

Fusions of Description Logics

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1 Motivation and outline

In order to ensure a reasonable and predictable behaviour of a Description Logic (DL) system, reasoning in the DL employed by the system should at least be decidable, and preferably of low complexity. Consequently, the expressive power of the DL in question must be restricted in an appropriate way. If the imposed restrictions are too severe, however, then the important notions of the application domain can no longer be expressed. Investigating this trade-off between the expressivity of DLs and the complexity of their inference problems has thus been one of the most important issues in DL research.

This paper investigates an approach for extending the expressivity of DLs that (in many cases) guarantees that reasoning remains decidable: the fusion of DLs. In order to explain the difference between the usual union and the fusion of DLs, let us consider a simple example. Assume that the DL \mathcal{D}_1 is \mathcal{ALC} , i.e., it provides for the Boolean operators \sqcap , \sqcup , \neg and the additional concept constructors value restriction $\forall r.C$ and existential restriction $\exists r.C$, and that the DL \mathcal{D}_2 provides for the Boolean operators and number restrictions ($\leq nr$) and ($\geq nr$). If an application requires concept constructors from both DLs for expressing its relevant concepts, then one would usually consider the *union* $\mathcal{D}_1 \cup \mathcal{D}_2$ of \mathcal{D}_1 and \mathcal{D}_2 , which allows for the unrestricted use of all constructors. For example, the concept description $C_1 := (\exists r.A) \sqcap (\exists r.\neg A) \sqcap (\leq 1r)$ is a legal $\mathcal{D}_1 \cup \mathcal{D}_2$ description. Note that this description is unsatisfiable, due to the interaction between constructors of \mathcal{D}_1 and \mathcal{D}_2 . The *fusion* $\mathcal{D}_1 \otimes \mathcal{D}_2$ of \mathcal{D}_1 and \mathcal{D}_2 prevents such interactions by imposing the following restriction: one assumes that the set of all role names is partitioned into two sets, one that can be used in constructors of \mathcal{D}_1 , and another one that can be used in constructors of \mathcal{D}_2 . Thus, the description C_1 from above is not a legal $\mathcal{D}_1 \otimes \mathcal{D}_2$ description since it uses the same role r both in the existential restrictions (which are \mathcal{D}_1 -constructors) and in the number restriction (which is a \mathcal{D}_2 -constructor). In contrast, the descriptions

$(\exists r_1.A) \sqcap (\exists r_1.\neg A) \sqcap (\leq 1r_2)$ and $(\exists r_1.(\leq 1r_2))$ are admissible in $\mathcal{D}_1 \otimes \mathcal{D}_2$ since they employ different roles in the \mathcal{D}_1 - and \mathcal{D}_2 -constructors. If the concepts that must be expressed are such that they require both constructors from \mathcal{D}_1 and \mathcal{D}_2 , but the ones from \mathcal{D}_1 for other roles than the ones from \mathcal{D}_2 , then one does not really need the union of \mathcal{D}_1 and \mathcal{D}_2 ; the fusion would be sufficient.

What is the advantage of taking the fusion instead of the union? In general, for the union of two DLs one must design new reasoning methods, whereas reasoning in the fusion can be reduced to reasoning in the component DLs. Indeed, reasoning in the union may even be undecidable whereas reasoning in the fusion is still decidable. As an example, we consider the DLs (i) \mathcal{ALCF} , which extends the basic DL \mathcal{ALC} by functional roles (features) and the same-as constructor (agreement) on chains of functional roles; and (ii) $\mathcal{ALC}^{+,o,\sqcup}$, which extends \mathcal{ALC} by transitive closure, composition, and union of roles. For both DLs, subsumption of concept descriptions is known to be decidable [9, 11, 1]. However, their *union* $\mathcal{ALCF}^{+,o,\sqcup}$ has an undecidable subsumption problem [2]. This undecidability result depends on the fact that, in $\mathcal{ALCF}^{+,o,\sqcup}$, the role constructors transitive closure, composition, and union can be applied to functional roles that also appear within the same-as constructor. This is not allowed in the fusion $\mathcal{ALCF} \otimes \mathcal{ALC}^{+,o,\sqcup}$. Of course, failure of a certain undecidability proof does not make the fusion decidable.

