The Complexity of Reasoning with Boolean Modal Logics

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ABSTRACT. In this paper, we investigate the complexity of reasoning with various Boolean Modal Logics. The main results are that (i) adding negation of modal parameters to (multi-modal) **K** makes reasoning ExpTime-complete and (ii) adding atomic negation and conjunction to **K** even yields a NExpTime-complete logic. The last result is relativized by the fact that it depends on an infinite number of modal parameters to be available. If the number of modal parameters is bounded, full Boolean Modal Logic becomes ExpTime-complete.

1 Motivation

Since Modal Logics are an extension of Propositional Logic, they provide Boolean operators for constructing complex formulae. However, most Modal Logics do not admit Boolean operators for constructing *complex modal parameters* to be used in the box and diamond operators. This asymmetry is not present in Boolean Modal Logics, in which box and diamond quantify over arbitrary Boolean combinations of atomic modal parameters; see Gargov and Passy 1987. Boolean Modal Logics have been considered in various forms and contexts:

1. "Pure" Boolean Modal Logic has been studied in Gargov and Passy 1987. Negation and intersection of modal parameters occur in some variants of Propositional Dynamic Logic, see, e.g., Danecki 1984, Harel 1984, Passy and Tinchev 1991.

2. The modal box operator can be thought of as expressing necessity. More precisely, when employing the usual Kripke Semantics, $\Box\varphi$ holds at a world w iff w' being accessible from w implies that φ holds at w'. Given this, it is obviously quite natural to define a symmetric

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operator $\boxbar$ (sometimes called "window operator") such that $\boxbar\varphi$ holds at a world w iff φ holding at a world w' implies that w' is accessible from w. Obviously, the window operator can be thought of as expressing sufficiency. Logics with this operator were investigated from different viewpoints by, e.g., Humberstone, Gargov et al., and Goranko Humberstone 1983, Gargov et al. 1987, Goranko 1987, Goranko 1990. If negation of modal parameters is available, the window operator comes for free since we can write $\boxbar_R\,\varphi$ as $[\neg R]\neg\varphi$.

3. There are several Description Logics that provide "negation of roles" which corresponds to the negation of modal parameters, see, e.g. Hustadt and Schmidt 2000. Union and intersection of modal parameters are also considered in Description Logics and other KR formalisms, as is the window operator; see Givan et al. 1991, Lutz and Sattler 2000a.

Although—as we just argued—logics involving Boolean operators on modal parameters or the window operator are widely used, to the best of our knowledge, complexity results for this class of logics have never been obtained. In this paper, we close the gap and determine the complexity of the satisfiability and validity problems for many Boolean Modal Logics. In the first part of this paper (Sections 2 and 3), we investigate the logic $\mathbf{K}_\omega$ ($\mathbf{K}$ with a countably infinite number of accessibility relations) enriched with negation of modal parameters and show that the afore mentioned inference problems are ExpTime-complete using an automata-theoretic approach. We then demonstrate the generality of our approach by extending this result to the logic $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, i.e., to the fusion of $\mathbf{K}_\omega$ with $\mathbf{K4}_\omega$ enriched with negation on relations. In the second part of this paper (Sections 4 and 5), we add other Boolean operators on roles. In doing so, one has the choice to either restrict negation to atomic relations or to allow for full negation of relations.

We give a complete list of complexity results for the logics obtained in this way, the central result being that the combination of (atomic) negation with intersection yields a logic whose inference problems are NExpTime-complete. The lower bound is obtained by a reduction of a NExpTime-complete variant of the domino problem. The mentioned result obviously implies that full Boolean Modal Logic $\mathbf{K}_\omega^{\neg,\cap,\cup}$ is also NExpTime-complete. However, the lower bound crucially depends on the number of relations to be unbounded. Inspired by this observation, in Section 5, we supplement our result by showing that, for any fixed finite number of relations, full Boolean Modal Logic is ExpTime-complete. The upper bound is proved by a reduction to multi-modal $\mathbf{K}$ (with finitely many relations) enriched with the universal modality.

To complete our investigation, in Section 6 we show that $\mathbf{K}_\omega$ with union and intersection of roles and without negation is of the same complexity as pure $\mathbf{K}_\omega$, i.e., PSpace-complete. Summing up, we thus have tight complexity bounds for $\mathbf{K}_\omega$ extended with any combination of Boolean operators on roles. This paper is accompanied by a technical report which contains all proofs and technical details (Lutz and Sattler 2000b).

2 Preliminaries

We define syntax and semantics of $\mathbf{K}_\omega^\neg$, introduce looping automata, and discuss some model- and complexity-theoretic properties of $\mathbf{K}_\omega^\neg$.

Definition 2.1 Given a countably infinite set of *propositional variables* Φ and a countably infinite set of *atomic modal parameters* $R_1, R_2, \ldots$, the set of $\mathbf{K}_\omega^\neg$*-formulae* is the smallest set that (i) contains the propositional variables in Φ, (ii) is closed under Boolean connectives $\wedge$, $\vee$, and $\neg$, and (iii) if it contains φ, then it also contains $\langle R_i\rangle\,\varphi$, $[R_i]\,\varphi$, $\langle\neg R_i\rangle\,\varphi$, and $[\neg R_i]\,\varphi$ for $i \geq 1$. The set of $\mathbf{K}_\omega^\neg$-modal parameters is the smallest set containing all atomic modal parameters and their negations (i.e., expressions of the form $\neg R_i$).

$\mathbf{K}_\omega^\neg$ semantics is given by *Kripke structures* $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots\rangle$, where W is a set of *worlds*, π is a mapping from the set of propositional variables into sets of worlds (i.e., for each $p \in \Phi$, $\pi(p)$ is the set of worlds in which p holds), and $\mathcal{R}_i$ is a binary relation on the worlds W, the so-called *accessibility relation* for the atomic modal parameter R_i.

The semantics is then given as follows, where, for a $\mathbf{K}_\omega^\neg$-formula φ, a Kripke structure $\mathcal{M}$, and a world $w \in W$; the expression $\mathcal{M}, w \models \varphi$ is read as "φ holds in $\mathcal{M}$ in world w".

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & w \in \pi(p) \qquad \text{for } p \in \Phi \\
\mathcal{M}, w \models \varphi_1 \wedge \varphi_2 & \text{iff} & \mathcal{M}, w \models \varphi_1 \text{ and } \mathcal{M}, w \models \varphi_2 \\
\mathcal{M}, w \models \varphi_1 \vee \varphi_2 & \text{iff} & \mathcal{M}, w \models \varphi_1 \text{ or } \mathcal{M}, w \models \varphi_2 \\
\mathcal{M}, w \models \neg\varphi & \text{iff} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \langle R_i\rangle\,\varphi & \text{iff} & \exists w' \in W \text{ with } (w, w') \in \mathcal{R}_i \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [R_i]\,\varphi & \text{iff} & \forall w' \in W, \text{ if } (w, w') \in \mathcal{R}_i, \text{ then } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models \langle\neg R_i\rangle\,\varphi & \text{iff} & \exists w' \in W \text{ with } (w, w') \notin \mathcal{R}_i \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [\neg R_i]\,\varphi & \text{iff} & \forall w' \in W, \text{ if } (w, w') \notin \mathcal{R}_i, \text{ then } \mathcal{M}, w' \models \varphi
\end{array}
$$

A $\mathbf{K}_\omega^\neg$-formula φ is *satisfiable* iff there is a Kripke structure $\mathcal{M}$ with a set of worlds W and a world $w \in W$ such that $\mathcal{M}, w \models \varphi$. Such a structure is called a *model* of φ. Two $\mathbf{K}_\omega^\neg$-formulae φ and ψ are *equivalent* (written $\varphi \equiv \psi$) iff $\mathcal{M}, w \models \varphi \iff \mathcal{M}, w \models \psi$ for all Kripke structures $\mathcal{M}$ with set of worlds W and all worlds $w \in W$. Let R be a modal parameter. We write $\mathcal{M}, (w, w') \models R$ to express that (i) $(w, w') \in \mathcal{R}_i$ if R is an atomic modal parameter R_i and (ii) $(w, w') \notin \mathcal{R}_i$ if $R = \neg R_i$ for an atomic modal parameter R_i.

