Combining Decision Procedures for Positive Theories Sharing Constructors

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Abstract. This paper addresses the following combination problem: given two equational theories E_1 and E_2 whose positive theories are decidable, how can one obtain a decision procedure for the positive theory of $E_1 \cup E_2$? For theories over disjoint signatures, this problem was solved by Baader and Schulz in 1995. This paper is a first step towards extending this result to the case of theories sharing constructors. Since there is a close connection between positive theories and unification problems, this also extends to the non-disjoint case the work on combining decision procedures for unification modulo equational theories.

1 Introduction

Built-in decision procedures for certain types of theories (like equational theories) can greatly speed up the performance of theorem provers. In many applications, however, the theories actually encountered are combinations of theories for which dedicated decision procedure are available. Thus, one must find ways to combine the decision procedures for the single theories into one for their combination. In the context of *equational theories over disjoint signatures*, this combination problem has been thoroughly investigated in the following three instances:[3] the word problem, the validity problem for universally quantified formulae, and the unification problem. For the word problem, i.e., the problem whether a single (universally quantified) equation $s \equiv t$ follows from the equational theory, the first solution to the combination problem was given by Pigozzi [9] in 1974. The problem of combining decision procedures for universally quantified formulae, i.e., arbitrary Boolean combinations of equations that are universally quantified, was solved by Nelson and Oppen [8] in 1979. Work on combining unification algorithms started also in the seventies with Stickel's investigation [12] of unification of terms containing several associative-commutative and free symbols. The first general result on how to combine *decision procedures* for unification was published by Baader and Schulz [1] in 1992. It turned out that decision procedures for unification (with constants) are not sufficient to allow for a combination result. Instead, one needs decision procedures for unification *with linear constant restrictions* in the theories to be combined. In 1995, Baader and Schulz

[3] Some of the work mentioned below can also handle more general theories. To simplify the presentation, we restrict our attention in this paper to the equational case.

[2] described a version of their combination procedure that applies to *positive theories*, i.e., positive Boolean combinations of equations with an arbitrary quantifier prefix. They also showed [3] that the decidability of the positive theory is equivalent to the decidability of unification with linear constant restrictions.

Since then, the main open problem in the area was how to extend these results to the combination of *theories having symbols in common*. In general, the existence of shared symbols may lead to undecidability results for the union theory (see, e.g., [6, 5] for some examples). This means that a controlled form of sharing of symbols is necessary. For the word problem and for universally quantified formulae, a suitable notion of *shared constructors* has proved useful. In [5], Pigozzi's combination result for the word problem was extended to theories all of whose shared symbols are constructors. A similar extension of the Nelson-Oppen combination procedure can be found in [13].

In a similar vein, we show in this paper that the combination results in [2] for positive theories (and thus for unification) can be extended to theories sharing constructors. We do that by extending the combination procedure in [2] with an extra step that deals with shared symbols and proving that the extended procedure is sound and complete. Since this extra step is not finitary, the new procedure in general yields only a semi-decision procedure for the combined theory. Under some additional assumptions on the equational theory of the shared symbols, the procedure can, however, be turned into a decision procedure. Although the combination procedure described here differs from the one in [2] by just one extra step, proving its correctness is considerably more challenging, due to the non-disjointness of the theories. A major contribution of this work is a novel algebraic construction of the free algebra of the combined theory. As in the non-disjoint case [2], this construction is vital for the correctness proof of the procedure, and we believe that it will prove helpful also in future research on non-disjoint combination.

The paper is organized as follows. Section 2 contains some formal preliminaries. Section 3 defines our notion of constructors and presents some of their properties, which will be used later to prove the correctness of the combination procedure. Section 4 describes our extension of the Baader-Schulz procedure to component theories sharing constructors. It then introduces a straightforward condition on the component theories under which the semi-decision procedure obtained this way can in fact be used to decide the positive consequences of their union. Finally, it proves that the general procedure is sound and complete. We conclude the paper with a comparison to related work and suggestions for further research. Space constraints prevent us from providing all the proofs of the results in the paper. The missing proofs can be found in [4].

2 Preliminaries

In this paper we will use standard notions from universal algebra such as formula, sentence, algebra, subalgebra, generators, reduct, entailment, model, homomorphism and so on. Notable differences are reported in the following.

We consider only first-order theories (with equality) over a functional signature. A *signature* Σ is a set of *function symbols*, each with an associated *arity*, an integer $n \geq 0$. A *constant* symbol is a function symbol of zero arity. We use the letters Σ, Ω, Δ to denote signatures. Throughout the paper, we fix a countably-infinite set V of *variables*, disjoint with any signature Σ. For any $X \subseteq V$, $T(\Sigma, X)$ denotes the set of Σ*-terms* over X, i.e., first-order terms with variables in X and function symbols in Σ. Formulae in the signature Σ are defined as usual. We use $\equiv$ to denote the equality symbol. We also use the standard notion of substitution, with the usual postfix notation. We call a substitution a *renaming* iff it is a bijection of V onto itself. We say that a subset T of $T(\Sigma, V)$ is *closed under renaming* iff $t\sigma \in T$ for all terms $t \in T$ and renamings σ.

If A is a set, we denote by A^* the set of all finite tuples made of elements of A. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two tuples, we denote by $\boldsymbol{a}, \boldsymbol{b}$ the tuple obtained as the concatenation of $\boldsymbol{a}$ and $\boldsymbol{b}$. If φ is a term or a formula, we denote by $\mathcal{V}ar(\varphi)$ the set of φ's free variables. We will often write $\varphi(\boldsymbol{v})$ to indicate, as usual, that $\boldsymbol{v}$ is a tuple of variables with no repetitions and all elements of $\mathcal{V}ar(\varphi)$ occur in $\boldsymbol{v}$. A formula is *positive* iff it is in prenex normal form and its matrix is obtained from atomic formulae using only conjunctions and disjunctions. A formula is *existential* iff it has the form $\exists \boldsymbol{u}.\ \varphi(\boldsymbol{u}, \boldsymbol{v})$ where $\varphi(\boldsymbol{u}, \boldsymbol{v})$ is a quantifier-free formula.

If $\mathcal{A}$ is an algebra of signature Ω, we denote by A the universe of $\mathcal{A}$ and by $\mathcal{A}^{\Sigma}$ the reduct of $\mathcal{A}$ to a given subsignature Σ of Ω. If $\varphi(\boldsymbol{v})$ is an Ω-formula and α is a valuation of $\boldsymbol{v}$ into A, we write $(\mathcal{A}, \alpha) \models \varphi(\boldsymbol{v})$ iff $\varphi(\boldsymbol{v})$ is satisfied by the interpretation $(\mathcal{A}, \alpha)$. Equivalently, where $\boldsymbol{a} = \alpha(\boldsymbol{v})$, we may also write $\mathcal{A} \models \varphi(\boldsymbol{a})$. If $t(\boldsymbol{v})$ is an Ω-term, we denote by $[\![t]\!]^{\mathcal{A}}_{\alpha}$ the interpretation of t in $\mathcal{A}$ under the valuation α of $\boldsymbol{v}$. Similarly, if T is a set of terms, we denote by $[\![T]\!]^{\mathcal{A}}_{\alpha}$ the set $\{[\![t]\!]^{\mathcal{A}}_{\alpha} \mid t \in T\}$.

