec{e}_1,\ldots,\vec{x}_k,\vec{e}_k)$.

Looking at the second condition of the lemma, one sees that here the positive sentence of the first condition is replaced by an *existential* positive sentence where the universally quantified variables are substituted by atoms, and additional restrictions are imposed on the values of the existentially quantified variables. For this reason, it is often not hard to extend decision procedures for the existential positive theory of a quasi-free structure to a decision procedure for the full positive theory. For example, this way of proceeding can be used to prove that the positive theories of the four domains of nested, hereditarily finite wellfounded or non-wellfounded lists or sets (as introduced in Examples 3.17) are decidable (for the case of lists, see [7] for proofs). This implies the following decidability result for constraint solving in a combination of such domains.

Corollary 6.3 *Simultaneous free amalgamated products have a decidable positive theory, if the components are non-ground rational feature structures with arity, finite or rational tree algebras, or nested, hereditarily finite wellfounded or non-wellfounded sets, or nested, hereditarily finite wellfounded or non-wellfounded lists, and if the signatures of the components are disjoint.*

Ground versus non-ground structures

In the definition of quasi-free structures, a countably infinite set of atoms was required. It would, of course, be possible to generalize this definition to atom sets of arbitrary (finite or infinite) cardinality. For most of the combination results presented in this paper, however, the presence of a countably infinite number of atoms ("variables") in the structures to be combined is an essential precondition. On the other hand, many constraint-based approaches consider ground structures as solution domains. In most cases, however, a corresponding non-ground structure containing the necessary atoms exists. Thus, our combination method can be applied to these non-ground variants. Of course, the combined structure obtained in this way is again non-ground. For *existential* positive formulae, however, it follows from Lemma 2.2 that validity in the non-ground combined structure is equivalent to validity in the ground substructure of the combined structure (cf. Definition 3.18).[16] This observation has the following interesting consequence. Even in cases where the (full) positive theory of a ground component structure is undecidable, our combination methods can be applied to show decidability of the existential positive theory even for the *ground* combined structure, provided that the (full) positive theories of the non-ground component structures are decidable. Our remark following Lemma 6.2 shows that decidability of the full positive theory of such a non-ground structure can sometimes be obtained by an easy modification of the decision method for the existential positive case. Free semigroups are an example for this situation: the positive theory of a free semigroup with a finite number $n \geq 2$ of generators is undecidable, whereas the positive theory of the countably generated free semigroup (which corresponds to our non-ground case) is decidable [41].

7 Conclusion

This paper provides an abstract framework for the combination of constraint languages and constraint solvers. It combines and simplifies the results of [5] and [6], emphasizing the rôle of universal algebra. The main questions that have been addressed are:

[16]It is trivial to see that there are homomorphisms between these two structures in both directions.

1. How can we capture—in an abstract algebraic setting—our intuition of what a reasonable combined solution structure should satisfy?

2. What are the essential algebraic and logical properties of free structures that

   - allow for an explicit construction of the combined solution structure, and
   - guarantee that the combination techniques for constraint solvers developed in unification theory can be applied?

3. Based on these insights, how can we define a more general class of interesting solution structures that behave like free structures with respect to combination?

As a possible answer to the first question, we have introduced the notion of a "free amalgamated product," which formalizes the intuitive idea of a most general combination of two given structures. For the case of free structures, the result of this algebraic construction coincides with what is obtained through the logical point of view: given two free structures defined by atomic theories G_1 and G_2, the free amalgamated product yields the free structure defined by $G_1 \cup G_2$.

As a result of analyzing the algebraic properties of free structures that are relevant in the combination context, we have introduced a more general class of structures—called quasi-free structures—that are equipped with structural properties that guarantee

1. that the free amalgamated product of quasi-free structures over disjoint signatures always exists, and can be obtained by an explicit amalgamation construction,

2. that validity of positive formulae in the free amalgamated product of quasi-free structures over disjoint signatures can be reduced to validity of positive formulae in the component structures with the help of the combination techniques developed in unification theory.

This class seems to be a very natural extension of the class of free structures. As we have seen, quasi-free structures have nice algebraic properties. For example, they allow for strong and very useful concepts like stable hulls and stabilizers, which generalize the notions of generated subalgebras and sets of variables occurring in a term. Moreover, the class of quasi-free structures contains many non-free structures that are used as solution domains in constraint solving. Hence, this class is an interesting object for further theoretical and practical studies. This claim is corroborated by the fact that a very similar class of structures has independently been introduced in [36, 43] in order to characterize a maximal class of algebras where equation (and constraint) solving essentially behaves like unification. The notion of a quasi-free structure can be considered as a sort-free version of the concepts that have been discussed in [36, 43].

For the case of general quasi-free structures, it is interesting to compare the concrete combined solution domains that can be found in the literature with the combined domains obtained by our amalgamation construction. It turns out that there can be differences, if the elements of the components have a tree-like structure that allows for infinite paths (as in the examples of non-wellfounded sets and rational trees). In these cases, frequently (see, e.g., [35, 15]) a combined solution structure is chosen where an infinite number of "signature changes" may occur when following an infinite path in an element of the combined domain. To be more precise, in the framework of the explicit amalgamation construction this would mean that one may obtain an infinite chain when starting with an element a of one component, then taking an atom x of its stabilizer, in turn taking the element b of the other component that is fibred with x, taking an element y of its stabilizer, etc. In contrast, our amalgamation construction yields a combined structure where elements allow for a finite number of signature changes only, i.e., the process described above always terminates. This indicates that the free amalgamated product, even if it exists, is not necessarily the only interesting combined domain. In [24] several interesting amalgamation constructions are investigated in more detail. In particular, an alternative combination called "rational amalgamation" has been introduced, and a new combination algorithm adapted to rational amalgamation has been given.