Throughout this paper, we denote modal parameters by R and S. For the sake of brevity, we will often omit the word "modal" when talking about modal parameters. As usual, we write $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. The semantics of the window operator discussed in the motivation can formally be defined as follows:

$$\mathcal{M}, w \models \boxminus_{R_i} \varphi \quad \text{iff} \quad \text{for all } w' \in W, \text{ if } \mathcal{M}, w' \models \varphi, \text{ then } (w, w') \in \mathcal{R}_i$$

It is easy to see that $\boxminus_{R_i} \varphi \equiv [\neg R_i]\neg\varphi$, and, hence, the window operator is expressible in $\mathbf{K}_\omega^\neg$.

It is not hard to see that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is ExpTime-hard and in NExpTime: (i) the logic $\mathbf{K}^u$, i.e., uni-modal $\mathbf{K}$ enriched with the universal modality, is a fragment of $\mathbf{K}_\omega^\neg$: Just replace

- every occurrence of $[u]\,\varphi$ by $[R]\,\varphi \wedge [\neg R]\,\varphi$ and
- every occurrence of $\langle u\rangle\,\varphi$ by $\langle R\rangle\,\varphi \vee \langle\neg R\rangle\,\varphi$

where $[u]$ and $\langle u\rangle$ denote the universal modality, and R is an arbitrary atomic modal parameter. This translation may clearly lead to an exponential blowup in the formula. However, in the class of formulae used to prove the ExpTime-hardness of $\mathbf{K}^u$ in Spaan 1993, $[u]$ occurs only once, and $\langle u\rangle$ does not occur. In this case, the translation is linear, and, thus, satisfiability of $\mathbf{K}_\omega^\neg$-concepts is ExpTime-hard; (ii) when using the standard translation of modal formulae into first order formulae (see, e.g, Blackburn et al. 2001, van Benthem 1983), $\mathbf{K}_\omega^\neg$-formulae are translated to first-order formulae with at most 2 variables. Since L^2, the two-variable fragment of first-order logic, is decidable in NExpTime (Grädel et al. 1997), this implies that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is also in NExpTime. However, these two complexity bounds are obviously not tight. One main contribution of this paper is to give an ExpTime-algorithm for the satisfiability of $\mathbf{K}_\omega^\neg$-formulae, thus tightening the complexity bounds.

For devising a satisfiability algorithm, it is interesting to know what kind of models need to be considered. In Gargov et al. 1987, it is proved that $\mathbf{K}_\omega^\neg$ has the finite model property. $\mathbf{K}_\omega^\neg$ does not have the tree model property since, e.g., the formula $p \wedge [\neg R]\neg p$ has no tree model. However, we will show that there exists a one-to-one correspondence between models and so-called Hintikka-trees which we then use to decide satisfiability (and thus validity) of $\mathbf{K}_\omega^\neg$-formulae. We do this by building, for each $\mathbf{K}_\omega^\neg$-formula φ, a looping automaton $\mathcal{A}_\varphi$ which accepts the empty (tree-) language iff φ is unsatisfiable. Hence we introduce trees, looping automata, and the language they accept here.

Definition 2.2 Let M be a set and $k \geq 1$. A *k-ary M-tree* is a mapping $T : \{1, \ldots, k\}^* \mapsto M$ that labels each node $\alpha \in \{1, \ldots, k\}^*$ with $T(\alpha) \in M$. Intuitively, the node αi is the i-th child of α. We use ϵ to denote the empty word (corresponding to the root of the tree).

A *looping automaton* $\mathcal{A} = (Q, M, I, \Delta)$ for k-ary M-trees is defined by a set Q of states, an alphabet M, a subset $I \subseteq Q$ of initial states, and a transition relation $\Delta \subseteq Q \times M \times Q^k$. A *run* of $\mathcal{A}$ on an M-tree T is a mapping $r : \{1, \ldots, k\}^* \mapsto Q$ with $(r(\alpha), T(\alpha), r(\alpha 1), \ldots, r(\alpha k)) \in \Delta$ for each $\alpha \in \{1, \ldots, k\}^*$.

A looping automaton accepts all those M-trees for which a run exists, i.e., the language $\mathcal{L}(\mathcal{A})$ of M-trees accepted by $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) = \{T \mid \text{There is a run from } \mathcal{A} \text{ on } T\}.$$

Since looping automata are special Büchi automata, emptiness of their language can effectively be tested using the well-known (quadratic) emptiness test for Büchi-automata Vardi and Wolper 1986. However, for looping automata, this algorithm can be specialized into a simpler (linear) one.

3 Negation of Modal Parameters

We show that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is decidable in exponential time. For this purpose, we first abstract from models of $\mathbf{K}_\omega^\neg$-formulae to Hintikka-trees, and then show how to construct a looping automaton that accepts exactly Hintikka-trees.

Notation: We assume all formulae to be in negation normal form (NNF), i.e., negation occurs only in front of atomic parameters and propositional variables. Each formula can easily be transformed into an equivalent one in NNF by pushing negation inwards, employing de Morgan's law and the duality between $[R]$ and $\langle R \rangle$ and between $[\neg R]$ and $\langle \neg R \rangle$. We use $\bar{\varphi}$ to denote the NNF of $\neg \varphi$.

Since we treat modalities with negated and unnegated modal parameters symmetrically, we introduce the notion

$$\langle \bar{R} \rangle \varphi = \begin{cases} \langle \neg R \rangle \varphi & \text{if } R \text{ is atomic,} \\ \langle S \rangle \varphi & \text{if } R = \neg S \text{ for some atomic parameter } S \end{cases}$$

and analogously $[\bar{R}] \varphi$. Let $\mathsf{cl}(\varphi)$ denote the set of φ's subformulae and the NNFs of their negations, i.e.,

$$\begin{aligned} \mathsf{cl}(\varphi) := \{\psi \mid \; & \psi \text{ is a subformula of } \varphi \text{ or} \\ & \psi = \bar{\rho} \text{ for a subformula } \rho \text{ of } \varphi\}. \end{aligned}$$

Obviously, the cardinality of $\mathsf{cl}(\varphi)$ is linear in the length of φ. We assume that diamond-formulae $\langle R \rangle \psi$ in $\mathsf{cl}(\varphi)$ are linearly ordered, and that $\mathcal{D}(i)$

yields the i-th diamond-formula in $\mathsf{cl}(\varphi)$.