A *theory* of signature Ω, or an Ω-theory, is any set of Ω-sentences, i.e., closed Ω-formulae. An algebra $\mathcal{A}$ is *a model of a theory* $\mathcal{T}$, or *models* $\mathcal{T}$, iff each sentence in $\mathcal{T}$ is satisfied by the interpretation $(\mathcal{A}, \alpha)$ where α is the empty valuation. Let $\mathcal{T}$ be an Ω-theory. We denote by $Mod(\mathcal{T})$ the class of all Ω-algebras that model $\mathcal{T}$. The theory $\mathcal{T}$ is *satisfiable* if it has a model, and *trivial* if it has only *trivial models*, i.e., models of cardinality 1. For all sentences φ (of any signature), we say as usual that $\mathcal{T}$ *entails* φ, or that φ *is valid in* $\mathcal{T}$, and write $\mathcal{T} \models \varphi$, iff $\mathcal{T} \cup \{\neg\varphi\}$ is unsatisfiable. We call *(existential) positive theory of* $\mathcal{T}$ the set of all (existential) positive sentences *in the signature of* $\mathcal{T}$ that are entailed by $\mathcal{T}$.

An *equational theory* is a set of (universally quantified) equations. If E is an equational theory of signature Ω and Σ is an arbitrary signature, we denote by E^{Σ} the set of all (universally quantified) Σ-equations entailed by E. When $\Sigma \subseteq \Omega$ we call E^{Σ} the Σ*-restriction of* E. For all Ω-terms $s(\boldsymbol{v}), t(\boldsymbol{v})$, we write $s =_E t$ and say that s and t are *equivalent in* E iff $E \models \forall \boldsymbol{v}.\ s \equiv t$.

We will later appeal to the two basic model theory results below about subalgebras (see [7] among others).

Lemma 1. *Let $\mathcal{B}$ be a Σ-algebra and $\mathcal{A}$ a subalgebra of $\mathcal{B}$. For all quantifier-free formulae $\varphi(v_1, \dots, v_n)$ and individuals $a_1, \dots, a_n \in A$, $\mathcal{A} \models \varphi(a_1, \dots, a_n)$ iff $\mathcal{B} \models \varphi(a_1, \dots, a_n)$.*

Lemma 2. *For all equational theories E, $Mod(E)$ is closed under subalgebras.*

Similarly to [2], our procedure's correctness proof will be based on free algebras. Instead of the usual definition of free algebras, we will rely on the following characterization [7].

Proposition 3. *Let E be a Σ-theory and $\mathcal{A}$ a Σ-algebra. Then, $\mathcal{A}$ is free in E over some set X iff the following holds:*

1. *$\mathcal{A}$ is a model of E generated by X;*
2. *for all $s, t \in T(\Sigma, V)$ and injections α of $\mathcal{V}ar(s \equiv t)$ into X, if $(\mathcal{A}, \alpha) \models s \equiv t$ then $s =_E t$.*

When $\mathcal{A}$ is free in E over X we will also say that $\mathcal{A}$ *is a free model of E (with basis X)*. We will implicitly rely on the well-known fact that every non-trivial equational theory E admits a free model with a countably infinite basis, namely the quotient term algebra $T(\Sigma, V)/{=_E}$. We will also use the following two results from [2] about free models and positive formulae.

Lemma 4. *Let $\mathcal{B}$ be an Ω-algebra free (in some theory E) over a countably infinite set X. For all positive Ω-formulae $\varphi(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{2m-1}, \boldsymbol{v}_{2m})$ the following are equivalent:*

1. *$\mathcal{B} \models \forall \boldsymbol{v}_1 \, \exists \boldsymbol{v}_2 \, \cdots \, \forall \boldsymbol{v}_{2m-1} \, \exists \boldsymbol{v}_{2m}. \; \varphi(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{2m-1}, \boldsymbol{v}_{2m})$;*
2. *there exist tuples $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X^*$ and $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m \in B^*$ and finite subsets $Z_1, \dots, Z_m$ of X such that*
   (a) *$\mathcal{B} \models \varphi(\boldsymbol{x}_1, \boldsymbol{b}_1, \dots, \boldsymbol{x}_m, \boldsymbol{b}_m)$,*
   (b) *all components of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are distinct,*
   (c) *for all $n \in \{1, \dots, m\}$, all components of $\boldsymbol{b}_n$ are generated by Z_n in $\mathcal{B}$,*
   (d) *for all $n \in \{1, \dots, m-1\}$, no components of $\boldsymbol{x}_{n+1}$ are in $Z_1 \cup \dots \cup Z_n$.*

Lemma 5. *For every equational theory E having a countable signature and a free model $\mathcal{A}$ with a countably infinite basis, the positive theory of E coincides with the set of positive sentences true in $\mathcal{A}$.*

In this paper, we will deal with *combined* equational theories, that is, theories of the form $E_1 \cup E_2$, where E_1 and E_2 are two *component* equational theories of (possibly non-disjoint) signatures Σ_1 and Σ_2, respectively. Where $\Sigma := \Sigma_1 \cap \Sigma_2$, we call *shared symbols* the elements of Σ and *shared terms* the elements of $T(\Sigma, V)$. Notice that, when Σ_1 and Σ_2 are disjoint, the only shared terms are the variables.

Most combination procedures, including the one described in this paper, work with $(\Sigma_1 \cup \Sigma_2)$-formulae by first "purifying" them into a set of Σ_1-formulae and a set of Σ_2-formulae. There is a standard *purification procedure* that, when Σ_1 and Σ_2 are disjoint, can convert any set S of equations of signature $\Sigma_1 \cup \Sigma_2$ into a set S' of *pure* equations (that is, each of signature Σ_1 or Σ_2) such that S' is satisfiable in a $(\Sigma_1 \cup \Sigma_2)$-algebra $\mathcal{A}$ iff S is satisfiable in $\mathcal{A}$. As we show in [5], a similar procedure also exists for the case in which Σ_1 and Σ_2 are not disjoint.

3 Theories with Constructors

The main requirement for our generalization of the combination procedure described in [2] to apply is that the symbols shared by the two theories are *constructors* as defined in [5, 13]. For the rest of the section, let E be an non-trivial equational theory of signature Ω. Also, let Σ be a subsignature of Ω.