In the free case, our results extend the combination results for unification algorithms in that we allow for predicate symbols other than equality. Combination of constraint solving techniques in the presence of such additional predicate symbols has independently been considered by H. Kirchner and Ch. Ringeissen [26]. Their approach is based on the more syntactic rewriting and abstraction techniques that have already been employed in the context of combining unification algorithms (see, e.g., [3, 10]). In particular, the interpretation of the predicate symbols in the combined structure is also defined with the help of these syntactic techniques. An advantage of this approach is that it is relatively easy to show that validity of atomic formulae (i.e., equations $s = t$ or relational formulae $p[s_1, \ldots, s_m]$) in the combined structure is decidable, provided that this problem is decidable for the single structures [26]. A disadvantage is that the combined structure is defined in a rather technical way, which means that it is not a priori clear what this definition means in an intuitive algebraic sense. Fortunately, it can be shown [9] that, for free structures, the combined structure defined in [26] coincides with our free amalgamated product, which provides this combined structure with an algebraic justification.

References

[1] P. Aczel, "Non-well-founded Sets," *CSLI Lecture Notes* **14**, Stanford University, 1988.

[2] H. Ait-Kaci, A. Podelski, and G. Smolka, "A feature-based constraint system for logic programming with entailment," *Theoretical Comp. Science* **122**, 1994, pp. 263–283.

[3] F. Baader, K.U. Schulz, "Unification in the union of disjoint equational theories: Combining decision procedures," *Journal of Symbolic Computation*, **21** (1996), pp. 211-243. A short version of this articel has appeared in Proc. CADE'92, Springer LNCS 607.

[4] F. Baader and K.U. Schulz, "Combination techniques and decision problems for disunification," in *Proc. RTA-93*, Springer LNCS 690, 1993, pp. 301–315.

[5] F. Baader and K.U. Schulz, "Combination of Constraint Solving Techniques: An Algebraic Point of View," Research Report CIS-Rep-94-75, University Munich, 1994; a short version has appeared in *Proc. RTA-95*, Springer LNCS 914, 1995, pp. 352–366.

[6] F. Baader and K.U. Schulz, "On the Combination of Symbolic Constraints, Solution Domains, and Constraint Solvers," in: Proceedings CP'95, U.Montanari, F.Rossi (Eds.), Springer LNCS 976, pp. 380-397.

[7] F. Baader and K.U. Schulz, "On the Combination of Symbolic Constraints, Solution Domains, and Constraint Solvers," extended version of [6], Research Report CIS-Rep-95-120, University Munich, 1995.

[8] R. Backofen and R. Treinen, "How to win a game with features," in *Constraints in Computational Logics, Proc. CCL-94*, Springer LNCS 845, 1994, pp. 320–335.

[9] K. Ben Khalifa, "Äquivalenzbeweis zweier Definitionen zur Kombination freier Strukturen," Diplomarbeit (Master Thesis), Dept. of Computer Science, RWTH Aachen, Germany, 1996.

[10] A. Boudet, "Combining unification algorithms," *J. Symbolic Computation* **16**, 1993, pp. 597–626.

[11] H.-J. Bürckert, "A Resolution Principle for a Logic with Restricted Quantifiers," Springer LNCS 568, 1991.

[12] G. Cherlin, "Model Theoretic Algebra: Selected Topics," Springer Lecture Notes in Mathematics 521, 1976.

[13] P.M. Cohn, "Universal Algebra," Harper & Row, New York, 1965.

[14] A. Colmerauer, "Equations and inequations on finite and infinite trees," in *Proc. 2nd Int. Conf. on Fifth Generation Computer Systems*, 1984, pp. 85–99.

[15] A. Colmerauer, "An introduction to PROLOG III," *C. ACM* **33**, 1990, pp.69–90.

[16] H. Comon and P. Lescanne. "Equational problems and disunification," *J. Symbolic Computation* **7**, 1989, pp.371–425.

[17] H. Comon and R. Treinen, "Ordering constraints on trees," in *Colloquium on Trees in Algebra and Programming (CAAP)*, Springer LNCS 787, 1994, pp. 1-14.

[18] N. Dershowitz, "Termination of rewriting," *J. Symbolic Computation* **3**, 1987, pp. 69–116.

[19] A. Dovier, E.G. Omodeo, E. Pontellio, G.F. Rossi, "{log}: A Logic Programming language with finite sets," in *Logic Programming: Proc. 8th International Conf.*, The MIT Press, 1991.

[20] A. Dovier, G. Rossi, "Embedding extensional finite sets in CLP," in *Proc. International Logic Programming Symposium*, 1993, pp. 540–556.

[21] M. Droste, R. Göbel, "Universal domains and the amalgamation property," *Math. Struct. in Comp. Science* **3**, 1993, pp. 137–159.

[22] F. Fages. "Associative-commutative unification," in *Proceedings of the 7th International Conference on Automated Deduction*, Springer LNCS 170, 1984, pp. 194–208.

[23] J. Jaffar and J.-L. Lassez, " Constraint logic programming," in *Proceedings of the 14th ACM Symposium on Principles of Programming Languages*, Munich, Germany, 1987, pp. 111–119.

[24] S. Kepser and K.U. Schulz, "Combination of Constraint Solvers II: Rational Amalgamation," in *Proceedings CP-96*, Eugene C. Freuder (Ed.), Springer LNCS 1118, pp. 282–296.