Why do we know that the fusion of decidable DLs is again decidable? Actually, in general we don't, and this was our main reason for writing this paper. The notion "fusion" was introduced and investigated in modal logic, basically to transfer results like finite axiomatizability, decidability, finite model property, etc. from uni-modal logics (with one pair of box and diamond operators) to multi-modal logics (with several such pairs, possibly satisfying different axioms). This has led to rather general transfer results (see, e.g., [13, 10, 8, 12] for results that concern decidability), which are sometimes restricted to so-called *normal* modal logics [6]. Since there is a close relationship between modal logics and DLs [11], it is clear that these transfer results also apply to some DLs. The question is, however, to which exactly. Some DLs allow for constructors that are not considered in modal logics (e.g., the same-as constructor mentioned above). In addition, some DL constructors that have been considered in modal logics (like qualified number restrictions $(\leq nr.C)$, $(\geq nr.C)$, which correspond to graded modalities) can easily be shown to be non-normal.

The purpose of this paper is to clarify for which DLs decidability of the component DLs transfers to their fusion. To this purpose, we will introduce so-called *abstract description systems* (ADSs), which can be seen as a common generalization of description and modal logics. We will define the fusion of ADSs, and state two theorems that say under which conditions decidability transfers from the component ADSs to their fusion. From the DL point of view, the two theorems are concerned with the following two decision problems: (i) decid-

ability of satisfiability and subsumption w.r.t. general concept inclusion axioms (Theorem 8); and (ii) decidability of satisfiability and subsumption without terminological axioms (Theorem 10). These theorems imply that decidability (for both types of problems) transfers to the fusion for most DLs considered in the literature. The main exceptions (which do not satisfy the prerequisites of the theorems) are DLs allowing for individuals (called nominals in modal logic) in concept descriptions, and DLs explicitly allowing for the universal role or for negation of roles. Results from modal logic for the second type of problems (ii) usually require the component modal logics to be normal. Our Theorem 10 is less restrictive, and thus also applies to DLs allowing for qualified number restrictions.

2 Description logics

Before defining abstract description systems, we recall the main features of DLs that must be covered by this definition. The *concept description language* may provide the following means of expressivity:

Concept constructors: We have already mentioned several of them in the previous section. They take concept and/or role descriptions and transform them into more complex concept descriptions. Concept constructors may also be nullary, like the top concept (\top) or individuals (which are just a name that must be interpreted as a singleton set).

Role constructors: We have mentioned composition, union, and transitive closure as well as role negation in the previous section. The complex role descriptions built this way can be used within concept constructors, though some restrictions may apply (e.g., in a DL with value and number restrictions one could allow the use of complex role descriptions in value, but not in number restrictions).

Restrictions on role interpretations: We have already mentioned functional roles, whose interpretation is restricted to partial functions, and the universal role, which must be interpreted as the universal relation. Other such restrictions are transitivity of roles, or inclusion relationships between roles enforced by role hierarchies.

We consider the most general form of *terminological axioms*, which are general inclusion axioms $C \sqsubseteq D$, where both C and D may be complex descriptions. A *TBox* is a finite set of such axioms. We will not consider *ABoxes* since they can be expressed using individuals in concepts. It should be noted, however, that Theorems 8 and 10 do not apply to DLs allowing for individuals. Thus we do not have transfer results for ABox reasoning.

Since all our DLs will be assumed to contain the Boolean operators \sqcap, \sqcup, \neg as concept constructors, subsumption can be reduced to satisfiability. Thus, it is sufficient to restrict the attention to satisfiability of concept descriptions. We consider the satisfiability problem both w.r.t. a TBox and without TBox (in the second case, we simply talk about satisfiability of concept descriptions).

3 Abstract description systems

In order to define the fusion of DLs and prove general results for fusions of DLs, one needs a formal definition of what are “Description Logics”. Since there exists a wide variety of DLs with very different characteristics, we introduce a very general formalization, which should cover all of the DLs considered in the literature, but also includes logics that would usually not be subsumed under the name DL.

In this formalization, concept descriptions will be represented by terms that are built using an abstract description language.

Definition 1. An *abstract description language* (ADL) is determined by a countably infinite set V of variables and a (possibly infinite) sequence $(f_i)_{i \in I}$ of functions symbols, which are equipped with arities $(n_i)_{i \in I}$. The *terms* t_j of this ADL are built using the follow syntax rules:

$$t_j \longrightarrow x, \neg t_1, t_1 \wedge t_2, t_1 \vee t_2, f_i(t_1, \dots, t_{n_i}),$$

where $x \in V$ and the Boolean operators \neg, \wedge, \vee are different from all f_i .