Definition 6 (Constructors). *The signature Σ is a* set of constructors for E *iff for every free model $\mathcal{A}$ of E with a countably infinite basis X, $\mathcal{A}^{\Sigma}$ is a free model of E^{Σ} with a basis Y including X.*

It is usually non-trivial to show that a signature Σ is a set of constructors for a given theory E by using just the definition above. Instead, using a syntactic characterization of constructors given in terms of certain subsets of $T(\Omega, V)$ is usually more helpful. Before we can give this characterization, we need a little more notation.

Given a subset G of $T(\Omega, V)$, we denote by $T(\Sigma, G)$ the set of Σ-terms over the "variables" G. More precisely, every member of $T(\Sigma, G)$ is obtained from a term $s \in T(\Sigma, V)$ by replacing the variables of s with terms from G. To express this construction, we will denote any such term by $s(\boldsymbol{r})$ where $\boldsymbol{r}$ is a tuple collecting the terms of G that replace the variables of s. Note that $G \subseteq T(\Sigma, G)$ and that $T(\Sigma, V) \subseteq T(\Sigma, G)$ whenever $V \subseteq G$.

Definition 7 (Σ-base). *A subset G of $T(\Omega, V)$ is a* Σ-base of E *iff*

1. *$V \subseteq G$;*
2. *for all $t \in T(\Omega, V)$, there is an $s(\boldsymbol{r}) \in T(\Sigma, G)$ such that $t =_E s(\boldsymbol{r})$;*
3. *for all $s_1(\boldsymbol{r}_1), s_2(\boldsymbol{r}_2) \in T(\Sigma, G)$, $s_1(\boldsymbol{r}_1) =_E s_2(\boldsymbol{r}_2)$ iff $s_1(\boldsymbol{v}_1) =_E s_2(\boldsymbol{v}_2)$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are tuples of fresh variables abstracting the terms of $\boldsymbol{r}_1, \boldsymbol{r}_2$ so that two terms in $\boldsymbol{r}_1, \boldsymbol{r}_2$ are abstracted by the same variable iff they are equivalent in E.*

We say that E admits a Σ-base *if some $G \subseteq T(\Omega, V)$ is a Σ-base of E.*

Theorem 8 (Characterization of constructors). *The signature Σ is a set of constructors for E iff E admits a Σ-base.*

A proof of this theorem and of the following corollary can be found in [5].

Corollary 9. *Where $\mathcal{A}$ is a free model of E with a countably-infinite basis X, let α be an arbitrary bijection of V onto X. If G is a Σ-base of E, then $\mathcal{A}^{\Sigma}$ is free in E^{Σ} over the superset $[\![G]\!]^{\mathcal{A}}_{\alpha}$ of X.*

In the following, we will assume that the theories we consider admit Σ-bases *closed under renaming.* This assumption is necessary for technical reasons. It is used in the long version of this paper in the proof of a lemma (Lemma 4.18 in [4]; omitted here) needed to prove the soundness of the combination procedure described later. Although we do not know whether this assumption can be made with no loss of generality, it is not clear how to avoid it and it seems to be satisfied by all "sensible" examples of theories admitting constructors. Also note that the same technical assumption was needed in our work on combining decision procedures for the word problem [5].

It is shown in [5] that, under the right conditions, constructors and the property of having Σ-bases closed under renaming are modular with respect to the union of theories.

Proposition 10. *For $i = 1, 2$ let E_i be a non-trivial equational Σ_i-theory. If $\Sigma := \Sigma_1 \cap \Sigma_2$ is a set of constructors for E_1 and for E_2 and ${E_1}^{\Sigma} = {E_2}^{\Sigma}$, then Σ is a set of constructors for $E_1 \cup E_2$. If both E_1 and E_2 admit a Σ-base closed under renaming, then $E_1 \cup E_2$ also admits a Σ-base closed under renaming.*

A useful consequence of Proposition 10 for us will be the following.

Proposition 11. *Let E be an Ω-theory and let E' be the empty Δ-theory for some signature Δ disjoint with Ω. If $\Sigma \subseteq \Omega$ is a set of constructors for E, then it is a set of constructors for $E \cup E'$. Furthermore, if E admits a Σ-base closed under renaming, then so does $E \cup E'$.*

4 Combining Decision Procedures

In this section, we generalize the Baader-Schulz procedure [2] for combining decision procedures for the validity of positive formulae in equational theories from theories over disjoint signatures to theories sharing constructors. More precisely, we will consider two theories E_1 and E_2 that satisfy the following assumptions for $i = 1, 2$, which we fix for the rest of the section:

- E_i is a non-trivial equational theory of some countable signature Σ_i;
- $\Sigma := \Sigma_1 \cap \Sigma_2$ is a set of constructors for E_i, and E_i admits a Σ-base closed under renaming;
- ${E_1}^{\Sigma} = {E_2}^{\Sigma}$.

Let $E := E_1 \cup E_2$. Under the assumptions above, $E^{\Sigma} = {E_1}^{\Sigma} = {E_2}^{\Sigma}$ (see [5]). In the following then, we will use E^{Σ} to refer indifferently to ${E_1}^{\Sigma}$ or ${E_2}^{\Sigma}$.

The combination procedure will use two kinds of substitutions that we call, after [13], *identifications* and *Σ-instantiations*. Given a set of variables U, an *identification of U* is a substitution defined by partitioning U, selecting a representative for each block in the partition, and mapping each element of U to the representative in its block. A *Σ-instantiation of U* is a substitution that maps some elements of U to non-variable Σ-terms and the other elements to themselves. For convenience, we will assume that the variables occurring in the terms introduced by a Σ-instantiation are *always fresh*.

4.1 The Combination Procedure

The procedure takes as input a positive existential $(\Sigma_1 \cup \Sigma_2)$-formula $\exists \boldsymbol{w}.\ \varphi(\boldsymbol{w})$ and outputs, non-deterministically, a pair of sentences: a positive Σ_1-sentence and a positive Σ_2-sentence. It consists of the following steps.

