$\mathcal{E}$-connections of abstract description systems

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Abstract

Combining knowledge representation and reasoning formalisms is an important and challenging task. It is important because non-trivial AI applications often comprise different aspects of the world, thus requiring suitable combinations of available formalisms modeling each of these aspects. It is challenging because the computational behavior of the resulting hybrids is often much worse than the behavior of their components.

In this paper, we propose a new combination method which is computationally robust in the sense that the combination of decidable formalisms is again decidable, and which, nonetheless, allows non-trivial interactions between the combined components.

The new method, called $\mathcal{E}$-connection, is defined in terms of abstract description systems (ADSs), a common generalization of description logics, many logics of time and space, as well as modal and epistemic logics. The basic idea of $\mathcal{E}$-connections is that the interpretation domains of n combined systems are disjoint, and that these domains are connected by means of n-ary 'link relations.' We define several natural variants of $\mathcal{E}$-connections and study in-depth the transfer of decidability from the component systems to their $\mathcal{E}$-connections.

Key words: description logics, temporal logics, spatial logics, combining logics, decidability.

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1 Introduction

Logic-based formalisms play a prominent role in modern Artificial Intelligence (AI) research. The numerous logical systems employed in various applications can roughly be divided into three categories:

(1) very expressive but undecidable logics, typically variants of first- or higher-order logics;
(2) quantifier-free formalisms of low computational complexity (typically P- or NP-complete), such as (fragments of) classical propositional logic and its non-monotonic variants;
(3) decidable logics with restricted quantification located 'between' propositional and first-order logics; typical examples are modal, description and propositional temporal logics.

The use of formalisms of the third kind is motivated by the fact that logics of category (2) are often not sufficiently expressive, e.g., for terminological, spatial, and temporal reasoning, while logics of the first kind are usually too complex to be used for efficient reasoning in realistic application domains.

Thus, the trade-off between expressiveness and effectiveness is the main design problem in the third approach, with decidability being an important indicator that the computational complexity of the language devised *might* be sufficiently low for successful applications. Over the last few years, an enormous progress has been made in the design and implementation of special purpose languages in this area—witness surprisingly fast representation and reasoning systems of description and temporal logics [58,40,65,44]. In contrast to first-order and propositional logics, however, these systems are useful only for very specific tasks, say, pure temporal, spatial, or terminological reasoning.

Since usually realistic application domains comprise various aspects of the world, the next target within this third approach is the design of suitable *combinations* of formalisms modeling each of these aspects. Following the underlying idea that to devise useful languages one has to search for a compromise between expressiveness and effectiveness, the problem then is to find combination methodologies which are sufficiently robust in the sense that the computational behavior of the resulting hybrids should not be much worse than that of the combined components. The need for such methodologies has been clearly recognized by the AI community (it suffices to mention the workshop series 'Frontiers of Combining Systems' FroCoS'96–02 and subsequent volumes [15,22,45,5]), and various approaches to combining logics have been proposed, e.g., description logics with concrete domains [56], multi-dimensional spatio-temporal logics [77,78], independent fusions and fibring [28,46,25,13], temporalized logics [26], temporal epistemic logic [23], or more general logics of rational agency [73,60].

In this paper we introduce and investigate a novel combination method with a wide range of applications and a very robust computational behavior (*in the sense that the combination is decidable whenever all of its components are decidable*).

This combination method can be applied in the following setting. Suppose that we have n mutually disjoint domains $D_1, \dots, D_n$ together with appropriate languages $L_1, \dots, L_n$ for speaking about them. Although the domains are disjoint, they can represent different aspects of the same objects (say, a concrete house as an instance of a general concept house, its spatial extension and life span). So we can assume that we have a set $\mathcal{E} = \{E_j \mid j \in J\}$ of links establishing certain relations $E_j \subseteq D_1 \times \cdots \times D_n$ among objects of the domains.

Now we form a new language L containing all of the L_i, $1 \leq i \leq n$, which is supposed to talk about the union $\bigcup_{i=1}^n D_i$, where the D_i are connected by the links in $\mathcal{E}$. The fragments L_i of L can still talk about each of the D_i, but the super-language L contains extra $(n-1)$-ary operators $\langle E_j \rangle^i$, $1 \leq i \leq n$, $j \in J$, which, given an input $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$, for $X_\ell \subseteq D_\ell$, return

$$\{x \in D_i \mid \forall \ell \neq i \; \exists x_\ell \in X_\ell \; (x_1, \dots, x_{i-1}, x, x_{i+1}, x_n) \in E_j\}.$$

In other words, the value of $\langle E_j \rangle^i (X_1, \dots, X_{i-1}, X_{i+1}, \dots X_n)$ is the i-th factor of

$$(X_1 \times \cdots \times X_{i-1} \times D_i \times X_{i+1} \times \cdots \times X_n) \cap E_j.$$

For instance, if $i = 2$ then, for all $X_1 \subseteq D_1$ and $X_2 \subseteq D_2$, we have

$$\begin{aligned} x_1 \in \langle E_j \rangle^1 (X_2) \quad &\text{iff} \quad \exists x_2 \in X_2 \; (x_1, x_2) \in E_j, \\ x_2 \in \langle E_j \rangle^2 (X_1) \quad &\text{iff} \quad \exists x_1 \in X_1 \; (x_1, x_2) \in E_j. \end{aligned}$$

We call the new system L the *basic $\mathcal{E}$-connection* of $L_1, \dots, L_n$. The operators $\langle E_j \rangle^i$ correspond to the exists-restrictions of standard description logics [9], or, in terms of first-order logic, to an E_j-guarded quantification over the members of foreign domains [2].

Here are four simple examples of $\mathcal{E}$-connections; in more detail they will be considered in Section 4.

Description Logic–Spatial Logic. A description logic L_1 (say, $\mathcal{ALC}$ or $\mathcal{SHIQ}$ [42]) talks about a domain D_1 of abstract objects. A spatial logic L_2 (say, qualitative $\mathsf{S4}_u$ [70,16,66,30] or quantitative $\mathcal{MS}$ [69,48]) talks about some spatial domain D_2. An obvious $\mathcal{E}$-connection is given by the relation $E \subseteq D_1 \times D_2$ defined by taking $(x, y) \in E$ iff y belongs to the spatial extension of x—whenever x occupies some space. Then, given an L_1-concept, say, river, the operator $\langle E \rangle^2$ (river) provides us with the spatial extension of all rivers. Conversely, given a spatial region of L_2, say, the Alps, $\langle E \rangle^1$ (Alps) provides the concept comprising all objects whose spatial extension has a non-empty intersection with the Alps. So the concept country$\sqcap\langle E \rangle^1$ (Alps) will then denote the set of all alpine countries.

Description Logic–Temporal Logic. Now let L_3 be a temporal logic (say, point-based PTL [29] or Halpern–Shoham's logic of intervals HS [34]) and let D_3 be a set of time points or, respectively, time intervals interpreting L_3. In this case, a natural relation $E \subseteq D_1 \times D_3$ is given by taking $(x, y) \in E$ iff y belongs to the life-span of x.

Description Logic–Description Logic. Besides the description logic L_1 talking about the domain D_1, another description logic L_4 may be given that is used to formalize knowledge about a domain D_4 closely related to D_1. For instance, if L_1 talks about countries and companies, while L_4 talks about people, we may have two relations $W, L \subseteq D_1 \times D_4$, where $(x, y) \in W$ iff y works in x (for x a company) and $(x, y) \in L$ iff y lives in x (for x a country). Typically, L_1 and L_4 will also use different sets of concept constructors.

Similar combinations, called *distributed description logics*, have been constructed by Borgida and Serafini [18] whose motivation was the integration of and logical reasoning in loosely federated information systems. In more detail the relationship between $\mathcal{E}$-connections of description logics and distributed description logics will be analyzed in Section 6, where we will show that distributed description logics can be thought of as special instances of $\mathcal{E}$-connections.

Description Logic–Spatial Logic–Temporal Logic. Further, we can combine the three logics L_1, L_2, L_3 above into a single formalism by defining a ternary relation $E \subseteq D_1 \times D_2 \times D_3$ such that $(x, y, z) \in E$ iff y belongs to the spatial extension of x at moment (interval) z.

This is a rough idea. To make it more precise and to provide evidence for the claim that this combination technique is computationally robust, we will use the framework of *abstract description systems* (ADSs, for short) introduced in [13]. Basically, all description, modal, temporal, epistemic and similar logics (in particular, modal logics of space) can be represented in the form of ADSs with the same computational behavior as the original formalisms. For this reason, ADSs appear to be a good level of abstraction for investigating $\mathcal{E}$-connections.

The next question is how we can 'prove' that the formation of $\mathcal{E}$-connections is a computationally robust operation. In this paper we adopt the idea that a proof of the decidability of the main reasoning services provided by a formalism is an important indication that the computational behavior of the formalism might be sufficiently good for applications.

Thus, our aim is to prove transfer results of the following form:

(1) *if a certain reasoning service for each of the component ADSs of an $\mathcal{E}$-connection is decidable, then this reasoning service for the $\mathcal{E}$-connection itself is decidable as well.*

On the other hand, to show that our results are in a sense optimal and that indeed we have found (at least on the theoretical level) a good compromise between expressivity and effectiveness, we provide examples which demonstrate that

(2) *the transfer results in* (1) *do not hold if we take more expressive $\mathcal{E}$-connections.*

All 'positive' decidability transfer theorems come with the following complexity result:

(3) *the time complexity of a reasoning service for an $\mathcal{E}$-connection is at most one non-deterministic exponential higher than the maximal time complexity of its components; in some cases this upper bound is optimal.*

The increase of the worst-case time complexity by one exponential shows that in general $\mathcal{E}$-connections are not given 'for free.' On the other hand, this result also shows that the formation of $\mathcal{E}$-connections is a 'relatively cheap' combination methodology compared, for instance, with the multi-dimensional approach (see Section 8.1). Of course, only experiments can show whether a *particular* $\mathcal{E}$-connection is of sufficiently low complexity to be useful in practice; this obviously cannot be done in a paper providing a formal framework. However, the idea underlying the decidability transfer theorems is not only to indicate that practical algorithms may exist for some particular cases, but also to help the designer of such algorithms by means of the insights provided by the proofs of these theorems.

The structure of the paper is as follows. Section 2 introduces abstract description systems and four logic-based knowledge representation (KR) formalisms that will be used in examples of $\mathcal{E}$-connections. In Section 3 we introduce the notion of a basic $\mathcal{E}$-connection [1] and discuss transfer results for this combination method. Examples illustrating basic $\mathcal{E}$-connections are provided in Section 4. In Section 5 we consider extended $\mathcal{E}$-connections which allow more interaction between the combined formalisms than basic $\mathcal{E}$-connections (for example, Boolean combinations of connecting relations or 'qualified number restrictions' on them). Decidability results as well as counterexamples for the

[1] Basic $\mathcal{E}$-connections were first introduced and investigated in [49].

transfer of decidability describe the trade-off between expressive power and computational behavior. In Section 6 we consider the relation between $\mathcal{E}$-connections and distributed description logics recently introduced in [18], and in Section 7 we discuss the extension of basic $\mathcal{E}$-connections by means of first-order constraints on the links between the domains. Finally, in Section 8 we briefly discuss $\mathcal{E}$-connections in the light of other combination methodologies. All of the proofs are collected in Appendices A–C.