[25] C. Kirchner and H. Kirchner, "Constrained equational reasoning," in *Proceedings of SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation*, ACM Press, 1989.

[26] H. Kirchner and Ch. Ringeissen, "Combining symbolic constraint solvers on algebraic domains," *J. Symbolic Computation*, **18**(2), 1994, pp. 113–155.

[27] M.J. Maher, "Complete axiomatizations of the algebras of finite, rational and infinite trees," in *Proceedings of Third Annual Symposium on Logic in Computer Science, LICS'88*, Edinburgh, Scotland, IEEE Computer Society, 1988, pp. 348–357.

[28] A.I. Mal'cev, "The Metamathematics of Algebraic Systems," volume 66 of *Studies in Logic and the Foundation of Mathematics*, North Holland, Amsterdam, London, 1971.

[29] A.I. Mal'cev, "Algebraic Systems," Volume 192 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, Springer–Verlag, Berlin, 1973.

[30] K. Meinke and J.V. Tucker, "Universal algebra," in *Handbook of Logic in Computer Science, Volume 1*, Oxford University Press, Oxford, UK, 1992, pp. 189–411.

[31] K. Mukai, "Constraint Logic Programming and the Unification of Information," Doctoral Thesis, Dept. of Comp. Science, Faculty of Engineering, Tokyo Institute of Technology, 1991.

[32] G. Nelson and D.C. Oppen, "Simplification by cooperating decision procedures," *ACM TOPLAS* **1**(2), 1979, pp. 245–257.

[33] D.C. Oppen, "Complexity, convexity and combination of theories," *Theoretical Computer Science* **12**, 1980, pp. 291–302.

[34] Ch. Ringeissen, "Unification in a combination of equational theories with shared constants and its application to primal algebras," in *Proc. LPAR'92*, Springer LNCS 624, 1992.

[35] W.C. Rounds, "Set Values for Unification Based Grammar Formalisms and Logic Programming," Research Report CSLI-88-129, Stanford, 1988.

[36] M. Schmidt-Schauß, "Unification Algebras: An Axiomatic Approach to Unification, Equation Solving and Constraint Solving," SEKI-Report SR-88-23, University of Kaiserslautern, 1988.

[37] M. Schmidt-Schauß, "Unification in a combination of arbitrary disjoint equational theories," *J. Symbolic Computation* **8**, 1989, pp. 51–99.

[38] G. Smolka, R. Treinen, "Records for Logic Programming," *J. of Logic Programming* **18**(3), 1994, pp. 229–258.

[39] R.E. Shostak, "A practical decision procedure for arithmetic with function symbols," *J. ACM* **26**(2), 1979, pp. 351–360.

[40] M.E. Stickel. "A unification algorithm for associative commutative functions," *J. ACM* **28**(3), 1981, pp. 423–434.

[41] Y.M. Vazhenin and B.V. Rozenblat, "Decidability of the positive theory of a free countably generated semigroup," *Math. USSR Sbornik* **44**, 1983, pp. 109–116.

[42] N. Weaver, "Generalized varieties," *Algebra Universalis* **30**, 1993, pp. 27–52.

[43] J.G. Williams, "Instantiation Theory: On the Foundation of Automated Deduction," Springer LNCS 518, 1991.

Appendix

Here we give the proofs for Lemma 3.28, Theorem 3.29, and Theorem 4.4. For convenience, we repeat the statements.

Lemma 3.28 *Let $(\mathcal{B}^\Sigma, Y)$ be a quasi-free structure. Let Z be an infinite subset of Y, and let $\mathcal{C}^\Sigma := SH^{\mathcal{B}}_\Sigma(Z)$. Then the following holds:*

1. *$(\mathcal{C}^\Sigma, Z)$ is quasi-free, and $(\mathcal{B}^\Sigma, Y)$ and $(\mathcal{C}^\Sigma, Z)$ are qf-isomorphic.*
2. *For each $c \in C$, we have $Stab^{\mathcal{B}}_\Sigma(c) = Stab^{\mathcal{C}}_\Sigma(c)$.*
3. *For each $U \subseteq Z$, $SH^{\mathcal{B}}_\Sigma(U) = SH^{\mathcal{C}}_\Sigma(U)$.*

Proof. By Lemma 3.19, $\mathcal{B}^\Sigma = SH^{\mathcal{B}}_\Sigma(Y)$. Let $h_0 : Y \to Z$ be a bijection between the two atom sets Y and Z. By Lemma 3.13, h_0 can be extended to an isomorphism h_{B-C} between $\mathcal{B}^\Sigma$ and $\mathcal{C}^\Sigma$. In order to prove the first part of the lemma it remains to show that $(\mathcal{C}^\Sigma, Z)$ is quasi-free. Let $h_{C-B} := h^{-1}_{B-C}$. We proceed in 4 steps.

(1.1) In this first step we introduce a useful isomorphism between $End^\Sigma_{\mathcal{B}}$ and $End^\Sigma_{\mathcal{C}}$. For $m \in End^\Sigma_{\mathcal{B}}$, let $m_\downarrow := h_{C-B} \circ m \circ h_{B-C}$. Obviously $m_\downarrow \in End^\Sigma_{\mathcal{C}}$. We consider the mapping

$$H_\downarrow : End^\Sigma_{\mathcal{B}} \to End^\Sigma_{\mathcal{B}} : m \mapsto m_\downarrow.$$

Since

$$\begin{aligned} m_\downarrow \circ m'_\downarrow &= h_{C-B} \circ m \circ h_{B-C} \circ h_{C-B} \circ m' \circ h_{B-C} \\ &= h_{C-B} \circ m \circ m' \circ h_{B-C} \\ &= (m \circ m')_\downarrow, \end{aligned}$$

$H_\downarrow$ is a homomorphism between the monoids $End^\Sigma_{\mathcal{B}}$ and $End^\Sigma_{\mathcal{C}}$. There exists a dual homomorphism $H_\uparrow : End^\Sigma_{\mathcal{C}} \to End^\Sigma_{\mathcal{B}}$ and it is easy to see that $H_\downarrow \circ H_\uparrow$ is the identity on $End^\Sigma_{\mathcal{B}}$, and $H_\uparrow \circ H_\downarrow$ is the identity on $End^\Sigma_{\mathcal{C}}$. Thus, both are isomorphisms that are inverse to each other.

