A NExpTime-complete Description Logic Strictly Contained in C^2

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Abstract. We examine the complexity and expressivity of the combination of the Description Logic $\mathcal{ALCQI}$ with a terminological formalism based on cardinality restrictions on concepts. This combination can naturally be embedded into C^2, the two variable fragment of predicate logic with counting quantifiers. We prove that $\mathcal{ALCQI}$ has the same complexity as C^2 but does not reach its expressive power.

Keywords. Description Logic, Counting, Complexity, Expressivity

1 Introduction

Description Logic (DL) systems can be used in knowledge based systems to represent and reason about taxonomical knowledge of problem domain in a semantically well-defined manner [WS92]. These systems usually consist at least of the following three components: a DL, a terminological component, and a reasoning service.

Description logics allow the definition of complex concepts (unary predicates) and roles (binary relations) to be built from atomic ones by the application of a given set of constructors; for example the following concept describes those fathers having at least two daughters:

$$\texttt{Parent} \sqcap \texttt{Male} \sqcap (\geq\ 2\ \texttt{hasChild}\ \texttt{Female})$$

The terminological component (TBox) allows for the organisation of defined concepts and roles. The TBox formalisms studied in the DL context range from weak ones allowing only for the introduction of abbreviations for complex concepts, over TBoxes capable of expressing various forms of axioms, to cardinality restrictions that can express restrictions on the number of elements a concept may have. Consider the following three TBox expressions:

$$\texttt{BusyParent} = \texttt{Parent} \sqcap (\geq\ 2\ \texttt{hasChild}\ \texttt{Toddler})$$
$$\texttt{Male} \sqcup \texttt{Female} = \texttt{Person} \sqcap (=\ 2\ \texttt{hasChild}^{-1}\ \texttt{Parent})$$
$$(\leq\ 2\ \texttt{Person} \sqcap (\leq\ 0\ \texttt{hasChild}^{-1}\ \texttt{Parent}))$$

The first introduces `BusyParent` as an abbreviation for a more complex concept, the second is an axiom stating that `Male` and `Female` are exactly those persons having two parents, the third is a cardinality restriction expressing that in the domain of discourse there are at most two earliest ancestors.

The reasoning service performs task like subsumption or consistency test for the knowledge stored in the TBox. There exist sound and complete algorithms for reasoning in a large number of DLs and different TBox formalisms that meet the known worst-case complexity of these problems (see [DLNN97] for an overview). Generally, reasoning for DLs can be performed in four different ways:

- by structural comparison of syntactical normal forms of concepts [BPS94].
- by tableaux algorithms that are hand-tailored to suit the necessities of the operators used to form the DL and the TBox formalism. Initially, these algorithms were designed to decide inference problems only for the DL without taking into account TBoxes, but it is possible to generalise these algorithms to deal with different TBox formalisms. Most DLs handled this way are at most PSPACE complete but additional complexity may arise from the TBox. The complexity of the tableaux approach usually meets the known worst-case complexity of the problem [SSS91,DLNN97].
- by perceiving the DL as a (fragment of a) modal logic such as PDL [GL96]; for many DLs handled in this manner already concept satisfiability is EXPTIME-complete, but axioms can be "internalised" [Baa91] into the concepts and hence do not increase the complexity.
- by translation of the problem into a fragment or first order other logic with a decidable decision problem [Bor96,OSH96].

From the fragments of predicate logic that are studied in the second context, only C^2, the two variable fragment of first order predicate logic augmented with counting quantifiers, is capable of dealing with counting expressions that are commonly used in DLs; similarly it is able to express cardinality restrictions. Another thing that comes "for free" when translating DLs into first order logic is the ability to deal with inverse roles.

Combining all these parts into a single DL, one obtains the DL $\mathcal{ALCQI}$—the well-known DL $\mathcal{ALC}$ [SSS91] augmented by qualifying number restrictions ($\mathcal{Q}$) and inverse roles ($\mathcal{I}$). In this work we study both complexity and expressivity of $\mathcal{ALCQI}$ combined with TBoxes based on cardinality restrictions.

Regarding the complexity we show that $\mathcal{ALCQI}$ with cardinality restrictions already is NEXPTIME-hard and hence has the same complexity as C^2 [PST97][1]. To our knowledge this is the first DL for which NEXPTIME-completeness has formally been proved. Since $\mathcal{ALCQI}$ with TBoxes consisting of axioms is still in EXPTIME, this indicates that cardinality restrictions are algorithmically hard to handle.

[1] The NEXPTIME-result is valid only if we assume unary coding of numbers in the counting quantifiers. This is the standard assumption made by most results concerning the complexity of DLs.

Despite the fact that both $\mathcal{ALCQI}$ and C^2 have the same worst-case complexity we show that $\mathcal{ALCQI}$ lacks some of the expressive power of C^2. Properties of binary predicates (e.g. reflexivity) that are easily expressible in C^2 can not be expressed in $\mathcal{ALCQI}$. We establish our result by giving an Ehrenfeucht-Fraïssé game that exactly captures the expressivity of $\mathcal{ALCQI}$ with cardinality restrictions. This is the first time in the area of DL that a game-theoretic characterisation is used to prove an expressivity result involving TBox formalisms. The game as it is presented here is not only applicable to $\mathcal{ALCQI}$ with cardinality restrictions; straightforward modifications make it applicable to both $\mathcal{ALCQ}$ as well as to weaker TBox formalisms such as terminological axioms.

In [Bor96] a DL is presented that has the same expressivity as C^2. This expressivity result is one of the main results of that paper and the DL combines a large number of constructs; the paper does not study the computational complexity of the presented logics. Our motivation is of a different nature: we study the complexity and expressivity of a DL consisting of only a minimal set of constructs that seem sensible when a reduction of that DL to C^2 is to be considered.