2 Abstract description systems

Abstract description systems (ADSs) have been proposed in [13] as a common generalization of description logics, modal logics, temporal logics, and some other formalisms. Our presentation of ADSs in this section will be brief, yet self-contained. As illustrating examples, we describe several logics that have been proposed in the literature for knowledge representation and reasoning, and show how these logics can be viewed as abstract description systems. For more details about ADSs the interested reader is referred to [13].

An abstract description system consists of an abstract description language and a class of admissible models specifying the intended semantics.

Definition 1 An *abstract description language* (ADL) $\mathcal{L}$ is determined by a countably infinite set $\mathcal{V}$ of *set variables*, a countably infinite set $\mathcal{X}$ of *object variables*, a countable set $\mathcal{R}$ of *relation symbols* R of arity m_R, and a countable set $\mathcal{F}$ of *function symbols* f of arity n_f such that $\neg, \wedge \notin \mathcal{F}$. The *terms* t_j of $\mathcal{L}$ are built in the following way:

$$t_j \quad ::= \quad x \quad | \quad \neg t_1 \quad | \quad t_1 \wedge t_2 \quad | \quad f(t_1, \dots, t_{n_f}),$$

where $x \in \mathcal{V}$ and $f \in \mathcal{F}$. The *term assertions* of $\mathcal{L}$ are of the form $t_1 \sqsubseteq t_2$, where t_1 and t_2 are terms, and the *object assertions* are

- $R(a_1, \dots, a_{m_R})$, for $a_1, \dots, a_{m_R} \in \mathcal{X}$ and $R \in \mathcal{R}$;
- $a : t$, for $a \in \mathcal{X}$ and t a term.

The sets of term and object assertions together form the set of $\mathcal{L}$-*assertions*. We will write $t_1 = t_2$ as an abbreviation for the two assertions $t_1 \sqsubseteq t_2,\ t_2 \sqsubseteq t_1$.

The semantics of ADLs is defined via abstract description models.

Definition 2 Given an ADL $\mathcal{L} = \langle \mathcal{V}, \mathcal{X}, \mathcal{R}, \mathcal{F} \rangle$, an *abstract description model* (ADM) for $\mathcal{L}$ is a structure of the form

$$\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}} = (x^{\mathfrak{W}})_{x \in \mathcal{V}}, \mathcal{X}^{\mathfrak{W}} = (a^{\mathfrak{W}})_{a \in \mathcal{X}}, \mathcal{F}^{\mathfrak{W}} = (f^{\mathfrak{W}})_{f \in \mathcal{F}}, \mathcal{R}^{\mathfrak{W}} = (R^{\mathfrak{W}})_{R \in \mathcal{R}} \right\rangle,$$

where W is a non-empty set, $x^{\mathfrak{W}} \subseteq W$, $a^{\mathfrak{W}} \in W$, each $f^{\mathfrak{W}}$ is a function mapping n_f-tuples $\left\langle X_1, \ldots, X_{n_f} \right\rangle$ of subsets of W to a subset of W, and the $R^{\mathfrak{W}}$ are m_R-ary relations on W.

The *value* $t^{\mathfrak{W}} \subseteq W$ of an $\mathcal{L}$-term t in $\mathfrak{W}$ is defined inductively by taking

- $(\neg t)^{\mathfrak{W}} = W \setminus (t)^{\mathfrak{W}}, \quad (t_1 \wedge t_2)^{\mathfrak{W}} = t_1^{\mathfrak{W}} \cap t_2^{\mathfrak{W}}$,
- $(f(t_1, \ldots, t_{m_f}))^{\mathfrak{W}} = f^{\mathfrak{W}}(t_1^{\mathfrak{W}}, \ldots, t_{m_f}^{\mathfrak{W}})$.

The *truth-relation* $\mathfrak{W} \models \varphi$ for an $\mathcal{L}$-assertion φ is defined in the obvious way:

- $\mathfrak{W} \models R(a_1, \ldots, a_{m_R}) \quad \text{iff} \quad R^{\mathfrak{W}}(a_1^{\mathfrak{W}}, \ldots, a_{m_R}^{\mathfrak{W}})$,
- $\mathfrak{W} \models a : t \quad \text{iff} \quad a^{\mathfrak{W}} \in t^{\mathfrak{W}}$,
- $\mathfrak{W} \models t_1 \sqsubseteq t_2 \quad \text{iff} \quad t_1^{\mathfrak{W}} \subseteq t_2^{\mathfrak{W}}$.

If $\mathfrak{W} \models \varphi$ holds, we say that φ is *satisfied* in $\mathfrak{W}$. For sets Γ of assertions, we write $\mathfrak{W} \models \Gamma$ if $\mathfrak{W} \models \varphi$ holds for all $\varphi \in \Gamma$.

ADSs become a powerful tool by providing a choice of an appropriate class of ADMs in which the ADL is interpreted. In this way, we can, e.g., ensure that a function symbol has the desired semantics, and that relation symbols are interpreted as relations having desired properties, say, transitivity.

Definition 3 An *abstract description system* (ADS) is a pair $(\mathcal{L}, \mathcal{M})$, where $\mathcal{L}$ is an ADL and $\mathcal{M}$ is a class of ADMs for $\mathcal{L}$ that is closed under the following operations:

(i) if $\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$ and $\mathcal{V}^{\mathfrak{W}'} = (x^{\mathfrak{W}'})_{x \in \mathcal{V}}$ is a new assignment of set variables in W, then $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}'}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$ as well;

(ii) for every finite $\mathcal{G} \subseteq \mathcal{F}$, there exists a finite set $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}$ such that, for every $\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ from $\mathcal{M}$ and every assignment $\mathcal{X}^{\mathfrak{W}'} = (a^{\mathfrak{W}'})_{a \in \mathcal{X}}$ of object variables in W such that $a^{\mathfrak{W}} = a^{\mathfrak{W}'}$ for all $a \in \mathcal{X}_{\mathcal{G}}$, there is an interpretation $\mathcal{F}^{\mathfrak{W}'} = (f^{\mathfrak{W}'})_{f \in \mathcal{F}}$ of the function symbols such that $f^{\mathfrak{W}'} = f^{\mathfrak{W}}$ for all $f \in \mathcal{G}$ and $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}'}, \mathcal{F}^{\mathfrak{W}'}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$.

The first closure condition imposed on the class of models $\mathcal{M}$ means that set variables are treated as *variables* in any ADS, i.e., their values are not fixed. Closure condition (ii) deals with object variables and is slightly weaker; it states that object variables behave almost like variables with the exception that the interpretation of a finite number of function symbols may determine the assignments of a finite number of object variables. This weakening is required to enable the representation of the important 'nominal-constructor' from modal and description logic (which associates with any object variable a nullary function symbol; see below for more details) in abstract description systems. Mostly, however, the example ADSs we are going to discuss satisfy the stronger condition:

(ii′) if $\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle \in \mathcal{M}$ and $\mathcal{X}^{\mathfrak{W}'} = (a^{\mathfrak{W}'})_{a \in \mathcal{X}}$ is a new assignment of object variables in W, then $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}'}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$ as well.

The main reasoning task for ADSs we are concerned with is the *satisfiability problem* for finite sets of assertions.

Definition 4 Let $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ be an ADS. A finite set Γ of $\mathcal{L}$-assertions is called *satisfiable* in $\mathcal{S}$ if there exists an ADM $\mathfrak{W} \in \mathcal{M}$ such that $\mathfrak{W} \models \Gamma$.

Note that the entailment of term assertions and object assertions of the form $a : t$—to decide, given such an assertion φ and a finite set of assertions Γ, whether $\mathfrak{W} \models \Gamma$ implies $\mathfrak{W} \models \varphi$ for all models $\mathfrak{W}$—is clearly reducible to the satisfiability problem. For example, Γ entails $a : t$ iff $\Gamma \cup \{a : \neg t\}$ is not satisfiable. The satisfiability problem for an ADS $\mathcal{S}$ restricted to sets Γ of *object assertions* will be called the *A-satisfiability problem* for $\mathcal{S}$ (here ‘A’ stands for ABox; see below).

We now introduce several logics that have been proposed for knowledge representation and reasoning in AI, and show how these logics can be viewed as ADSs. Again, our presentation will be brief but self-contained. For readers not familiar with the presented formalisms we give pointers to the literature. Moreover, examples of the use of these formalisms can be found in Section 4 illustrating $\mathcal{E}$-connections.

2.1 Description logics

Description logics (DLs) are formalisms devised for the representation of and reasoning about conceptual knowledge. Such knowledge is represented in terms of compound concepts which are composed from atomic concepts (unary predicates) and roles (binary predicates) using the concept and role constructors provided by the given DL. Description logic knowledge bases consist of

- a TBox containing concept inclusion statements of the form $C_1 \sqsubseteq C_2$, where both C_1 and C_2 are concepts, and
- an ABox containing assertions of the form $a : C$ and $(a, b) : R$, where a, b are object names, C is a concept, and R is a role.

Description logics have found applications in various fields of Artificial Intelligence, for example, as languages for describing ontologies in the context of the semantic web. More information on DLs can be found in the recent handbook [9]. It has been shown in [13] that almost all description logics can be regarded as ADSs. Here we briefly describe three description logics and their translations into ADSs. We start with the basic description logic $\mathcal{ALC}$.

The alphabet of $\mathcal{ALC}$ is comprised of concept names $A_1, A_2, \dots$, role names $R_1, R_2, \dots$, object names $a_1, a_2, \dots$, the Boolean constructors $\neg$ and $\sqcap$, and the existential and the universal restrictions $\exists$ and $\forall$, respectively. $\mathcal{ALC}$-*concepts* C_i are built according to the following rule:

$$C_i \quad ::= \quad A_i \quad | \quad \neg C_1 \quad | \quad C_1 \sqcap C_2 \quad | \quad \exists R.C \quad | \quad \forall R.C$$

As usual, we use $C_1 \sqcup C_2$ as an abbreviation for $\neg(\neg C_1 \sqcap \neg C_2)$, and $\exists R.C$ as an abbreviation for $\neg\forall R.\neg C$. An $\mathcal{ALC}$-*model* is a structure of the form

$$\mathcal{I} = \left\langle \Delta, A_1^{\mathcal{I}}, \dots, R_1^{\mathcal{I}}, \dots, a_1^{\mathcal{I}}, \dots \right\rangle,$$

where Δ is a non-empty set, the $A_i^{\mathcal{I}}$ are subsets of Δ, the $R_i^{\mathcal{I}}$ are binary relations on Δ, and the $a_i^{\mathcal{I}}$ are elements of Δ. The interpretation of complex concepts is defined by setting

$$(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}, \;\; (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}},$$
$$(\exists R.C)^{\mathcal{I}} = \{w \in \Delta \mid \exists v \in \Delta((w,v) \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}})\},$$
$$(\forall R.C)^{\mathcal{I}} = \{w \in \Delta \mid \forall v \in \Delta((w,v) \in R^{\mathcal{I}} \to v \in C^{\mathcal{I}})\}.$$

