A New Combination Procedure for the Word Problem that Generalizes Fusion Decidability Results in Modal Logics

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Abstract. Previous results for combining decision procedures for the word problem in the non-disjoint case do not apply to equational theories induced by modal logics—whose combination is not disjoint since they share the theory of Boolean algebras. Conversely, decidability results for the fusion of modal logics are strongly tailored towards the special theories at hand, and thus do not generalize to other equational theories. In this paper, we present a new approach for combining decision procedures for the word problem in the non-disjoint case that applies to equational theories induced by modal logics, but is not restricted to them. The known fusion decidability results for modal logics are instances of our approach. However, even for equational theories induced by modal logics our results are more general since they are not restricted to so-called normal modal logics.

1 Introduction

The combination of decision procedures for logical theories arises in many areas of logic in computer science, such as constraint solving, automated deduction, term rewriting, modal logics, and description logics. In general, one has two first-order theories T_1 and T_2 over the signatures Σ_1 and Σ_2, for which validity of a certain type of formulae (e.g., universal, existential positive, etc.) is decidable. The question is then whether one can combine the decision procedures for T_1 and T_2 into one for their union $T_1 \cup T_2$. The problem is usually much easier (though not at all trivial) if the theories do not share symbols, i.e., if $\Sigma_1 \cap \Sigma_2 = \emptyset$. For non-disjoint signatures, the combination of theories can easily lead to undecidability, and thus one must find appropriate restrictions on the theories to be combined.

In automated deduction, the Nelson-Oppen combination procedure [17, 16] as well as the problem of combining decision procedures for the word problem [19, 21, 18, 6] have drawn considerable attention. The Nelson-Oppen method combines decision procedures for the validity of quantifier-free formulae in so-called stably infinite theories. If we restrict the attention to equational theories,[1] then

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[1] Equational theories are stably infinite if one adds the axiom $\exists x, y.\ x \not\approx y$ [6].

it is easy to see that the validity of arbitrary quantifier-free formulae can be reduced to the validity of formulae of the form $s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \rightarrow s \approx t$ where $s_1, \ldots, t$ are terms. Thus, in this case the Nelson-Oppen method combines decision procedures for the *conditional word problem* (i.e., for validity of conditional equations of the above form). Though this may at first sight seem surprising, combining decision procedures for the *word problem* (i.e., for validity of equations $s \approx t$) is a harder task: the known combination algorithms for the word problem are more complicated than the Nelson-Oppen method, and the same applies to their proofs of correctness. The reason is that the algorithms for the component theories are then less powerful. For example, if one applies the Nelson-Oppen method to a word problem $s \approx t$, then it will generate as input for the component procedures conditional word problems, not word problems (see [6] for a more detailed discussion). Both the Nelson-Oppen method and the methods for combining decision procedures for the word problem have been generalized to the non-disjoint case [11, 24, 7, 12]. The main restriction on the theories to be combined is that they share only so-called constructors.

In modal logics, one is interested in whether properties (like decidability, finite axiomatizability) of uni-modal logics transfer to multi-modal logics that are obtained as the fusion of uni-modal logics. For the decidability transfer, one usually considers two different decision problems, the *validity* problem (Is the formula φ a theorem of the logic?) and the *relativized validity* problem (Does the formula φ follow from the global assumption ψ?). There are strong combination results that show that in many cases decidability transfers from two modal logics to their fusion [15, 23, 25, 4]. Again, transfer results for the harder decision problem, relativized validity,[2] are easier to show than for the simpler one, validity. In fact, for validity the results only apply to so-called *normal* modal logics,[3] whereas this restriction is not necessary for relativized validity.

There is a close connection between the (conditional) word problem and the (relativized) validity problem in modal logics. In fact, in so-called *classical* modal logics (which encompass most well-known modal logics), modal formulae can be viewed as terms, on which equivalence of formulae induces an equational theory. The fusion of modal logics then corresponds to the union of the corresponding equational theories, and the (relativized) validity problem to the (conditional) word problem. The union of the equational theories corresponding to two modal logics is over non-disjoint signatures since the Boolean operators are shared. Unfortunately, in this setting the Boolean operators are not shared constructors in the sense of [24, 7] (see [12]), and thus the decidability transfer results for modal logics cannot be obtained as special cases of the results in [24, 7, 12].

Recently, a new generalization of the Nelson-Oppen combination method to non-disjoint theories was developed in [13, 14]. The main restriction on the theories T_1 and T_2 to be combined is that they are *compatible* with their shared theory T_0, and that their shared theory is *locally finite* (i.e., its finitely generated

[2] This is in fact a harder problem since in modal logics the deduction theorem typically does not hold.

[3] An exception is [4], where only the existence of "covering normal terms" is required.

models are finite). A theory T is compatible with a theory T_0 iff (i) $T_0 \subseteq T$; (ii) T_0 has a model completion T_0^*; and (iii) every model of T embeds into a model of $T \cup T_0^*$. It is well-known that the theory BA of Boolean algebras is locally finite and that the equational theories induced by classical modal logics are compatible with BA (see [2] for details). Thus, the combination method in [14, 13] applies to (equational theories induced by) classical modal logics. However, since it generalizes the Nelson-Oppen method, it only yields transfer results for decidability of the conditional word problem (i.e., the relativized validity problem).

In the present paper, we address the harder problem of designing a combination method for the word problem in the non-disjoint case that has the known transfer results for decidability of validity in modal logics as instances.

In fact, we will see that our approach strictly generalizes these results since it does not require the modal logics to be normal. The question of whether such transfer results hold also for non-normal modal logics was a long-standing open problem in modal logics. In addition to the conditions imposed in [13, 14], our method needs the shared theory T_0 to have *local solvers*. Roughly speaking, this is the case if in T_0 one can solve an arbitrary equation with respect to any of its variables (see Definition 3 for details).

In the next section, we introduce some basic notions for equational theories, and define the restrictions under which our combination approach applies. In Section 3, we describe the new combination procedure, and show that it is sound and complete. Section 4 shows that the restrictions imposed by our procedure are satisfied by all classical modal logics. In particular, we show there that the theory of Boolean algebras has local solvers. In this section, we also comment on the complexity of our combination procedure if applied to modal logics, and illustrate the working of the procedure on an example.