2 The Logic $\mathcal{ALCQI}$

Definition 1. *A* signature *is a pair* $\tau = (N_C, N_R)$ *where* N_C *is a finite set of* concepts names *and* N_R *is a finite set of* role names. Concepts *in* $\mathcal{ALCQI}$ *are built inductively from these using the following rules: All* $A \in N_C$ *are concepts, and, if* C, C_1, *and* C_2 *are concepts, then also* $\neg C$, $C_1 \sqcap C_2$, *and* $(\geq\ n\ S\ C)$ *with* $n \in \mathbb{N}$, *and* $S = R$ *or* $S = R^{-1}$ *for some* $R \in N_R$ *are concepts. We define* $C_1 \sqcup C_2$ *as an abbreviation for* $\neg(\neg C_1 \sqcap \neg C_2)$ *and* $(\leq\ n\ S\ C)$ *as an abbreviation for* $\neg(\geq\ (n+1)\ S\ C)$. *We also use* $(=\ n\ S\ C)$ *as an abbreviation for* $(\leq\ n\ S\ C) \sqcap (\geq\ n\ S\ C)$.

A cardinality restriction *of* $\mathcal{ALCQI}$ *is an expression of the form* $(\geq n\ C)$ *or* $(\leq n\ C)$ *where* C *is a concept and* $n \in \mathbb{N}$; *a* TBox T *of* $\mathcal{ALCQI}$ *is a finite set of cardinality restrictions.*

The semantics of a concept is defined relative to an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, *which consists of a domain* $\Delta^{\mathcal{I}}$ *and a valuation* $(\cdot^{\mathcal{I}})$ *which maps each concept name* A *to a subset* $A^{\mathcal{I}}$ *of* $\Delta^{\mathcal{I}}$ *and each role name* R *to a subset* $R^{\mathcal{I}}$ *of* $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. *This valuation is inductively extended to arbitrary concept definitions using the following rules, where* $\sharp M$ *denotes the cardinality of a set* M:

$$(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \quad (C_1 \sqcap C_2)^{\mathcal{I}} := C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}},$$
$$(\geq n\ R\ C)^{\mathcal{I}} := \{a \in \Delta^{\mathcal{I}} \mid \sharp\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\} \geq n\},$$
$$(\geq n\ R^{-1}\ C)^{\mathcal{I}} := \{a \in \Delta^{\mathcal{I}} \mid \sharp\{b \in \Delta^{\mathcal{I}} \mid (b,a) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\} \geq n\}.$$

An interpretation $\mathcal{I}$ satisfies *a cardinality restriction* $(\geq n\ C)$ *iff* $\sharp(C^{\mathcal{I}}) \geq n$ *and it satisfies* $(\leq n\ C)$ *iff* $\sharp(C^{\mathcal{I}}) \leq n$. *It satisfies a TBox* T *iff it satisfies all cardinality restrictions in* T; *in this case,* $\mathcal{I}$ *is called a* model *of* T *and we will denote this fact by* $\mathcal{I} \models T$. *A TBox that has a model is called* consistent.

$$\begin{array}{lll}
\Psi_x(A) & := Ax & \text{for } A \in N_C \\
\Psi_x(\neg C) & := \neg\Psi_x(C) & \\
\Psi_x(C_1 \sqcap C_2) & := \Psi_x(C_1) \wedge \Psi_x(C_2) & \\
\Psi_x(\geq\ n\ R\ C) & := \exists^{\geq n} y.(Rxy \wedge \Psi_y(C)) & \\
\Psi_x(\geq\ n\ R^{-1}\ C) & := \exists^{\geq n} y.(Ryx \wedge \Psi_y(C)) & \\
\Psi(\bowtie\ n\ C) & := \exists^{\bowtie n} x.\Psi_x(C) & \text{for } \bowtie \in \{\geqslant, \leqslant\} \\
\Psi(T) & := \bigwedge\{\Psi(\bowtie\ n\ C) \mid (\bowtie\ n\ C) \in T\} &
\end{array}$$

Fig. 1. The translation from $\mathcal{ALCQI}$ into C^2 adopted from [Bor96]

With $\mathcal{ALCQ}$ we denote the fragment of $\mathcal{ALCQI}$ that does not contain any inverse roles R^{-1}.

TBoxes consisting of cardinality restrictions have first been studied in [BBH96] for the DL $\mathcal{ALCQ}$. They can express terminological axioms of the form $C = D$ that are the most expressive TBox formalisms usually studied in the DL context [GL96] as follows: obviously, two concepts C, D have the same extension in an interpretation iff it satisfies the cardinality restriction $(\leq\ 0\ (C \sqcap \neg D) \sqcup (\neg C \sqcap D))$. One standard inference service for DL systems is satisfiability of a concept C with respect to a TBox T (i.e., is there an interpretation $\mathcal{I}$ such that $\mathcal{I} \models T$ and $C^{\mathcal{I}} \neq \emptyset$). For a TBox formalism based on cardinality restrictions this is easily reduced to TBox consistency, because obviously C is satisfiable with respect to T iff $T \cup \{(\geq\ 1\ C)\}$ is a consistent TBox. To this the reason we will restrict our attention to TBox consistency; other standard inferences such as concept subsumption can be reduced to consistency as well.

Until now there does not exist a tableaux based decision procedure for $\mathcal{ALCQI}$ TBox consistency. Nevertheless this problem can be decided with the help of a well-known translation of $\mathcal{ALCQI}$-TBoxes to C^2 [Bor96] given in Fig. 1. The logic C^2 is fragment of predicate logic that allows only two variables but is enriched with counting quantifiers of the form $\exists^{\geq l}$. The translation Ψ yields a satisfiable sentence of C^2 if and only if the translated TBox is consistent. Since the translation from $\mathcal{ALCQI}$ to C^2 can be performed in linear time, the NExpTime upper bound [GOR97,PST97] for satisfiability of C^2 directly carries over to $\mathcal{ALCQI}$-TBox consistency:

Lemma 1. *Consistency of an $\mathcal{ALCQI}$-TBox T can be decided in* NExpTime.

Please note that the NExpTime-completeness result from [PST97] is only valid if we assume unary coding of numbers in the input; this implies that a large number like 1000 may not be stored in logarithmic space in some k-ary representation but consumes 1000 units of storage. This is the standard assumption made by most results concerning the complexity of DLs. We will come back to this issue later in this paper.