(1.2) In the second step we show that Z is an atom set of $\mathcal{C}^\Sigma$. Let $g_{Z-C} : Z \to C$ be a mapping. There is a corresponding mapping

$$g_{Y-B} : Y \to B : y \mapsto h_{C-B}(g_{Z-C}(h_{B-C}(y))).^{17}$$

Since $(\mathcal{B}^\Sigma, Y)$ is quasi-free, there exists an extension g_{B-B} of g_{Y-B} to an endomorphism of $\mathcal{B}^\Sigma$. Its image $(g_{B-B})_\downarrow$ is an endomorphism of $\mathcal{C}^\Sigma$, and it is easy to see that this endomorphism extends g_{Z-C}.

(1.3) Third, we show that every element c of C is stabilized by the set $h_{B-C}(Stab_\Sigma^\mathcal{B}(h_{C-B}(c)))$. Let $m_\downarrow$ and $m'_\downarrow$ be two endomorphisms of $\mathcal{C}^\Sigma$ that coincide on $h_{B-C}(Stab_\Sigma^\mathcal{B}(h_{C-B}(c)))$. For $y \in Stab_\Sigma^\mathcal{B}(h_{C-B}(c))$ we have

$$\begin{aligned} m(y) &= h_{C-B}(m_\downarrow(h_{B-C}(y))) \\ &= h_{C-B}(m'_\downarrow(h_{B-C}(y))) = m'(y), \end{aligned}$$

which shows that m and m' coincide on $Stab_\Sigma^\mathcal{B}(h_{C-B}(c))$. Thus m and m' coincide on $h_{C-B}(c)$. We obtain

$$\begin{aligned} m_\downarrow(c) &= h_{B-C}(m(h_{C-B}(c))) \\ &= h_{B-C}(m'(h_{C-B}(c))) \\ &= m'_\downarrow(c). \end{aligned}$$

(1.4) Since $Stab_\Sigma^\mathcal{B}(h_{C-B}(c))$ is a finite subset of Y, we know that the set $h_{B-C}(Stab_\Sigma^\mathcal{B}(h_{C-B}(c)))$ is a finite subset of Z. Thus, every element of C is stabilized by a finite subset of Z, which shows that $(\mathcal{C}^\Sigma, Z)$ is quasi-free and completes the proof of the first part of the lemma.

In order to prove the second part we show in (2.1) that $Stab_\Sigma^\mathcal{B}(c) \subseteq Stab_\Sigma^\mathcal{C}(c)$, for all $c \in C$. In (2.2) we show that $Stab_\Sigma^\mathcal{C}(c) \subseteq Stab_\Sigma^\mathcal{B}(c)$, for all $c \in C$.

(2.1) Let m and m' be two endomorphisms of $\mathcal{B}^\Sigma$ that coincide on $Stab_\Sigma^\mathcal{C}(c)$. Thus $m_\downarrow = h_{C-B} \circ m \circ h_{B-C}$ and $m'_\downarrow = h_{C-B} \circ m' \circ h_{B-C}$ coincide on the set $h_{B-C}(Stab_\Sigma^\mathcal{C}(c))$. Clearly $Stab_\Sigma^\mathcal{C}(c) \subseteq C$. The restriction $\breve{h}_{B-C}$ of h_{B-C} to $\mathcal{C}^\Sigma$ is an endomorphism of $\mathcal{C}^\Sigma$, and $m_\downarrow$ and $m'_\downarrow$ coincide on $\breve{h}_{B-C}(Stab_\Sigma^\mathcal{C}(c))$. Hence, by Lemma 3.23, $m_\downarrow$ and $m'_\downarrow$ coincide on $\breve{h}_{B-C}(c) = h_{B-C}(c)$. But then

$$\begin{aligned} m(c) &= h_{C-B}(m_\downarrow(h_{B-C}(c))) \\ &= h_{C-B}(m'_\downarrow(h_{B-C}(c))) \\ &= m'(c). \end{aligned}$$

It follows that $Stab_\Sigma^\mathcal{C}(c)$ stabilizes c in $\mathcal{B}^\Sigma$, which shows that $Stab_\Sigma^\mathcal{B}(c) \subseteq Stab_\Sigma^\mathcal{C}(c)$.