For space constraints we must forgo most of the proofs of the results presented here. The interested reader can find them in [2].

2 Preliminaries

In this paper we will use standard notions from equational logic, universal algebra and term rewriting (see, e.g., [5]). We consider only first-order theories (with equality $\approx$) over a functional signature. We use the letters Σ, Ω, possibly with subscripts, to denote signatures. Throughout the paper, we fix a countably-infinite set V of *variables* and a countably-infinite set C of *free constants*, both disjoint with any signature Σ and with each other. For any $X \subseteq V \cup C$, $T(\Sigma, X)$ denotes the set of Σ-*terms* over X, i.e., first-order terms with variables and free constants in X and function symbols in Σ.[4] First-order Σ-*formulae* are defined in the usual way, using equality as the only predicate symbol. A Σ-*sentence* is a Σ-formula without *free* variables, and a *ground* Σ-*formula* is a Σ-formula without variables. An equational theory E over Σ is a set of (implicitly

[4] Note that Σ may also contain constants.

universally quantified) Σ-identities of the form $s \approx t$, where $s, t \in T(\Sigma, V)$. As usual, first-order interpretations of Σ are called Σ-algebras. We denote algebras by calligraphic letters $(\mathcal{A}, \mathcal{B}, \dots)$, and their carriers by the corresponding Roman letter $(A, B, \dots)$. A Σ-algebra $\mathcal{A}$ is a *model* of a set T of Σ-sentences iff it satisfies every sentence in T. For a set Γ of sentences and a sentence φ, we write $\Gamma \models_E \varphi$ if every model of E that satisfies Γ also satisfies φ. When Γ is the empty set, we write just $\models_E \varphi$, as usual. We denote by $\approx_E$ the equational consequences of E, i.e., the relation $\approx_E = \{(s,t) \in T(\Sigma, V \cup C) \times T(\Sigma, V \cup C) \mid \models_E s \approx t\}$. The *word problem* for E is the problem of deciding the relation $\approx_E$.

If $\mathcal{A}$ is an Ω-algebra and $\Sigma \subseteq \Omega$, we denote by $\mathcal{A}^{\Sigma}$ the Σ-*reduct* of $\mathcal{A}$, i.e., the algebra obtained from $\mathcal{A}$ by ignoring the symbols in $\Omega \setminus \Sigma$. An *embedding* of a Σ-algebra $\mathcal{A}$ into a Σ-algebra $\mathcal{B}$ is an injective Σ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If such an embedding exists then we say that $\mathcal{A}$ can be embedded into $\mathcal{B}$. If $\mathcal{A}$ is Σ-algebra and $\mathcal{B}$ is an Ω-algebra with $\Sigma \subseteq \Omega$, we say that $\mathcal{A}$ can be Σ-embedded into $\mathcal{B}$ if there is an embedding of $\mathcal{A}$ into $\mathcal{B}^{\Sigma}$. We call the corresponding embedding a Σ-embedding of $\mathcal{A}$ into $\mathcal{B}$. If this embedding is the inclusion function, then we say that $\mathcal{A}$ is a Σ-subalgebra of $\mathcal{B}$.

Given a signature Σ and a set X disjoint with $\Sigma \cup V$, we denote by $\Sigma(X)$ the signature obtained by adding the elements of X as constant symbols to Σ. When X is included in the carrier of a Σ-algebra $\mathcal{A}$, we can view $\mathcal{A}$ as a $\Sigma(X)$-algebra by interpreting each $x \in X$ by itself. The Σ-*diagram* $\Delta_X^{\Sigma}(\mathcal{A})$ of $\mathcal{A}$ (w.r.t. X) consists of all ground $\Sigma(X)$-literals that hold in $\mathcal{A}$. We write just $\Delta^{\Sigma}(\mathcal{A})$ when X coincides with the whole carrier of $\mathcal{A}$. By a result known as Robinson's Diagram Lemma [9] embeddings and diagrams are related as follows.

Lemma 1. *Let $\mathcal{A}$ be a Σ-algebra generated by a set X, and let $\mathcal{B}$ be an Ω-algebra for some $\Omega \supseteq \Sigma(X)$. Then $\mathcal{A}$ can be $\Sigma(X)$-embedded into $\mathcal{B}$ iff $\mathcal{B}$ is a model of $\Delta_X^{\Sigma}(\mathcal{A})$.*

A consequence of the lemma above, which we will use later, is that if two Σ-algebras $\mathcal{A}, \mathcal{B}$ are both generated by a set X and if one of them, say $\mathcal{B}$, satisfies the other's diagram w.r.t. X, then they are isomorphic.

Ground formulae are invariant under embeddings in the following sense.

Lemma 2. *Let $\mathcal{A}$ be a Σ-algebra that can be Σ-embedded into an algebra $\mathcal{B}$. For all ground $\Sigma(A)$-formulae φ, $\mathcal{A}$ satisfies φ iff $\mathcal{B}$ satisfies φ where $\mathcal{B}$ is extended to a $\Sigma(A)$-algebra by interpreting $a \in A$ by its image under the embedding.*

Given equational theories E_1, E_2 over their respective signatures Σ_1, Σ_2, we want to define conditions under which the decidability of the word problem for E_1 and E_2 implies the decidability of the word problem for their union.

First restriction: We will require that both E_1 and E_2 be *compatible* with a shared subtheory E_0 over the signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$. The definition of compatibility depends on the notion of a model completion. A first-order Σ-theory E^* is a *model completion* of an equational Σ-theory E iff it extends E and for every model $\mathcal{A}$ of E (i) $\mathcal{A}$ can be embedded into a model of E^*, and (ii) $E^* \cup \Delta^{\Sigma}(\mathcal{A})$ is a complete $\Sigma(A)$-theory, i.e., $E^* \cup \Delta^{\Sigma}(\mathcal{A})$ is satisfiable and for any $\Sigma(A)$-sentence φ, either φ or its negation follows from $E^* \cup \Delta^{\Sigma}(\mathcal{A})$.