From the DL point of view, the variables correspond to concept names and the Boolean operators as well as the function symbols correspond to concept constructors. As an example, let us view concept descriptions of the DL \mathcal{ALCN}^\sqcap , i.e., \mathcal{ALC} extended with number restrictions and conjunction of roles, as terms of an ADL. Value restrictions and existential restrictions can be seen as unary function symbols: for each role description r , we have the function symbols $f_{\forall r}$ and $f_{\exists r}$, which take a term t_C (corresponding to the concept description C) and transform it into the more complex terms $f_{\forall r}(t_C)$ and $f_{\exists r}(t_C)$ (corresponding to the concept descriptions $\forall r.C$ and $\exists r.C$). Similarly, number restrictions can be seen as nullary function symbols: for each role description r and each $n \in \mathbb{N}$, we have the function symbols $f_{\geq nr}$ and $f_{\leq nr}$. Hence, the \mathcal{ALCN}^\sqcap concept description $A \sqcap \forall(r_1 \sqcap r_2). \neg(B \sqcap (\geq 2r_1))$ corresponds to the term $x_A \wedge f_{\forall(r_1 \sqcap r_2)}(\neg(x_B \wedge f_{(\geq 2r_1)}))$.

Other concept constructors can be translated analogously. For example, qualified number restrictions $(\leq nr.C)$, $(\geq nr.C)$ correspond to unary function symbols, individuals in concept descriptions and the same-as constructor mentioned in the motivation correspond to nullary function symbols.

The semantics of abstract description systems is defined based on abstract description models.

Definition 2. An *abstract description model* (ADM) is of the form

$$\mathfrak{W} = \langle W, F^{\mathfrak{W}} = (f_i^{\mathfrak{W}})_{i \in I} \rangle,$$

where W is a nonempty set and the $f_i^{\mathfrak{W}}$ are functions mapping every sequence $\langle X_1, \dots, X_{n_i} \rangle$ of subsets of W to a subset of W .

Since ADMs do not interpret variables, we need an assignment \mathcal{A} , which assigns a subset of W to each variable, before we can evaluate terms in an ADM.

Definition 3. Let \mathcal{L} be an ADL and $\mathfrak{W} = \langle W, F^{\mathfrak{W}} \rangle$ be an ADM for \mathcal{L} . An *assignment* for \mathfrak{W} is a mapping \mathcal{A} from the set of variables V to 2^W . The value an assignment \mathcal{A} assigns to a variable x is denoted by $x^{\mathcal{A}}$. Let \mathfrak{W} be an ADM and \mathcal{A} be an assignment for \mathfrak{W} . With each \mathcal{L} -term t , we inductively associate a value $t^{\mathfrak{W}, \mathcal{A}}$ in 2^W as follows:

- $x^{\mathfrak{W}, \mathcal{A}} := x^{\mathcal{A}}$ for all variables $x \in V$,
- $(\neg t)^{\mathfrak{W}, \mathcal{A}} := W \setminus (t)^{\mathfrak{W}, \mathcal{A}}$, $(t_1 \wedge t_2)^{\mathfrak{W}, \mathcal{A}} := t_1^{\mathfrak{W}, \mathcal{A}} \cap t_2^{\mathfrak{W}, \mathcal{A}}$, $(t_1 \vee t_2)^{\mathfrak{W}, \mathcal{A}} := t_1^{\mathfrak{W}, \mathcal{A}} \cup t_2^{\mathfrak{W}, \mathcal{A}}$,
- $f(t_1, \dots, t_k)^{\mathfrak{W}, \mathcal{A}} := f^{\mathfrak{W}}(t_1^{\mathfrak{W}, \mathcal{A}}, \dots, t_k^{\mathfrak{W}, \mathcal{A}})$.

If x_1, \dots, x_n are the variables occurring in t , then we often write $t^{\mathfrak{W}}(X_1, \dots, X_n)$ as shorthand for $t^{\mathfrak{W}, \mathcal{A}}$, where \mathcal{A} is an assignment with $x_i^{\mathcal{A}} = X_i$ for $1 \leq i \leq n$.

A model in the DL sense interprets both role and concept names. The interpretation of the role names fixes the interpretation of the function symbols corresponding to concept constructors that involve roles (like value restrictions, number restrictions, etc.). The interpretation of the concept names corresponds to an assignment. Thus, a DL model is an ADM together with an assignment, whereas an ADM alone corresponds to what is called frame in modal logics. Since individuals in DLs correspond to nullary function symbols, their interpretation must also be fixed in the ADM. We will call a non-empty domain together with an interpretation of the role names and (if any) individual names a *DL frame*. We introduce abstract description systems before giving examples of ADMs.