(2.2) Let $m_\downarrow$ and $m'_\downarrow$ be two endomorphisms of $\mathcal{C}^\Sigma$ that coincide on $Stab_\Sigma^\mathcal{B}(c)$ (recall here that $Stab_\Sigma^\mathcal{B}(c) \subseteq Z$, by Lemma 3.22). Then $m = h_{B-C} \circ m_\downarrow \circ h_{C-B}$

[17] Recall that h_{B-C} maps Y to Z.

and $m' = h_{B-C} \circ m'_\downarrow \circ h_{C-B}$ coincide on $h_{C-B}(Stab^{\mathcal{B}}_\Sigma(c))$. As an intermediate step we show

(2.2.1) The mapping h_{C-B} can be extended to an endomorphism $\widehat{h}_{C-B}$ of $\mathcal{B}^\Sigma$. In fact, let $h_1 : Y \to B$ be any mapping that coincides with h_{C-B} on Z, and let $\widehat{h}_{C-B}$ be the endomorphism of $\mathcal{B}^\Sigma$ that extends h_1. Now h_{C-B} and the restriction of $\widehat{h}_{C-B}$ to $\mathcal{C}^\Sigma$ are two homomorphisms from $\mathcal{C}^\Sigma$ to $\mathcal{B}^\Sigma$ that coincide on Z. By Lemma 3.27 and part 1, $(\mathcal{C}^\Sigma, Z)$ is quasi-free for $\mathcal{B}^\Sigma$. Now Lemma 3.25 shows that h_{C-B} and the restriction of $\widehat{h}_{C-B}$ to $\mathcal{C}^\Sigma$ are identical mappings. Hence $\widehat{h}_{C-B}$ extends h_{C-B}.

We have seen that m and m' coincide on $\widehat{h}_{C-B}(Stab^{\mathcal{B}}_\Sigma(c))$. By Lemma 3.23, m and m' coincide on $\widehat{h}_{C-B}(c) = h_{C-B}(c)$. But then

$$\begin{aligned} m_\downarrow(c) &= h_{B-C}(m(h_{C-B}(c))) \\ &= h_{B-C}(m'(h_{C-B}(c))) \\ &= m'_\downarrow(c). \end{aligned}$$

It follows that $Stab^{\mathcal{B}}_\Sigma(c)$ stabilizes c in $\mathcal{C}^\Sigma$, which shows that $Stab^{\mathcal{C}}_\Sigma(c) \subseteq Stab^{\mathcal{B}}_\Sigma(c)$. This concludes the proof of the second part of the lemma.

To see that the third part holds as well, note that for $U \subseteq Z$ and $b \in B$ we know by Lemma 3.22 that $Stab^{\mathcal{B}}_\Sigma(b) \subseteq U$ implies $b \in C$. By Lemma 3.22 and part 2,

$$\begin{aligned} SH^{\mathcal{B}}_\Sigma(U) &= \{b \in B \mid Stab^{\mathcal{B}}_\Sigma(b) \subseteq U\} \\ &= \{c \in C \mid Stab^{\mathcal{B}}_\Sigma(c) \subseteq U\} \\ &= \{c \in C \mid Stab^{\mathcal{C}}_\Sigma(c) \subseteq U\} \\ &= SH^{\mathcal{C}}_\Sigma(U). \end{aligned}$$

This completes the proof of Lemma 3.28. □

Theorem 3.29 *Let $(\mathcal{A}^\Sigma, X)$ be a quasi-free structure. Then there exists a quasi-free superstructure $(\mathcal{B}^\Sigma, Y)$ with the following properties:*

1. *$Y \setminus X$ is infinite.*
2. *$X \subseteq Y$, and $\mathcal{A}^\Sigma = SH^{\mathcal{B}}_\Sigma(X)$.*
3. *$(\mathcal{A}^\Sigma, X)$ and $(\mathcal{B}^\Sigma, Y)$ are qf-isomorphic.*
4. *If $X \subseteq Z \subseteq Y$, and if $\mathcal{C}^\Sigma = SH^{\mathcal{B}}_\Sigma(Z)$, then $\mathcal{A}^\Sigma = SH^{\mathcal{C}}_\Sigma(X)$, and $(\mathcal{A}^\Sigma, X)$ and $(\mathcal{C}^\Sigma, Z)$ are qf-isomorphic.*

Proof. (1) In the first part of the proof, we define the structure $\mathcal{B}^\Sigma$. Let X_0 be an infinite subset of X such that $X \setminus X_0$ is infinite, and let $(\mathcal{A}^\Sigma_0, X_0) = SH^{\mathcal{A}}_\Sigma(X_0)$

be the quasi-free substructure satisfying the properties stated in Lemma 3.28. Let $h_{A_0-A} : \mathcal{A}_0^\Sigma \to \mathcal{A}^\Sigma$ be an isomorphism that extends a bijection between the atom sets X_0 and X.

As carrier of the superstructure to be constructed, we take an arbitrary countably infinite superset B of A such that $B \setminus A$ is infinite. Let Y be a subset of B such that

1. $X \subseteq Y$ and $Y \setminus X$ is infinite,
2. $Y \cap A = X$,
3. the sets $A \setminus (A_0 \cup X)$ and $B \setminus (A \cup Y)$ have the same cardinality.

We extend h_{A_0-A} to a bijection $h_{A-B} : A \to B$ such that $h_{A-B}(X) = Y$. This is possible because of our choice of h_{A_0-A} and of Y. In fact, by Lemma 3.19, $A = A_0 \uplus (X \setminus X_0) \uplus (A \setminus (A_0 \cup X))$ is a partitioning of A, and our assumptions ensure that $B = A \uplus (Y \setminus X) \uplus (B \setminus (A \cup Y))$ is a partitioning of B. In addition, both $X \setminus X_0$ and $Y \setminus X$ are countably infinite, and $A \setminus (A_0 \cup X)$ and $B \setminus (A \cup Y)$ have the same cardinality by assumption.