1. **Convert into DNF.** Convert the input's matrix φ into the disjunctive normal form $\psi_1 \vee \cdots \vee \psi_n$ and choose a disjunct ψ_j.
2. **Convert into Separate Form.** Let S be the set obtained by purifying, as mentioned in Section 2, the set of all the equations in ψ_j. For $i = 1, 2$, let $\varphi_i(\boldsymbol{v}, \boldsymbol{u}_i)$ be the conjunction of all Σ_i-equations in S,[4] with $\boldsymbol{v}$ listing the variables in $\mathcal{V}ar(\varphi_1) \cap \mathcal{V}ar(\varphi_2)$ and $\boldsymbol{u}_i$ listing the remaining variables of φ_i.
3. **Instantiate Shared Variables.** Choose a Σ-instantiation ρ of $\mathcal{V}ar(\boldsymbol{v}) = \mathcal{V}ar(\varphi_1) \cap \mathcal{V}ar(\varphi_2)$.
4. **Identify Shared Variables.** Choose an identification ξ of $\mathcal{V}ar(\varphi_1\rho) \cap \mathcal{V}ar(\varphi_2\rho) = \mathcal{V}ar(\boldsymbol{v}\rho)$. For $i = 1, 2$, let $\varphi_i' := \varphi_i\rho\xi$.
5. **Partition Shared Variables.** Group the elements of $V_{\mathrm{s}} := \mathcal{V}ar(\boldsymbol{v}\rho\xi) = \mathcal{V}ar(\varphi_1') \cap \mathcal{V}ar(\varphi_2')$ into the tuples $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2m}$, with $2 \leq 2m \leq |V_{\mathrm{s}}| + 1$, so that each element of V_{s} occurs exactly once in the tuple $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2m}$.[5]
6. **Generate Output Pair.** Output the pair of sentences

$$(\exists \boldsymbol{v}_1\, \forall \boldsymbol{v}_2\, \cdots\, \exists \boldsymbol{v}_{2m-1}\, \forall \boldsymbol{v}_{2m}\, \exists \boldsymbol{u}_1.\, \varphi_1',\ \forall \boldsymbol{v}_1\, \exists \boldsymbol{v}_2\, \cdots\, \forall \boldsymbol{v}_{2m-1}\, \exists \boldsymbol{v}_{2m}\, \exists \boldsymbol{u}_2.\, \varphi_2').$$

Ignoring inessential differences and our restriction to functional signatures, this combination procedure differs from Baader and Schulz's only for the presence of Step 3. Note however that, for component theories with disjoint signatures (the case considered in [2]), Step 3 is vacuous because Σ is empty. In that case then the procedure above reduces to that in [2]. Correspondingly, our requirements on the two component theories also reduce to that in [2], which simply asks that E_1 and E_2 be non-trivial. In fact, when Σ is empty it is always a set of constructors for E_i ($i = 1, 2$), with $T(\Sigma_i, V)$ being a Σ-base closed under renaming. Moreover, ${E_1}^{\Sigma} = {E_2}^{\Sigma}$ because they both coincide with the theory $\{v \equiv v \mid v \in V\}$.

As will be shown in Section 4.3, our combination procedure is sound and complete in the following sense.

Theorem 12 (Soundness and Completeness). *For all possible input sentences $\exists \boldsymbol{w}.\, \varphi(\boldsymbol{w})$ of the combination procedure, $E_1 \cup E_2 \models \exists \boldsymbol{w}.\, \varphi(\boldsymbol{w})$ iff there is a possible output (γ_1, γ_2) such that $E_1 \models \gamma_1$ and $E_2 \models \gamma_2$.*

Unlike the procedure in [2], the combination procedure above does not necessarily yield a decision procedure. The reason is that the non-determinism in Step 3 of the procedure is not finitary since in general there are infinitely-many possible Σ-instantiations to choose from. One viable, albeit strong, restriction for obtaining a decision procedure is described in the next subsection.

4.2 Decidability Results

In order to turn the combination procedure from above into a decision procedure, we require that the equivalence relation defined by the theory $E^{\Sigma} = {E_1}^{\Sigma} = {E_2}^{\Sigma}$ be *bounded* in a sense described below.

[4] Where Σ-equations are considered arbitrarily as either Σ_1- or Σ_2-equations.

[5] Note that some of the subtuples $\boldsymbol{v}_i$ may be empty.

Definition 13. *Let E be an equational Ω-theory. We say that equivalence in E is* finitary modulo renaming *iff there is a finite subset R of $T(\Omega, V)$ such that for all $s \in T(\Omega, V)$ there is a term $t \in R$ and a renaming σ such that $s =_E t\sigma$. We call R a set of E*-representatives.

When Ω in the above definition is empty, equivalence in E is trivially finitary—with any singleton set of variables being a set of E-representatives. A non-trivial example is provided at the end of this section.

If E^{Σ} is finitary modulo renaming, then it is easy to see that it suffices to consider only finitely many instantiations in Step 3 of the procedure, which leads to the following decidability result.

Proposition 14. *Assume that Σ, E_1, E_2 satisfy the assumptions stated at the beginning of Section 4, and that equivalence in E^{Σ} is finitary modulo renaming. If the positive theories of E_1 and of E_2 are both decidable, then the positive existential theory of $E_1 \cup E_2$ is also decidable.*

Using a Skolemization argument together with Proposition 11, the result above can be extended from positive existential input sentences to arbitrary positive input sentences. The main idea is to Skolemize the universal quantifiers of the input sentence and then expand the signature of one the theories, E_2 say, to the newly introduced Skolem symbols. Proposition 11 and the combination result in [2] for the disjoint case imply that the pair E_1, E_2', where E_2' is the conservative extension of E_2 to the expanded signature, satisfies the assumptions of Proposition 14.

Theorem 15. *Assume that E_1, E_2 satisfy the assumptions of Proposition 14. If the positive theories of E_1 and of E_2 are both decidable, then the positive theory of $E := E_1 \cup E_2$ is also decidable.*

The following example describes one theory satisfying all the requirements on the component theories imposed by Theorem 15.

Example 16. Consider the signature $\Omega := \{0, s, +\}$ and, for some $n > 1$, the equational theory E_n axiomatized by the identities

$$\begin{array}{lll} x + (y + z) \equiv (x + y) + z, & x + y \equiv y + x, & \\ x + \mathsf{s}(y) \equiv \mathsf{s}(x + y), & x + 0 \equiv x, & \mathsf{s}^n(x) \equiv x. \end{array}$$

where as usual $\mathsf{s}^n(x)$ stands for the n-fold application of s to x. We show in [4] that, for E_n and the subsignature $\Sigma := \{0, \mathsf{s}\}$ of Ω, all the assumptions of Theorem 15 on the component theories are satisfied.