3 $\mathcal{ALCQI}$ is NExpTime-complete

To show that NExpTime is also the lower bound for the complexity of TBox consistency we use a bounded version of the domino problem. Domino problems [Wan63,Ber66] have successfully been employed to establish undecidability and complexity results for various description and modal logics [Spa93,BS99].

3.1 Domino Systems

Definition 2. *For an $n \in \mathbb{N}$ let $\mathbb{Z}_n$ denote the set $\{0, \dots, n-1\}$ and $\oplus_n$ denote the addition modulo n. A* domino system *is a triple $\mathcal{D} = (D, H, V)$, where D is a finite set (of tiles) and $H, V \subseteq D \times D$ are relations expressing horizontal and vertical compatibility constraints between the tiles. For $s, t \in \mathbb{N}$ let $U(s,t)$ be the torus $\mathbb{Z}_s \times \mathbb{Z}_t$ and $w = w_0, \dots, w_{n-1}$ be an n-tuple of tiles (with $n \leq s$). We say that $\mathcal{D}$* tiles *$U(s,t)$* with initial condition *w iff there exists a mapping $\tau : U(s,t) \to D$ such that, for all $(x,y) \in U(s,t)$,*

- *if $\tau(x,y) = d$ and $\tau(x \oplus_s 1, y) = d'$ then $(d, d') \in H$ (horizontal constraint);*
- *if $\tau(x,y) = d$ and $\tau(x, y \oplus_t 1) = d'$ then $(d, d') \in V$ (vertical constraint);*
- *$\tau(i, 0) = w_i$ for $0 \leq i < n$ (initial condition).*

Bounded domino systems are capable of expressing the computational behaviour of restricted, so called *simple*, Turing Machines (TM). This restriction is non-essential in the following sense: Every language accepted in time $T(n)$ and space $S(n)$ by some one-tape TM is accepted within the same time and space bounds by a simple TM, as long as $S(n), T(n) \geq 2n$ [BGG97].

Theorem 1 ([BGG97], Theorem 6.1.2). *Let M be a simple TM with input alphabet Σ. Then there exists a domino system $\mathcal{D} = (D, H, V)$ and a linear time reduction which takes any input $x \in \Sigma^*$ to a word $w \in D^*$ with $|x| = |w|$ such that*

- *If M accepts x in time t_0 with space s_0, then $\mathcal{D}$ tiles $U(s,t)$ with initial condition w for all $s \geq s_0 + 2, t \geq t_0 + 2$;*
- *if M does not accept x, then $\mathcal{D}$ does not tile $U(s,t)$ with initial condition w for any $s, t \geq 2$.*

Corollary 1. *Let M be a (w.l.o.g. simple) non-deterministic TM with time- (and hence space-) bound 2^{n^d} (d constant) deciding an arbitrary* NExpTime-*complete language $\mathcal{L}(M)$ over the alphabet Σ. Let $\mathcal{D}$ be the according domino system and and trans the reduction from Theorem 1. The following is a* NExpTime-*hard problem:*

> *Given an initial condition $w = w_0, \dots, w_{n-1}$ of length n. Does $\mathcal{D}$ tile $U(2^{n^d+1}, 2^{n^d+1})$ with initial condition w?*

Proof. The function *trans* is a linear reduction from $\mathcal{L}(M)$ to the problem above: For $v \in \Sigma^*$ with $|v| = n$ it holds that $v \in \mathcal{L}(M)$ iff M accepts v in time and space $2^{|v|^d}$ iff $\mathcal{D}$ tiles $U(2^{n^d+1}, 2^{n^d+1})$ with initial condition *trans*(v). □

3.2 Defining a Torus of Exponential Size

Just as defining infinite grids is the key problem in proving undecidability by reduction of unbounded domino problems, defining a torus of exponential size is the key to obtaining a NExpTime-completeness proof by reduction of bounded domino problems.

To be able to apply Corollary 1 to TBox consistency for $\mathcal{ALCQI}$ we must characterise the torus $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ with a TBox of polynomial size. To characterise this torus we will use $2n$ concepts $X_0, \dots, X_{n-1}$ and $Y_0, \dots, Y_{n-1}$, where X_i codes the ith bit of the binary representation of the X-coordinate of an element a:

For an interpretation $\mathcal{I}$ and an element $a \in \Delta^{\mathcal{I}}$, we define $pos(a)$ by

$$pos(a) := (xpos(a), ypos(a)) := \left(\sum_{i=0}^{n-1} x_i \cdot 2^i, \quad \sum_{i=0}^{n-1} y_i \cdot 2^i \right), where$$

$$x_i = \begin{cases} 0, & \text{if } a \notin X_i^{\mathcal{I}} \\ 1, & \text{otherwise} \end{cases} \qquad y_i = \begin{cases} 0, & \text{if } a \notin Y_i^{\mathcal{I}} \\ 1, & \text{otherwise} \end{cases}.$$

We use a well-known characterisation of binary addition (e.g. [BGG97]) to relate the positions of the elements in the torus:

Lemma 2. *Let x, x' be natural numbers with binary representations*

$$x = \sum_{i=0}^{n-1} x_i \cdot 2^i \quad and \quad x' = \sum_{i=0}^{n-1} x'_i \cdot 2^i.$$

This implies:

$$x' \equiv x + 1 \pmod{2^n} \quad \textit{iff} \quad \bigwedge_{k=0}^{n-1} (\bigwedge_{j=0}^{k-1} x_j = 1) \rightarrow (x_k = 1 \leftrightarrow x'_k = 0)$$
$$\wedge \bigwedge_{k=0}^{n-1} (\bigvee_{j=0}^{k-1} x_j = 0) \rightarrow (x_k = x'_k)$$

where the empty conjunction and disjunction are interpreted as true and false respectively.