Definition 1 (Compatibility). *Let E be an equational theory over the signature Σ, and let E_0 be an equational theory over a subsignature $\Sigma_0 \subseteq \Sigma$. We say that E is* E_0-compatible *iff (1) $\approx_{E_0} \subseteq \approx_E$; (2) E_0 has a model completion E_0^*; (3) every model of E embeds into a model of $E \cup E_0^*$.*

Examples of theories that satisfy this definition can be found in [2, 13, 14] and in Section 4. Here we just show two consequences that will be important when proving completeness of our combination procedure.

Lemma 3. *Assume that E_1 and E_2 are two equational theories over the respective signatures Σ_1 and Σ_2 that are both* E_0-compatible *for some equational theory E_0 with signature $\Sigma_0 = \Sigma_1 \cap \Sigma_2$. For $i = 0, 1, 2$, let $\mathcal{A}_i$ be a model of E_i such that $\mathcal{A}_0$ is a Σ_0-subalgebra of both $\mathcal{A}_1$ and $\mathcal{A}_2$. Then there are a model $\mathcal{A}$ of $E_1 \cup E_2$ and Σ_i-embeddings f_i of $\mathcal{A}_i$ into $\mathcal{A}$ whose restrictions to A_0 coincide.*

In the following, we call conjunctions of Σ-identities *e-formulae*. We will write $\varphi(\boldsymbol{x})$ to denote an e-formula φ all of whose variables are included in the tuple $\boldsymbol{x}$. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ we will write $\varphi(\boldsymbol{a})$ to denote that $\boldsymbol{a}$ is a tuple of free constants of the form $(a_1, \ldots, a_n)$ and $\varphi(\boldsymbol{a})$ is the formula obtained from φ by replacing every occurrence of x_i by a_i for $i = 1, \ldots, n$.

Lemma 4. *Let E_1 be* E_0-compatible *where E_1 and E_0 are equational theories over the respective signatures Σ_1 and Σ_0 with $\Sigma_1 \supseteq \Sigma_0$. Let $\psi_1(\boldsymbol{x}, \boldsymbol{y})$ be an e-formula in the signature Σ_1 and $\psi_2(\boldsymbol{y}, \boldsymbol{z})$ an e-formula in the signature Σ_0 such that $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0) \models_{E_1} \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$, where $\boldsymbol{a}_1$, $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$ are pairwise disjoint tuples of fresh constants. Then, there is an e-formula $\psi_0(\boldsymbol{y})$ in the signature Σ_0, such that $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0) \models_{E_1} \psi_0(\boldsymbol{a}_0)$ and $\psi_0(\boldsymbol{a}_0) \models_{E_0} \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$.*

Second restriction: We will require that all the finitely generated models of E_0 be finite. From a more syntactical point of view this means that if C_0 is a finite subset of C, then there are only finitely many E_0-equivalence classes of terms in $T(\Sigma_0, C_0)$. For our combination procedure to be effective, we must be able to compute representatives of these equivalence classes.

Definition 2. *An an equational theory E_0 over the signature Σ_0 is* effectively locally finite *iff for every (finite) tuple $\boldsymbol{c}$ of constants from C we can effectively compute a finite set of terms $R_{E_0}(\boldsymbol{c}) \subseteq T(\Sigma_0, \boldsymbol{c})$ such that*

1. *$s \not\approx_{E_0} t$ for all distinct $s, t \in R_{E_0}(\boldsymbol{c})$;*
2. *for all terms $s \in T(\Sigma_0, \boldsymbol{c})$, there is some $t \in R_{E_0}(\boldsymbol{c})$ such that $s \approx_{E_0} t$.*

Example 1. A well-known example of an effectively locally finite theory is the usual (equational) theory BA of Boolean algebras over the signature $\Sigma_{BA} := \{\cap, \cup, \overline{(\_)}, 1, 0\}$. In fact, if $\boldsymbol{c} = (c_1, \ldots, c_n)$, every ground Boolean term over the constants in $\boldsymbol{c}$ is equivalent in BA to a term in "conjunctive normal form," a meet of terms of the kind $d_1 \cup \cdots \cup d_n$, where each d_i is either c_i or $\overline{c}_i$. It is easy to see that the set $R_{BA}(\boldsymbol{c})$ of such normal forms is isomorphic to the powerset of the powerset of $\boldsymbol{c}$, which is effectively computable and has cardinality 2^{2^n}.

Third restriction: We will require that E_1 and E_2 be each a *conservative extensions* of E_0, i.e., for $i = 1, 2$ and for all $s, t \in T(\Sigma_0, V)$, $s \approx_{E_0} t$ iff $s \approx_{E_i} t$.

Fourth restriction: Finally, we will require the theory E_0 to have local solvers, in the sense that any finite set of equations can be *solved* with respect to any of its variables.

Definition 3 (Gaussian). *The equational theory E_0 is* Gaussian *iff for every e-formula $\varphi(\boldsymbol{x}, y)$ it is possible to compute an e-formula $C(\boldsymbol{x})$ and a term $s(\boldsymbol{x}, \boldsymbol{z})$ with fresh variables $\boldsymbol{z}$ such that*

$$\models_{E_0} \varphi(\boldsymbol{x}, y) \Leftrightarrow (C(\boldsymbol{x}) \wedge \exists \boldsymbol{z}.(y = s(\boldsymbol{x}, \boldsymbol{z}))) \tag{1}$$

We call the formula C the solvability condition *of φ w.r.t. y, and the term s a* (local) solver *of φ w.r.t. y in E_0.*

There is a close connection between the above definition and Gaussian elimination, which is explained in the following example.