Definition 3.1 Hintikka-set and Hintikka-tree

Let φ be a $\mathbf{K}_\omega^{\neg}$-formula and k the number of diamond-formulae in $\mathsf{cl}(\varphi)$. A set $\Psi \subseteq \mathsf{cl}(\varphi)$ is a *Hintikka-set* iff it satisfies the following conditions:

(H1) if $\varphi_1 \wedge \varphi_2 \in \Psi$, then $\{\varphi_1, \varphi_2\} \subseteq \Psi$,
(H2) if $\varphi_1 \vee \varphi_2 \in \Psi$, then $\{\varphi_1, \varphi_2\} \cap \Psi \neq \emptyset$, and
(H3) $\{\psi, \bar{\psi}\} \not\subseteq \Psi$ for all $\mathbf{K}_\omega^{\neg}$-formulae ψ.

A k-ary $2^{\mathsf{cl}(\varphi)}$-tree T is a *Hintikka-tree for* φ iff $T(\alpha)$ is a Hintikka-set for each node α in T, and T satisfies, for all nodes $\alpha, \beta \in \{1, \ldots, k\}^*$, the following conditions:

(H4) $\varphi \in T(\epsilon)$,
(H5) if $\{\langle R \rangle \psi, [R]\rho_1, \ldots, [R]\rho_m\} \subseteq T(\alpha)$ and $\mathcal{D}(i) = \langle R \rangle \psi$, then $\{\psi, \rho_1, \ldots, \rho_m\} \subseteq T(\alpha i)$
(H6) if $\mathcal{D}(i) \notin T(\alpha)$, then $T(\alpha i) = \emptyset$,
(H7) if $[R]\rho \in T(\alpha)$, then $\rho \in T(\beta)$, $\bar{\rho} \in T(\beta)$, or $T(\beta) = \emptyset$, and
(H8) if $\{[R]\rho, [\bar{R}]\psi\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$, then $\psi \in T(\beta)$.

For **(H5)**, **(H7)**, and **(H8)**, recall that R denotes an atomic parameter or the negation of an atomic parameter. The following lemma shows the connection between models and Hintikka trees.

Lemma 3.2 *A* $\mathbf{K}_\omega^{\neg}$*-formula* φ *is satisfiable iff* φ *has a Hintikka-tree.*

Thus, we have that Hintikka-trees are appropriate abstractions of models of $\mathbf{K}_\omega^{\neg}$-formulae. Hintikka-trees enjoy the nice property that they are trees, and we can thus define, for a $\mathbf{K}_\omega^{\neg}$-formula φ, a tree-automaton $\mathcal{A}_\varphi$ that accepts exactly the Hintikka-trees for φ.

Definition 3.3 For a $\mathbf{K}_\omega^{\neg}$-formula φ with k diamond-formulae in $\mathsf{cl}(\varphi)$, the looping automaton $\mathcal{A}_\varphi = (Q, 2^{\mathsf{cl}(\varphi)}, \Delta, I)$ is defined as follows:

- Let $P = \{\{[R]\psi, [\bar{R}]\rho\} \mid [R]\psi, [\bar{R}]\rho \in \mathsf{cl}(\varphi)\}$,
  $S = \{[R]\psi \mid [R]\psi \in \mathsf{cl}(\varphi)\}$,
  Q is the set of all those elements (Ψ, p, s) of
  $$\{\Psi \in 2^{\mathsf{cl}(\varphi)} \mid \Psi \text{ is a Hintikka-set}\} \times 2^P \times 2^S$$
  satisfying the following conditions:
  1. if $\{[R]\rho, [\bar{R}]\psi\} \in p$ and $\bar{\rho} \in \Psi$, then $\psi \in \Psi$,
  2. if $[R]\rho \in s$, then $\Psi = \emptyset$ or $\{\rho, \bar{\rho}\} \cap \Psi \neq \emptyset$,
  3. if $[R]\rho \in \Psi$, then $[R]\rho \in s$, and

4. if $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq \Psi$, then $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$.

- $I = \{(\Psi, p, s) \mid \varphi \in \Psi\}$.
- $((\Psi, p, s), \Psi', (\Psi_1, p_1, s_1), \ldots, (\Psi_k, p_k, s_k)) \in \Delta$ iff

$\Psi = \Psi'$, $\quad p_i = p$, $\quad s_i = s$ for all $1 \leq i \leq k$, and
$\mathcal{D}(i) = \langle R \rangle\,\psi \in \Psi$ implies $\psi \in \Psi_i$ and $\rho \in \Psi_i$ for each $[R]\,\rho \in \Psi$
$\mathcal{D}(i) = \langle R \rangle\,\psi \notin \Psi$ implies $\Psi_i = \emptyset$.

Note that, since $\mathcal{A}_\varphi$ is a looping automata, every run is accepting. As a consequence of the following lemma and Lemma 3.2, we can reduce satisfiability of $\mathbf{K}_\omega^\neg$-formulae to the emptyness of the language accepted by looping automata.

Lemma 3.4 *T is a Hintikka-tree for a $\mathbf{K}_\omega^\neg$-formula φ iff $T \in \mathcal{L}(\mathcal{A}_\varphi)$.*

Obviously, the cardinality of $\mathsf{cl}(\varphi)$ is linear in the length of φ. Hence, by definition of $\mathcal{A}_\varphi$, the cardinality of each component of $\mathcal{A}_\varphi$ is exponential in the length of φ, and thus the size of $\mathcal{A}_\varphi$ is also exponential in the length of φ. This fact together with Lemma 3.2, Lemma 3.4, and the fact that emptiness of the language accepted by a looping automaton $\mathcal{A}_\varphi$ can be tested in time linear in the size of $\mathcal{A}_\varphi$ (Vardi and Wolper 1986) implies that satisfiability of $\mathbf{K}_\omega^\neg$-concepts is in ExpTime. We already noted in Section 2 that satisfiability of $\mathbf{K}_\omega^\neg$-concepts is also ExpTime-hard and, hence, we obtain the following theorem:

Theorem 3.5 *Satisfiability of $\mathbf{K}_\omega^\neg$-formulae is ExpTime-complete.*

Is our approach still of use if we replace $\mathbf{K}_\omega$ by some logic with a restricted class of frames? In the following, we perform a case study by extending the presented algorithm to deal with $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formulae, where $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ denotes the fusion of $\mathbf{K}_\omega$ and $\mathbf{K4}_\omega$ enriched with the negation of modal parameters. More precisely, $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ provides two disjoint sets of atomic modal parameters $R_1, R_2, \ldots$ and $S_1, S_2, \ldots,$ where the latter are called *transitive modal parameters*. Moreover, accessibility relations corresponding to transitive modal parameters are required to be transitive. Apart from demonstrating the generality of our approach, the logic $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ is very natural if viewed as a Description Logic (Lutz and Sattler 2000b, Sattler 1996).

To define Hintikka trees for $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, we introduce counterparts for **(H5)** and **(H8)** which deal with transitive modal parameters.