The concepts of $\mathcal{ALC}$ can be regarded as terms $C^\sharp$ of an ADS $\mathcal{ALC}^\sharp$. Indeed, we can associate with each concept name A_i a set variable $A_i^\sharp$, with each role name R_i two unary function symbols $f_{\forall R_i}$ and $f_{\exists R_i}$, and then set inductively:

$$(\neg C)^\sharp = \neg C^\sharp, \qquad (C \sqcap D)^\sharp = C^\sharp \wedge D^\sharp,$$
$$(\exists R_i.C)^\sharp = f_{\exists R_i}(C^\sharp), \;\; (\forall R_i.C)^\sharp = f_{\forall R_i}(C^\sharp).$$

The object names of $\mathcal{ALC}$ are treated as object variables of $\mathcal{ALC}^\sharp$ and the role names as its binary relations. Thus, $\mathcal{ALC}^\sharp$-term assertions correspond to concept inclusion statements, while object assertions correspond to ABox assertions. The class $\mathcal{M}$ of ADMs for $\mathcal{ALC}^\sharp$ is defined as follows. For every $\mathcal{ALC}$-model $\mathcal{I} = \left\langle \Delta, A_1^{\mathcal{I}}, \dots, R_1^{\mathcal{I}}, \dots, a_1^{\mathcal{I}}, \dots \right\rangle$, the class $\mathcal{M}$ contains the model

$$\mathfrak{M} = \left\langle \Delta, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}}, \mathcal{R}^{\mathfrak{M}} \right\rangle,$$

where, for every concept name A, role name R, and every object name a,

$$(A^\sharp)^{\mathfrak{M}} = A^{\mathcal{I}}, \qquad R^{\mathfrak{M}} = R^{\mathcal{I}}, \qquad a^{\mathfrak{M}} = a^{\mathcal{I}},$$
$$f^{\mathfrak{M}}_{\exists R}(X) = \{w \in \Delta \mid \exists v \; ((w,v) \in R^{\mathcal{I}} \wedge v \in X)\},$$
$$f^{\mathfrak{M}}_{\forall R}(X) = \{w \in \Delta \mid \forall v \; ((w,v) \in R^{\mathcal{I}} \to v \in X)\}.$$

Observe that the semantics of the function symbols $f_{\exists R}$ and $f_{\forall R}$ is obtained in a straightforward way from the semantics of the DL constructors $\exists R.C$ and $\forall R.C$. Since the interpretations of concept and object names can be changed arbitrarily, $\mathcal{M}$ satisfies the closure conditions (i) and (ii′) (and therefore (ii) as well). Now, considering this translation, it is easily seen that

- the satisfiability problem of $\mathcal{ALC}^\sharp$ corresponds to the problem of whether an $\mathcal{ALC}$-ABox is satisfiable with respect to a TBox; [2]
- the A-satisfiability problem of $\mathcal{ALC}^\sharp$ corresponds to the problem of whether an $\mathcal{ALC}$-ABox is satisfiable without any reference to TBoxes.

Our second description logic $\mathcal{SHIQ}$ extends $\mathcal{ALC}$ by various additional constructors. For brevity, we define here only those that will be used in the examples later on, viz., inverse roles and qualified number restrictions. The *inverse roles* allow us to use roles of the form R^{-1} (where R is a role name) in place of role names, and the *qualified number restrictions* are concept constructors of the form $(\geq nR.C)$ and $(\leq nR.C)$; their semantics is almost obvious:

$$(R^{-1})^{\mathcal{I}} = \{(w,v) \mid (v,w) \in R^{\mathcal{I}}\},$$
$$(\geq nR.C)^{\mathcal{I}} = \{w \in \Delta \mid \left|\{v \in \Delta \mid (w,v) \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}}\}\right| \geq n\},$$
$$(\leq nR.C)^{\mathcal{I}} = \{w \in \Delta \mid \left|\{v \in \Delta \mid (w,v) \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}}\}\right| \leq n\}.$$

More details on $\mathcal{SHIQ}$ can be found in [42,43]. By extending the translation $^\sharp$ of $\mathcal{ALC}$ above in a straightforward way, one can transform $\mathcal{SHIQ}$ into the corresponding ADS $\mathcal{SHIQ}^\sharp$. Details of this translation can be found in [13].

The third description logic we deal with is called $\mathcal{ALCO}$; it extends $\mathcal{ALC}$ with the nominal constructor $\{a\}$, where a is an object name; cf. [63,41]. The semantics of the concepts $\{a\}$ is as expected: $\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$. Thus, the difference between $\mathcal{ALC}$ and $\mathcal{ALCO}$ is that $\mathcal{ALCO}$ allows the use of object names *in concepts* rather than only in ABox assertions. The corresponding ADS $\mathcal{ALCO}^\sharp$ is obtained from $\mathcal{ALC}^\sharp$ by introducing, for every object variable a of $\mathcal{ALC}^\sharp$, the nullary function symbol f_a such that, for every model $\mathfrak{M}$, $f_a^{\mathfrak{M}} = \{a^{\mathfrak{M}}\}$, and by setting $\{a\}^\sharp = f_a$. While $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ satisfy

[2] Note that in the literature the TBoxes we are concerned with are usually called *general* TBoxes.

the closure condition (ii′) following Definition 3—simply observe that there is no interaction between the interpretation of function symbols and object variables—this is obviously not the case for $\mathcal{ALCO}^\sharp$, since, by changing the assignment of an object variable a, we also change the interpretation of the nullary function symbol f_a. However, $\mathcal{ALCO}^\sharp$ does satisfy (ii). Indeed, given a finite set $\mathcal{G}$ of function symbols of $\mathcal{ALCO}^\sharp$, let $\mathcal{X}_\mathcal{G}$ be the set of all object variables a such that $f_a \in \mathcal{G}$. Now, for any new assignment of the variables in $\mathcal{X} \setminus \mathcal{X}_\mathcal{G}$, the new interpretation of the function symbols not occurring in $\mathcal{G}$ is obtained by interpreting every nominal f_a, $a \in \mathcal{X} \setminus \mathcal{X}_\mathcal{G}$, as the singleton set containing the object newly assigned to a. The remaining function symbols are interpreted as before.

To determine the computational complexity of reasoning with the ADSs defined above, let us recall that, for $\mathcal{ALC}$, $\mathcal{SHIQ}$, and $\mathcal{ALCO}$, ABox-satisfiability with respect to TBoxes is EXPTIME-complete [20,71,3]. It follows immediately that we have the following:

Proposition 5 *The satisfiability problem for $\mathcal{ALC}^\sharp$, $\mathcal{SHIQ}^\sharp$, and $\mathcal{ALCO}^\sharp$ is EXPTIME-complete.*

In what follows, it will turn out that the difference between $\mathcal{ALC}$ and $\mathcal{ALCO}$ is rather important, also on the level of ADSs. To be precise about the notion of 'nominal' within the framework of ADSs, we require one more definition.

Definition 6 An ADS $\mathcal{S} = (\mathcal{L}, \mathcal{M})$, where $\mathcal{L} = \langle \mathcal{V}, \mathcal{X}, \mathcal{R}, \mathcal{F} \rangle$, is said to *have nominals* if $\mathcal{F}$ contains a nullary function symbol f_a, for each $a \in \mathcal{X}$, such that, for every $\mathfrak{W} = \left\langle W, \mathcal{V}^\mathfrak{W}, \mathcal{X}^\mathfrak{W}, F^\mathfrak{W}, R^\mathfrak{W} \right\rangle$ in $\mathcal{M}$, we have $f_a^\mathfrak{M} = \{a^\mathfrak{M}\}$. Usually, we will denote the function symbols f_a by $\{a\}$ and call them *nominals*.

The ADS $\mathcal{ALCO}^\sharp$ obviously has nominals in the sense of this definition, while the ADSs $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ do not.

Remark 7 There is a close connection between nominals and object assertions: for an ADS with nominals, object assertions of the form $a : t$ can be reformulated as $\{a\} \sqsubseteq t$. On the other hand, in general object assertions of the form $R(a_1, \ldots, a_m)$ cannot be rephrased in this style. Yet, for some ADSs they are equivalent to assertions of the form $\{a_1\} \sqsubseteq f(a_2, \ldots, a_m)$, as will be clear from examples below. We could give a more general definition of 'to have nominals' by replacing nullary function symbols f_a with terms t_a. The results we are going to obtain for ADSs with nominals hold true under this more general definition as well.

In the examples below, some expressive means provided by the ADSs have no direct counterparts in the corresponding logics. For instance, none of these logics has explicit term and object assertions. However, we will see that this additional expressivity can be regarded just as 'syntactic sugar.'

The modal logic $\mathsf{S4}_u$, i.e., Lewis's modal system $\mathsf{S4}$ enriched with the universal modality, is an important formalism for reasoning about spatial knowledge. Tarski [70] interpreted the basic $\mathsf{S4}$ (without the universal modality) in topological spaces as early as 1938. Later, the universal box was added in order to allow the representation of and reasoning about the well-known $\mathsf{RCC\text{-}8}$ set of relations between two regions in a topological space [59,61,16,62,66,78,30]. We discuss the encoding of the $\mathsf{RCC\text{-}8}$ relations in $\mathsf{S4}_u$ in Section 4.2.

The language of $\mathsf{S4}_u$ is built from *region variables* $X_1, X_2, \dots$ (in the modal context, *propositional variables*), the Boolean operators, the *interior operator* $\boldsymbol{I}$ (the *necessity operator*), and the *universal quantifier* $\boxed{\forall}$ (the *universal box*). More precisely, $\mathsf{S4}_u$-*formulas* φ_i are defined as follows:

$$\varphi_i \quad ::= \quad X_j \quad | \quad \neg\varphi_1 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \boldsymbol{I}\varphi_1 \quad | \quad \boxed{\forall}\varphi_1.$$

As usual, we use $\varphi_1 \vee \varphi_2$ as an abbreviation for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\Diamond\!\!\!\exists\,\varphi$ (the *universal diamond*) as an abbreviation for $\neg\boxed{\forall}\neg\varphi$, and the *closure operator* $\boldsymbol{C}\varphi$ (the *possibility operator*) as an abbreviation for $\neg\boldsymbol{I}\neg\varphi$. A (*topological*) $\mathsf{S4}_u$-*model*

$$\mathcal{I} = \left\langle T, \mathbb{I}, \mathbb{C}, X_1^{\mathcal{I}}, X_2^{\mathcal{I}}, \dots \right\rangle$$

consists of a topological space $\langle T, \mathbb{I}\rangle$, where $\mathbb{I}$ is an *interior operator* mapping subsets X of T to their *interior* $\mathbb{I}(X) \subseteq T$ and satisfying Kuratowski's axioms

$$\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y), \quad \mathbb{I}\mathbb{I}(X) = \mathbb{I}(X), \quad \mathbb{I}(X) \subseteq X,$$

for all $X, Y \subseteq T$, $\mathbb{C}$ is the closure operator defined by $\mathbb{C}(X) = T \setminus \mathbb{I}(T \setminus X)$, and the $X_i^{\mathcal{I}}$ are subsets of T (interpreting the region variables of $\mathsf{S4}_u$). The *value* $\varphi^{\mathcal{I}}$ of an $\mathsf{S4}_u$-formula φ in $\mathcal{I}$ is defined inductively in the natural way:

$$\begin{aligned}
&(\neg\psi)^{\mathcal{I}} = T \setminus \psi^{\mathcal{I}}, \ \ (\chi \wedge \psi)^{\mathcal{I}} = \chi^{\mathcal{I}} \cap \psi^{\mathcal{I}},\\
&(\boldsymbol{I}\psi)^{\mathcal{I}} = \mathbb{I}\psi^{\mathcal{I}}, \qquad (\boxed{\forall}\psi)^{\mathcal{I}} = \begin{cases} \emptyset & \text{if } \psi^{\mathcal{I}} \neq T,\\ T & \text{if } \psi^{\mathcal{I}} = T.\end{cases}
\end{aligned}$$

We say that φ is *satisfiable* if there is an $\mathsf{S4}_u$-model $\mathcal{I}$ such that $\varphi^{\mathcal{I}} \neq \emptyset$.