We define the TBox T_n to consist of the following cardinality restrictions:

$$(\forall\ (\geq\ 1\ east\ \top)), \quad (\forall\ (\geq\ 1\ north\ \top)),$$
$$(\forall\ (=1\ east^{-1}\ \top)), \quad (\forall\ (=\ 1\ north^{-1}\ \top)),$$
$$(\geq 1\ C_{(0,0)}), \quad (\geq 1\ C_{(2^n-1,2^n-1)}), \quad (\leq 1\ C_{(2^n-1,2^n-1)}), \quad (\forall\ D_{east} \sqcap D_{north}),$$

where we use the following abbreviations: the expression $(\forall\ C)$ is an abbreviation for the cardinality restriction $(\leq\ 0\ \neg C)$, the concept $\forall R.C$ stands for

$(\leq\ 0\ R\ \neg C)$, and $\top$ stands for an arbitrary concept that is satisfied in all interpretations (e.g. $A \sqcup \neg A$).

The concept $C_{(0,0)}$ is satisfied by all elements a of the domain for which $pos(a) = (0,0)$ holds. $C_{(2^n-1,2^n-1)}$ is a similar concept, which is satisfied if $pos(a) = (2^n-1, 2^n-1)$:

$$C_{(0,0)} = \prod_{k=0}^{n-1} \neg X_k \sqcap \prod_{k=0}^{n-1} \neg Y_k, \quad C_{(2^n-1,2^n-1)} = \prod_{k=0}^{n-1} X_k \sqcap \prod_{k=0}^{n-1} Y_k.$$

The concept D_{east} (resp. D_{north}) enforces that along the role *east* (resp. *north*) the value of *xpos* (resp. *ypos*) increases by one while the value of *ypos* (resp. *xpos*) stays the same. They exactly resemble the formula from Lemma 2:

$$\begin{aligned} D_{east} = & \prod_{k=0}^{n-1} (\prod_{j=0}^{k-1} X_j) \rightarrow ((X_k \rightarrow \forall east.\neg X_k) \sqcap (\neg X_k \rightarrow \forall east.X_k)) \\ \sqcap & \prod_{k=0}^{n-1} (\bigsqcup_{j=0}^{k-1} \neg X_j) \rightarrow ((X_k \rightarrow \forall east.X_k) \sqcap (\neg X_k \rightarrow \forall east.\neg X_k)) \\ \sqcap & \prod_{k=0}^{n-1} ((Y_k \rightarrow \forall east.Y_k) \sqcap (\neg Y_k \rightarrow \forall east.\neg Y_k)). \end{aligned}$$

The concept D_{north} is similar to D_{east} where the role *north* has been substituted for *east* and variables X_i and Y_i have been swapped.

The following lemma is a consequence of the definition of *pos* and Lemma 2.

Lemma 3. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation and $a, b \in \Delta^{\mathcal{I}}$.*

$$\begin{aligned} (a,b) \in east^{\mathcal{I}} \text{ and } a \in D_{east}^{\mathcal{I}} \text{ implies: } & xpos(b) \equiv xpos(a) + 1 \pmod{2^n} \\ & ypos(b) = ypos(a) \\ (a,b) \in north^{\mathcal{I}} \text{ and } a \in D_{north}^{\mathcal{I}} \text{ implies: } & xpos(b) = xpos(a) \\ & ypos(b) \equiv ypos(a) + 1 \pmod{2^n} \end{aligned}$$

The TBox T_n defines a torus of exponential size in the following sense:

Lemma 4. *Let T_n be the TBox as introduced above. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that $\mathcal{I} \models T_n$. This implies*

$$(\Delta^{\mathcal{I}}, east^{\mathcal{I}}, north^{\mathcal{I}}) \cong (U(2^n, 2^n), S_1, S_2)$$

where $U(2^n, 2^n)$ is the torus $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ and S_1, S_2 are the horizontal and vertical successor relations on the torus.

Proof. We will only sketch the proof of this lemma. It is established by showing that the function *pos* is an isomorphism from $\Delta^{\mathcal{I}}$ to $U(2^n, 2^n)$. That *pos* is a homomorphism follows immediately from Lemma 3. Injectivity of *pos* is established by showing that each element $(x, y) \in U(2^n, 2^n)$ is the image of at most one element of $\Delta^{\mathcal{I}}$ by induction over the Manhattan distance of (x, y) to the upper right corner $(2^n - 1, 2^n - 1)$ of the torus. The base case is trivially satisfied because T_n contains the cardinality restrictions $(\leq\ 1\ C_{(2^n-1,2^n-1)})$. The induction step follows from the fact that each element $a \in \Delta^{\mathcal{I}}$ has exactly one *east*- and *north*-predecessor (since $(\forall\ (=1\ \mathit{east}^{-1}\ \top)), (\forall\ (=\ 1\ \mathit{north}^{-1}\ \top)) \in T_n$) and Lemma 3. Surjectivity is established similarly starting from the corner $(0, 0)$. □

It is interesting to note that we need inverse roles only to guarantee that *pos* is injective. The same can be achieved by adding the cardinality restriction $(\leq (2^n \cdot 2^n)\ \top)$ to T_n, from which the injectivity of *pos* follows from its surjectivity and simple cardinality considerations. Of course the size of this cardinality restriction would only be polynomial in n if we allow binary coding of numbers. Also note that we have made explicit use of the special expressive power of cardinality restrictions by stating that, in any model of T_n, the extension of $C_{(2^n-1,2^n-1)}$ must have *at most* one element. This can not be expressed with a TBox consisting of terminological axioms.

3.3 Reducing Domino Problems to TBox Consistency

Once Lemma 4 has been proved, it is easy to reduce the bounded domino problem to TBox consistency. We use the standard reduction that has been applied in the DL context, e.g., in [BS99].

Lemma 5. *Let* $\mathcal{D} = (D, V, H)$ *be a domino system. Let* $w = w_0, \ldots, w_{n-1} \in D^*$. *There is a TBox* $T(n, \mathcal{D}, w)$ *such that:*

- $T(n, \mathcal{D}, w)$ *is consistent iff* $\mathcal{D}$ *tiles* $U(2^n, 2^n)$ *with initial condition* w.
- $T(n, \mathcal{D}, w)$ *can be computed in time polynomial in* n.