Definition 3.6 A $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-Hintikka-tree is a Hintikka-tree as in Definition 3.1 extended with the following two conditions:

(H5b) if, for a trans. parameter S_j, we have $\{\langle S_j \rangle \psi, [S_j] \rho_1, \ldots, [S_j] \rho_m\} \subseteq T(\alpha)$ and $\mathcal{D}(i) = \langle S_j \rangle \psi$, then $\{\psi, \rho_1, \ldots, \rho_m, [S_j] \rho_1, \ldots, [S_j] \rho_m\} \subseteq T(\alpha i)$.

(H8b) if, for a trans. parameter S_j, we have $\{[S_j] \psi, [\neg S_j] \rho\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$, then $\{[S_j] \psi, \psi\} \subseteq T(\beta)$.

We can now "lift" Lemma 3.2 to the $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ case.

Lemma 3.7 *A $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ is satisfiable iff φ has a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-Hintikka-tree.*

It remains to construct a looping automaton that accepts exactly the Hintikka-trees for a given $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ. This construction is the same as the one in Defintion 3.3, with an additional fifth condition in the definition of Q as a translation of **(H8b)**, and an additional implication in the definition of Δ as a translation of **(H5b)**.

Definition 3.8 Let $\mathcal{A}_\varphi = (Q, 2^{\mathsf{cl}(\varphi)}, \Delta, I)$ be the looping automaton corresponding to a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ as defined in Lemma 3.3. Define a new looping automaton $\mathcal{A}'_\varphi := (Q', 2^{\mathsf{cl}(\varphi)}, \Delta', I)$ by setting

- Q' to the maximal subset of Q such that, for all $(\Psi, p, s) \in Q'$, if $\{[S_j] \psi, [\neg S_j] \rho\} \in p$ and $\bar{\rho} \in \Psi$ for a transitive parameter S_j, then $\{\psi, [S_j] \psi\} \in \Psi$.
- Δ' to the maximal subset of Q such that, for all

$$((\Psi, p, s), \Psi', (\Psi_1, p_1, s_1), \ldots, (\Psi_k, p_k, s_k)) \in \Delta',$$

  if $\mathcal{D}(i) = \langle S_j \rangle \psi \in \Psi$ for a transitive parameter S, then $[S_j] \rho \in \Psi_i$ for each $[S_j] \rho \in \Psi$.

Proving an analogon to Lemma 3.4, we obtain the following theorem:

Theorem 3.9 *Satisfiability of $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formulae is ExpTime-complete.*

4 Adding Intersection and Union of Modal Parameters

In this section, we investigate the complexity of adding intersection and union of modal parameters to the logic $\mathbf{K}_\omega^\neg$. In doing this, one has the choice to either restrict the applicability of negation to atomic modal parameters or allowing for full negation w.r.t. modal parameters. In the latter case, adding union is obviously equivalent to adding intersection or both. We start with the smallest extension, i.e., we add either intersection or union on modal parameters while restricting negation to atomic parameters.

Definition 4.1 A $\mathbf{K}_\omega^{(\neg),\cup}$-formula ($\mathbf{K}_\omega^{(\neg),\cap}$-formula) is a $\mathbf{K}_\omega^{\neg}$-formula which, additionally, allows for modal parameters of the form $S_1 \cup \cdots \cup S_k$ ($S_1 \cap \cdots \cap S_k$), where each S_i is an atomic or a negated atomic parameter. The semantics of the new modal operators is defined as follows:

$$\begin{array}{lll}
\mathcal{M}, w \models \langle S_1 \cup \cdots \cup S_k \rangle\, \varphi & \text{iff} & \exists w' \in W \text{ with } \mathcal{M}, (w, w') \models S_i \text{ for} \\
 & & \text{some } i \in \{1, \ldots, k\} \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [S_1 \cup \cdots \cup S_k]\, \varphi & \text{iff} & \forall w' \in W, \text{ if } \mathcal{M}, (w, w') \models S_i \text{ for} \\
 & & \text{some } i \in \{1, \ldots, k\}, \text{ then } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models \langle S_1 \cap \cdots \cap S_k \rangle\, \varphi & \text{iff} & \exists w' \in W \text{ with } \mathcal{M}, (w, w') \models S_i \\
 & & \text{for all } 1 \leq i \leq k \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [S_1 \cap \cdots \cap S_k]\, \varphi & \text{iff} & \forall w' \in W, \text{ if } \mathcal{M}, (w, w') \models S_i \\
 & & \text{for all } 1 \leq i \leq k, \text{ then } \mathcal{M}, w' \models \varphi
\end{array}$$

Let us first investigate the logic $\mathbf{K}_\omega^{(\neg),\cup}$. It is not hard to see that

$$\begin{array}{rcl}
[S_1 \cup \cdots \cup S_k]\, \varphi & \equiv & [S_1]\, \varphi \wedge \cdots \wedge [S_k]\, \varphi \text{ and} \\
\langle S_1 \cup \cdots \cup S_k \rangle\, \varphi & \equiv & \langle S_1 \rangle\, \varphi \vee \cdots \vee [S_k]\, \varphi,
\end{array}$$

i.e., satisfiability of $\mathbf{K}_\omega^{(\neg),\cup}$-formulae can be reduced to satisfiability of $\mathbf{K}_\omega^{\neg}$-formulae. However, this naive reduction might lead to an exponential blow-up of the formula. In order to avoid this blow-up, we can proceed as follows to transform a $\mathbf{K}_\omega^{(\neg),\cup}$-formula ψ into an equivalent $\mathbf{K}_\omega^{\neg}$-formula $\widehat{\psi}$ whose length is linear in the length of ψ: As the first step, recursively apply the following substitutions to ψ from the inside to the outside (i.e., no union on modal parameters occurs in φ)

$$\begin{array}{rcl}
[S_1 \cup \cdots \cup S_k]\, \varphi & \rightsquigarrow & [S_1]\, p_\varphi \wedge \cdots \wedge [S_k]\, p_\varphi \text{ and} \\
\langle S_1 \cup \cdots \cup S_k \rangle\, \varphi & \rightsquigarrow & \langle S_1 \rangle\, p_\varphi \vee \cdots \vee [S_k]\, p_\varphi
\end{array}$$

where p_φ is a new propositional variable. Call the result of these substitutions ψ'. Secondly, use a new modal parameter R and define

$$\widehat{\psi} \quad := \quad \psi' \wedge \bigwedge_{p_\varphi \text{ occurs in } \psi'} [R](p_\varphi \leftrightarrow \varphi) \wedge [\neg R](p_\varphi \leftrightarrow \varphi)$$

It can easily be seen that this gives the following result.

Theorem 4.2 *Satisfiability of $\mathbf{K}_\omega^{(\neg),\cup}$-formulae is ExpTime-complete.*

Next, we show that satisfiability of $\mathbf{K}_\omega^{(\neg),\cap}$-formulae is NExpTime-hard. The proof is given by a reduction of a NExpTime-complete variant of the well-known, undecidable domino problem.

A domino problem (Berger 1966, Knuth 1968) is given by a finite set of *domino types*. All domino types are of the same size, each type has a quadratic shape and colored edges. Of each type, an unlimited number of dominoe is available. The problem in the original domino problem is to arrange these dominoe to cover the plane without holes or overlapping such that adjacent dominoe have identical colors on their touching edges (rotation of the dominoe is not allowed). In the NExpTime-complete variant of the domino problem that we use, the task is not to tile the whole plane, but to tile a $2^{n+1} \times 2^{n+1}$-torus, i.e., a $2^{n+1} \times 2^{n+1}$-rectangle whose edges are “glued” together. See, e.g., Berger 1966, Knuth 1968 for undecidable versions of the domino problem and Börger et al. 1997 for bounded variants.