Let us see now how $\mathsf{S4}_u$ can be represented as an ADS $\mathsf{S4}_u^\sharp$. The corresponding ADL contains the set variables $X_1^\sharp, X_2^\sharp, \dots$, the unary function symbols $f_{\boldsymbol{I}}$ and $f_{\boxed{\forall}}$, but no relation symbols. Besides, according to the definition, $\mathsf{S4}_u^\sharp$ must contain a countably infinite set of object variables a_i. The translation $^\sharp$ of $\mathsf{S4}_u$-formulas into $\mathsf{S4}_u^\sharp$-terms is obvious, e.g., $(\Box\varphi)^\sharp = f_\Box(\varphi^\sharp)$, where $\Box \in \{\boldsymbol{I}, \boxed{\forall}\}$.

Define a class $\mathcal{M}$ of ADMs for $\mathsf{S4}_u^\sharp$ by taking, for every $\mathsf{S4}_u$-model $\mathcal{I}$ as above, the ADMs

$$\mathfrak{M} = \left\langle T, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, f_{\boldsymbol{I}}^{\mathfrak{M}}, f_{\boxed{\forall}}^{\mathfrak{M}} \right\rangle,$$

where $(X_i^\sharp)^{\mathfrak{M}} = X^{\mathcal{I}_i}$, $a^{\mathfrak{M}} \in T$, for every $a \in \mathcal{X}$, $f_{\boldsymbol{I}}^{\mathfrak{M}} = \mathbb{I}$, and, for every $Y \subseteq T$,

$$f_{\boxed{\forall}}^{\mathfrak{M}}(Y) = \begin{cases} \emptyset & \text{if } Y \neq T, \\ T & \text{if } Y = T. \end{cases}$$

Obviously, $\mathsf{S4}_u^\sharp$ satisfies the closure conditions (i) and (ii) of Definition 3 (it even satisfies (ii′)); so it is an ADS. Unlike $\mathsf{S4}_u^\sharp$, the logic $\mathsf{S4}_u$ does not have assertions of the form $t_1 \sqsubseteq t_2$ or $a : t$. So we have to be careful when relating the computational complexity of $\mathsf{S4}_u^\sharp$ to that of $\mathsf{S4}_u$. The proof of the following proposition can be found in Appendix A.

Proposition 8 *The satisfiability problem for* $\mathsf{S4}_u^\sharp$ *is PSPACE-complete.*

Note that $\mathsf{S4}_u^\sharp$ does not have nominals.

2.3 A logic of metric spaces

Formalisms like $\mathsf{S4}_u$ allow the representation of qualitative spatial knowledge using, e.g., the RCC-8 relations. Motivated by the fact that many spatial AI applications also require representations of *quantitative* information, a family of logics of metric spaces has been introduced in [69,48,47,79]. Here, we consider a member of this family called $\mathcal{MS}$ and define a corresponding ADS. [3]

The language of $\mathcal{MS}$ consists of *region terms* constructed from *region variables* X_i and *location variables* a_i using the Booleans, the operators $\mathsf{E}_{\leq r}$ and $\mathsf{E}_{>r}$, for $r \in \mathbb{Q}^+$, and the *nominal constructor* giving the region term $\{a_i\}$ for every location variable a_i. More precisely, $\mathcal{MS}$*-formulas* φ_i are defined as follows:

$$\varphi_i \quad ::= \quad X_j \quad | \quad \{a_k\} \quad | \quad \neg\varphi_1 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \mathsf{E}_{\leq r}\varphi_1 \quad | \quad \mathsf{E}_{>r}\varphi_1,$$

Intuitively, given a set X in a metric space, $\mathsf{E}_{\leq r}X$ is the set of all points in the space located at distance $\leq r$ from (at least one point in) X. We use $\mathsf{A}_{\leq r}X$ as abbreviation for $\neg\mathsf{E}_{\leq r}\neg X$ and $\mathsf{A}_{>r}X$ for $\neg\mathsf{E}_{>r}\neg X$. Thus, a point is in $\mathsf{A}_{>r}X$ iff the complement of its r-neighborhood is in X. An $\mathcal{MS}$*-model*

$$\mathcal{I} = \left\langle W, \delta, X_1^{\mathcal{I}}, \ldots, a_1^{\mathcal{I}}, \ldots \right\rangle$$

consists of a metric space $\langle W, \delta\rangle$ together with interpretations of set variables X_i as subsets $X_i^{\mathcal{I}}$ of W and location variables a_i as elements $a_i^{\mathcal{I}}$ of W. We

[3] The logic we consider here is called $\mathcal{MS}_2$ in [69] and $\mathcal{MS}^\sharp$ in [48].

remind the reader that δ is a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying the axioms

$$\begin{aligned}
&\delta(x,y) = 0 \quad \text{iff} \quad x = y,\\
&\delta(x,z) \leq \delta(x,y) + \delta(y,z),\\
&\delta(x,y) = \delta(y,x),
\end{aligned}$$

for all $x, y, z \in W$. The value $\delta(x,y)$ is called the *distance* from x to y.

The semantics of complex concepts is defined in the usual way, the only interesting cases being:

$$\begin{aligned}
(\mathsf{E}_{\leq r}\varphi)^{\mathcal{I}} &= \{w \in W \mid \exists v\ (\delta(w,v) \leq r \wedge v \in \varphi^{\mathcal{I}})\},\\
(\mathsf{E}_{> r}\varphi)^{\mathcal{I}} &= \{w \in W \mid \exists v\ (\delta(w,v) > r \wedge v \in \varphi^{\mathcal{I}})\}.
\end{aligned}$$

To define a corresponding ADS $\mathcal{MS}^\sharp$, we reserve a set variable $X_i^\sharp$ for each region variable X_i, an object variable $a_i^\sharp$ for each location variable a_i, and take unary function symbols $f_{\mathsf{E}_{\leq r}}$ and $f_{\mathsf{E}_{>r}}$ for each $r \in \mathbb{Q}^+$. Again, the set of relation symbols is empty. It should now be clear how to devise a translation $^\sharp$ of $\mathcal{MS}$-formulas into $\mathcal{MS}^\sharp$-set terms and to describe the class of ADMs similarly to what was done in the preceding two sections. Note that the semantics of the function symbols $f_{\mathsf{E}_{\leq r}}$ and $f_{\mathsf{E}_{>r}}$ can be derived from the semantics of the $\mathsf{E}_{\leq r}$ and $\mathsf{E}_{>r}$ operators in a straightforward way. As a consequence of the decidability and complexity results from [48,79], we obtain:

Proposition 9 *The satisfiability problem for $\mathcal{MS}^\sharp$ is EXPTIME-complete (even if the parameters r are represented in binary).*

The proof is similar to the proof of Proposition 8 because in $\mathcal{MS}$ we can define the universal box $\boxed{\forall}\varphi$ as, e.g., $\mathsf{A}_{>1}\varphi \wedge \mathsf{A}_{\leq 1}\varphi$.

$\mathcal{MS}^\sharp$ does have nominals.

2.4 Propositional temporal logic

Finally, we consider the propositional temporal logic PTL [29,31,23] which is a well-known tool for reasoning about time. PTL-*formulas* φ_i are composed from *propositional variables* p_i by means of the Booleans and the binary *temporal operators* $\mathcal{U}$ ('until') and $\mathcal{S}$ ('since'):

$$\varphi_i \quad ::= \quad p_j \quad | \quad \neg\varphi_1 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \mathcal{U} \varphi_2 \quad | \quad \varphi_1 \mathcal{S} \varphi_2.$$

We introduce $\Diamond_F\varphi$ ('eventually φ'), $\Box_F\varphi$ ('always in the future φ'), $\Diamond_P\varphi$ ('sometime in the past φ'), $\Box_P\varphi$ ('always in the past φ') as abbreviations for

$\top\mathcal{U}\varphi$, $\neg\Diamond_F\neg\varphi$, $\top\mathcal{S}\varphi$, and $\neg\Diamond_P\neg\varphi$, respectively. A PTL-*model* is a structure of the form

$$\mathcal{I} = \left\langle \mathbb{N}, <, p_0^{\mathcal{I}}, p_1^{\mathcal{I}}, \dots \right\rangle,$$

where $\langle \mathbb{N}, < \rangle$ is the intended *flow of time*, and $p_i^{\mathcal{I}} \subseteq \mathbb{N}$. The *temporal extension* $\varphi^{\mathcal{I}}$ of a PTL-formula φ is defined inductively in the standard way, the interesting cases being:

$$(\varphi_1 \mathcal{U} \varphi_2)^{\mathcal{I}} = \{u \in \mathbb{N} \mid \exists z > u\ (z \in \varphi_2^{\mathcal{I}} \wedge \forall y \in (u,z)\ y \in \varphi_1^{\mathcal{I}})\},$$
$$(\varphi_1 \mathcal{S} \varphi_2)^{\mathcal{I}} = \{u \in \mathbb{N} \mid \exists z < u\ (z \in \varphi_2^{\mathcal{I}} \wedge \forall y \in (z,u)\ y \in \varphi_1^{\mathcal{I}})\},$$

where $(u,v) = \{w \in \mathbb{N} \mid u < w < v\}$.

To obtain the corresponding ADS $\mathsf{PTL}^\sharp$, we associate with $\mathcal{U}$ and $\mathcal{S}$ *binary* function symbols $f_{\mathcal{U}}$ and $f_{\mathcal{S}}$. It is not hard now to define a translation $\sharp$ from PTL-formulas to $\mathsf{PTL}^\sharp$-terms. We represent individual time points and the precedence relation $<$ by adding nominals and the relation symbol $<$ to PTL, i.e., the language $\mathsf{PTL}^\sharp$ has the function symbols $f_{\mathcal{U}}$, $f_{\mathcal{S}}$ and $\{a\}$, for any object variable a, and the binary relation symbol $<$ interpreted by the precedence relation on $\mathbb{N}$. Note that although PTL itself contains none of these explicitly, nominals $\{a\}$ (and so object variables) can be simulated as PTL-formulas $p_a \wedge \neg\Diamond_F p_a \wedge \neg\Diamond_P p_a$, and the assertion $a < b$ can be simulated as

$$(p_a \wedge \neg\Diamond_F p_a \wedge \neg\Diamond_P p_a) \wedge \Diamond_F (p_b \wedge \neg\Diamond_F p_b \wedge \neg\Diamond_P p_b).$$

The definition of the class of ADMs for $\mathsf{PTL}^\sharp$ is now straightforward. The proof of the following proposition can be found in Appendix A.