The bijection h_{A-B} and its inverse $h_{B-A} := h_{A-B}^{-1}$ can be used to define a Σ-structure $\mathcal{B}^\Sigma$ on the carrier B as follows: Let $f \in \Sigma$ be an n-ary function symbol, and $a_1, \ldots, a_n \in B$. We define the interpretation of f in $\mathcal{B}^\Sigma$ by

$$f_\mathcal{B}(a_1, \ldots, a_n) := h_{A-B}(f_\mathcal{A}(h_{B-A}(a_1), \ldots, h_{B-A}(a_n))).$$

Let $p \in \Sigma$ be an m-ary predicate symbol, and $a_1, \ldots, a_m \in B$. We define the interpretation of p in $\mathcal{B}^\Sigma$ by

$$p_\mathcal{B}[a_1, \ldots, a_n] :\iff p_\mathcal{A}[h_{B-A}(a_1), \ldots, h_{B-A}(a_n)].$$

Note that this definition is compatible with the given Σ-structure on $A \subset B$ since h_{A_0-A}, i.e., the restriction of h_{A-B} to A_0, is a Σ-isomorphism. With this definition, the mapping h_{A-B} becomes an isomorphism between the Σ-structures $\mathcal{B}^\Sigma$ and $\mathcal{A}^\Sigma$, and h_{B-A} is the inverse isomorphism.

For $m \in End_\mathcal{A}^\Sigma$, let $m_\infty := h_{B-A} \circ m \circ h_{A-B}$. As in the proof of Lemma 3.28 we can show that the mapping $m \mapsto m_\infty$ is an isomorphism between the monoids $End_\mathcal{A}^\Sigma$ and $End_\mathcal{B}^\Sigma$.

(2) In the second part of the proof we show that $(\mathcal{B}^\Sigma, Y)$ is quasi-free.

To this purpose, we show that Y is an atom set of $\mathcal{B}^\Sigma$. Let $g_{Y-B} : Y \to B$ be a mapping. There is a corresponding mapping

$$g_{X-A} : X \to A : x \mapsto h_{B-A}(g_{Y-B}(h_{A-B}(x))).$$

Since $(\mathcal{A}^\Sigma, X)$ is quasi-free, there exists an extension g_{A-A} of g_{X-A} to an endomorphism of $\mathcal{A}^\Sigma$. Its image $(g_{A-A})_\infty$ is an endomorphisms of $\mathcal{B}^\Sigma$, and it is easy to see that this endomorphism extends g_{Y-B}. Thus, Y is in fact an atom set of $\mathcal{B}^\Sigma$. For a given $b \in B$ is also straightforward to verify that the finite set $h_{A-B}(Stab_\Sigma^{\mathcal{A}}(h_{B-A}(b)) \subseteq Y$ stabilizes b. Thus we have shown that $(\mathcal{B}^\Sigma, Y)$ is quasi-free.

It remains to show that $(\mathcal{B}^\Sigma, Y)$ has the properties stated in the theorem. We have seen that, by construction, $Y \setminus X$ is infinite and $X \subseteq Y$. Hence the first property is satisfied.

(3) In order to prove the second property, it remains to be shown that $\mathcal{A}^\Sigma = SH_\Sigma^{\mathcal{B}}(X)$. We know that $\mathcal{A}_0^\Sigma = SH_\Sigma^{\mathcal{A}}(X_0)$.

First, assume that $a \in A$. Since h_{A-B} maps A_0 bijectively onto A, there exists $a_0 \in A_0$ such that $a = h_{A-B}(a_0)$. Now assume that the endomorphisms m_∞ and m'_∞ of $\mathcal{B}^\Sigma$ coincide on X. It follows that m, m' coincide on X_0. In fact, let $x_0 \in X_0$. Then $h_{A-B}(x_0) \in X$, and thus

$$\begin{aligned} m(x_0) &= h_{B-A}(m_\infty(h_{A-B}(x_0))) \\ &= h_{B-A}(m'_\infty(h_{A-B}(x_0))) \\ &= m'(x_0). \end{aligned}$$

Thus, we know that m, m' coincide on $\mathcal{A}_0^\Sigma = SH_\Sigma^{\mathcal{A}}(X_0)$. It follows that

$$\begin{aligned} m_\infty(a) &= h_{A-B}(m(h_{B-A}(a))) \\ &= h_{A-B}(m(a_0)) \\ &= h_{A-B}(m'(a_0)) \\ &= h_{A-B}(m'(h_{B-A}(a))) \\ &= m'_\infty(a), \end{aligned}$$

and thus we have proved $a \in SH_\Sigma^{\mathcal{B}}(X)$.

Second, assume that $a \in SH_\Sigma^{\mathcal{B}}(X)$. We show that this implies that its image $h_{B-A}(a) \in SH_\Sigma^{\mathcal{A}}(X_0) = \mathcal{A}_0^\Sigma$. Since the restriction of h_{A-B} to A_0 maps A_0 onto A, it follows that $a = h_{A-B}(h_{B-A}(a)) \in A$. Thus, assume that the endomorphisms m, m' of $\mathcal{A}^\Sigma$ coincide on X_0. It is easy to see that this implies that m_∞, m'_∞ coincide on X, and thus they coincide on $a \in SH_\Sigma^{\mathcal{B}}(X)$. It follows that

$$\begin{aligned} m(h_{B-A}(a)) &= h_{B-A}(m_\infty(a)) \\ &= h_{B-A}(m'_\infty(a)) \\ &= m'(h_{B-A}(a)), \end{aligned}$$

which proves $h_{B-A}(a) \in SH_\Sigma^{\mathcal{A}}(X_0)$.

(4) The isomorphism $h_{A-B} : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$, which maps X to Y, shows that $(\mathcal{A}^\Sigma, X)$ and $(\mathcal{B}^\Sigma, Y)$ are qf-isomorphic.