4.3 Soundness and Completeness of the Procedure

The soundness and completeness proof for the disjoint case in [2] relies on an explicit construction of the free model of $E = E_1 \cup E_2$ as an *amalgamated product* of the free models of the component theories. A direct adaptation of the

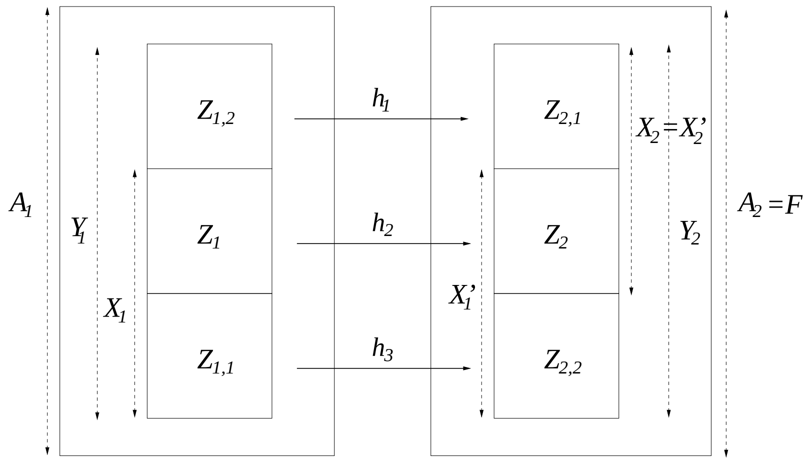

Fig. 1. The Fusion $\mathcal{F}$ of $\mathcal{A}_1$ and $\mathcal{A}_2$.

free amalgamation construction of [2] to the non-disjoint case has so far proven elusive. An important technical contribution of the present work is to provide an alternative way to obtain an appropriate amalgamated free model in the case of theories sharing constructors. We obtain this model for the union theory E indirectly, by first building a simpler sort of amalgamated model as a *fusion* (defined below) of the free models of the two component theories. Contrary to Baader and Schulz's free amalgamated product, the fusion model we construct is not free in E. However, it has a subalgebra that is so. That subalgebra will serve as the free amalgamated model of E.

Definition 17 (Fusion [5, 13]). *A $(\Omega_1 \cup \Omega_2)$-algebra $\mathcal{F}$ is a* fusion *of a Ω_1-algebra $\mathcal{A}_1$ and a Ω_2-algebra $\mathcal{A}_2$ iff $\mathcal{F}^{\Omega_1}$ is Ω_1-isomorphic to $\mathcal{A}_1$ and $\mathcal{F}^{\Omega_2}$ is Ω_2-isomorphic to $\mathcal{A}_2$.*

It is shown in [13] that two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ have fusions exactly when they are isomorphic over their shared signature, and that every fusion of a model of a theory $\mathcal{T}_1$ with a model of a theory $\mathcal{T}_2$ is a model of the theory $\mathcal{T}_1 \cup \mathcal{T}_2$.

In the following, we will construct a model of $E = E_1 \cup E_2$ as a fusion of free models of the theories E_1 and E_2 fixed earlier, whose shared signature Σ was a set of constructors for both. We start by fixing, for $i = 1, 2$,

- a free model $\mathcal{A}_i$ of E_i with a countably infinite basis X_i,
- a bijective valuation α_i of V onto X_i,
- a Σ-base G_i of E_i closed under renaming, and
- the set $Y_i := [\![G_i]\!]^{\mathcal{A}_i}_{\alpha_i}$.

We know from Corollary 9 that $X_i \subseteq Y_i$ and ${\mathcal{A}_i}^{\Sigma}$ is free in $E^{\Sigma} = {E_1}^{\Sigma} = {E_2}^{\Sigma}$ over Y_i. Observe that $\mathcal{A}_i$ is countably infinite, given our assumption that X_i is countably infinite and Σ_i is countable. As a consequence, Y_i is countably infinite as well.

Now let $Z_{i,2} := Y_i \setminus X_i$ for $i = 1, 2$, and let $\{Z_{1,1}, Z_1\}$ be a partition of X_1 such that Z_1 is countably infinite and $|Z_{1,1}| = |Z_{2,2}|$.[6] Similarly, let $\{Z_{2,1}, Z_2\}$ be a partition of X_2 such that $|Z_{2,1}| = |Z_{1,2}|$ and Z_2 is countably infinite (see Figure 1). Then consider 3 arbitrary bijections

$$h_1 : Z_{1,2} \longrightarrow Z_{2,1}, \;\; h_2 : Z_1 \longrightarrow Z_2, \;\; h_3 : Z_{1,1} \longrightarrow Z_{2,2},$$

as shown in Figure 1. Observing that $\{Z_{i,1}, Z_i, Z_{i,2}\}$ is a partition of Y_i for $i = 1, 2$, it is immediate that $h_1 \cup h_2 \cup h_3$ is a well-defined bijection of Y_1 onto Y_2. Since ${\mathcal{A}_i}^{\Sigma}$ is free in E^{Σ} over Y_i for $i = 1, 2$, we have that $h_1 \cup h_2 \cup h_3$ extends uniquely to a (Σ)-isomorphism h of ${\mathcal{A}_1}^{\Sigma}$ onto ${\mathcal{A}_2}^{\Sigma}$. The isomorphism h induces a fusion of $\mathcal{A}_1$ and $\mathcal{A}_2$ whose main properties are listed in the following lemma, taken from [5].

Lemma 18. *There is a fusion $\mathcal{F}$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ having the same universe as $\mathcal{A}_2$ and such that*

1. *h is a (Σ_1)-isomorphism of $\mathcal{A}_1$ onto $\mathcal{F}^{\Sigma_1}$;*
2. *the identity map of A_2 is a (Σ_2)-isomorphism of $\mathcal{A}_2$ onto $\mathcal{F}^{\Sigma_2}$;*
3. *$\mathcal{F}^{\Sigma_i}$ is free in E_i over $X_i' := Z_{2,j} \cup Z_2$ for $i, j = 1, 2$, $i \neq j$;*
4. *$\mathcal{F}^{\Sigma}$ is free in E^{Σ} over $Y_2 = Z_{2,1} \cup Z_2 \cup Z_{2,2}$;*
5. *$Y_2 = [\![G_2]\!]^{\mathcal{F}^{\Sigma_2}}_{\alpha_2} = [\![G_1]\!]^{\mathcal{F}^{\Sigma_1}}_{h \circ \alpha_1}$.*

We will now consider the theory $E = E_1 \cup E_2$ again, together with the algebras $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{A}$ where:

- $\mathcal{F}$ is the fusion of $\mathcal{A}_1$ and $\mathcal{A}_2$ from Lemma 18;
- $\mathcal{F}_i := \mathcal{F}^{\Sigma_i}$ for $i = 1, 2$;[7]
- $\mathcal{A}$ is the subalgebra of $\mathcal{F}$ generated by Z_2.

Both $\mathcal{F}$ and $\mathcal{A}$ are models of E. In fact, $\mathcal{F}$ is a model of $E = E_1 \cup E_2$ for being a fusion of a model of E_1 and a model of E_2, whereas $\mathcal{A}$ is a model of E by Lemma 2. We prove in [4] that $\mathcal{A}$ is in fact a free model of E. To do that we use the following sets of terms, which will come in handy later as well.