Proposition 10 *The satisfiability problem for* $\mathsf{PTL}^\sharp$ *is PSPACE-complete.*

3 Connections of abstract description systems

In this section, we introduce the basic variant of $\mathcal{E}$-connections and show that decidability transfers from the component formalisms to their combination, whereas A-satisfiability does not.

Suppose that we want to connect n ADSs $\mathcal{S}_1, \dots, \mathcal{S}_n$, where $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$ for $1 \le i \le n$. Without loss of generality we assume that, for $1 \le i < j \le n$, the alphabets of the ADSs $\mathcal{S}_i$ and $\mathcal{S}_j$ (i.e., the sets of set variables, object variables, function symbols, and relation symbols) are disjoint apart from the Boolean operators. To connect $\mathcal{S}_1, \dots, \mathcal{S}_n$, we take (i) a non-empty set of n-ary relation symbols

$$\mathcal{E} = \{E_j \mid j \in J\},$$

and (ii) for $1 \leq i \leq n$ and each $j \in J$, function symbols $\langle E_j \rangle^i$ of arity $n-1$ that are distinct from the function symbols of $\mathcal{S}_1, \ldots, \mathcal{S}_n$. In what follows, we will call the elements of $\mathcal{E}$ *link relations* (or *links*, for short) and the function symbols $\langle E_j \rangle^i$ *link operators*.

We define the *$\mathcal{E}$-connection* $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ of $\mathcal{S}_1, \ldots, \mathcal{S}_n$ following the definition of ADSs: first we introduce terms of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$, then assertions, and finally define a class of models and a truth-relation between these models and assertions. The set of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$-terms is partitioned into n sets, each of which contains *i-terms* for some i, $1 \leq i \leq n$. Intuitively, i-terms are the terms of $\mathcal{L}_i$ enriched with the new function symbols $\langle E_j \rangle^i$ for each $j \in J$. Here is a formal inductive definition:

- every set variable of $\mathcal{L}_i$ is an i-term;
- the set of i-terms is closed under $\neg$, $\wedge$ and the function symbols of $\mathcal{L}_i$;
- if $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ is a sequence of k-terms t_k for $k \neq i$, then

$$\langle E_j \rangle^i (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$$

  is an i-term, for every $j \in J$.

There are three types of assertions of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. Two of these types are the term assertions and object assertions of the component ADSs. Additionally, to be able to speak about the new ingredients of $\mathcal{E}$-connections, link relations, we require so-called *link assertions*. A formal definition is as follows: for $1 \leq i \leq n$,

- the *i-term assertions* are of the form $t_1 \sqsubseteq t_2$, where both t_1 and t_2 are i-terms;
- the *i-object assertions* are of the form $a : t$ or $R(a_1, \ldots, a_{m_R})$, where a and $a_1, \ldots, a_{m_R}$ are object variables of $\mathcal{L}_i$, t is an i-term, and R is a relation symbol of $\mathcal{L}_i$.
- the *link assertions* are of the form $(a_1, \ldots, a_n) : E_j$, where the a_i are object variables of $\mathcal{L}_i$, $1 \leq i \leq n$, and $j \in J$.

Taken together, the sets of all link assertions, i-term assertions, and i-object assertions form the set of *assertions* of the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. A finite set of assertions is also called a *knowledge base* of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$.

We now introduce the semantics of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. A structure

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} = (E_j^{\mathfrak{M}})_{j \in J}) \right\rangle,$$

where $\mathfrak{W}_i \in \mathcal{M}_i$ for $1 \leq i \leq n$ and $E_j^{\mathfrak{M}} \subseteq W_1 \times \cdots \times W_n$ for each $j \in J$, is called a *model for* $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. The *extension* $t^{\mathfrak{M}} \subseteq W_i$ of an i-term t is defined by induction. For set and object variables X and a of $\mathcal{L}_i$, we put $X^{\mathfrak{M}} = X^{\mathfrak{W}_i}$ and $a^{\mathfrak{M}} = a^{\mathfrak{W}_i}$. The inductive steps for the Booleans and function symbols of $\mathcal{L}_i$ are the same as in Definition 2:

- $(\neg t_1)^{\mathfrak{M}} = W_i \setminus t_1^{\mathfrak{M}}$, $\quad (t_1 \wedge t_2)^{\mathfrak{M}} = t_1^{\mathfrak{M}} \wedge t_2^{\mathfrak{M}}$,
- $(f(t_1, \ldots, t_{m_f}))^{\mathfrak{M}} = f^{\mathfrak{W}_i}(t_1^{\mathfrak{M}}, \ldots, t_{m_f}^{\mathfrak{M}})$.

Now let $\overline{t}_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ be a sequence of j-terms t_j. Then set

$$(\langle E_j \rangle^i (\overline{t}_i))^{\mathfrak{M}} = \{x \in W_i \mid \underset{\ell \neq i}{\exists} x_\ell \in t_\ell^{\mathfrak{M}} \ (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}}\}.$$

Finally, the extension $R^{\mathfrak{M}}$ of a relation symbol R of $\mathcal{L}_i$ is just $R^{\mathfrak{W}_i}$.

The *truth-relation* $\models$ between models $\mathfrak{M}$ for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and assertions of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is defined in the obvious way:

- $\mathfrak{M} \models t_1 \sqsubseteq t_2 \quad \text{iff} \quad t_1^{\mathfrak{M}} \subseteq t_2^{\mathfrak{M}}$,
- $\mathfrak{M} \models a : t \quad \text{iff} \quad a^{\mathfrak{M}} \in t^{\mathfrak{M}}$,
- $\mathfrak{M} \models R(a_1, \ldots, a_{m_R}) \quad \text{iff} \quad R^{\mathfrak{M}}(a_1^{\mathfrak{M}}, \ldots, a_{m_R}^{\mathfrak{M}})$,
- $\mathfrak{M} \models (a_1, \ldots, a_n) : E_j \quad \text{iff} \quad E_j^{\mathfrak{M}}(a_1^{\mathfrak{M}}, \ldots, a_n^{\mathfrak{M}})$.

As in the case of ADSs, we say that φ is *satisfied* in $\mathfrak{M}$ if $\mathfrak{M} \models \varphi$. A set Γ of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$-assertions is *satisfiable* if there exists a model $\mathfrak{M}$ for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ which satisfies all assertions in Γ. In this case we write $\mathfrak{M} \models \Gamma$. If Γ contains only object assertions then, as before, we use the term *A-satisfiability* instead of satisfiability. As in the case of ADSs, the entailment of term assertions and object assertions of the form $a : t$ can be reduced to the satisfiability problem.

Observe that, technically, the $\mathcal{E}$-connection of ADSs is not an ADS itself because the structure of models for $\mathcal{E}$-connections is different from the structure of models for ADSs. This approach was taken on purpose. Since we define the $\mathcal{E}$-connection as an n-ary operation, there is hardly any need to connect $\mathcal{E}$-connections. An alternative would be to extend the definition of ADSs in order to capture $\mathcal{E}$-connections. Although this is not a problem in general, it would further complicate the definition of ADSs and, in turn, also of $\mathcal{E}$-connections.

Several examples of $\mathcal{E}$-connections are given in the next section. For now, we refer the reader to Fig. 1 for an illustration of the semantics of $\mathcal{E}$-connections: the figure displays the connection of two ADSs by means of a single link relation E, highlighting the extensions of two 1-terms and two 2-terms (one of the latter is a nominal and thus has a singleton extension).

Our central result on $\mathcal{E}$-connections is that they preserve decidability of the satisfiability problem:

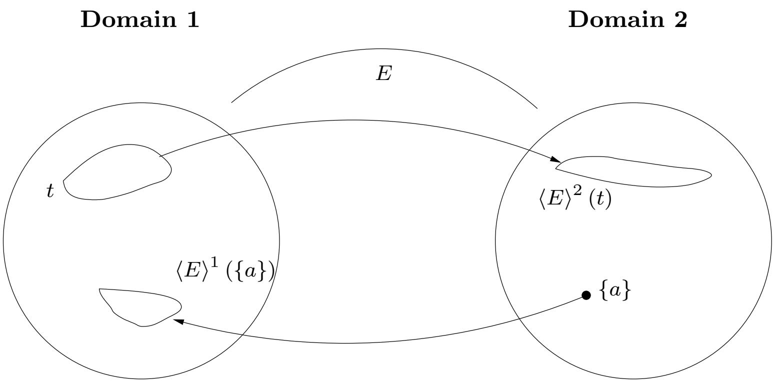

Fig. 1. A two-dimensional connection.

Theorem 11 *Let $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ be an $\mathcal{E}$-connection of ADSs $\mathcal{S}_1, \ldots, \mathcal{S}_n$. If the satisfiability problem for each of $\mathcal{S}_1, \ldots, \mathcal{S}_n$ is decidable, then it is decidable for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ as well.*

A proof of this theorem can be found in Appendix B.1. Intuitively, the decision procedure for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ works as follows (for simplicity, we confine ourselves to the connection of *two* ADSs and a single connection relation). To check whether there exists a model $\mathfrak{M} = \langle \mathfrak{W}_1, \mathfrak{W}_2, E \rangle$ of a given set of assertions Γ, the algorithm non-deterministically 'guesses'

(1) the 1-types that are realized in $\mathfrak{W}_1$ and the 2-types that are realized in $\mathfrak{W}_2$, where an i-type is a set of i-terms satisfied by a domain element of $\mathfrak{W}_i$; and
(2) a binary relation e between the guessed sets of 1-types and 2-types.

Then it checks whether the guessed sets satisfy a set of integrity conditions. This check involves satisfiability tests of certain sets of $\mathcal{S}_i$-assertions ($i = 1, 2$) constructed from Γ—here we use the fact that the satisfiability problems for $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable. If the integrity conditions are satisfied, then it is possible to construct a model of Γ using models of the constructed sets of $\mathcal{S}_i$-assertions. If the integrity conditions are not satisfied, Γ has no model.

This algorithm also provides an upper complexity bound for the satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$: the time complexity of our algorithm is one exponential higher than the time complexity of the original decision procedures for $\mathcal{S}_1, \ldots, \mathcal{S}_n$. Moreover, the combined decision procedure is non-deterministic. It is an open problem whether this complexity result is optimal. We can, however, show that there indeed exist cases where the complexity of the $\mathcal{E}$-connection is higher than the complexity of the combined formalisms, namely, growing from NP to EXPTIME. Let $\mathcal{B} = (\mathcal{L}_{\mathcal{B}}, \mathcal{M}_{\mathcal{B}})$ be the ADS, where

- $\mathcal{L}_{\mathcal{B}}$ is the abstract description language *without* any function and relation symbols (but, by definition, with the Booleans, infinitely many set variables

and infinitely many object variables);
- $\mathcal{M}_{\mathcal{B}}$ consists of all ADMs for $\mathcal{L}_{\mathcal{B}}$.