Proof. We define $T(n, \mathcal{D}, w) := T_n \cup T_{\mathcal{D}} \cup T_w$, where T_n is defined as above, $T_{\mathcal{D}}$ captures the vertical and horizontal compatibility constraints of the domino system $\mathcal{D}$, and T_w enforces the initial condition. We use an atomic concept C_d for each tile $d \in D$. $T_{\mathcal{D}}$ consists of the following cardinality restrictions:

$$(\forall \bigsqcup_{d \in D} C_d), \quad (\forall \bigsqcap_{d \in D} \bigsqcap_{d' \in D \setminus \{d\}} \neg(C_d \sqcap C_{d'})),$$

$$(\forall \bigsqcap_{d \in D} (D_d \rightarrow (\forall \mathit{east}. \bigsqcup_{(d,d') \in H} C_{d'}))), \quad (\forall \bigsqcap_{d \in D} (D_d \rightarrow (\forall \mathit{north}. \bigsqcup_{(d,d') \in V} C_{d'}))).$$

$\mathcal{T}_w$ consists of the cardinality restrictions

$$(\forall\ (C_{(0,0)} \rightarrow C_{w_0})), \ldots, (\forall\ (C_{(n-1,0)} \rightarrow C_{w_{n-1}})$$

where, for each x, y, $C_{(x,y)}$ is a concept that is satisfied by an element a iff $\mathit{pos}(a) = (x, y)$, similar to $C_{(0,0)}$ and $C_{(2^n-1,2^n-1)}$.

From the definition of $T(n, \mathcal{D}, w)$ and Theorem 4, it follows that each model of $T(n, \mathcal{D}, w)$ immediately induces a tiling of $U(2^n, 2^n)$ and vice versa. Also, for a fixed domino system $\mathcal{D}$, $T(n, \mathcal{D}, w)$ is obviously polynomially computable. □

The next theorem is an immediate consequence of Lemma 5 and Corollary 1:

Theorem 2. *Consistency of $\mathcal{ALCQI}$-TBoxes is* NExpTime*-hard, even if unary coding of numbers is used in the input.*

Recalling the note below Lemma 4, we see that the same argument also applies to $\mathcal{ALCQ}$ if we allow binary coding of numbers.

Corollary 2. *Consistency of $\mathcal{ALCQ}$-TBoxes is* NExpTime*-hard, if binary coding is used to represent numbers in cardinality restrictions.*

Note that for unary coding we needed both inverse roles and cardinality restrictions for the reduction. This is consistent with the fact that satisfiability for $\mathcal{ALCQI}$ concepts with respect to TBoxes consisting of terminological axioms is still in ExpTime, which can be shown by a reduction to Converse-PDL [GM99]. This shows that cardinality restrictions on concepts are an additional source of complexity; one reason for this might be that $\mathcal{ALCQI}$ with cardinality restrictions no longer has a tree-model property in the modal logic sense.

4 Expressiveness of $\mathcal{ALCQI}$

Since reasoning for $\mathcal{ALCQI}$ has the same (worst-case) complexity as for C^2, naturally the question arises how the two logics are related with respect to their expressivity. We show that $\mathcal{ALCQI}$ is strictly less expressive than C^2.

4.1 A Definition of Expressiveness

There are different approaches to define the expressivity of Description Logics [Baa96,Bor96,AdR98], but only the one presented in [Baa96] is capable of handling TBoxes. We will use a definition that is equivalent to the one given in [Baa96] restricted to a special case. It bases the notion of expressivity on the classes of interpretations definable by a sentence (or TBox).

Definition 3. *Let $\tau = (N_C, N_R)$ be a finite signature. A class $\mathcal{C}$ of τ-interpretations is called* characterisable *by a logic $\mathcal{L}$ iff there is a sentence $\varphi_\mathcal{C}$ over τ such that $\mathcal{C} = \{\mathcal{I} \mid \mathcal{I} \models \varphi_\mathcal{C}\}$.*

The class $\mathcal{C}$ is called projectively characterisable *iff there is a sentence $\varphi'_\mathcal{C}$ over a signature $\tau' \supseteq \tau$ such that $\mathcal{C} = \{\mathcal{I}|_\tau \mid \mathcal{I} \models \varphi'_\mathcal{C}\}$, where $\mathcal{I}|_\tau$ denotes the τ-reduct of $\mathcal{I}$.*

A logic $\mathcal{L}_1$ is called as expressive *as another logic $\mathcal{L}_2$ ($\mathcal{L}_1 \geq \mathcal{L}_2$) iff, for any finite signature τ, any $\mathcal{L}_2$-characterisable class $\mathcal{C}$ can be projectively characterised in $\mathcal{L}_1$.*

Since C^2 is usually restricted to a relational signature with relation symbols of arity at most two, this definition is appropriate to relate the expressiveness of $\mathcal{ALCQI}$ and C^2. It is worth noting that $\mathcal{ALCQI}$ is strictly more expressive than $\mathcal{ALCQ}$, because $\mathcal{ALCQ}$ has the finite model property [BBH96], while the following $\mathcal{ALCQI}$ TBox has no finite models:

$$\mathcal{T}_{\text{inf}} = \{(\forall\ (\geq\ 1\ R\ \top)), (\forall\ (\leq\ 1\ R^{-1}\ \top)), (\geq\ 1\ (=\ 0\ R^{-1}\ \top))\}.$$

The first cardinality restriction requires an outgoing R-edge for every element of a model and thus each R-path in the model in infinite. The second and third restriction require the existence of an R-path in the model that contains no cycle, which implies the existence of infinitely many elements in the model. Since $\mathcal{ALCQ}$ has the finite model property, the class $\mathcal{C}_{\text{inf}} := \{\mathcal{I} \mid \mathcal{I} \models \mathcal{T}_{\text{inf}}\}$, which contains only models with infinitely many elements, can not be projectively characterised by an $\mathcal{ALCQ}$-TBox.

The translation Ψ from $\mathcal{ALCQI}$-TBoxes to C^2 sentences given in Fig. 1 not only preserves satisfiability, but the translation also has exactly the same models as the initial TBox. This implies that $\mathcal{ALCQI} \leq C^2$.