Example 2. Let K be a fixed field (e.g., the field of rational or real numbers). We consider the theory of vector spaces over K whose signature consists of a symbol for addition, a symbol for additive inverse and, for every scalar $k \in K$, a unary function symbol $k \cdot (-)$. Axioms are the usual vector spaces axioms (namely, the Abelian group axioms plus the axioms for scalar multiplication). In this theory, terms are equivalent to linear homogeneous polynomials (with non-zero coefficients) over K. Every e-formula $\varphi(\boldsymbol{x}, y)$ can be transformed into a homogeneous system $t_1(\boldsymbol{x}, y) = 0 \wedge \cdots \wedge t_k(\boldsymbol{x}, y) = 0$ of linear equations with unknowns $\boldsymbol{x}, y$. If y does not occur in φ, then φ is its own solvability condition and any fresh variable z is a local solver of φ w.r.t. y.[5] If y occurs in φ, then (modulo easy algebraic transformations) we can assume that φ contains an equation of the form $y = t(\boldsymbol{x})$; this equation gives the local solver, which is $t(\boldsymbol{x})$ (the sequence of existential quantifiers $\exists \boldsymbol{z}$ in (1) is empty), whereas the solvability condition is the e-formula obtained from φ by eliminating y, i.e., replacing y by $t(\boldsymbol{x})$ everywhere in φ.

In Section 4 we will see that the theory of Boolean algebras introduced in Example 1 is not only Gaussian but also satisfies our other restrictions.

3 The combination procedure

In the following, we assume that E_1, E_2 are equational theories over the signatures Σ_1, Σ_2 with decidable word problems, and that there exists an equational theory E_0 over the signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$ such that

- E_0 is Gaussian and effectively locally finite;

[5] Note that φ is trivially equivalent to $\varphi \wedge \exists z.(y = z)$.

– for $i = 1, 2$, E_i is E_0-compatible and a conservative extension of E_0.

Abstraction rewrite systems. Our combination procedure works on the following data structure (where C is again a set of free constants disjoint with Σ_1 and Σ_2).

Definition 4. *An* abstraction rewrite system (ARS) *is a finite ground rewrite system R that can be partitioned into $R = R_1 \cup R_2$ so that*

- *for $i = 1, 2$, the rules of R_i have the form $a \to t$ where $a \in C$, $t \in T(\Sigma_i, C)$, and every constant a occurs at most once as a left-hand side in R_i;*
- *$R = R_1 \cup R_2$ is terminating.*

The ARS R is an initial *ARS iff every constant a occurs at most once as a left-hand side in the whole R.*

In particular, for $i = 1, 2$, R_i is also terminating, and the restriction that every constant occurs at most once as a left-hand side in R_i implies that R_i is confluent. We denote the unique normal form of a term s w.r.t. R_i by $s{\downarrow_{R_i}}$.

Given a ground rewrite system R, an equational theory E, and an e-formula ψ, we write $R \models_E \psi$ to express that $\{l \approx r \mid l \to r \in R\} \models_E \psi$.

Lemma 5. *Let $R = R_1 \cup R_2$ be an ARS, and $s, t \in T(\Sigma_i, C)$ for some $i \in \{1, 2\}$. Then $R_i \models_{E_i} s \approx t$ iff $s{\downarrow_{R_i}} \approx_{E_i} t{\downarrow_{R_i}}$.*

If we want to decide the word problem in $E_1 \cup E_2$, it is sufficient to consider ground terms with free constants, i.e., terms $s, t \in T(\Sigma_1 \cup \Sigma_2, C)$. Given such terms s, t we can employ the usual abstraction procedures that replace subterms by new constants in C (see, e.g., [7]) to generate terms $u, v \in T(\Sigma_0, C)$ and an initial ARS $R = R_1 \cup R_2$ such that $s \approx_{E_1 \cup E_2} t$ iff $R \models_{E_1 \cup E_2} u \approx v$. Thus, to decide $\approx_{E_1 \cup E_2}$, it is sufficient to devise a procedure that can solve problems of the form "$R \models_{E_1 \cup E_2} u \approx v$?" where R is an initial ARS and $u, v \in T(\Sigma_0, C)$.

The combination procedure. The input of the procedure is an initial ARS $R = R_1 \cup R_2$ and two terms $u, v \in T(\Sigma_0, C)$. Let $>$ be a total ordering of the left-hand side (lhs) constants of R such that for all $a \to t \in R$, t contains only lhs constants smaller than a (this ordering exists since R is terminating). Given this ordering, we can assume that $R = \{a_i \to t_i \mid i = 1, \ldots, n\}$ for some $n \geq 0$ where $a_n > a_{n-1} > \cdots > a_1$.

Note that u, v and each t_i may also contain free constants from C that are not left-hand side constants. In the following, we use $\boldsymbol{c}$ to denote a tuple of all these constants. Furthermore, for $j = 1, 2$ and $i = 0, \ldots, n$, we denote by $R_j^{(i)}$ the restriction of R_j to the rules whose left-hand sides are smaller or equal to a_i—where, by convention, $R_j^{(0)}$ is the empty system.

The combination procedure is described in Figure 1. First, note that all of the steps of the procedure are effective. Step 1 of the for loop is trivially effective; Step 2 is effective because E_0 is effectively locally finite by assumption. Step 3 is effective because the test that $R_j^{(i)} \models_{E_j} t \approx t'$ can be reduced by

Input: an initial ARS $R = R_1 \cup R_2 = \{a_i \to t_i \mid i = 1, \ldots, n\}$ and terms $u, v \in T(\Sigma_0, C)$.

Let $\boldsymbol{c}$ collect the free constants in R, u, v that are not in $\{a_1, \ldots, a_n\}$.

for $i = 1$ to n do

1. Let j be such that $a_i \to t_i \in R_j$ and k such that $\{j, k\} = \{1, 2\}$.
2. Let $T = R_{E_0}(a_1, \ldots, a_i, \boldsymbol{c})$ (see Definition 2).
3. For each pair of distinct terms $t, t' \in T$, test whether $R_j^{(i)} \models_{E_j} t \approx t'$.
4. Let $\varphi(a_1, \ldots, a_i, \boldsymbol{c})$ be the conjunction of those identities $t \approx t'$ for which the test succeeds.
5. Let $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ be a local solver of φ w.r.t. a_i in E_0.
6. Add to R_k the new rule $a_i \to s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$.

done

Output: "yes" if $R_1 \models_{E_1} u \approx v$, and "no" otherwise.