Definition 4.3 Let $\mathcal{D} = (D, H, V)$ be a *domino system*, where D is a finite set of *domino types* and $H, V \subseteq D \times D$ represent the horizontal and vertical matching conditions. For $s, t \in \mathbb{N}$, let $U(s,t)$ be the torus $\mathbb{Z}_s \times \mathbb{Z}_t$, where $\mathbb{Z}_n$ denotes the set $\{0, \ldots, n-1\}$. Let $a = a_0, \ldots, a_{n-1}$ be an n-tuple of dominoe (with $n \leq s$). We say that $\mathcal{D}$ *tiles* $U(s,t)$ *with initial condition* a iff there exists a mapping $\tau : U(s,t) \to D$ such that, for all $(x, y) \in U(s,t)$:

- if $\tau(x,y) = d$ and $\tau(x \oplus_s 1, y) = d'$, then $(d, d') \in H$
- if $\tau(x,y) = d$ and $\tau(x, y \oplus_t 1) = d'$, then $(d, d') \in V$
- $\tau(i, 0) = a_i$ for $0 \leq i < n$.

where $\oplus_n$ denotes addition modulo n. Such a mapping τ is called a *solution* for $\mathcal{D}$ w.r.t. a.

The following is a consequence of Theorem 6.1.2 in Börger et al. 1997 (see also Lutz and Sattler 2000b).

Theorem 4.4 *There exists a domino system $\mathcal{D}$ such that the following is a NExpTime-hard problem: Given an initial condition $a = a_0 \cdots a_{n-1}$ of length n, does $\mathcal{D}$ tile the torus $U(2^{n+1}, 2^{n+1})$ with initial condition a?*

We reduce the NExpTime-complete variant of the domino problem from Theorem 4.4 to the satisfiability of $\mathbf{K}_\omega^{(\neg),\cap}$-formulae. Given a domino system $\mathcal{D} = (D, H, V)$ and an initial condition $a = a_0, \ldots, a_{n-1}$, we define a reduction formula $\varphi_{(\mathcal{D},a)}$ such that $\varphi_{(\mathcal{D},a)}$ is satisfiable iff $\mathcal{D}$ tiles the torus $U(2^{n+1}, 2^{n+1})$ with initial condition a. The subformulae of the reduction formula

$$\varphi(\mathcal{D}, a) = \mathit{Count}_x \wedge \mathit{Count}_y \wedge \mathit{Stable} \wedge \mathit{Unique} \wedge \mathit{Tiling} \wedge \mathit{Init}$$

can be found in Figure 1, where Count_y is Count_x with R_x replaced by R_y, x_j by y_j, and x_k by y_k. In this figure, $[u]\varphi$ is an abbreviation for $[R]\varphi \wedge [\neg R]\varphi$, where R is an arbitrary atomic modal parameter. Obviously, in each model of $[u]\varphi$, each world satisfies φ. In *Init*, we write

$$
\begin{array}{lcl}
\mathit{Count}_x & = & [u]\Big[\bigwedge_{k=0}^{n}\Big((\bigwedge_{j=0}^{k-1} x_j) \to (x_k \leftrightarrow [R_x]\,\neg x_k)\Big) \wedge \\
& & \bigwedge_{k=0}^{n}\Big((\bigvee_{j=0}^{k-1} \neg x_j) \to (x_k \leftrightarrow [R_x]\,x_k)\Big) \wedge \langle R_x\rangle\,\mathsf{true}\Big] \\
\mathit{Stable} & = & [u]\Big[\bigwedge_{k=0}^{n}(x_k \to [R_y]\,x_k) \wedge \bigwedge_{k=0}^{n}(\neg x_k \to [R_y]\,\neg x_k) \wedge \\
& & \bigwedge_{k=0}^{n}(y_k \to [R_x]\,y_k) \wedge \bigwedge_{k=0}^{n}(\neg y_k \to [R_x]\,\neg y_k)\Big] \\
\mathit{Unique} & = & [u]\Big[\bigwedge_{k=0}^{n}\Big((x_k \to [\neg R_k]\,\neg x_k) \wedge (\neg x_k \to [\neg R_k]\,x_k)\Big) \wedge \\
& & \bigwedge_{k=0}^{n}\Big((y_k \to [\neg S_k]\,\neg y_k) \wedge (\neg y_k \to [\neg S_k]\,y_k)\Big) \wedge \\
& & \bigwedge_{d\in D} p_d \to [R_0 \cap \cdots \cap R_n \cap S_0 \cap \cdots \cap S_n]\,p_d\Big] \\
\mathit{Tiling} & = & [u]\Big[(\bigvee_{d\in D} p_d) \wedge \bigwedge_{d\in D}\ \bigwedge_{d'\in D\setminus\{d\}} \neg(p_d \wedge p_{d'}) \wedge \\
& & \bigwedge_{d\in D} p_d \to \big([R_x] \bigvee_{(d,d')\in H} p_{d'}\big) \wedge \\
& & \bigwedge_{d\in D} p_d \to \big([R_y] \bigvee_{(d,d')\in G} p_{d'}\big)\Big] \\
\mathit{Init} & = & \bigwedge_{k=0}^{n}(\neg x_i \wedge \neg y_i) \wedge p_{w_0} \wedge [R_x]\,p_{w_1} \wedge \cdots \wedge [R_x]^{n-1} p_{w_{n-1}}
\end{array}
$$

FIGURE 1 Sub-formulae of $\varphi_{(\mathcal{D},a)}$ for $\mathcal{D} = (D, H, V)$ and $a = a_0, \ldots, a_{n-1}$.

$[R]^n\varphi$ to denote the n-fold nesting of $[R]$. The strategy of the reduction is to define the reduction formula $\varphi_{(\mathcal{D},a)}$ such that, for every model $\mathcal{M}$ of $\varphi_{(\mathcal{D},a)}$ with set of worlds W,

1. there exists a propositional variable p_d for every domino type $d \in D$ such that each $w \in W$ is in the extension of p_d for exactly one $d \in D$ (first line of *Tiling*),
2. for each point (i, j) in the torus $U(2^{n+1}, 2^{n+1})$, there exists a corresponding set of worlds $\{w_1, \ldots, w_k\} \subseteq W$ with $k \geq 1$ and a

$d \in D$ such that all $w_1, \ldots, w_k$ are in the extension of p_d (*Count*$_x$, *Count*$_y$, *Stable*, and *Unique* formulae),

3. the horizontal and vertical conditions V and H are satisfied w.r.t. sets of worlds representing points in the plane (second and third line of *Tiling*), and
4. the initial condition is satisfied (*Init*).