4.2 A Game for $\mathcal{ALCQI}$

Usually, the separation of two logics with respect to their expressivity is a hard task and not as easily accomplished as we have just done with $\mathcal{ALCQ}$ and $\mathcal{ALCQI}$. Even for logics of very restricted expressivity, proofs of separation results may become involved and complex [Baa96] and usually require a detailed analysis of the classes of models a logic is able to characterise. Valuable tools for these analyses are Ehrenfeucht-Fraïssé games. In this section we present an Ehrenfeucht-Fraïssé game that exactly captures the expressivity of $\mathcal{ALCQI}$.

Definition 4. *For an $\mathcal{ALCQI}$ concept C, the* role depth $rd(C)$ *counts the maximum number of nested cardinality restrictions. Formally we define rd as follows:*

$$\begin{aligned} rd(A) &:= 0 \quad \text{for } A \in N_C \\ rd(\neg C) &:= rd(C) \\ rd(C_1 \sqcap C_2) &:= max\{rd(C_1), rd(C_2)\} \\ rd(\geq n\ R\ C) &:= 1 + rd(C) \end{aligned}$$

The set $\mathcal{C}^n_m$ is defined to consist of exactly those $\mathcal{ALCQI}$ concepts that have a role depth of at most m, and in which the numbers appearing in number restrictions are bounded by n; the set $\mathcal{L}^n_m$ is defined to consist of all $\mathcal{ALCQI}$-TBoxes T that contain only cardinality restrictions of the form $(\bowtie\ k\ C)$ with $k \leq n$ and $C \in \mathcal{C}^n_m$.

Two interpretations $\mathcal{I}$ and $\mathcal{J}$ are called n-m-equivalent *($\mathcal{I} \equiv^n_m \mathcal{J}$) iff, for all TBoxes T in $\mathcal{L}^n_m$, it holds that $\mathcal{I} \models T$ iff $\mathcal{J} \models T$. Similarly, for $x \in \Delta^{\mathcal{I}}$ and $y \in \Delta^{\mathcal{J}}$ we say that $\mathcal{I}, x$ and $\mathcal{J}, y$ are* n-m-equivalent *($\mathcal{I}, x \equiv^n_m \mathcal{J}, y$) iff, for all $C \in \mathcal{C}^n_m$ it holds that, $x \in C^{\mathcal{I}}$ iff $y \in C^{\mathcal{J}}$.*

Two elements $x \in \Delta^{\mathcal{I}}$ and $y \in \Delta^{\mathcal{J}}$ are called locally equivalent *($\mathcal{I}, x \equiv_l \mathcal{J}, y$), iff for all $A \in N_C$: $x \in A^{\mathcal{I}}$ iff $y \in A^{\mathcal{J}}$.*

Note that, since we assume τ to be finite, there are only finitely many pairwise inequivalent concepts in each class $\mathcal{C}_m^n$.

We will now define an Ehrenfeucht-Fraisse game for $\mathcal{ALCQI}$ to capture the expressivity of concepts in the classes $\mathcal{C}_m^n$: The game is played by two players. Player I is called the spoiler while Player II is called the duplicator. The spoiler's aim is to prove two structures not to be n-m-equivalent, while Player II tries to prove the contrary. The game consists of a number of rounds in which the players move pebbles on the elements of the two structures.

Definition 5. *Let Δ be a nonempty set. Let x be an element of Δ and X a subset of Δ. For any binary relation $\mathcal{R} \subseteq \Delta \times \Delta$ we write $x\mathcal{R}X$ to denote the fact that $(x, x') \in \mathcal{R}$ holds for all $x' \in X$. For the set N_R of role names let $\overline{N_R}$ be the union of N_R and $\{R^{-1} \mid R \in N_R\}$.*

A configuration *captures the state of a game in progress. It is of the form $G_m^n(\mathcal{I}, x, \mathcal{J}, y)$, where $n \in \mathbb{N}$ is a limit on the size of set that may be chosen during the game, m denotes the number of moves which still have to be played, and x and y are the elements of $\Delta^{\mathcal{I}}$ resp. $\Delta^{\mathcal{J}}$ on which the pebbles are placed.*

For the configuration $G_m^n(\mathcal{I}, x, \mathcal{J}, y)$ the rules are as follows:

1. *If $\mathcal{I}, x \not\equiv_l \mathcal{J}, y$, then Player II loses; if $m = 0$ and $\mathcal{I}, x \equiv_l \mathcal{J}, y$, then Player II wins.*
2. *If $m > 0$, then Player I selects one of the interpretations; assume this is $\mathcal{I}$ (the case $\mathcal{J}$ is handled dually). He then picks a role $S \in \overline{N_R}$ and a number $l \leq n$. He picks a set $X \subseteq \Delta^{\mathcal{I}}$ such that $xS^{\mathcal{I}}X$ and $\sharp X = l$. The duplicator has to answer with a set $Y \subseteq \Delta^{\mathcal{J}}$ with $yS^{\mathcal{J}}Y$ and $\sharp Y = l$. If there is no such set, then she loses.*
3. *If Player II was able to pick such a set Y, then Player I picks an element $y' \in Y$. Player II has to answer with an element $x' \in X$.*
4. *The game continues with $G_{m-1}^n(\mathcal{I}, x', \mathcal{J}, y')$.*

We say that Player II has a winning strategy *for $G_m^n(\mathcal{I}, x, \mathcal{J}, y)$ iff she can always reach a winning position no matter which moves Player I plays. We write $\mathcal{I}, x \cong_m^n \mathcal{J}, y$ to denote this fact.*

Theorem 3. *For two structures $\mathcal{I}, \mathcal{J}$ and two elements $x \in \Delta^{\mathcal{I}}, y \in \Delta^{\mathcal{J}}$ it holds that $\mathcal{I}, x \cong_m^{n+1} \mathcal{J}, y$ iff $\mathcal{I}, x \equiv_m^n \mathcal{J}, y$.*

We omit the proof of this and the next theorem. These employ the same techniques that are used to show the appropriateness of the known Ehrenfeucht-Fraïssé games for C^2 and for modal logics, please refer to [Tob99] for details.