Fig. 1. The combination procedure.

Lemma 5 to testing that $t{\downarrow}_{R_j^{(i)}} \approx_{E_j} t'{\downarrow}_{R_j^{(i)}}$. The latter test is effective because, (i) the word problem in E_j is decidable by assumption and (ii) $R_j^{(i)}$ is confluent and terminating at each iteration of the loop. In Step 4 the formula φ can be computed because T is finite and the local solver in Step 5 can be computed by the algorithm provided by the definition of a Gaussian theory. Step 6 is trivial and for the final test after the loop, the same observations as for Step 3 apply.

A few more remarks on the procedure are in order. In the fifth step of the loop, $\boldsymbol{d}$ is a tuple of new constants introduced by the solver s. In the definition of a local solver, we have used variables instead of constants, but this difference will turn out to be irrelevant since free constants behave like variables. One may wonder why the procedure ignores the solvability condition for the local solver. The reason is that this condition follows from both R_1 and R_2, as will be shown in the proof of completeness.

Adding the new rule to R_k in the sixth step of the loop does not destroy the property of $R_1 \cup R_2$ being an ARS—although it will make it non-initial. In fact, $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ contains only lhs constants smaller than a_i, and R_k before did not contain a rule with lhs a_i because the input was an *initial* ARS.

The test after the loop is performed using R_1, E_1. The choice R_1 and E_1 versus R_2 and E_2 is arbitrary (see Lemma 6).

The correctness proof. Since the combinations procedure obviously terminates on any input, it is sufficient to show soundness and completeness. In the proofs, we will use $R_{1,i}, R_{2,i}$ to denote the updated rewrite systems obtained after step i in the loop ($R_{1,0}$ and $R_{2,0}$ are the input systems R_1 and R_2). Soundness is not hard to show (see [2]).

Proposition 1 (Soundness). *If the combination procedure answers "yes", then* $R_1 \cup R_2 \models_{E_1 \cup E_2} u \approx v$.

The following lemma, which is used in the completeness proof, depends on our definition of a Gaussian theory (see [2] for details).

Lemma 6. *For every ground e-formula* ψ *in the signature* $\Sigma_0 \cup \{a_1, .., a_n\} \cup \boldsymbol{c}$, $R_{1,n} \models_{E_1} \psi$ *iff* $R_{2,n} \models_{E_2} \psi$.

Proposition 2 (Completeness). *If* $R_1 \cup R_2 \models_{E_1 \cup E_2} u \approx v$, *then the combination procedure answers "yes".*

Proof. Since the procedure is terminating, it is enough to show that $R_{1,0} \cup R_{2,0} \not\models_{E_1 \cup E_2} u \approx v$ whenever the combination procedure answers "no". We do that by building a model of $R_{1,0} \cup R_{2,0} \cup E_1 \cup E_2$ that falsifies $u \approx v$. Let $\boldsymbol{a} := (a_1, \ldots, a_n)$ and let $k \in \{1, 2\}$. Where $\boldsymbol{c}$ is defined as in Figure 1 and $\boldsymbol{d}_k$ is a tuple collecting all the new constants introduced in the rewrite system R_k during execution of the procedure (see Step 4 of the loop), let $\mathcal{A}_{k,0}$ be the initial model (see, e.g., [5] for a definition) of E_k over the signature $\Sigma_k \cup \boldsymbol{c} \cup \boldsymbol{d_k}$.

Observe that the final rewrite system $R_{k,n}$ contains (exactly) one rule of the form $a_i \to u_i$ for all $i = 1, \ldots, n$. This is because either the rule $a_i \to t_i$ was already in $R_{k,0}$ to begin with (then $u_i = t_i$), or a rule of the form $a_i \to s_i$ for some solver s_i was added to $R_{k,i-1}$ at step i to produce $R_{k,i}$ (in which case $u_i = s_i$). Thus, we can use the rewrite rules of $R_{k,n}$ to define by induction on $i = 1, \ldots, n$ an expansion $\mathcal{A}_{k,i}$ of $\mathcal{A}_{k,0}$ to the constants $a_1, \ldots, a_i$. Specifically, $\mathcal{A}_{k,i}$ is defined as the expansion of $\mathcal{A}_{k,i-1}$ that interprets a_i as $u_i^{\mathcal{A}_{k,i-1}}$ where u_i is the term such that $a_i \to u_i \in R_{k,n}$. Note that $u_i^{\mathcal{A}_{k,i-1}}$ is well defined because u_i does not contain any of the constants $a_i, \ldots, a_n$.

By induction on i it is easy to show (see [2]) for every ground e-formula $\varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k)$ in the signature $\Sigma_k \cup \{a_1, \ldots, a_i\} \cup \boldsymbol{c} \cup \boldsymbol{d}_k$, that

$$\mathcal{A}_{k,i} \text{ satisfies } \varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k) \text{ iff } R_{k,n}^{(i)} \models_{E_k} \varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k). \tag{2}$$

Let $\mathcal{A}_k = \mathcal{A}_{k,n}^{\Omega_k}$ where $\Omega_k = \Sigma_k \cup \boldsymbol{a} \cup \boldsymbol{c}$. As a special case of (2) above, we have that for every ground e-formula $\varphi(\boldsymbol{a}, \boldsymbol{c})$ in the signature $\Sigma_0 \cup \boldsymbol{a} \cup \boldsymbol{c}$,

$$\mathcal{A}_k \text{ satisfies } \varphi \text{ iff } R_{k,n} \models_{E_k} \varphi. \tag{3}$$

For $k = 1, 2$ let $\mathcal{B}_k$ be the subalgebra of $\mathcal{A}_k^{\Sigma_0}$ generated by (the interpretations in $\mathcal{A}_k$ of) the constants $\boldsymbol{a} \cup \boldsymbol{c}$. We claim that the algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy each other's diagram. To see that, let ψ be a ground identity of signature $\Sigma_0 \cup \boldsymbol{a} \cup \boldsymbol{c}$. Then, $\psi \in \Delta^{\Sigma_0}_{\boldsymbol{a} \cup \boldsymbol{c}}(\mathcal{B}_k)$ iff $\mathcal{B}_k$ satisfies ψ (by definition of $\Delta^{\Sigma_0}_{\boldsymbol{a} \cup \boldsymbol{c}}(\mathcal{B}_k)$) iff $\mathcal{A}_k$ satisfies ψ (by construction of $\mathcal{B}_k$ and Lemma 2) iff $R_{k,n} \models_{E_k} \psi$ (by (3) above).