Properties 1, 3, and 4 are enforced in a standard way using **K** formulae. Property 2, however, needs some explanation. Usually, domino-reductions axiomatize a "grid" in order to capture the structure of the torus. As Property 2 indicates, we employ a different strategy: Each world in each model of $\varphi_{(\mathcal{D},a)}$ corresponds to a point (i, j) in the torus. The number i is binarily encoded by the propositional variables $x_0, \ldots, x_n$ while the number j is encoded by the propositional variables $y_0, \ldots, y_n$. We use standard binary incrementation modulo 2^{n+1} to ensure that, for every world w corresponding to a position (i, j), there exists a world w_1 such that w_1 corresponds to $(i \oplus_{2^{n+1}} 1, j)$ and $\mathcal{M}, (w, w_1) \models R_x$, and a world w_2 such that w_2 corresponds to $(i, j \oplus_{2^{n+1}} 1)$ and $\mathcal{M}, (w, w_2) \models R_y$. The *Count*$_x$ and *Count*$_y$ formulae encode the incrementation of the one dimension while the *Stable* formula ensures that the other dimension does not change. It remains to guarantee that every two worlds corresponding to the same position are labeled with the same domino. This task is accomplished by the *Unique* formula which is the only one to use conjunction of modal parameters and the only one to use negation for a purpose different from expressing the universal modality. In order to understand the *Unique* formula, it may be helpful to read subformulae of the form $[\neg R]\neg\varphi$ as $\boxminus_R \varphi$.

Proposition 4.5 *A domino system $\mathcal{D}$ tiles the torus $U(2^{n+1}, 2^{n+1})$ with initial condition $a = a_0, \ldots, a_{n-1}$ iff $\varphi_{(\mathcal{D},a)}$ is satisfiable.*

Together with Theorem 4.4, we obtain a NExpTime lower bound for $\mathbf{K}_\omega^{(\neg),\cap}$-formulae. The corresponding upper bound follows from the fact that the translation of $\mathbf{K}_\omega^{\neg}$-formulae to L^2-formulae mentioned in Section 2 can also be applied to $\mathbf{K}_\omega^{(\neg),\cap}$-formulae.

Theorem 4.6 *Satisfiability of $\mathbf{K}_\omega^{(\neg),\cap}$-formulae is NExpTime-complete.*

5 Full Boolean Modal Logic

In this section, we investigate the complexity of full Boolean Modal Logic. Let us start with introducing this logic formally.

Definition 5.1 A *complex modal parameter* is a Boolean formula of atomic modal parameters. We use $\mathbf{K}_\omega^{\neg,\cap,\cup}$ to denote the extension of $\mathbf{K}_\omega$ with complex modal parameters. Let $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \rangle$ be a Kripke structure, and S a (possibly complex) modal parameter. Then the extension $\mathcal{E}(S)$ is inductively defined as follows:

if $S = R_i$ (i.e., S is atomic)	then $\mathcal{E}(S) = \mathcal{R}_i$
if $S = \neg S'$	then $\mathcal{E}(S) = (W \times W) \setminus \mathcal{E}(S')$
if $S = S_1 \cap S_2$	then $\mathcal{E}(S) = \mathcal{E}(S_1) \cap \mathcal{E}(S_2)$
if $S = S_1 \cup S_2$	then $\mathcal{E}(S) = \mathcal{E}(S_1) \cup \mathcal{E}(S_2)$

The semantics of formulae is extended as follows:

$$\mathcal{M}, w \models \langle S \rangle \varphi \quad \text{iff} \quad \exists w' \in W \text{ with } (w, w') \in \mathcal{E}(S) \text{ and } \mathcal{M}, w' \models \varphi$$
$$\mathcal{M}, w \models [S] \varphi \quad \text{iff} \quad \forall w' \in W, \text{ if } (w, w') \in \mathcal{E}(S), \text{ then } \mathcal{M}, w' \models \varphi$$

We write $\mathcal{M}, (w, w') \models S$ iff $(w, w') \in \mathcal{E}(S)$.

From Theorem 4.6 and the standard tranlation of $\mathbf{K}_\omega^{\neg,\cap,\cup}$ into L^2, we easily obtain the following result:

Theorem 5.2 *Satisfiability of* $\mathbf{K}_\omega^{\neg,\cap,\cup}$*-formulae is NExpTime-complete.*

However, it is interesting to note that the NExpTime reduction used to prove Theorem 4.6 crucially depends on the fact that an infinite number of modal parameters is available: Since the size of the torus to be tiled is not bounded, there exists no upper bound for the number of the R_i and S_i parameters used for the reduction either. Although Boolean Modal Logics usually provide an infinite number of modal parameters (see, e.g., Gargov and Passy 1987), the question whether NExpTime-hardness can still be obtained if only a bounded number of modal parameters is available is natural. In the remainder of this section, we answer this question by showing that satisfiability and validity of $\mathbf{K}_m^{\neg,\cap,\cup}$, i.e., full Boolean Modal Logic with a fixed number m of modal parameters, is ExpTime-complete. The upper bound is proved by a reduction to multi-modal $\mathbf{K}$ enriched with the universal modality.

We show that satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae can be reduced to satisfiability of $\mathbf{K}_n^u$-formulae (i.e., formulae of multi-modal $\mathbf{K}$ enriched with the universal modality) by giving a series of polynomial reduction steps. We do not introduce $\mathbf{K}_n^u$ formally but refer the reader to, e.g., Spaan 1993. The following notions are central to several of the reduction steps.

Definition 5.3 A Kripke structure $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \mathcal{R}_m \rangle$ is called *simple* iff we have $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for all $1 \leq i < j \leq m$. $\mathcal{M}$ is called *complete* iff, for all $w, w' \in W$, there exists a unique i with $1 \leq i \leq m$ such that $(w, w') \in \mathcal{R}_i$. A formula (of any logic defined in this paper) is called *s-satisfiable* iff it has a model which is a simple Kripke structure. Similarly, a formula is called *c-satisfiable* iff it has a model which is a complete Kripke structure.

Note that every complete Kripke structure is also simple. We now describe the reduction steps in detail. Let φ be a $\mathbf{K}_m^{\neg,\cap,\cup}$-formula whose satisfiability is to be decided and let $R_1, \ldots, R_m$ be the modal parameters of $\mathbf{K}_m^{\neg,\cap,\cup}$.

Step 1. Convert all modal parameters in φ to disjunctive normal form using a truth table and one disjunct for each line in the truth table that yields *true*. If the "empty disjunction" is obtained when converting a modal parameter S, then replace every occurrence of $\langle S \rangle \psi$ with *false* and every occurrence of $[S]\psi$ with *true*. Call the result of the conversion φ_1. The length of φ_1 is linear in the length of φ since the number m of atomic modal parameters is fixed (and the conversion can be done in linear time). It is easy to see that φ_1 is satisfiable iff φ is satisfiable.

Since the conversion to DNF was done using a truth table, each disjunct occurring in a modal parameter in φ_1 is a *relational type*, i.e., of the form

$$S_1 \cap \cdots \cap S_m \text{ with } S_i = R_i \text{ or } S_i = \neg R_i \text{ for } 1 \leq i \leq m.$$

Let Γ be the set of all relational types. As is easily seen, if $\mathcal{M}, (w, w') \models S$ for some Kripke structure $\mathcal{M}$ with set of worlds W, $w, w' \in W$, and $S \in \Gamma$, then, for every atomic modal parameters R_i, this determines whether $\mathcal{M}, (w, w') \models R_i$ holds. Hence, for every $w, w' \in W$, we have $\mathcal{M}, (w, w') \models S$ for exactly one $S \in \Gamma$.