Definition 19 **($G_1^\infty, G_2^\infty, G^\infty$).** *Let $G^\infty := G_1^\infty \cup G_2^\infty$ where for $i = 1, 2$, $G_i^\infty := \bigcup_{n=0}^{\infty} G_i^n$ and $\{G_i^n \mid n \geq 0\}$ is the family of sets defined as follows:*

$$\begin{aligned}
G_i^0 \;\; &:= V,\\
G_i^{n+1} &:= G_i^n \cup \{r(r_1, \ldots, r_m) \mid r(v_1, \ldots, v_m) \in G_i \setminus V,\ r \neq_E v \text{ for all } v \in V,\\
&\qquad\qquad\qquad\qquad\qquad\; r_j \in G_k^n \text{ with } k \neq i, \text{ for all } j = 1, \ldots, m,\\
&\qquad\qquad\qquad\qquad\qquad\; r_j \neq_E r_{j'} \text{ for all distinct } j, j' = 1, \ldots, m\ \}.
\end{aligned}$$

As proved in [5], the sets $G_1^\infty, G_2^\infty, G^\infty$ satisfy the following two properties.

[6] This is possible because $Z_{2,2}$ is countable (possibly finite).

[7] These algebras are defined just for notational convenience.

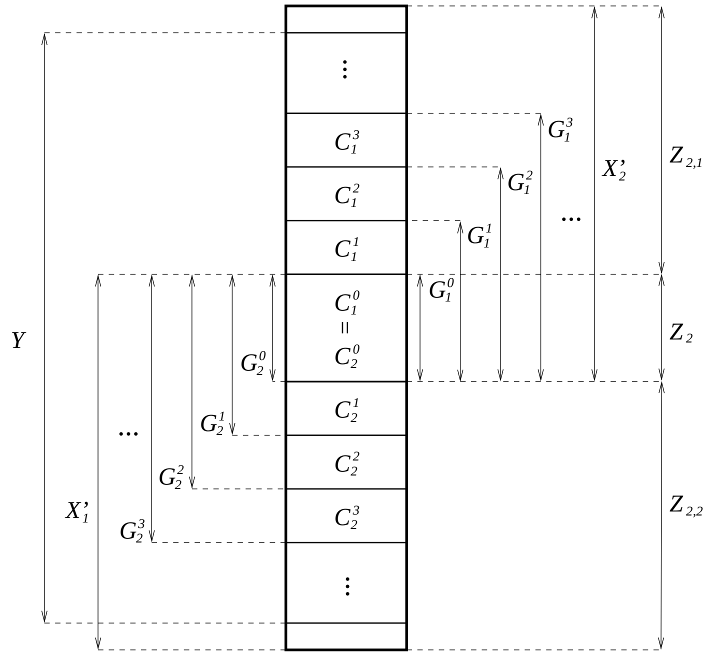

Fig. 2. The families $\{[\![G_i^n]\!] \mid n \geq 0\}$ and $\{C_i^n \mid n \geq 0\}$.

Lemma 20. *Let $i \in \{1,2\}$. For any bijection α of V onto Z_2 the following holds:*

1. $[\![G_i^\infty \setminus V]\!]_\alpha^{\mathcal{F}} \subseteq Z_{2,i}$;
2. *for all $t_1, t_2 \in G_i^\infty \setminus V$, if $[\![t_1]\!]_\alpha^{\mathcal{F}} = [\![t_2]\!]_\alpha^{\mathcal{F}}$ then $t_1 =_E t_2$.*

Proposition 21. *The set G^∞ is Σ-base of $E = E_1 \cup E_2$.*

Note that this proposition entails by Theorem 8 that Σ is a set of constructors for E. Using these two properties (and Proposition 3) we can show the following.

Proposition 22. *$\mathcal{A}$ is free in E over Z_2.*

Corollary 23. *For every bijection α of V onto Z_2, $\mathcal{A}^\Sigma$ is free in E^Σ over $Y := [\![G^\infty]\!]_\alpha^{\mathcal{A}}$, and $Y \subseteq Y_2$.*

For the rest of the section, let us fix a bijection α of V onto Z_2 and the corresponding set $Y := [\![G^\infty]\!]_\alpha^{\mathcal{A}}$.

To prove the completeness of the combination procedure we will need two families $\{C_1^n \mid n \geq 0\}$ and $\{C_2^n \mid n \geq 0\}$ of sets partitioning the set Y above. To build these families we use the denotations in $\mathcal{A}$ of the sets G_1^n and G_2^n introduced in Definition 19. More precisely, for $i = 1, 2$, we consider the family

$\{[\![G_i^n]\!]_\alpha^{\mathcal{A}} \mid n \geq 0\}$ of subsets of Y. Since $\mathcal{A}$ is the subalgebra of $\mathcal{F}$ generated by Z_2 and α is a valuation of V into Z_2, it is easy to see that $[\![G_i^n]\!]_\alpha^{\mathcal{A}} = [\![G_i^n]\!]_\alpha^{\mathcal{F}}$ for all $n \geq 0$. Therefore, we will write just $[\![G_i^n]\!]$ in place of either $[\![G_i^n]\!]_\alpha^{\mathcal{A}}$ or $[\![G_i^n]\!]_\alpha^{\mathcal{F}}$.

Observe that $[\![G_1^0]\!] = [\![G_2^0]\!] = Z_2$ and $[\![G_i^n]\!] \subseteq [\![G_i^{n+1}]\!]$ for all $n \geq 0$ and $i = 1, 2$. Given that $[\![G_i^n]\!] \setminus Z_2 \subseteq [\![G_i^n \setminus V]\!]_\alpha^{\mathcal{A}}$, we can conclude by Lemma 20 that $[\![G_i^n]\!] \setminus Z_2 \subseteq Z_{2,i}$.[8] By Corollary 23 we have that

$$\bigcup_{n \geq 0} ([\![G_1^n]\!] \cup [\![G_2^n]\!]) = [\![\bigcup_{n \geq 0} (G_1^n \cup G_2^n)]\!] = [\![G_1^\infty \cup G_2^\infty]\!] = [\![G^\infty]\!] = Y.$$

Now consider the family of sets $\{C_i^n \mid n \geq 0\}$, depicted in Figure 2 along with $\{[\![G_i^n]\!] \mid n \geq 0\}$ and defined as follows:

$$C_i^0 := [\![G_i^0]\!] \quad \text{and} \quad C_i^{n+1} := [\![G_i^{n+1}]\!] \setminus [\![G_i^n]\!] \quad \text{for all } n \geq 0.$$

First note that $\bigcup_{n \geq 0} (C_1^n \cup C_2^n) = \bigcup_{n \geq 0} ([\![G_1^n]\!] \cup [\![G_2^n]\!]) = Y$. Then note that, for all $n \geq 0$ and $i = 1, 2$, the elements of C_i^n are individuals of the algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ (which have the same universe). By Lemma 20, $C_1^n \subseteq [\![G_1^n]\!] \subseteq Z_{2,1} \cup Z_2 = X_2'$; in other words, every element of C_1^n is a generator of $\mathcal{F}_2$. Similarly, $C_2^n \subseteq [\![G_2^n]\!] \subseteq Z_{2,2} \cup Z_2 = X_1'$, that is, every element of C_2^n is a generator of $\mathcal{F}_1$. In addition, we have the following.