By Lemma 6, we can conclude that $\psi \in \Delta^{\Sigma_0}_{\boldsymbol{a} \cup \boldsymbol{c}}(\mathcal{B}_1)$ iff $\psi \in \Delta^{\Sigma_0}_{\boldsymbol{a} \cup \boldsymbol{c}}(\mathcal{B}_2)$. It follows from the observation after Lemma 1 that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Σ_0-isomorphic, hence they can be identified with no loss of generality. Therefore, let $\mathcal{A}_0 = \mathcal{B}_1 = \mathcal{B}_2$ and observe that (i) $\mathcal{A}_k^{\Sigma_k}$ is a model of E_k by construction; (ii) $\mathcal{A}_0$ is a Σ_0-subalgebra of $\mathcal{A}_k^{\Sigma_k}$; and (iii) $\mathcal{A}_0$ is a model of E_0 because $\mathcal{A}_k^{\Sigma_0}$ is a model of E_0 and the set of models of an equational theory is closed under subalgebras.

By Lemma 3 it follows that there is a model $\mathcal{A}$ of $E_1 \cup E_2$ such that there are Σ_k-embeddings f_k of $\mathcal{A}_k^{\Sigma_k}$ into $\mathcal{A}$ $(i = 1, 2)$ satisfying $f_1(c^{\mathcal{A}_1}) = f_2(c^{\mathcal{A}_2})$ for all $c \in \boldsymbol{a} \cup \boldsymbol{c}$. Let then $\mathcal{A}'$ be the expansion of $\mathcal{A}$ to the signature $\Sigma_1 \cup \Sigma_2 \cup \boldsymbol{a} \cup \boldsymbol{c}$ such that $c^{\mathcal{A}'} = f_1(c^{\mathcal{A}_1})$ for every $c \in \boldsymbol{a} \cup \boldsymbol{c}$. It is not difficult to see that f_k is an Ω_k-embedding of $\mathcal{A}_k$ into $\mathcal{A}'$ for $k = 1, 2$. Observe that $\mathcal{A}'$, which is clearly a model of $E_1 \cup E_2$, is also a model of $R_{1,0} \cup R_{2,0}$. In fact, by construction of $R_{1,n}$ and $R_{2,n}$, for all $a \to t \in R_{1,0} \cup R_{2,0}$, there is a $k \in \{1, 2\}$ such that $a \to t \in R_{k,n}$. It follows immediately that $R_{k,n} \models_{E_k} a \approx t$, which implies by (3) above that $\mathcal{A}_k$ satisfies $a \approx t$. But then $\mathcal{A}'$ satisfies $a \approx t$ as well by Lemma 2.

In conclusion, we have that $\mathcal{A}'$ is a model of $R_{1,0} \cup R_{2,0} \cup E_1 \cup E_2$. Since the procedure returns "no" by assumption, it must be that $R_{1,n} \not\models_{E_1} u \approx v$. We then have that $\mathcal{A}_1$ falsifies $u \approx v$ by (3) above and $\mathcal{A}'$ falsifies $u \approx v$ by Lemma 2. □

From the total correctness of the combination procedure, we then obtain the following modular decidability result.

Theorem 1. *Let E_0, E_1, E_2 be three equational theories of respective signature $\Sigma_0, \Sigma_1, \Sigma_2$ such that*

- *$\Sigma_0 = \Sigma_1 \cap \Sigma_2$;*
- *E_0 is Gaussian and effectively locally finite;*
- *for $i = 1, 2$, E_i is E_0-compatible and a conservative extension of E_0.*

If the word problem in E_1 and in E_2 is decidable, then the word problem in $E_1 \cup E_2$ is also decidable.

4 Fusion decidability in modal logics

First, we define the modal logics to which our combination procedure applies. Basically, these are modal logics that corresponds to equational extensions of the theory of Boolean algebras. A *modal signature* Σ_M is a set of operation symbols endowed with corresponding arities; from Σ_M propositional formulae are built up using countably many propositional variables, the operation symbols in Σ_M, the Boolean connectives, and the constant $\top$ for truth and $\bot$ for falsity. We use letters $x, x_1, \ldots, y, y_1, \ldots$ for propositional variables and letters $t, t_1, \ldots, u, u_1, \ldots$ as metavariables for propositional formulae. The following definition is adapted from [22].

Definition 5. *A* classical modal logic *L based on a modal signature Σ_M is a set of propositional formulae that (i) contains all classical tautologies; (ii) is closed under uniform substitution of propositional variables by propositional formulae; (iii) is closed under the modus ponens rule (from t and $t \Rightarrow u$ infer u); (iv) for each n-ary $o \in \Sigma_M$, is closed under the following replacement rule:*

$$\text{from} \quad t_1 \Leftrightarrow u_1, \; \ldots, \; t_n \Leftrightarrow u_n \quad \text{infer} \quad o(t_1, \ldots, t_n) \Leftrightarrow o(u_1, \ldots, u_n).$$

As classical modal logics (based on a given modal signature) are closed under intersections, it makes sense to speak of the least classical modal logic $[S]$ containing a certain set of propositional formulae S. If $L = [S]$, we say that S is a set of axiom schemata for L and write $S \vdash t$ for $t \in [S]$.

We say that a classical modal logic L is *decidable* iff L is a recursive set of propositional formulae; the *decision problem* for L is just the membership problem for L.