Step 2. We reduce satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae of the form of φ_1 (i.e, the modal parameters are in DNF and hence $\cup$ does not appear nested inside other operators) to the satisfiability of $\mathbf{K}_m^{(\neg),\cap}$-formulae in which all modal parameters are relational types. It is not hard to see that this can be done as in Section 4, where $\mathbf{K}_\omega^{(\neg),\cup}$ is reduced to $\mathbf{K}_\omega^{\neg}$: In the reduction, just replace the formula $[R](p_\varphi \leftrightarrow \varphi) \wedge [\neg R](p_\varphi \leftrightarrow \varphi)$ with $\bigwedge_{S \in \Gamma} [S](p_\varphi \leftrightarrow \varphi)$.[1] The reduction can again be done in linear time since m is fixed. The $\mathbf{K}_m^{(\neg),\cap}$-formula obtained by converting φ_1 is called φ_2.

Step 3. We reduce satisfiability of $\mathbf{K}_m^{(\neg),\cap}$-formulae of the form of φ_2 to c-satisfiability of $\mathbf{K}_{2^m}$-formulae. Set $n := 2^m$ and let $K_1, \ldots, K_n$ be the

[1] This reduction ensures that all modal parameters in the resulting formula are relational types.

atomic modal parameters of the logic $\mathbf{K}_n$. Let r be some bijection between Γ and the set $\{K_1, \ldots, K_n\}$. The formula φ_3 is obtained from φ_2 by replacing each element S of Γ in φ_2 with $r(S)$. Considering the special syntactic form of φ_2 and the definitions of Γ and of c-satisfiability, it is easy to see that φ_2 is satisfiable iff φ_3 is c-satisfiable. Furthermore, the reduction is obviously linear. Note that using 2^m instead of m modal parameters does not spoil the reduction since, ultimately, our reduction goes to satisfiability of multi-modal $\mathbf{K}$ enriched with the universal modality, and this logic is known to be in ExpTime for *any* fixed number of modalities (Spaan 1993).

Step 4. We reduce c-satisfiability of $\mathbf{K}_n$-formulae to s-satisfiability of $\mathbf{K}_n^u$-formulae. Define φ_4 as the conjunction of φ_3 with the formula

$$\chi := [u]\big(\bigwedge_{\psi_1,\ldots,\psi_n \text{ subformulae of } \varphi_3} [K_1]\psi_1 \wedge \cdots \wedge [K_n]\psi_n \rightarrow [u](\psi_1 \vee \cdots \vee \psi_n)\big)$$

Note that the length of φ_4 is polynomial in the length $|\varphi_3|$ of φ_3: The number of subformulae of φ_3 is bounded by $|\varphi_3|$; hence, χ consists of at most $|\varphi_3|^\ell$ conjuncts, where ℓ is a constant since the number of modal parameters is fixed. Let us prove that φ_3 is c-satisfiable iff φ_4 is s-satisfiable. The "only if" direction is straightforward: Let $\mathcal{M}$ be a complete model for φ_3. Obviously, $\mathcal{M}$ is also simple. Moreover, using the fact that $\mathcal{M}$ is complete, it is straightforward to check that $\mathcal{M}$ is a model for φ_4. It remains to prove the "if" direction. Let $\mathcal{M} = \langle W, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle$ be a simple model for φ_4. We first show that

Claim. For each $w, w' \in W$, there exists an ℓ with $1 \leq \ell \leq n$ such that, for all subformulae ψ of φ_3, $\mathcal{M}, w \models [K_\ell]\psi$ implies $\mathcal{M}, w' \models \psi$.

Assume to the contrary that, for some $w, w' \in W$, there exist no ℓ as in the claim. Hence, for each i with $1 \leq i \leq n$, there exists a subformula ρ_i of φ_3 such that $\mathcal{M}, w \models [K_i]\rho_i$ and $\mathcal{M}, w' \not\models \rho_i$. Since $\mathcal{M}$ is a model for χ, we clearly have

$$\mathcal{M}, w \models [K_1]\rho_1 \wedge \cdots \wedge [K_n]\rho_n \rightarrow [u](\rho_1 \vee \cdots \vee \rho_n).$$

This is obviously a contradiction to the fact that $\mathcal{M}, w \not\models \rho_1 \vee \cdots \vee \rho_n$ which proves the claim.

Extend the Kripke structure $\mathcal{M}$ to $\mathcal{M}' = \langle W, \pi, \mathcal{K}_1', \ldots, \mathcal{K}_n' \rangle$ as follows: For any $w, w' \in W$ with $(w, w') \notin \mathcal{K}_i$ for all i with $1 \leq i \leq n$, augment $\mathcal{K}_\ell$ with the tuple (w, w'), where ℓ is as in the claim. Obviously, $\mathcal{M}'$ is complete. It is now a matter of routine to prove that $\mathcal{M}, w \models \psi$ implies $\mathcal{M}', w \models \psi$ for all subformulae ψ of φ_3. The proof is by induction over the structure of ψ. The only interesting case is:

$\psi = [K_i]\psi'$. Let $(w, w') \in \mathcal{K}'_i$. We need to show that $\mathcal{M}', w' \models \psi'$. First assume that $(w, w') \in \mathcal{K}_i$. Since $\mathcal{M}, w \models \psi$, this implies $\mathcal{M}, w' \models \psi'$. By induction, we have $\mathcal{M}', w' \models \psi'$ and are done. Now assume $(w, w') \in \mathcal{K}'_i \setminus \mathcal{K}_i$. By definition of $\mathcal{K}'_i$, we have that $\mathcal{M}, w \models [K_i]\rho$ implies $\mathcal{M}, w' \models \rho$ for all subformulae ρ of φ_3. Since ψ is a subformula of φ_3, we have $\mathcal{M}, w' \models \psi'$. It remains to apply the induction hypothesis.

Since $\mathcal{M}$ is a model for φ_4, we have that $\mathcal{M}'$ is a model for φ_3. ❏

Step 5. It remains to prove that s-satisfiability of $\mathbf{K}_n^u$-formulae is decidable in ExpTime. This is, however, an easy consequence of the facts that satisfiability of $\mathbf{K}_n^u$-formulae is in ExpTime and that $\mathbf{K}_n^u$ has the tree model property: since every tree model is obviously simple, satisfiability coincides with s-satisfiability.

The sequence of reductions given above yields an ExpTime upper bound for the satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae. Since the lower bound for $\mathbf{K}_\omega^{\neg}$ already holds if we have only a single modal parameter available (again, see Spaan 1993), we obtain the following theorem.

Theorem 5.4 *Satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae (i.e., $\mathbf{K}_\omega^{\neg,\cap,\cup}$ with a bounded bumber of modal parameters) is ExpTime-complete.*

6 Boolean Modal Logics without Negation

So far, we have only considered logics with negation of modal parameters. We will complete our investigation by showing that adding intersection and union of modal parameters does not increase the complexity of $\mathbf{K}_\omega$ (and thus neither the complexity of $\mathbf{K}_m$ is increased by this extension). The fact that the extension of $\mathbf{K}_\omega$ with intersection of modal parameters (i.e., $\mathbf{K}_\omega^{\cap}$) is still in PSpace is an immediate consequence of PSpace-completeness of the Description Logic $\mathcal{ALCR}$ (Donini et al. 1991) and the fact that $\mathcal{ALCR}$ is a notational variant of $\mathbf{K}_\omega^{\cap}$ (Schild 1991). Moreover, it is folklore that $\mathbf{K}_\omega$ extended with union of modal parameters (i.e., $\mathbf{K}_\omega^{\cup}$) is also in PSpace (however, the reduction from Section 4 cannot be applied since the universal modality is not available). For both union and intersection, we go into more detail.