$\mathcal{B}$ can be regarded as the basic ADS from which all others are obtained by adding function and relation symbols and/or constraints on the ADMs. Obviously, the satisfiability problems for $\mathcal{B}$ and classical propositional logic are mutually reducible to each other. So we have:

Lemma 12 *The satisfiability problem for $\mathcal{B}$ is NP-complete.*

On the other hand, the $\mathcal{E}$-connection of $\mathcal{B}$ with itself is quite powerful:

Theorem 13 *The satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$ is EXPTIME-hard for any infinite set $\mathcal{E}$ of links.*

This result is proved in Appendix C.3 by reduction of the satisfiability of $\mathcal{ALC}$-concepts with respect to (general) TBoxes, which is known to be EXPTIME-hard [64]. Intuitively, $\mathcal{B}$ is used for the Boolean part of $\mathcal{ALC}$, while the link relations and link operators simulate roles and value- and exists-restrictions, respectively.

In contrast to full satisfiability, the decidability of A-satisfiability is not preserved under the formation of $\mathcal{E}$-connections. Consider the description logic $\mathcal{ALCF}$ which is the extension of $\mathcal{ALC}$ with functional roles and the feature agreement and disagreement constructors. More precisely, the set of role names of $\mathcal{ALCF}$ is partitioned into two sets R and F, where the elements of F (called *features*) are interpreted as partial functions. For any two sequences of features $p = f_1 \cdots f_k$ and $q = f'_1 \cdots f'_\ell$, $\mathcal{ALCF}$ provides the additional concept constructors $p \downarrow q$ (*feature agreement*) and $p \uparrow q$ (*feature disagreement*) with the following semantics:

$$(p \downarrow q)^{\mathcal{I}} = \{w \in \Delta \mid \exists v(v = f_k(\cdots(f_1(w))) = f'_\ell(\cdots(f'_1(w))))\},$$
$$(p \uparrow q)^{\mathcal{I}} = \{w \in \Delta \mid \exists v, v'(v = f_k(\cdots(f_1(w))) \wedge v' = f'_\ell(\cdots(f'_1(w))) \wedge v \neq v')\}.$$

It is now straightforward to define a corresponding ADS $\mathcal{ALCF}^\sharp$ (see [13] for more details). The satisfiability of ABoxes with respect to (general) TBoxes is undecidable for $\mathcal{ALCF}$, while satisfiability of ABoxes (without TBoxes) is decidable [39,52,8]. Hence, for $\mathcal{ALCF}^\sharp$ the satisfiability problem is undecidable, while the A-satisfiability problem is decidable. Interestingly, in the $\mathcal{E}$-connection of $\mathcal{ALCF}^\sharp$ and $\mathcal{ALCO}^\sharp$ we can simulate general TBoxes, even in the case of A-satisfiability. Thus, we obtain the following theorem, a proof of which can be found in Appendix C.1:

Theorem 14 *Let $\mathcal{E}$ be an arbitrary non-empty set of link relations. Then the A-satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$ is undecidable.*

4 Examples of $\mathcal{E}$-connections

In this section we give four examples of $\mathcal{E}$-connections using the knowledge representation formalisms introduced in Section 2. Our aim is to demonstrate the versatility of the new combination technique and to outline its limits. The first three examples are 'two-dimensional,' while the fourth one connects three ADSs. To simplify notation, we will not distinguish between description, spatial, metric, or temporal logics and the corresponding ADSs.

4.1 $\mathcal{C}^{\mathcal{E}}(\mathcal{ALC}^{\sharp}, \mathcal{MS}^{\sharp})$

Suppose that you are developing a KR&R system for an estate agency. You imagine yourself to be a customer hunting for a house in London. What kind of requirements (constraints) could you have? Perhaps something like this:

(A) The house should not be too far from King's College, not more than 5 miles.

(B) The house should be close to a shop selling newspapers, say, within 0.5 mile.

(C) There should be a 'green zone' around the house, at least within 2 miles in each direction.

(D) There must be a sports center around, and moreover, all sports centers of the district should be reachable on foot, i.e., they should be within, say, 3 miles.

(E) Public transport should easily be accessible: whenever you are not more than 8 miles away from home, the nearest bus stop or tube station should be reachable within 1 mile.

(F) The house should have a telephone.

(G) The neighbors should not have children.

The terminology usually requires some background ontology; in this case you may also need statements like:

(H) Supermarkets are shops which provide no service and sell cheese, newspapers, etc.

(I) Newsagents are shops which sell magazines and newspapers.

The resulting constraints (A)–(I) contain two kinds of knowledge. (F)–(I) can be classified as conceptual knowledge which is captured by almost any description logic, say, $\mathcal{ALC}$:

(F) $\mathit{house} : \exists\mathsf{has.Telephone}$
(G) $\mathit{house} : \forall\mathsf{neighbor}.\forall\mathsf{child}.\bot$
(H) $\mathsf{Supermarket} \sqsubseteq \mathsf{Shop} \sqcap \forall\mathsf{service}.\bot \sqcap \exists\mathsf{sell.Newspaper} \sqcap \exists\mathsf{sell.Cheese}$
(I) $\mathsf{Newsagent} \sqsubseteq \mathsf{Shop} \sqcap \exists\mathsf{sell.Magazine} \sqcap \exists\mathsf{sell.Newspaper}$

(A)–(E) speak about distances and can be represented in the logic $\mathcal{MS}$ of metric spaces:

(A) $\mathit{house} : \mathsf{E}_{\leq 5}\{\mathit{King's\_college}\}$
(B) $\mathit{house} : \mathsf{E}_{\leq 0.5}\mathsf{Newspaper\_shop}$
(C) $\mathit{house} : \mathsf{A}_{\leq 2}\mathsf{Green\_zone}$
(D) $\mathit{house} : (\mathsf{E}_{\leq 3}\mathsf{Sports\_center}) \sqcap (\mathsf{A}_{>3}\neg\mathsf{Sports\_center})$
(E) $\mathit{house} : \mathsf{A}_{\leq 8}\mathsf{E}_{\leq 1}\mathsf{Public\_transport}$

Note that *house* and *King's_college* are location constants of $\mathcal{MS}$, while Newspaper_shop, Green_zone, etc. are set variables.

However, we cannot just join these two knowledge bases together without connecting them. They speak about the same things, but from different points of view. For instance, in (H) 'shop' is used as a *concept*, while (B) deals with the *space* occupied by 'shops selling newspapers.' Without connecting these different aspects we cannot deduce from the knowledge base that a supermarket or a news agent within 0.5 mile is sufficient to satisfy constraint (B). Moreover, it is obviously not too natural for the spatial part of the knowledge base to deal with primitive set variables for regions occupied by 'shops selling newspapers.'

The required interaction can easily be captured by an $\mathcal{E}$-connection between $\mathcal{ALC}^\sharp$ and $\mathcal{MS}^\sharp$, where $\mathcal{E} = \{E\}$ and the relation E is intended to relate abstract points of an $\mathcal{ALC}$-model with points in a metric space understood as the abstract point's spatial extension. Indeed, take relations has, neighbor, child, sell, service and set variables Telephone, Supermarket, Shop, Green_zone etc. from $\mathcal{ALC}^\sharp$, and the object variable *King's_college* from $\mathcal{MS}^\sharp$. Now, using the constructors $\langle E\rangle^1$ and $\langle E\rangle^2$ connecting $\mathcal{ALC}$- and $\mathcal{MS}$-models, we can represent constraints (A)–(I) as the concept Good_house defined by the following knowledge base in $\mathcal{C}^{\mathcal{E}}(\mathcal{ALC}^\sharp, \mathcal{MS}^\sharp)$:[4]

$$\mathsf{Good\_house} = \mathsf{House} \sqcap \mathsf{Well\_located} \sqcap \exists\mathsf{has.Telephone} \sqcap \forall\mathsf{neighbor}.\forall\mathsf{child}.\bot$$

[4] To enhance readability, here and in further examples we use the syntax of the underlying logical formalism rather than the syntax of the corresponding ADS.

$$\begin{aligned}
\mathsf{Well_located} = {} & \langle E\rangle^1 \Big(\mathsf{E}_{\leq 5}\{\mathit{King's_college}\} \sqcap \mathsf{E}_{\leq 0.5}\,\langle E\rangle^2\,(\exists \mathsf{sell.Newspaper}) \sqcap \\
& \mathsf{A}_{\leq 2}\,\langle E\rangle^2\,(\mathsf{Green_zone}) \sqcap \mathsf{E}_{\leq 3}\,\langle E\rangle^2\,(\mathsf{Sports_center}) \sqcap \\
& \mathsf{A}_{>3}\neg\,\langle E\rangle^2\,(\mathsf{Sports_center}) \sqcap \mathsf{A}_{\leq 8}\mathsf{E}_{\leq 1}\,\langle E\rangle^2\,(\mathsf{Public_transport})\Big) \\
\mathsf{Supermarket} \sqsubseteq {} & \mathsf{Shop} \sqcap \forall \mathsf{service}.\bot \sqcap \exists \mathsf{sell.Newspaper} \sqcap \exists \mathsf{sell.Cheese} \\
\mathsf{Newsagent} \sqsubseteq {} & \mathsf{Shop} \sqcap \exists \mathsf{sell.Magazine} \sqcap \exists \mathsf{sell.Newspaper}
\end{aligned}$$

If we also want to specify that the house should be available at a reasonable price, $\mathcal{ALC}$ can be extended with a suitable 'concrete domain' dealing with (natural or rational) numbers such that the resulting description logic is still decidable [56,55]. As shown in [13], description logics with concrete domains can still be regarded as ADSs and, therefore, the decidability of the $\mathcal{E}$-connection is preserved as well.

As discussed in Section 3, we can combine satisfiability checking algorithms for $\mathcal{ALC}^\sharp$ and $\mathcal{MS}^\sharp$ to obtain an algorithm for their $\mathcal{E}$-connection. This algorithm can then be used to check whether the formulated requirements are consistent. However, we can go one step further: to answer the query whether such a house really exists *in London*, we should not perform reasoning with respect to arbitrary metric spaces, but rather take a suitable map of London as our metric space. This scenario can be represented by an $\mathcal{E}$-connection of $\mathcal{ALC}^\sharp$ with the following ADS. Suppose that our map is a structure

$$\mathfrak{D} = \langle D, \delta, P_1, \ldots, P_n, c_1, \ldots, c_m\rangle\,,$$

where D is a finite set, δ a distance function on D, the P_i are subsets of D representing spatial extensions of concepts like House, Sports_center, etc., and the c_i are elements of D representing objects such as *King's_college*.[5] Then we define an ADS $\mathcal{MAP} = (\mathcal{MAP}_l, \mathcal{MAP}_m)$: here the ADL $\mathcal{MAP}_l$ extends the language of $\mathcal{MS}^\sharp$ by 0-ary function symbols $f_{P_1}, \ldots, f_{P_n}$ and $f_{c_1}, \ldots, f_{c_m}$, and $\mathcal{MAP}_m$ contains models of the form

$$\mathfrak{M} = \left\langle D, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}}, f^{\mathfrak{M}}_{P_1}, \ldots, f^{\mathfrak{M}}_{P_n}, f^{\mathfrak{M}}_{c_1}, \ldots, f^{\mathfrak{M}}_{c_m}\right\rangle,$$

where $\left\langle D, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}}\right\rangle$ is an $\mathcal{MS}^\sharp$-model corresponding to $\langle D, \delta\rangle$ as defined in Section 2.3, $f^{\mathfrak{M}}_{P_1} = P_1, \ldots, f^{\mathfrak{M}}_{P_n} = P_n$, and $f^{\mathfrak{M}}_{c_1} = \{c_1\}, \ldots, f^{\mathfrak{M}}_{c_m} = \{c_m\}$. Note that $\mathcal{MAP}_m$ contains more than one model since, according to Definition 3, the class of ADMs of any ADS is closed under arbitrary variations of the extensions of set variables. For this reason, we have to take 0-ary function symbols rather than set variables to represent the sets P_i and 0-ary function symbols rather than object variables to represent the constants c_i. However, since all models in $\mathcal{MAP}_m$ agree on $\mathcal{F}^{\mathfrak{M}}$, the $f^{\mathfrak{M}}_{P_i}$, and the $f^{\mathfrak{M}}_{c_i}$, the ADS $\mathcal{MAP}$ uniquely describes a single map.