The game as it has been presented so far is suitable only if we have already placed pebbles on the interpretations. To obtain a game that characterises $\equiv_m^n$ as a relation between interpretations, we have to introduce an additional rule that governs the placement of the first pebbles. Since a TBox consists of cardinality restrictions which solely talk about concept membership, we introduce an unconstrained set move as the first move of the game $G_m^n(\mathcal{I}, \mathcal{J})$.

Definition 6. *For two interpretations $\mathcal{I}, \mathcal{J}$, $G_m^n(\mathcal{I}, \mathcal{J})$ is played as follows:*

1. *Player I picks one of the structures; assume he picks $\mathcal{I}$ (the case $\mathcal{J}$ is handled dually). He then picks a set $X \subseteq \Delta^{\mathcal{I}}$ with $\sharp X = l$ where $l \leq n$. Player II must pick a set $Y \subseteq \Delta^{\mathcal{J}}$ of equal size. If this is impossible then she loses.*
2. *Player I picks an element $y \in Y$, Player II must answer with an $x \in X$.*
3. *The game continues with $G^n_m(\mathcal{I}, x, \mathcal{I}, y)$.*

Again we say that Player II has a winning strategy for $G^n_m(\mathcal{I}, \mathcal{J})$ iff she can always reach a winning positions no matter which moves Player I chooses. We write $\mathcal{I} \cong^n_m \mathcal{J}$ do denote this fact.

Theorem 4. *For two structures $\mathcal{I}, \mathcal{J}$ it holds that $\mathcal{I} \equiv^n_m \mathcal{J}$ iff $\mathcal{I} \cong^{n+1}_m \mathcal{J}$.*

Similarly, it would be possible to define a game that captures the expressivity of $\mathcal{ALCQI}$ with TBoxes consisting of terminological axioms by replacing the unconstrained set move from Def. 6 by a move where Player I picks a structure and one element from that structure; Player II then has to answer accordingly and the game continues as described in Def. 5.

4.3 The Expressivity Result

We will now use this characterisation of the expressivity of $\mathcal{ALCQI}$ to prove that $\mathcal{ALCQI}$ is less expressive than C^2. Even though we have introduced the powerful tool of Ehrenfeucht-Fraïssé games, the proof is still rather complicated. This is mainly due to the fact that we use a general definition of expressiveness that allows for the introduction of arbitrary additional role- and concept-names into the signature.

Theorem 5. *$\mathcal{ALCQI}$ is not as expressive as C^2.*

Proof. To prove this theorem we have to show that there is a class $\mathcal{C}$ that is characterisable in C^2 but that cannot be projectively characterised in $\mathcal{ALCQI}$:
CLAIM 1: For an arbitrary $R \in N_R$ the class $\mathcal{C}_R := \{\mathcal{I} \mid R^{\mathcal{I}} \text{ is reflexive}\}$ is not projectively characterisable in $\mathcal{ALCQI}$. Obviously, $\mathcal{C}_R$ is characterisable in C^2.
PROOF OF CLAIM 1: Assume Claim 1 does not hold and that $\mathcal{C}_R$ is projectively characterised by the TBox $\mathcal{T}_R \in \mathcal{L}^n_m$ over an arbitrary (but finite) signature $\tau = (N_C, N_R)$ with $R \in N_R$. We will have derived a contradiction once we have shown that there are two τ-interpretations $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \in \mathcal{C}_R$, $\mathcal{B} \notin \mathcal{C}_R$, but $\mathcal{A} \equiv^n_m \mathcal{B}$. In fact, $\mathcal{A} \equiv^n_m \mathcal{B}$ implies $\mathcal{B} \models \mathcal{T}_R$ and hence $\mathcal{B} \in \mathcal{C}_R$, a contradiction.

In particular, $\mathcal{C}_R$ contains all interpretations $\mathcal{A}$ with $R^{\mathcal{A}} = \{(x, x) \mid x \in \Delta^{\mathcal{A}}\}$, i.e. interpretations in which R is interpreted as equality. Since $\mathcal{C}^n_m$ contains only finitely many pairwise inequivalent concepts and $\mathcal{C}_R$ contains interpretations of arbitrary size, there is also such an $\mathcal{A}$ such that there are two elements $x_1, x_2 \in \Delta^{\mathcal{A}}$ with $x_1 \neq x_2$ and $\mathcal{A}, x_1 \equiv^n_m \mathcal{A}, x_2$. We define $\mathcal{B}$ from $\mathcal{A}$ as follows:

$$
\begin{aligned}
\Delta^{\mathcal{B}} &:= \Delta^{\mathcal{A}}, \\
A^{\mathcal{B}} &:= A^{\mathcal{A}} \quad \text{for each } A \in N_C, \\
S^{\mathcal{B}} &:= S^{\mathcal{A}} \quad \text{for each } S \in N_R \setminus \{R\}, \\
R^{\mathcal{B}} &:= (R^{\mathcal{A}} \setminus \{(x_1, x_1), (x_2, x_2)\}) \cup \{(x_1, x_2), (x_2, x_1)\}.
\end{aligned}
$$

Since $R^{\mathcal{B}}$ is no longer reflexive, as desired $\mathcal{B} \notin \mathcal{C}_R$ holds. It remains to be shown that $\mathcal{A} \equiv^n_m \mathcal{B}$ holds. We prove this by showing that $\mathcal{A} \cong^{n+1}_m \mathcal{B}$ holds, which is equivalent to $\mathcal{A} \equiv^n_m \mathcal{B}$ by Theorem 4.

Any opening move of Player I can be answered by Player II in a way that leads to the configuration $G^{n+1}_m(\mathcal{A}, x, \mathcal{B}, x)$, where x depends on the choices of Player I. We have to show that, for any configuration of this type, Player II has a winning strategy. Since certainly $\mathcal{A}, x \cong^{n+1}_m \mathcal{A}, x$ this follows from Claim 2:

Claim 2: For all $k \leq m$: If $\mathcal{A}, x \cong^{n+1}_k \mathcal{A}, y$ then $\mathcal{A}, x \cong^{n+1}_k \mathcal{B}, y$.