(5) In order to verify the last statement, assume that Z is a set with $X \subseteq Z \subseteq Y$, and let $\mathcal{C}^\Sigma := SH^{\mathcal{B}}_\Sigma(Z)$. Above we have seen that $\mathcal{A}^\Sigma = SH^{\mathcal{B}}_\Sigma(X)$. It follows from Lemma 3.28, part 3, that $SH^{\mathcal{B}}_\Sigma(X) = SH^{\mathcal{C}}_\Sigma(X)$, hence $\mathcal{A}^\Sigma = SH^{\mathcal{C}}_\Sigma(X)$. By Lemma 3.28, part 1, $(\mathcal{B}^\Sigma, Y)$ and $(\mathcal{C}^\Sigma, Z)$ are qf-isomorphic. As we have seen above, $(\mathcal{A}^\Sigma, X)$ and $(\mathcal{B}^\Sigma, Y)$ are qf-isomorphic. This implies that $(\mathcal{A}^\Sigma, X)$ and $(\mathcal{C}^\Sigma, Z)$ are qf-isomorphic. □

Finally, it remains to prove Theorem 4.4. This theorem is an immediate consequence of the following lemma (and its dual).

Lemma *Let $\Gamma \subseteq \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$, and let $\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}$ be structures with fixed homomorphic embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Gamma$, $h^\Gamma_{A-B_2} : \mathcal{A}^\Gamma \to \mathcal{B}_2^\Gamma$, and $h^\Gamma_{A-B_3} : \mathcal{A}^\Gamma \to \mathcal{B}_3^\Gamma$. Assume that the free amalgamated product $\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}$ of $\mathcal{B}_2^{\Sigma_2}$ and $\mathcal{B}_3^{\Sigma_3}$, and the free amalgamated product $\mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$ of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}$ exist, and that the classes of admissible structures satisfy*

$$\mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}) \in Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \subseteq Adm(\mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \cap Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}).$$

Then $\mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$ is the free simultaneous amalgamated product of $\mathcal{B}_1^{\Sigma_1}$, $\mathcal{B}_2^{\Sigma_2}$, and $\mathcal{B}_3^{\Sigma_3}$ over $\mathcal{A}^\Gamma$.

Proof. In the sequel, if J is a subsequence of 123, Σ_J denotes the union of the signatures Σ_i with $i \in J$. Let $\mathcal{B}_{23}^{\Sigma_{23}} := \mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}$ denote the free amalgamated product of $\mathcal{B}_2^{\Sigma_2}$ and $\mathcal{B}_3^{\Sigma_3}$, and let $h^{\Sigma_i}_{B_i-B_{23}}$ $(i = 2, 3)$ be the corresponding embeddings. Thus, we have

$$h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-B_{23}} \quad = \quad h^\Gamma_{A-B_3} \circ h^{\Sigma_3}_{B_3-B_{23}}. \tag{8.1}$$

Now, we consider $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_{23}^{\Sigma_{23}})$ with the embeddings $h^\Gamma_{A-B_1} : \mathcal{A}^\Gamma \to \mathcal{B}_1^\Gamma$ and $h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-B_{23}} : \mathcal{A}^\Gamma \to \mathcal{B}_{23}^\Gamma$ as amalgamation base. Let $\mathcal{B}_{123}^{\Sigma_{123}} := \mathcal{B}_1^{\Sigma_1} \odot \mathcal{B}_{23}^{\Sigma_{23}}$ be the corresponding free amalgamated product with embeddings $h^{\Sigma_1}_{B_1-B_{123}}$ and $h^{\Sigma_{23}}_{B_{23}-B_{123}}$. By definition of the amalgamated product, these embeddings satisfy

$$h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-B_{123}} \quad = \quad (h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-B_{23}}) \circ h^{\Sigma_{23}}_{B_{23}-B_{123}}. \tag{8.2}$$

We show that $\mathcal{B}_{123}^{\Sigma_{123}}$ closes the simultaneous amalgamation base $(\mathcal{A}^\Gamma, \mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$. To this purpose, we define

$$h^{\Sigma_i}_{B_i-B_{123}} \quad := \quad h^{\Sigma_i}_{B_i-B_{23}} \circ h^{\Sigma_{23}}_{B_{23}-B_{123}} \ (i = 2, 3). \tag{8.3}$$

It is easy to see that, with this definition, (8.1) and (8.2) imply

$$h^\Gamma_{A-B_1} \circ h^{\Sigma_1}_{B_1-B_{123}} = h^\Gamma_{A-B_2} \circ h^{\Sigma_2}_{B_2-B_{123}} = h^\Gamma_{A-B_3} \circ h^{\Sigma_3}_{B_3-B_{123}},$$

i.e., $\mathcal{B}_{123}^{\Sigma_{123}}$ indeed closes the simultaneous amalgamation base. Because of the assumption that $\mathcal{B}_1^{\Sigma_1} \odot (\mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3}) \in Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$, we know that $\mathcal{B}_{123}^{\Sigma_{123}} \in$

$Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$. Thus, it remains to be shown that the simultaneous amalgamated product $\mathcal{B}_{123}^{\Sigma_{123}}$ is in fact free.