Definition 4. An *abstract description system* (ADS) is a pair $(\mathcal{L}, \mathcal{M})$, where \mathcal{L} is an ADL and \mathcal{M} is a class of ADMs for \mathcal{L} that is closed under isomorphic copies.

From the DL point of view, the choice of the class \mathcal{M} defines the semantics of the concept and role constructors, and it allows us, e.g., to incorporate restrictions on role interpretations. As mentioned above, an ADM is given by the interpretation of the role names, and thus one can, for example, restrict the class to ADMs that interpret a certain role as a transitive relation. Another restriction that can be realized by the choice of \mathcal{M} is that individuals (corresponding to nullary function symbols) must be interpreted as singleton sets.

As an example, let us view the DL \mathcal{ALCN}^\square as an ADS. The ADL \mathcal{L} corresponding to \mathcal{ALCN}^\square has already been discussed. Thus, we concentrate on the class of ADMs \mathcal{M} induced by the frames of \mathcal{ALCN}^\square . Assume that \mathcal{F} is such a frame, i.e., \mathcal{F} consists of a nonempty domain and interpretations $r^{\mathcal{F}}$ of the role names r . The ADM $\mathfrak{W} = \langle W, F^{\mathfrak{W}} \rangle$ induced by \mathcal{F} is defined as follows. The set W is identical to the domain of \mathcal{F} . It remains to define the interpretation of the function symbols. We illustrate this on two examples. First, consider the (unary) function symbol $f_{\forall(r_1 \sqcap r_2)}$. Given a subset X of W , the function $f_{\forall(r_1 \sqcap r_2)}^{\mathfrak{W}}$ maps X to

$$f_{\forall(r_1 \sqcap r_2)}^{\mathfrak{W}}(X) := \{w \in W \mid v \in X \text{ for all } v \text{ with } (w, v) \in r_1^{\mathcal{F}} \cap r_2^{\mathcal{F}}\},$$

i.e., the interpretation of the concept description $\forall(r_1 \sqcap r_2).A$ in the interpretations based on \mathcal{F} interpreting A by X . Accordingly, the value of the constant symbol $f_{(\geq 2r)}$ in \mathfrak{W} is given by the interpretation of $(\geq 2r)$ in the interpretations based on \mathcal{F} .

For our transfer results to hold, we must restrict ourselves to so-called local ADSs.

Definition 5. Given two ADMs $\mathfrak{W}_1 = \langle W_1, F^{\mathfrak{W}_1} \rangle$ and $\mathfrak{W}_2 = \langle W_2, F^{\mathfrak{W}_2} \rangle$ with $W_1 \cap W_2 = \emptyset$, we say that $\mathfrak{W} = \langle W, F^{\mathfrak{W}} \rangle$ is the *disjoint union* of \mathfrak{W}_1 and \mathfrak{W}_2 iff $W = W_1 \cup W_2$ and

$$f_i^{\mathfrak{W}}(X_1, \dots, X_{n_i}) = f_i^{\mathfrak{W}_1}(X_1 \cap W_1, \dots, X_{n_i} \cap W_1) \cup f_i^{\mathfrak{W}_2}(X_1 \cap W_2, \dots, X_{n_i} \cap W_2)$$

for all $X_1, \dots, X_{n_i} \subseteq W$ and $i \in I$. An ADS $S = (\mathcal{L}, \mathcal{M})$ is called *local* if the set \mathcal{M} is closed under disjoint unions.

Given two DL frames (without individuals) over disjoint domains, one can build their union by just taking the union of the domains and, for each role name, the union of its interpretations. It is easy to see that the function symbols corresponding to the usual concept constructors (such as value restrictions on role names) satisfy the requirements for the disjoint union. There are, however, also examples of DLs whose associated ADSs are not local: (i) The restriction that individuals must be interpreted as singleton sets would be violated in the disjoint union of two DL frames with individuals. (ii) If one has negation of roles, then the function symbol $f_{\neg r}$ does not satisfy the requirement for the

disjoint union. (iii) The union of two universal relations on respective domains W_1 and W_2 is not the universal relation on $W_1 \cup W_2$.

Finally, let us define the reasoning problems for which we will investigate transfer of decidability.