Proof of Claim 2: We prove Claim 2 by induction over k. Denote Player II's strategy for the configuration $G^{n+1}_k(\mathcal{A}, x, \mathcal{A}, y)$ by S.

For $k = 0$, Claim 2 follows immediately from the construction of $\mathcal{B}$: $\mathcal{A}, x \cong^{n+1}_0 \mathcal{A}, y$ implies $\mathcal{A}, x \equiv_l \mathcal{A}, y$ and $\mathcal{A}, y \equiv_l \mathcal{B}, y$ since $\mathcal{B}$ agrees with $\mathcal{A}$ on the interpretation of all atomic concepts. It follows that $\mathcal{A}, x \equiv_l \mathcal{B}, y$, which means that Player II wins the game $G^{n+1}_0(\mathcal{A}, x, \mathcal{B}, y)$. For $0 < k \leq m$, assume that Player I selects an arbitrary structure and a legal subset of the respective domain. Player II tries to answer that move according to S which provides her with a move for the game $G^{n+1}_k(\mathcal{A}, x, \mathcal{A}, y)$. There are two possibilities:

- The move provided by S is a valid move also for the game $G^{n+1}_k(\mathcal{A}, x, \mathcal{B}, y)$: Player II can answer the choice of Player I according to S without violating the rules, which yields a configuration $G^{n+1}_{k-1}(\mathcal{A}, x', \mathcal{B}, y')$ such that for x', y' it holds that $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{A}, y'$ (because Player II moved according to S). From the induction hypothesis it follows that $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{B}, y'$.
- The move provided by S is not a valid move for the game $G^{n+1}_k(\mathcal{A}, x, \mathcal{B}, y)$ This requires a more detailed analysis: Assume Player I has chosen to move in $\mathcal{A}$ and has chosen an $S \in \overline{N_R}$ and a set X of size $l \leq n+1$ such that $xS^{\mathcal{A}}X$. Let Y be the set that Player II would choose according S. This implies that Y has also l elements and that $yS^{\mathcal{A}}Y$. That this choice is not valid in the game $G^{n+1}_k(\mathcal{A}, x, \mathcal{B}, y)$ implies that there is an element $z \in Y$ such that $(y, z) \notin S^{\mathcal{B}}$. This implies $y \in \{x_1, x_2\}$ and $S \in \{R, R^{-1}\}$, because these are the only elements and relations that are different in $\mathcal{A}$ and $\mathcal{B}$. W.l.o.g. assume $y = x_1$ and $S = R$. Then also $z = x_1$ must hold, because this is the only element such that $(x_1, z) \in R^{\mathcal{A}}$ and $(x_1, z) \notin R^{\mathcal{B}}$. Thus, the choice $Y' := (Y \setminus \{x_1\}) \cup \{x_2\}$ is a valid one for Player II in the game $G^{n+1}_m(\mathcal{A}, x, \mathcal{B}, y)$: $x_1 R^{\mathcal{B}} Y'$ and $|Y'| = l$ because $(x_1, x_2) \notin R^{\mathcal{A}}$.

  There are two possibilities for Player I to choose an element $y' \in Y'$:
  1. $y' \neq x_2$: Player II chooses $x' \in X$ according to S. This yields a configuration $G^{n+1}_{k-1}(\mathcal{A}, x', \mathcal{B}, y')$ such that $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{A}, y'$.
  2. $y' = x_2$: Player II answers with the $x' \in X$ that is the answer to the move x_1 of Player I according to S. For the obtained configuration $G^{n+1}_{k-1}(\mathcal{A}, x', \mathcal{B}, y')$ also $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{A}, y'$ holds: By the choice of x_1, x_2, $\mathcal{A}, x_1 \equiv^n_m \mathcal{A}, x_2$ is satisfied and since $k - 1 < m$ also $\mathcal{A}, x_1 \equiv^n_{k-1} \mathcal{A}, x_2$ holds which implies $\mathcal{A}, x_1 \cong^{n+1}_{k-1} \mathcal{A}, x_2$ by Theorem 4. Since Player II chose x' according to S it holds that $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{A}, x_1$ and hence $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{A}, x_2$ since $\cong^{n+1}_{k-1}$ is transitive.

In both cases we can apply the induction hypothesis which yields $\mathcal{A}, x' \cong^{n+1}_{k-1} \mathcal{B}, y'$ and hence Player II has a winning strategy for $G^{n+1}_k(\mathcal{A}, x, \mathcal{B}, y)$. The case that Player I chooses from $\mathcal{B}$ instead of $\mathcal{A}$ can be handled dually. □

By adding constructs to $\mathcal{ALCQI}$ that allow to form more complex role expressions one can obtain a DL that has the same expressive power as C^2, such a DL is presented in [Bor96]. The logic presented there has the ability to express a universal role that makes it possible to internalise both TBoxes based on terminological axioms and cardinality restrictions on concepts.

5 Conclusion

We have shown that, with a rather limited set of constructors, one can define a DL whose reasoning problems are as hard as those of C^2 without reaching the expressive power of the latter. This shows that cardinality restrictions, although interesting for knowledge representation, are inherently hard to handle algorithmically. At a first glance, this makes $\mathcal{ALCQI}$ with cardinality restrictions on concepts obsolete for knowledge representation, because C^2 delivers more expressive power at the same computational price. Yet, is is likely that a dedicated algorithm for $\mathcal{ALCQI}$ may have better average complexity than the C^2 algorithm; such an algorithm has yet to be developed. An interesting question lies in the coding of numbers: If we allow binary coding of numbers, the translation approach together with the result from [PST97] leads to a 2-NExpTime algorithm. As for C^2, it is an open question whether this additional exponential blow-up is necessary. A positive answer would settle the same question for C^2 while a proof of the negative answer might give hints how the result for C^2 might be improved.

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