[5] This representation depends, of course, on the size or granularity of the map.

Now, returning to our example, let us assume that the map $\mathfrak{D}$ contains subsets $P_1 = \mathit{Green\_zone}$, $P_2 = \mathit{Sports\_center}$, $P_3 = \mathit{Public\_transport}$, $P_4 = \mathit{Super\_market}$, $P_5 = \mathit{Newsagent}$, and a point $c_1 = \mathit{King's\_College}$ (but no subset marked by *shop*). Then we can modify the knowledge base above by replacing *King's_College* with f_{c_1} and by adding the following equations to the knowledge base in order to fix the spatial extensions of certain concepts:

$$\begin{array}{ll} \langle E \rangle^2 \, (\mathsf{Green\_zone}) = f_{P_1}, & \langle E \rangle^2 \, (\mathsf{Sports\_center}) = f_{P_2}, \\ \langle E \rangle^2 \, (\mathsf{Public\_transport}) = f_{P_3}, & \langle E \rangle^2 \, (\mathsf{Supermarket}) = f_{P_4}, \\ \langle E \rangle^2 \, (\mathsf{Newsagent}) = f_{P_5}. & \end{array}$$

Although shops selling newspapers are not marked in the map, it will follow from the subsumption relations (H) and (I) of the $\mathcal{ALC}^\sharp$-part of the knowledge base that any supermarket or shop at distance ≤ 0.5 in the map is sufficient to satisfy the constraint on shops selling newspapers.

Finally, by adding $\mathit{house} : \mathsf{Good\_house}$ to the knowledge base and checking its satisfiability, we can find out whether London has the house of our dreams.

4.2 $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^\sharp, \mathsf{S4}_u^\sharp)$

Now imagine that you are employed by the EU parliament to develop a geographical information system about Europe. One part of the task is easy. You take the description logic $\mathcal{ALCO}$ and, using concepts Country, Treaty, etc., object names *EU*, *Schengen_treaty*, *Spain*, *Luxembourg*, *UK*, etc., and a role member, write

$$\begin{array}{l} \mathit{Luxembourg} : \exists \mathsf{member}.\{\mathit{EU}\} \sqcap \exists \mathsf{member}.\{\mathit{Schengen\_treaty}\} \\ \mathit{Iceland} : \exists \mathsf{member}.\{\mathit{Schengen\_treaty}\} \sqcap \neg\exists \mathsf{member}.\{\mathit{EU}\} \\ \mathit{France} : \mathsf{Country} \\ \mathit{Schengen\_treaty} : \mathsf{Treaty} \\ \exists \mathsf{member}.\{\mathit{Schengen\_treaty}\} \sqsubseteq \mathsf{Country}, \quad \text{etc.} \end{array}$$

After that you have to say something about the geography of Europe. To this end you can use the spatial logic $\mathsf{S4}_u$ in which, as we have mentioned already, the topological meaning of the RCC-8 predicates can be encoded as follows, where X, Y are set variables and $\top = Z \vee \neg Z$:

$$\begin{array}{ll} \mathsf{DC}(X,Y): & \top = \neg \diamondminus (X \wedge Y), \\ \mathsf{EQ}(X,Y): & \top = (X \leftrightarrow Y), \\ \mathsf{EC}(X,Y): & \top = \diamondminus (X \wedge Y) \wedge \neg \diamondminus (\boldsymbol{I}X \wedge \boldsymbol{I}Y), \\ \mathsf{PO}(X,Y): & \top = \diamondminus (\boldsymbol{I}X \wedge \boldsymbol{I}Y) \wedge \diamondminus (\boldsymbol{I}X \wedge \neg Y) \wedge \diamondminus (\boldsymbol{I}Y \wedge \neg X), \end{array}$$

$$\begin{aligned}
&\mathsf{TPP}(X,Y): && \top = (\neg X \vee Y) \wedge \Diamond(X \wedge \neg \boldsymbol{I} Y) \wedge \Diamond(\neg X \wedge Y),\\
&\mathsf{NTPP}(X,Y): && \top = \Box(\neg X \vee \boldsymbol{I} Y) \wedge \Diamond(\neg X \wedge Y)
\end{aligned}$$

($\mathsf{TPPi}(X,Y) = \mathsf{TPP}(Y,X)$ and $\mathsf{NTPPi}(X,Y) = \mathsf{NTPP}(Y,X)$). To ensure that $\mathsf{RCC\text{-}8}$ predicates are only applied to regular closed sets, one can add the assertions $\boldsymbol{CI}X = X$ and $\boldsymbol{CI}Y = Y$ to the knowledge base.

Now, using an $\mathcal{E}$-connection between $\mathcal{ALCO}^\sharp$ and $\mathsf{S4}_u^\sharp$ you can continue:

$$\begin{aligned}
&\mathsf{EQ}(\langle E\rangle^2(\{EU\}), \langle E\rangle^2(\{Portugal\} \sqcup \{Spain\} \sqcup \dots \sqcup \{UK\})\\
&\mathsf{EC}(\langle E\rangle^2(\{France\}), \langle E\rangle^2(\{Luxembourg\}))\\
&\mathsf{NTPP}(\langle E\rangle^2(\{Luxembourg\}), \langle E\rangle^2(\exists \mathsf{member}.\{Schengen_Treaty\}))\\
&Austria : \langle E\rangle^1(Alps)
\end{aligned}$$

i.e., 'the space occupied by the EU is the space occupied by its members,' 'France and Luxembourg have a common border' (see Fig. 2), 'if you cross the border of Luxembourg, then you enter a member of the Schengen Treaty,' 'Austria is an alpine country' ($Alps$ is a set variable of $\mathsf{S4}_u^\sharp$). You can even say that Germany, Austria and Switzerland meet at one point:

$$\begin{aligned}
&\Diamond(\langle E\rangle^2(\{Austria\}) \sqcap \langle E\rangle^2(\{Germany\}) \sqcap \langle E\rangle^2(\{Switzerland\})) \wedge\\
&\quad \wedge \neg\Diamond(\boldsymbol{I}\langle E\rangle^2(\{Austria\}) \sqcap \boldsymbol{I}\langle E\rangle^2(\{Germany\})) \wedge\\
&\quad \wedge \neg\Diamond(\boldsymbol{I}\langle E\rangle^2(\{Austria\}) \sqcap \boldsymbol{I}\langle E\rangle^2(\{Switzerland\})) \wedge\\
&\quad \wedge \neg\Diamond(\boldsymbol{I}\langle E\rangle^2(\{Switzerland\}) \sqcap \boldsymbol{I}\langle E\rangle^2(\{Germany\})).
\end{aligned}$$

Of course, to ensure that the spatial extensions of the EU, $France$, etc. are not degenerate and to comply with requirements of $\mathsf{RCC\text{-}8}$ you should guarantee that all mentioned spatial regions are interpreted by regular closed sets, i.e.,

$$\begin{aligned}
\langle E\rangle^2(\{EU\}) &= \boldsymbol{CI}\langle E\rangle^2(\{EU\})\\
\langle E\rangle^2(\{France\}) &= \boldsymbol{CI}\langle E\rangle^2(\{France\})\\
&\text{etc.}
\end{aligned}$$

Suppose now that you want to test your system and ask whether France is a member of the Schengen treaty, i.e., $France : \exists\mathsf{member}.\{\mathsf{Schengen\_treaty}\}$. The answer will be 'Don't know!' because you did not tell your system that the spatial extensions of any two countries do not overlap. If you add, for example,

$$\begin{aligned}
\neg\Diamond(\boldsymbol{I}(\langle E\rangle^2(\mathsf{Country} \sqcap \neg\exists\mathsf{member}.\{Schengen_treaty\}) \wedge\\
\wedge\ \boldsymbol{I}(\langle E\rangle^2(\exists\mathsf{member}.\{Schengen_treaty\}))
\end{aligned}$$

('the members of the Schengen treaty do not overlap with the non-Schengen countries') to the knowledge base, then the answer to the query will be 'Yes!'

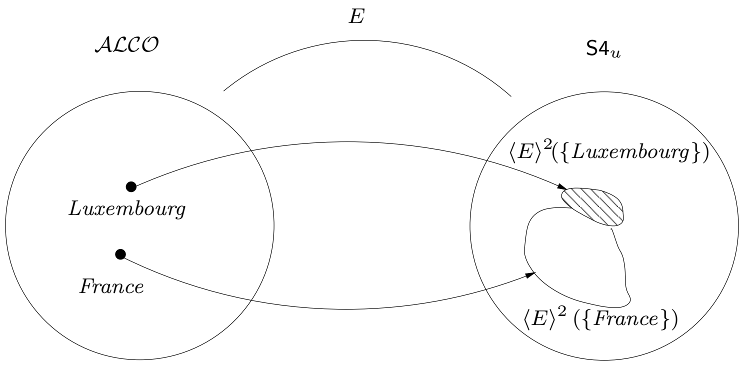

Fig. 2. France and Luxembourg have a common border.

Clearly, the representation task is much easier if complete knowledge about the geography of Europe is available. Then you could have taken an existing spatial database describing the RCC-8 relations between the European countries, mountains, etc., and thus use a fixed model of $\mathsf{S4}_u$ with a fixed connection E. This database can be conceived of as an ADS in the same manner as the map of London in the previous example.

4.3 $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$

Having satisfied your boss in the EU parliament with the constructed GIS, you get a new task: to develop a knowledge base regulating relations between people in the EU (citizenship, jobs, etc.). On the one hand, you already have the $\mathcal{ALCO}$ knowledge base describing countries in the EU from the previous example. But on the other hand, you must also be able to express laws like (i) 'no citizen of the EU may have more than one spouse,' (ii) 'all children of UK citizens are UK citizens,' or (iii) 'a person living in the UK is either a child of somebody who is a UK citizen or has a work permit in the UK, or the person is a UK citizen or has a work permit in the UK herself.' This means, in particular, that you need more constructors than $\mathcal{ALCO}$ can provide, say, qualified number restrictions and inverse roles. It is known, however, that inverse roles, number restrictions, and nominals are difficult to handle algorithmically in one system [41]. The fusion of $\mathcal{ALCO}$ with, say, $\mathcal{SHIQ}$ of [42], having the required constructors, does not help either, because transfer results for fusions are available so far only for DLs whose models are closed under disjoint unions [13] which is not the case if nominals are allowed as concept constructors. It seems that a perspective way to attack this problem is to *connect* $\mathcal{SHIQ}^\sharp$ with $\mathcal{ALCO}^\sharp$.