Definition 6. Given an ADS $(\mathcal{L}, \mathcal{M})$, the \mathcal{L} -term s is called *satisfiable* iff there exists an ADM $\mathfrak{W} \in \mathcal{M}$ and an assignment \mathcal{A} such that $s^{\mathfrak{W}, \mathcal{A}} \neq \emptyset$. We say that the *satisfiability problem* for $(\mathcal{L}, \mathcal{M})$ is decidable iff there is an algorithm that, given an \mathcal{L} -term s , answers “yes” if s is satisfiable and “no” otherwise.

An inclusion axiom is of the form $t_1 \sqsubseteq t_2$, and it is satisfied by the ADM \mathfrak{W} and the assignment \mathcal{A} iff $t_1^{\mathfrak{W}, \mathcal{A}} \sqsubseteq t_2^{\mathfrak{W}, \mathcal{A}}$. The \mathcal{L} -term s is called *satisfiable relative* to the finite set of inclusion axioms \mathcal{S} iff there exists an ADM $\mathfrak{W} \in \mathcal{M}$ and an assignment \mathcal{A} satisfying all inclusion axioms of \mathcal{S} such that $s^{\mathfrak{W}, \mathcal{A}} \neq \emptyset$. We will call this problem the *relativized satisfiability problem*.

The satisfiability problem just introduced corresponds to satisfiability of concept descriptions without TBox, whereas the relativized satisfiability problem corresponds to satisfiability of concept descriptions w.r.t. a TBox.

4 The fusion of abstract description systems

Now, we can formally define the fusion of abstract description systems.

Definition 7. The *fusion* $S_1 \otimes S_2 = (\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$ of two abstract description systems $S_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $S_2 = (\mathcal{L}_2, \mathcal{M}_2)$ over disjoint sets of function symbols $(f_i)_{i \in I}$ and $(g_j)_{j \in J}$ is defined as follows: $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the ADL based on the union f_1, \dots, g_1, \dots of the function symbols of \mathcal{L}_1 and \mathcal{L}_2 , and

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := \{ \langle W, (f_i^{\mathfrak{W}})_{i \in I} \cup (g_j^{\mathfrak{W}})_{j \in J} \rangle \mid \langle W, (f_i^{\mathfrak{W}})_{i \in I} \rangle \in \mathcal{M}_1 \text{ and } \langle W, (g_j^{\mathfrak{W}})_{j \in J} \rangle \in \mathcal{M}_2 \}.$$

As an example, consider the ADSs S_1 and S_2 corresponding to the DLs \mathcal{ALCF} and $\mathcal{ALC}^{+, \circ, \sqcup}$ introduced in the first section. The ADS S_1 is based on the following function symbols: (i) unary functions symbol $f_{\forall r}$ and $f_{\exists r}$ for every role name r , (ii) nullary functions symbols corresponding to the same-as constructor for every pair of chains of functional roles; and the ADS S_2 is based on the following function symbols: (iii) unary functions symbol $f_{\forall r}$ and $f_{\exists r}$ for every role description r built from role names using union, composition, and transitive closure. If we assume that the set of role names employed by \mathcal{ALCF} and $\mathcal{ALC}^{+, \circ, \sqcup}$ are disjoint, then these sets of function symbols are disjoint. The union of these sets provides us both with the symbols for the same-as constructor and with the symbols for value and existential restrictions on role descriptions involving union, composition, and transitive closure. However, the

role descriptions contain only role names from $\mathcal{ALC}^{+,o,\sqcup}$, and thus none of the functional roles of \mathcal{ALCF} occurs in such descriptions. Thus, the fusion of \mathcal{ALCF} and $\mathcal{ALC}^{+,o,\sqcup}$ yields a strict fragment of their union $\mathcal{ALCF}^{+,o,\sqcup}$.

Our first transfer result [3] is concerned with the relativized satisfiability problem.