Lemma 24. *For all distinct $m, n \geq 0$ and distinct $i, j \in \{1, 2\}$,*

1. *$C_i^m \cap C_i^n = \emptyset$ and*
2. *C_i^{n+1} is Σ_i-generated by $[\![G_j^n]\!]$ in $\mathcal{F}_i$.*

Now, Theorem 12 is an easy consequence of the following proposition.

Proposition 25. *For $i = 1, 2$, let $\varphi_i(\boldsymbol{v}, \boldsymbol{u}_i)$ be a conjunction of Σ_i-equations where $\boldsymbol{v}$ lists the elements of $\mathcal{V}ar(\varphi_1) \cap \mathcal{V}ar(\varphi_2)$ and $\boldsymbol{u}_i$ lists the elements of $\mathcal{V}ar(\varphi_i)$ not in $\boldsymbol{v}$. The following are equivalent:*

1. *There is a Σ-instantiation ρ of $\boldsymbol{v}$, an identification ξ of $\mathcal{V}ar(\boldsymbol{v}\rho)$ and a grouping $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2m}$ of $\mathcal{V}ar(\boldsymbol{v}\rho\xi)$ with each element of $\mathcal{V}ar(\boldsymbol{v}\rho\xi)$ occurring exactly once in $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2m}$ such that*

$$\mathcal{A}_1 \models \exists \boldsymbol{v}_1 \forall \boldsymbol{v}_2 \cdots \exists \boldsymbol{v}_{2m-1} \forall \boldsymbol{v}_{2m} \exists \boldsymbol{u}_1. (\varphi_1 \rho \xi) \quad \text{and}$$
$$\mathcal{A}_2 \models \forall \boldsymbol{v}_1 \exists \boldsymbol{v}_2 \cdots \forall \boldsymbol{v}_{2m-1} \exists \boldsymbol{v}_{2m} \exists \boldsymbol{u}_2. (\varphi_2 \rho \xi).$$

2. *$\mathcal{A} \models \exists \boldsymbol{v} \exists \boldsymbol{u}_1 \exists \boldsymbol{u}_2. (\varphi_1 \wedge \varphi_2)$.*

Proof. The proof of $(1 \Rightarrow 2)$ is similar to the corresponding proof in [2], although it requires some additional technical lemmas (see [4] for details). We concentrate here on the proof of $(2 \Rightarrow 1)$.

Assume that $\mathcal{A} \models \exists \boldsymbol{v}, \boldsymbol{u}_1, \boldsymbol{u}_2. (\varphi_1(\boldsymbol{v}, \boldsymbol{u}_1) \wedge \varphi_2(\boldsymbol{v}, \boldsymbol{u}_2))$. Let α be the bijection of V onto Z_2 and Y the subset of Y_2 that we fixed after Corollary 23. Since

[8] This entails that $[\![G_1^m]\!] \setminus Z_2$ is disjoint with $[\![G_2^n]\!] \setminus Z_2$ for all $m, n > 0$.

the reduct $\mathcal{A}^{\Sigma}$ of $\mathcal{A}$ is Σ-generated by Y by the same corollary, there is a Σ-instantiation ρ of $\boldsymbol{v}$, an identification ξ of $\mathcal{V}ar(\boldsymbol{v}\rho)$, and an injective valuation β of $\boldsymbol{v}'$ into Y such that, for $\varphi_i' := \varphi_i\rho\xi$ $(i = 1, 2)$ and $\boldsymbol{v}'$ listing the variables of $\boldsymbol{v}\rho\xi$, we have

$$(\mathcal{A}, \beta) \models \exists \boldsymbol{u}_1, \boldsymbol{u}_2.\ (\varphi_1'(\boldsymbol{v}', \boldsymbol{u}_1) \wedge \varphi_2'(\boldsymbol{v}', \boldsymbol{u}_2)).$$

From this, recalling that $\mathcal{A}$ is $(\Sigma_1 \cup \Sigma_2)$-generated by Z_2 by construction and Y is included in Y_2, we can conclude that there is a tuple $\boldsymbol{a}$ of *pairwise distinct* elements of Y_2, all $(\Sigma_1 \cup \Sigma_2)$-generated by Z_2, such that

$$\mathcal{A} \models \exists \boldsymbol{u}_1, \boldsymbol{u}_2.\ \varphi_1'(\boldsymbol{a}, \boldsymbol{u}_1) \wedge \varphi_2'(\boldsymbol{a}, \boldsymbol{u}_2).$$

Since $\mathcal{A}$ is a subalgebra of $\mathcal{F}$ and $\varphi_1' \wedge \varphi_2'$ is quantifier-free, it follows by Lemma 1 that $\mathcal{F} \models \exists \boldsymbol{u}_1, \boldsymbol{u}_2.\ \varphi_1'(\boldsymbol{a}, \boldsymbol{u}_1) \wedge \varphi_2'(\boldsymbol{a}, \boldsymbol{u}_2)$ as well. Given that each φ_i' is a Σ_i-formula and $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are disjoint, we have then that

$$\mathcal{F}_1 \models \exists \boldsymbol{u}_1.\ \varphi_1'(\boldsymbol{a}, \boldsymbol{u}_1) \ \text{ and } \ \mathcal{F}_2 \models \exists \boldsymbol{u}_2.\ \varphi_2'(\boldsymbol{a}, \boldsymbol{u}_2). \tag{1}$$

We construct a partition of the elements of $\boldsymbol{a}$ that will induce a grouping of $\boldsymbol{v}'$ having the properties listed in Point 1 of the proposition. For that, we will use the families $\{C_1^n \mid n \geq 0\}$ and $\{C_2^n \mid n \geq 0\}$ defined before Lemma 24.