A classical modal logic L is said to be *normal* iff for every n-ary modal operator o in the signature of L and every argument position $i = 1, \ldots, n$, L contains the formulae $o(\boldsymbol{x}, \top, \boldsymbol{x}')$ and $o(\boldsymbol{x}, (y \Rightarrow z), \boldsymbol{x}') \Rightarrow (o(\boldsymbol{x}, y, \boldsymbol{x}') \Rightarrow o(\boldsymbol{x}, z, \boldsymbol{x}'))$. The least normal (classical modal, unary, unimodal) logic is the modal logic usually called **K** [8]. Most well-known modal logics considered in the literature (both normal and non-normal) fit Definition 5 (see [2] for some examples).

Let us call an equational theory *Boolean-based* if its signature includes the signature Σ_{BA} of Boolean algebras and its axioms include the Boolean algebras axioms BA (see Example 1). For notational convenience, we will assume that Σ_{BA} also contains the binary symbol $\supset$, defined by the axiom $x \supset y \approx \overline{x} \cup y$.

Given a classical modal logic L we can associate with it a Boolean-based equational theory E_L. Conversely, given a Boolean-based equational theory E we can associate with it a classical modal logic L_E. In fact, given a classical modal logic L with modal signature Σ_M, we define E_L as the theory having as signature $\Sigma_M \cup \Sigma_{BA}$ and as a set of axioms the set $BA \cup \{t_{BA} \approx 1 \mid t \in L\}$ where t_{BA} is obtained from t by replacing t's logical connectives $(\neg, \wedge, \vee, \Rightarrow)$ by the corresponding Boolean algebra operators $(\overline{(\_)}, \cap, \cup, \supset)$, and the logical constants $\top$ and $\bot$ by 1 and 0, respectively. Vice versa, given a Boolean-based equational theory E over the signature Σ, we define L_E as the classical modal logic over the modal signature $\Sigma \setminus \Sigma_{BA}$ axiomatized by the formulae $\{t_L \mid \models_E t \approx 1\}$ where t_L is obtained from t by the inverse of the replacement process above.

Classical modal logics (in our sense) and Boolean-based equational theories are equivalent formalisms, as is well-known from algebraic logic [20]. In particular, for our purposes, the following standard proposition is crucial, as it reduces the decision problem for a classical modal logic L to the word problem in E_L.

Proposition 3. *For every classical modal logic L and for every propositional formula t, we have that $t \in L$ iff $\models_{E_L} t_{BA} \approx 1$.*

Given two classical modal logics L_1, L_2 over two *disjoint* modal signatures Σ_M^1, Σ_M^2, the *fusion* of L_1 and L_2 is the classical modal logic $L_1 \oplus L_2$ over the signature $\Sigma_M^1 \cup \Sigma_M^2$ defined as $[L_1 \cup L_2]$. As $E_{L_1 \oplus L_2}$ is easily seen to be deductively equivalent to the theory $E_{L_1} \cup E_{L_2}$ (i.e., $\approx_{E_{L_1 \oplus L_2}} = \approx_{E_{L_1} \cup E_{L_2}}$), it is clear that the decision problem $L_1 \cup L_2 \vdash t$ reduces to the word problem $E_{L_1} \cup E_{L_2} \models t_{BA} \approx 1$. Our goal in the remainder of this section is to show that, thanks to the combination result in Theorem 1, this combined word problem for $E_{L_1} \cup E_{L_2}$ reduces to the single word problems for E_{L_1} and E_{L_2}, and thus to the decision problems for L_1 and L_2.

Note that, although the modal signatures Σ_M^1 and Σ_M^2 are disjoint, the signatures of E_{L_1} and E_{L_2} are no longer disjoint, because they share the Boolean operators. To show that our combination theorem applies to E_{L_1} and E_{L_2}, we thus must establish that the common subtheory BA of Boolean algebras matches the requirements for our combination procedure. To this end, we will restrict ourselves to component modal logics L_1 and L_2 that are *consistent*, that is, do not include $\bot$, (or, equivalently, do not contain all modal formulae over their signature). This restriction is without loss of generality because when either L_1 or L_2 are inconsistent $L_1 \oplus L_2$ is inconsistent as well, which means that its decision problem is trivial.

We have already shown in Section 2 that BA satisfies one of our requirements, namely effective local finiteness. As for the others, for every consistent classical modal logic L, the theory E_L is guaranteed to be a conservative extension of BA. The main reason is that there are no non-trivial equational extensions of the theory of Boolean algebras. In fact, as soon as one extends BA with an axiom $s \approx t$ for any s and t such that $s \not\approx_{BA} t$, the equation $0 \approx 1$ becomes valid.[6] By Proposition 3, this entails that if an equational theory E_L induced by a classical modal logic L is not a conservative extension of BA then $L \vdash \bot$. Hence L cannot be consistent.

Thus, it remains to be shown that BA is Gaussian and that E_L is BA-compatible for every consistent classical modal logic L. For space constraints we cannot do this here, but we refer the interested reader to [2] for complete proofs. Here, we just point out how the local solver looks like for BA. For each e-formula of the form $u(\boldsymbol{x}, y) \approx 1$ (and fresh variable z), the term

$$s(\boldsymbol{x}, z) := (u(\boldsymbol{x}, 1) \supset u(\boldsymbol{x}, z)) \supset (z \cap (u(\boldsymbol{x}, 0) \supset u(\boldsymbol{x}, z))) \tag{4}$$

is a local solver for $u(\boldsymbol{x}, y) \approx 1$ in BA w.r.t. y. Note that $s(\boldsymbol{x}, z)$ can be computed in linear time from $u(\boldsymbol{x}, y)$ and that the restriction to formulae of the form $u(\boldsymbol{x}, y) \approx 1$ can be made with no loss of generality because every Boolean *e*-formula can be (effectively) converted in linear time into a BA-equivalent *e*-formula of that form.

Combining Theorem 1 with the results above on the theories BA and E_L, we get the following general modular decidability result.