With $\mathbf{K}_\omega^{\cap,\cup}$, we denote the variant of $\mathbf{K}_\omega^{\neg,\cap,\cup}$ obtained by disallowing the use of negation of modal parameters. In the following, we will present a slight extension of the standard PSpace tableau algorithm for $\mathbf{K}$, $\mathbf{K}$-*World* (Ladner 1977), to decide satisfiability of $\mathbf{K}_\omega^{\cap,\cup}$-formulae. Please note that we cannot adapt the reduction from the previous section since the disjunctive normal form of a complex modal parameter can yield an exponential blow-up if the number of boolean parameters is not bounded.

When started with an input formula φ, $\mathbf{K}$-World decides φ's satisfiability by recursively searching a finite tree-model of φ in a depth-first manner. For each world w in this tree model, it checks whether the set Δ of formulae that w must satisfy is not contradictory. Then, for each $\Diamond\psi$ in Δ, $\mathbf{K}$-World is called recursively with ψ and all ρ with $\Box\rho$ in Δ.

To extend $\mathbf{K}$-World to $\mathbf{K}_\omega^{\cap,\cup}$, it is comfortable to view the semantics of roles in a different way. For S a complex modal parameter and s a set of atomic modal parameters, we say $s \models S$ iff s, when viewed as the valuation that maps each $R_i \in s$ to true and each $R_j \notin s$ to false, evaluates the Boolean expression S to true. Then $(w, w') \in \mathcal{E}(S)$ iff there is a set s of atomic modal parameters with $s \models S$ and $(w, w') \in \mathcal{R}_i$ for each $R_i \in s$. The only modifications to $\mathbf{K}$-World concern the recursive calls for diamond formulae which are more elaborate in the presence of complex modal parameters. For each $\langle S\rangle\,\psi$ in the set Δ of formulae currently considered, we guess an s with $s \models S$, and then consider ψ together with all ρ where $[S']\,\rho$ is in Δ and $s \models S'$.

For the sake of a succinct presentation, we assume the input formula φ to contain no disjunction and no diamond-formulae. For Δ and S sets of $\mathbf{K}_\omega^{\cap,\cup}$-formulae where S is closed under subformulae and single negations, $\mathbf{K}_\omega^{\cap,\cup}$-World$(\Delta, S)$ returns true iff

- Δ is a maximally propositionally consistent subset of S, i.e.,
  - $\Delta \subseteq S$,
  - for each $\neg\psi \in S$, $\psi \in \Delta$ iff $\neg\psi \notin \Delta$, and
  - for each $\psi_1 \wedge \psi_2 \in S$, $\psi_1 \wedge \psi_2 \in \Delta$ iff $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$.
- For each subformula $\neg\,[S]\,\psi \in \Delta$, there exists a set s of modal parameters with $s \models S$ and a set $\Delta_{\psi,s}$ such that
  - $\neg\psi \in \Delta_{\psi,s}$,
  - for each S' and ρ, if $[S']\,\rho \in \Delta$ and $s \models S'$, then $\rho \in \Delta_{\psi,s}$,
  - $\mathbf{K}_\omega^{\cap,\cup}-\mathsf{World}(\Delta_{\psi,s}, S')$ returns true, where S' is the closure under subformulae and single negation of $\{\rho \mid [S']\,\rho \in \Delta$ and $s \models S'\} \cup \{\neg\psi\}$.

Let $\mathsf{cl}(\varphi)$ be the smallest set of formulae containing φ that is closed under subformulae and single negation. The proof that a $\mathbf{K}_\omega^{\cap,\cup}$-formula φ is satisfiable iff there exists a $\Delta \subseteq \mathsf{cl}(\varphi)$ with $\varphi \in \Delta$ such that

$$\mathbf{K}_\omega^{\cap,\cup}\text{-World}(\Delta, \mathsf{cl}(\varphi)\})$$

returns true is analogous to the one for $\mathbf{K}$-World. Just like $\mathbf{K}$-World, $\mathbf{K}_\omega^{\cap,\cup}$-World runs in PSpace (since PSpace = NPSpace (Savitch 1970), the additional non-deterministic guessing of the set of modal parameters s does not matter). Moreover, $\mathbf{K}$ is known to be PSpace-hard (Ladner 1977), and we thus have the following result.

Theorem 6.1 *Satisfiability of $\mathbf{K}_\omega^{\cup,\cap}$-formulae is PSpace-complete.*

7 Conclusion

We have given a complete picture of the complexity of Boolean Modal Logics, both with and without a bound on the number of modal parameters. The results for (fragments of) Boolean Modal Logic with an unbounded number of modal parameters are summarised in Figure 2, showing known results in grey.

NExpTime-hardness of $\mathbf{K}_\omega^{(\neg),\cap}$ was rather surprising since so far, intersection of atomic modal parameters (not of chainings/composition of modal parameters) is mostly considered to be "harmless" w.r.t. complexity. Interestingly, we were able to show that, if a bound m is imposed on the number of atomic modal parameters, then full Boolean Modal Logic $\mathbf{K}_m^{\neg,\cap,\cup}$ becomes ExpTime-complete. For this proof, we did not use the automata-based approach because we considered that extending it to take care of complex modal parameters was more involved than the reduction to $\mathbf{K}_n^u$ that we used.

As future work, it may be interesting to extend our techniques to more expressive logics. For example, one may consider arbitrary combinations of the Boolean operators on modal parameters with composition and converse. Several results for such logics are known from the area of Propositional Dynamic Logics (PDL). For example, Harel proves that PDL extended with negation of modal parameters is undecidable using a reduction to the equivalence problem for relation algebra (Harel 1984). It is not hard to see that a similar reduction (of the equivalence problem of boolean algebras of relations with composition only, see, e.g., Andreka et al. 2001) can be used to show that Boolean Modal Logic extended with composition of modal parameters is undecidable. On the contrary, it follows from Danecki's results on PDL with intersection that $\mathbf{K}_\omega^{\cap,\cup}$ extended with composition is decidable in double ExpTime (Danecki 1984). As we demonstrated by extending our results to $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, our automata-based approach to proving ExpTime-bounds can be considered flexible. As a first step towards more expressive

	no negation	atomic negation	full negation
–	PSpace-compl.	ExpTime-compl.	
$\cup$	PSpace-compl.	ExpTime-compl.	NExpTime-compl.
$\cap$	PSpace-compl.	NExpTime-compl.	NExpTime-compl.
$\cap$ and $\cup$	PSpace-compl.	NExpTime-compl.	NExpTime-compl.

FIGURE 2 Complexity of $\mathbf{K}_\omega$ extended with various role constructors.

logics, we hope that our approach can be "married" with the standard automata-based decidability procedure for PDL thus yielding a decidability result for PDL extended with atomic negation of modal parameters.

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