Theorem 8. *Let S_1 and S_2 be local ADSs, and suppose that the relativized satisfiability problems for S_1 and S_2 are decidable. Then the relativized satisfiability problem for $S_1 \otimes S_2$ is also decidable.*

In the next section we will give an example for the application of this theorem. Note that this theorem does not yield a transfer result for the (unrelativized) satisfiability problem. Of course, if the relativized satisfiability problems for S_1 and S_2 are decidable, then the theorem implies that the satisfiability problem for $S_1 \otimes S_2$ is also decidable (since it is a special case of the relativized satisfiability problem). However, to be able to apply Theorem 8 to obtain decidability of the satisfiability problem in the fusion, the component ADSs must satisfy the stronger requirement that the relativized satisfiability problem is decidable, which may not always be the case. For example, the theorem cannot be applied for the fusion of \mathcal{ALCF} and $\mathcal{ALC}^{+,o,\sqcup}$ since the relativized satisfiability problem for \mathcal{ALCF} is already undecidable [2]. However, the satisfiability problem is decidable for both DLs. Before we can formulate a transfer result for the satisfiability problem, we need to introduce an additional notion, which generalizes the notion of a normal modal logics.

Definition 9. Let $(\mathcal{L}, \mathcal{M})$ be an ADS and f be a function symbol of \mathcal{L} of arity n . The term $t_f(x)$ (with one variable x) is a *covering normal term* for f iff the following holds for all $\mathfrak{W} \in \mathcal{M}$:

- $t_f^{\mathfrak{W}}(W) = W$,
- for all $X, Y \subseteq W$, $t_f^{\mathfrak{W}}(X \cap Y) = t_f^{\mathfrak{W}}(X) \cap t_f^{\mathfrak{W}}(Y)$,
- for all $X, X_1, \dots, Y_n \subseteq W$: $X \cap X_i = X \cap Y_i$ for $1 \leq i \leq n$ implies $t_f^{\mathfrak{W}}(X) \cap f^{\mathfrak{W}}(X_1, \dots, X_n) = t_f^{\mathfrak{W}}(X) \cap f^{\mathfrak{W}}(Y_1, \dots, Y_n)$.

An ADS $(\mathcal{L}, \mathcal{M})$ is said to *have covering normal terms* iff one can effectively determine a covering normal term t_f for every f of \mathcal{L} .

For example, consider the term $f_{\forall r}(x)$, where $f_{\forall r}$ is the function symbol corresponding to the value restriction constructor for the role r . Then $f_{\forall r}(x)$ obviously satisfies the first two requirements for covering normal terms. In fact, it is easy to see that $f_{\forall r}(x)$ is a covering normal term for the function symbols corresponding to the value, existential, and (qualified) number restrictions on the role r .

Theorem 10. *Let S_1 and S_2 be local ADSs having covering normal terms, and suppose that the satisfiability problems for S_1 and S_2 are decidable. Then the satisfiability problem for $S_1 \otimes S_2$ is also decidable.*

It is easy to see that the prerequisites for this second theorem are satisfied by \mathcal{ALCF} and $\mathcal{ALC}^{+,o,\sqcup}$, and thus satisfiability is decidable for their fusion.

5 Transfer results for the fusion of DLs

Let us start with recalling for which DLs our transfer theorems do not apply. As mentioned above, the locality requirement is violated by DLs providing for individuals in concept descriptions, role negation, or the universal role. Since the Boolean operators are always available in ADLs, we can only treat DLs whose set of concept constructors is Boolean closed. Finally, to get covering normal terms, we need value restrictions. Taking the last two points together means that we consider extensions of \mathcal{ALC} . We can show that the transfer results apply to any DL that extends \mathcal{ALC} by some of the following:

- the concept constructors number restriction, qualified number restriction, feature agreement and disagreement, and concrete domain predicates,
- the role constructors composition, conjunction, disjunction, converse, and transitive closure,
- the restrictions on role interpretations that respectively enforce functional roles, transitive roles, and role hierarchies.

The above list is not exhaustive; it is just intended to give an impression of the generality of the transfer results.

We can now give an example for the application of Theorem 8. Consider the DLs (i) \mathcal{ALCN} , which extends \mathcal{ALC} by (unqualified) number restrictions, and (ii) $\mathcal{ALC}^{o,\sqcap}$, which extends \mathcal{ALC} by composition and conjunction of roles. For both DLs, satisfiability w.r.t. TBoxes is decidable [5, 7], whereas for the union $\mathcal{ALCN}^{o,\sqcap}$ of the two logics, satisfiability is already undecidable [4]. As in our previous examples, we can take the fusion $\mathcal{ALCN} \otimes \mathcal{ALC}^{o,\sqcap}$ to obtain a decidable fragment of the undecidable union $\mathcal{ALCN}^{o,\sqcap}$. It is easy to see that this fragment is obtained from $\mathcal{ALCN}^{o,\sqcap}$ by restricting the roles in number restrictions to role names that do not occur in complex role descriptions.

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