Assume that $\mathcal{D}^{\Sigma_{123}} \in Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$ is a simultaneous amalgamated product with embeddings $g^{\Sigma_i}_{B_i-D} : \mathcal{B}_i^{\Sigma_i} \to \mathcal{D}^{\Sigma_i}$ $(i = 1, 2, 3)$, which thus satisfy

$$h^{\Gamma}_{A-B_1} \circ g^{\Sigma_1}_{B_1-D} = h^{\Gamma}_{A-B_2} \circ g^{\Sigma_2}_{B_2-D} = h^{\Gamma}_{A-B_3} \circ g^{\Sigma_3}_{B_3-D}. \tag{8.4}$$

Equation (8.4), together with our assumption that the classes of admissible structures satisfy $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \subseteq Adm(\mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3})$, implies that $\mathcal{D}^{\Sigma_{123}}$ is also an amalgamated product of $\mathcal{B}_2^{\Sigma_2}$ and $\mathcal{B}_3^{\Sigma_3}$. Since $\mathcal{B}_{23}^{\Sigma_{23}}$ is the free amalgamated product of $\mathcal{B}_2^{\Sigma_2}$ and $\mathcal{B}_3^{\Sigma_3}$, there exists a unique homomorphism $f^{\Sigma_{23}}_{B_{23}-D} : \mathcal{B}_{23}^{\Sigma_{23}} \to \mathcal{D}^{\Sigma_{23}}$ such that

$$g^{\Sigma_i}_{B_i-D} \;=\; h^{\Sigma_i}_{B_i-B_{23}} \circ f^{\Sigma_{23}}_{B_{23}-D} \;(i = 2, 3). \tag{8.5}$$

Because of our assumption $Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2}, \mathcal{B}_3^{\Sigma_3}) \subseteq Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$, we know that $\mathcal{D}^{\Sigma_{123}} \in Adm(\mathcal{B}_1^{\Sigma_1}, \mathcal{B}_2^{\Sigma_2} \odot \mathcal{B}_3^{\Sigma_3})$. In addition, we have $h^{\Gamma}_{A-B_1} \circ g^{\Sigma_1}_{B_1-D} = h^{\Gamma}_{A-B_2} \circ g^{\Sigma_2}_{B_2-D} = h^{\Gamma}_{A-B_2} \circ h^{\Sigma_2}_{B_2-B_{23}} \circ f^{\Sigma_{23}}_{B_{23}-D}$ (the first identity holds because of (8.4) and the second because of (8.5)). This shows that $\mathcal{D}^{\Sigma_{123}}$ with the embeddings $g^{\Sigma_1}_{B_1-D}$ and $f^{\Sigma_{23}}_{B_{23}-D}$ is an amalgamated product of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_{23}^{\Sigma_{23}}$. Since $\mathcal{B}_{123}^{\Sigma_{123}}$ is the free amalgamated product of $\mathcal{B}_1^{\Sigma_1}$ and $\mathcal{B}_{23}^{\Sigma_{23}}$, there exists a unique homomorphism $f^{\Sigma_{123}}_{B_{123}-D} : \mathcal{B}_{123}^{\Sigma_{123}} \to \mathcal{D}^{\Sigma_{123}}$ such that

$$g^{\Sigma_1}_{B_1-D} \;=\; h^{\Sigma_1}_{B_1-B_{123}} \circ f^{\Sigma_{123}}_{B_{123}-D}, \tag{8.6}$$

$$f^{\Sigma_{23}}_{B_{23}-D} \;=\; h^{\Sigma_{23}}_{B_{23}-B_{123}} \circ f^{\Sigma_{123}}_{B_{123}-D}. \tag{8.7}$$

We must show that $g^{\Sigma_i}_{B_i-D} = h^{\Sigma_i}_{B_i-B_{123}} \circ f^{\Sigma_{123}}_{B_{123}-D}$ for $i = 1, 2, 3$. For $i = 1$, this is just identity (8.6). For $i = 2, 3$, we have $h^{\Sigma_i}_{B_i-B_{123}} \circ f^{\Sigma_{123}}_{B_{123}-D} = h^{\Sigma_i}_{B_i-B_{23}} \circ h^{\Sigma_{23}}_{B_{23}-B_{123}} \circ f^{\Sigma_{123}}_{B_{123}-D} = h^{\Sigma_i}_{B_i-B_{23}} \circ f^{\Sigma_{23}}_{B_{23}-D} = g^{\Sigma_i}_{B_i-D}$ (the first identity holds by (8.3), the second by (8.7), and the third by (8.5)).

It remains to be shown that $f^{\Sigma_{123}}_{B_{123}-D}$ is unique with this property. Thus, assume that $e^{\Sigma_{123}}_{B_{123}-D} : \mathcal{B}_{123}^{\Sigma_{123}} \to \mathcal{D}^{\Sigma_{123}}$ is a homomorphism satisfying

$$g^{\Sigma_i}_{B_i-D} \;=\; h^{\Sigma_i}_{B_i-B_{123}} \circ e^{\Sigma_{123}}_{B_{123}-D} \;(i = 1, 2, 3). \tag{8.8}$$

The identity (8.8) together with (8.3) yields

$$g^{\Sigma_i}_{B_i-D} = h^{\Sigma_i}_{B_i-B_{23}} \circ h^{\Sigma_{23}}_{B_{23}-B_{123}} \circ e^{\Sigma_{123}}_{B_{123}-D} \;(i = 2, 3).$$

Since $f^{\Sigma_{23}}_{B_{23}-D}$ is the unique morphism satisfying (8.5), this implies

$$f^{\Sigma_{23}}_{B_{23}-D} \;=\; h^{\Sigma_{23}}_{B_{23}-B_{123}} \circ e^{\Sigma_{123}}_{B_{123}-D}. \tag{8.9}$$

Now, consider (8.8) for $i = 1$ and (8.9): Since $f^{\Sigma_{123}}_{B_{123}-D}$ is the unique homomorphism satisfying (8.6) and (8.7), these two identities imply $f^{\Sigma_{123}}_{B_{123}-D} = e^{\Sigma_{123}}_{B_{123}-D}$.
□