Theorem 2. *If L_1, L_2 are decidable classical modal logics, so is $L_1 \oplus L_2$.*

In [2] we also show that the complexity upper-bounds for the combined decision procedures obtained by applying our combination procedure to classical modal logics are not worse than the ones given in [4] for the case of normal modal logics. If the decision procedures for L_1 and for L_2 are in PSPACE, we get an EXPSPACE combined decision procedure for $L_1 \oplus L_2$. If instead the procedures are in EXPTIME, we get a 2EXPTIME combined decision procedure.

We close this section by giving an examples of our combination procedure at work.

[6] This is can be shown by a proper instantiation of the variables of $s \approx t$ by 0 and 1, followed by simple Boolean simplifications.

Example 3. Consider the classical modal logic **KT** with modal signature $\{\Box\}$ and obtained by adding to **K** the axiom schema $\Box\, x \Rightarrow x$. Now let $\mathbf{KT}_1$ and $\mathbf{KT}_2$ be two signature disjoint renamings of **KT** in which $\Box_1$ and $\Box_2$, respectively, replace $\Box$, and consider the fusion logic $\mathbf{KT}_1 \oplus \mathbf{KT}_2$. We can use our combination procedure to show that $\mathbf{KT}_1 \oplus \mathbf{KT}_2 \vdash \Box_2 x \Rightarrow \Diamond_1 x$ (where as usual $\Diamond_1 x$ abbreviates $\neg\,\Box_1\,\neg x$). For $i = 1, 2$, let E_i be the equational theory corresponding to $\mathbf{KT}_i$. It is enough to show that

$$\models_{E_1 \cup E_2} \ (\Box_2(x) \supset \Diamond_1(x)) \approx 1$$

where now $\Diamond_1 x$ abbreviates $\overline{\Box_1(\overline{x})}$. After the abstraction process, we get the two rewrite systems $R_1 = \{a_1 \to \Diamond_1(c)\}$ and $R_2 = \{a_2 \to \Box_2(c)\}$ and the goal equation $(a_2 \supset a_1) \approx 1$ where a_1, a_2 and c are fresh constants.

As explained in [2], for the test in Step 3 of the procedure's loop we need to consider only identities of the form $t \approx 1$ where t is a term-clause over the set of constants under consideration.[7] During the first execution of the procedure's loop the constants in question are a_1 and c, therefore there are only four identities to consider: $\overline{a}_1 \cup \overline{c} \approx 1$, $\overline{a}_1 \cup c \approx 1$, $a_1 \cup \overline{c} \approx 1$, and $a_1 \cup c \approx 1$. The only identity for which the test is positive is $a_1 \cup \overline{c}$. In fact, $a_1 \cup \overline{c}$ rewrites to $\Diamond_1(c) \cup \overline{c}$, which is equivalent to $c \supset \Diamond_1(c)$. This is basically the contrapositive of (the translation of) the axiom schema $\Box_1(c) \supset c$.[8]

Using the formula (4) seen earlier, we can produce a solver for that identity, which reduces to $c \cup d_1$ after some simplifications, where d_1 is a fresh free constant. Hence, the following rewrite rule is added to R_2 in Step 6 of the loop: $a_1 \to c \cup d_1$.

Continuing the execution of the loop with the second—and final—iteration, we get the following. Among the eight term-clauses involving a_1, a_2, c, the test in Step 3 is positive for four of them. The conjunction of such term-clauses gives a Boolean e-formula that is equivalent to $(a_2 \supset c) \cap (c \supset a_1) \approx 1$. This e-formula, once solved with respect to a_2, gives (after simplifications) the rewrite rule $a_2 \to d_2 \cap ((c \supset a_1) \supset (d_2 \supset c))$, which is added to R_1 before quitting the loop. Using this R_1, the final test of the procedure ($R_1 \models_{E_1} a_2 \supset a_1 \approx 1$) succeeds because the modal formula $d_2 \wedge ((c \Rightarrow \Diamond_1 c) \Rightarrow (d_2 \Rightarrow c)) \Rightarrow \Diamond_1 c$ is a theorem of $\mathbf{KT}_1$.

5 Conclusion

In this paper, we have described a new approach for combining decision procedures for the word problem in equational theories over *non-disjoint* signatures. Unlike the previous combination methods for the word problem [7, 12] in the

[7] For a given set of constants $c_1, \ldots, c_m$, a term-clause is a term of the form $b_1 \cup \cdots \cup b_m$ where each b_j is either c_j or $\overline{c}_j$.

[8] Another approach for checking this, and also that the tests for the other term-clauses are negative, is to translate the rewritten term-clauses into the corresponding modal formulae, and then check whether their complement is unsatisfiable in all Kripke structures with a reflexive accessibility relation (see [10], Fig. 5.1).

non-disjoint case, this approach has the known decidability transfer results for *validity* in the fusion of modal logics [15, 25] as consequences. Our combination result is however more general than these transfer results since it applies also to *non-normal* modal logics—thus answering in the affirmative a long-standing open question in modal logics—and to equational theories not induced by modal logics (see, e.g., Example 2). Nevertheless, for the modal logic application, the complexity upper-bounds obtained through our approach are the same as for the more restricted approaches [25, 4].

Our results are not consequences of combination results for the conditional word problem (the relativized validity problem) recently obtained by generalizing the Nelson-Oppen combination method [13, 14]. In fact, there are modal logics (obtained by translating certain description logics into modal logic notation) for which the validity problem is decidable, but the relativized validity problem is not. This is, e.g, the case for description logics with feature agreements [1] or with concrete domains [3].

Our new combination approach is orthogonal to the previous combination approaches for the word problem in equational theories over non-disjoint signatures [7, 12]. On the one hand, the previous results do not apply to theories induced by modal logics [12]. On the other hand, there are equational theories that (i) satisfy the restrictions imposed by the previous approaches, and (ii) are not locally finite [7], and thus do not satisfy our restrictions. Both the approach described in this paper and those in [7, 12] have the combination results for the case of disjoint signatures as a consequence. For the previous approaches, this was already pointed out in [7, 12]. For our approach, this is not totally obvious since some minor technical problems have to be overcome (see [2] for details).

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