First, let $\boldsymbol{a}_1$ be a tuple collecting the components of $\boldsymbol{a}$ that are in $C_1^0 \cup C_1^1$. Then, for all $n > 1$, let $\boldsymbol{a}_n$ be a tuple collecting the components of $\boldsymbol{a}$ that are in C_1^n. Finally, for all $n > 0$, let $\boldsymbol{b}_n$ be a tuple collecting the components of $\boldsymbol{a}$ that are in C_2^n.[9]

Since $\boldsymbol{a}$ is a (finite) tuple of Y^* and $Y = \bigcup_{n\geq 0} (C_1^n \cup C_2^n)$ as observed earlier, there is a smallest $m > 0$ such that every component of $\boldsymbol{a}$ is in $\bigcup_{n=0}^{m}(C_1^n \cup C_2^n)$. Let $n \in \{0, \ldots, m-1\}$. By Lemma 24(2), $\boldsymbol{b}_{n+1}$ is Σ_2-generated by $[\![G_1^n]\!]$ in $\mathcal{F}_2$. Let Z_{n+1} be any finite subset of $[\![G_1^n]\!]$ that generates $\boldsymbol{b}_{n+1}$. Now recall that $\mathcal{F}_2$ is free over the countably-infinite set X_2'. We prove that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$, and $Z_1, \ldots, Z_m$ satisfy Lemma 4(2).

To start with, we have that $\boldsymbol{a}_n \in (X_2')^*$ for all $n \in \{1, \ldots, m\}$ because $C_1^n \subseteq [\![G_1^n]\!] \subseteq X_2'$ by construction of C_1^n. From Lemma 24(1) it follows that the tuples $\boldsymbol{a}_n$ and $\boldsymbol{a}_{n'}$ are pairwise disjoint for all distinct $n, n' \in \{1, \ldots, m\}$, which means that all components of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are distinct. Now let $n \in \{1, \ldots, m-1\}$. Observe that the set $Z_1 \cup \cdots \cup Z_n$ is included in $[\![G_1^{n-1}]\!] = C_1^0 \cup \cdots \cup C_1^{n-1}$ whereas every component of $\boldsymbol{a}_{n+1}$ belongs to C_1^{n+1}. It follows that no components of $\boldsymbol{a}_{n+1}$ are in $Z_1 \cup \cdots \cup Z_n$. Finally, where f is the bijection that maps, in order, the components of $\boldsymbol{a}$ to those of $\boldsymbol{v}'$, let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{2m-1}, \boldsymbol{v}_{2m}$ be the rearrangement of $\boldsymbol{v}'$ corresponding to $\boldsymbol{a}_1, \boldsymbol{b}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_m$ according to f. From (1) above we know that $\mathcal{F}_2 \models \exists \boldsymbol{u}_2.\ \varphi_2'(\boldsymbol{a}_1, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_m, \boldsymbol{a}_m, \boldsymbol{u}_2)$. By Lemma 4 we can then conclude that $\mathcal{F}_2 \models \forall \boldsymbol{v}_1\, \exists \boldsymbol{v}_2 \cdots \forall \boldsymbol{v}_{2m-1}\, \exists \boldsymbol{v}_{2m}\, \exists \boldsymbol{u}_2.\ \varphi_2'$.

Almost symmetrically, we can prove $\mathcal{F}_1 \models \exists \boldsymbol{v}_1\, \forall \boldsymbol{v}_2 \cdots \exists \boldsymbol{v}_{2m-1}\, \forall \boldsymbol{v}_{2m}\, \exists \boldsymbol{u}_1.\ \varphi_1'$. The claim then follows from the fact that $\mathcal{F}_i$ is Σ_i-isomorphic to $\mathcal{A}_i$ for $i = 1, 2$ by Lemma 18. □

[9] Each tuple above is meant to have no repeated components, and may be empty.

5 Related Research

From a technical point of view, this work strongly depends on previous research on combining decision procedures for unification in the disjoint case and on research on combining decision procedures for the word problem in the non-disjoint case. The combination procedure as well as the proof of correctness are modeled on the corresponding procedure and proof in [2]. The only extension to the procedure is Step 3, which takes care of the shared symbols. In the proof, one of the main obstacles to overcome was to find an amalgamation construction that worked in the non-disjoint case. Several of the hard technical results used in the proof depend on results from our previous work on combining decision procedures for the word problem [5]. The definition of the sets G_i, which are vital for proving that the constructed algebra $\mathcal{A}$ is indeed free, is also borrowed from there. It should be noted, however, that this definition can also be seen as a generalization to the non-disjoint case of a syntactic amalgamation construction originally due to Schmidt-Schauß [11]. As already mentioned in the introduction, the notion of constructors used here is taken from [5, 13].

The only other work on combination methods for unification in the non-disjoint case is due to Domenjoud, Ringeissen and Klay [6]. The main differences with our work are that (i) their notion of constructors is considerably more restrictive than ours; and (ii) they combine algorithms computing complete sets of unifiers, and so their method cannot be used to combine decision procedures. On the other hand, Domenjoud, Ringeissen and Klay do not impose the strong restriction that the component theories be finitary modulo renaming, which we need for our decidability result. However, it was recently discovered [10] that termination of the combination algorithm in [6] is actually not guaranteed with the conditions given in that paper.

6 Conclusion

We have extended the Baader-Schulz combination procedure [2] for positive theories to the case of component theories over non-disjoint signatures. The main contribution of this paper is the formulation of appropriate restrictions under which this procedure is sound and complete, and the proof of soundness and completeness itself. This proof depends on a novel construction of the free model of the combined theory, which is *not* just a straightforward extension of the free amalgamation construction used in [2] in the disjoint case. Regarding the generality of our restriction to theories sharing constructors, we believe that the notion of constructors is as general as one can get, a conviction that is supported by the work on combining decision procedures for the word problem and for universal theories [5, 13].

Unfortunately, our combination procedure yields only a semi-decision procedure since it incorporates an infinitary step. The restriction to equational theories that are finitary modulo renaming overcomes this problem, but it is probably too strong to be useful in applications. Thus, the main thrust of further research will be to remove or at least relax this restriction. We believe that the overall framework introduced in this paper and the proof of soundness and completeness of the semi-decision procedure (or at least the tools used in this proof) will help us obtain more interesting decidability results in the near future. One direction to follow could be to try to impose additional algorithmic requirements on the theories to be combined or on the constructor theory, and exploit those requirements to transform the infinitary step into a series of finitary ones. For this, the work in [6], which assumes algorithms computing complete sets of unifiers for the component theories, could be a starting point. Since the combination algorithm presented there has turned out to be non-terminating [10], that work needs to be reconsidered anyway.

Another direction for extending the results presented here is to withdraw the restriction to functional signatures. As a matter of fact, the combination results in [2] apply not just to equational theories, but to arbitrary atomic theories, i.e., theories over signatures also containing relation symbols and axiomatized by a set of (universally quantified) atomic formulae. Since the algebraic apparatus employed in the present paper (in particular, free algebras) is also available in this more general case (in the form of free structures), it should be easy to generalize our results to atomic theories.

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