Let $\mathcal{E}$ contain three binary relations between the domains of $\mathcal{SHIQ}$ (people, companies, etc.) and $\mathcal{ALCO}$ (countries): xSy means that x is a citizen of y, xLy means that x lives in y, and xWy means that x has a work permit in y. For example, $\langle L\rangle^1\,(UK)$ denotes all people living in the UK, while $\langle S\rangle^1\,(UK)$

all UK citizens. The subsumptions below represent the regulations (i)–(iii):

$$\langle S\rangle^1(\{EU\}) \sqsubseteq \neg(\geq 2\mathsf{married}.\top)$$
$$\exists\mathsf{child\_of}.\langle S\rangle^1(\{UK) \sqsubseteq \langle S\rangle^1(\{UK\})$$
$$\langle L\rangle^1(\{UK\}) \sqsubseteq \exists\mathsf{child\_of}^{-1}.\big(\langle S\rangle^1(\{UK\}) \sqcup \langle W\rangle^1(\{UK\})\big) \sqcup \langle S\rangle^1(\{UK\}) \sqcup \langle W\rangle^1(\{UK\})$$

4.4 $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^\sharp, \mathsf{S4}_u^\sharp, \mathsf{PTL}^\sharp)$

'The EU is developing,' said your boss, 'we are going to have new members by 2005.' So you extend the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^\sharp, \mathsf{S4}_u^\sharp)$ with one more ADS—propositional temporal logic $\mathsf{PTL}^\sharp$. Now, besides object variables *EU*, *Germany*, etc. of $\mathcal{ALCO}^\sharp$ and set variables *Alps*, *Basel*, etc. of $\mathsf{S4}_u^\sharp$, we use the terms $\{0\}, \{1\}, \ldots$ as abbreviations for $(\neg \bigcirc_P^n \top \wedge \bigcirc_P^{n-1}\top)^\sharp$, where $\bigcirc_P\varphi$ stands for $\bot\mathcal{S}\varphi$. We then have $\{n\}^{\mathfrak{W}} = \{n\}$, for any $\mathsf{PTL}^\sharp$-model $\mathfrak{W}$. The ternary relation $E(x, y, z)$ means now that at moment z (from the domain of PTL) point y (in the domain of $\mathsf{S4}_u$) belongs to the spatial region occupied by object x (in the domain of $\mathcal{ALCO}$).

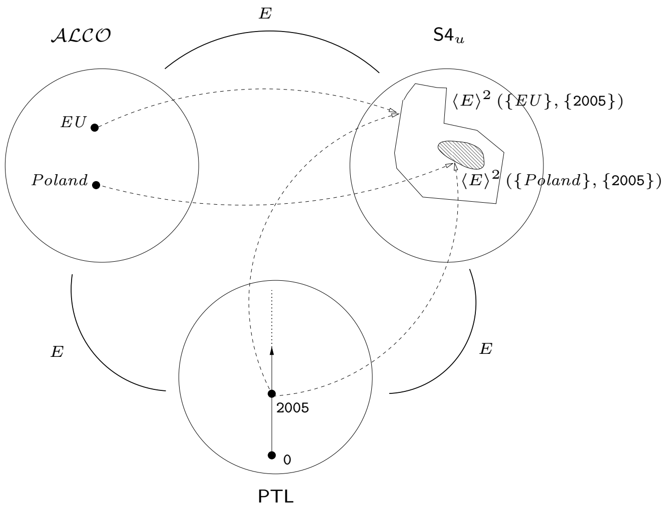

Fig. 3. In 2005 Poland will be part of the EU.

Then we can say, for example:

$$\langle E\rangle^2(\{Poland\}, \{2005\}) \sqsubseteq \langle E\rangle^2(\{EU\}, \{2005\}),$$
$$\mathsf{PO}(\langle E\rangle^2(\{Austria\}, \{1914)\}, \langle E\rangle^2(\{Italy\}, \{1950\})),$$
$$\Box_F \neg \langle E\rangle^3(\{Basel\}, \{EU\}),$$

i.e., 'in 2005, the territory of Poland will belong to the territory occupied by the EU' (see Fig. 3), 'the territory of Austria in 1914 partially overlaps the territory of Italy in 1950,' 'no part of Basel will ever belong to the EU.'

5 Extensions

In this section, we introduce several variants of $\mathcal{E}$-connections that allow an even closer interaction of the combined formalisms than the original version. These variants are obtained by extending basic $\mathcal{E}$-connections with more powerful link operators: those that can be applied to object variables even though the connected ADSs do not have nominals; those that can talk about Boolean combinations of links; and 'qualified number restrictions' on links (we use description logic terminology here) which can be used to say, e.g., that a given link operator is a partial function. We provide (brief) examples illustrating the expressive power of the new constructors and study the computational properties of the resulting formalisms.

5.1 Applications of link operators to object variables

In some of the examples from Section 4, the connected ADSs have nominals. According to Definition 6, this means that, for each object name a, they provide terms $\{a\}$ such that, for every model $\mathfrak{W}$, we have $\{a\}^{\mathfrak{M}} = \{a^{\mathfrak{M}}\}$. This is the case, e.g., for $\mathcal{MS}^\sharp$, $\mathcal{ALCO}^\sharp$, and $\mathsf{PTL}^\sharp$ (see Section 2). In connections where the components do have nominals, it is often convenient to form terms such as $\langle E\rangle^i(\{a\})$ to state that the current element is connected to a *particular element* of the other component, namely, the one denoted by a. However, not all $\mathcal{E}$-connections considered in Section 4 are of this type, e.g., $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ from Section 4.3. In this combination, we are *not* allowed to build, say, the term comprising all of the countries where some person Bob has citizenship: since $\mathcal{SHIQ}^\sharp$ has no nominals, we cannot use

$$\mathsf{country} \sqcap \langle S\rangle^2 (\{Bob\}),$$

where Bob is an object variable of $\mathcal{SHIQ}^\sharp$. An addition of the nominal constructor to $\mathcal{SHIQ}$ does not seem to be a promising solution because, despite considerable efforts of the description logic community, no 'implementable' algorithms are known for $\mathcal{SHIQ}$ extended with nominals. A better idea is to allow applications of link operators directly to objects, even if nominals are not available in the component ADS. Indeed, we can show that this kind of $\mathcal{E}$-connection is still computationally robust.

Definition 15 Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, are abstract description systems and $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. Denote by

$$\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection in which the definition of i-term is extended with the following clause, for $1 \leq i \leq n$:

- if $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is a sequence of object variables a_j from $\mathcal{L}_j$, $j \neq i$, then $\langle E_k \rangle^i (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is an *i-term*, for $k \in J$.

As for the semantics, given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle$$

and a tuple $\overline{a}_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, we set

$$(\langle E_j \rangle^i (\overline{a}_i))^{\mathfrak{M}} = \{x \in W_i \mid (a_1^{\mathfrak{M}}, \ldots, a_{i-1}^{\mathfrak{M}}, x, a_{i+1}^{\mathfrak{M}}, \ldots, a_n^{\mathfrak{M}}) \in E_j^{\mathfrak{M}}\}.$$

The following result, to be proved in Appendix B.1, shows that applications of link operators to object variables do not influence the decidability of $\mathcal{E}$-connections:

Theorem 16 *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be ADSs with decidable satisfiability problems. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well.*

This result is somewhat surprising, since the addition of nominals to an arbitrary ADS with a decidable satisfiability problem sometimes results in an undecidable one; for an example see Lemma 52 in Appendix C.2.

Theorem 16 is proved similarly to the basic transfer theorem (Theorem 11), and thus the same discussion and the same notes concerning the computational complexity apply. Indeed, Appendix B contains only a proof of Theorem 16 from which Theorem 11 follows immediately.

In Theorem 14, we connected the ADSs $\mathcal{ALCF}^{\sharp}$ and $\mathcal{ALCO}^{\sharp}$ to obtain a counterexample for the transfer of decidability of A-satisfiability. The choice of $\mathcal{ALCO}^{\sharp}$ was motived by the fact that this ADS has nominals. Now that we are allowed to apply the link operators to object variables, we can strengthen this result: *any* connection (of the type considered in this section) involving $\mathcal{ALCF}^{\sharp}$ as one of its components has an undecidable A-satisfiability problem.

Theorem 17 *Let $\mathcal{E}$ be an arbitrary non-empty set of link relations and $\mathcal{S}$ an ADS. Then the A-satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{ALCF}^{\sharp}, \mathcal{S})$ is undecidable.*

The proof of this result can be found in Appendix C.1.

The two variants of $\mathcal{E}$-connections introduced so far do not allow any interaction between links, which is a rather severe restriction. To illustrate this, we again consider the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ from Section 4.3. Recall that $\mathcal{E} = \{S, L, W\}$, where the link S represents citizenship (of people in EU countries) and L represents the place of living. In the $\mathcal{E}$-connections $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ and $\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$, we cannot describe the concept of 'all people that live in the country of their citizenship.' To do this we need the intersection of the links S and L:

$$\mathsf{Human\_being} \sqcap \langle S \cap L \rangle^1 (\mathsf{Country}).$$

Similarly, suppose that we are in the estate agent's framework of Section 4.1 and want to describe the set of points in space (say, London) which are served by *all* mobile phone providers. This can be naturally done using the complement operator on a link S (this time representing 'serves'):

$$\neg \langle \neg S \rangle^2 (\mathsf{Mobile\_phone\_provider}).$$

Note that $\langle \neg S \rangle^2 (\mathsf{Mobile\_phone\_provider})$ is the set of points that are not served by some mobile phone provider.

These simple examples motivate the following definition:

Definition 18 Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, are ADSs and that $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. Denote by

$$\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection with the smallest set $\overline{\mathcal{E}}$ of links such that

- $\mathcal{E} \subseteq \overline{\mathcal{E}}$;
- if $F \in \overline{\mathcal{E}}$, then $\neg F \in \overline{\mathcal{E}}$;
- if $F, G \in \overline{\mathcal{E}}$, then $F \wedge G \in \overline{\mathcal{E}}$.

Given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle,$$

we interpret the links $F \in \overline{\mathcal{E}}$ as relations $F^{\mathfrak{M}} \subseteq W_1 \times \cdots \times W_n$ (with W_i being the domain of $\mathfrak{W}_i$) in the obvious way:

$$(F \wedge G)^{\mathfrak{M}} = F^{\mathfrak{M}} \cap G^{\mathfrak{M}}, \quad (\neg F)^{\mathfrak{M}} = (W_1 \times \cdots \times W_n) \setminus F^{\mathfrak{M}}.$$

The Boolean operations on links allow us to express *link inclusion assertions* of the form $F \sqsubseteq G$, where F and G are links, and $\mathfrak{M} \models F \sqsubseteq G$ iff $F^{\mathfrak{M}} \subseteq G^{\mathfrak{M}}$. Such assertions are called *role hierarchies* in the area of description logics.

Indeed, $F \sqsubseteq G$ can be equivalently rewritten as $\top_1 \sqsubseteq \neg \langle F \wedge \neg G \rangle^1 \top_2$, where $\top_i = x_i \vee \neg x_i$, for some set variable x_i of $\mathcal{L}_i$.

We denote by $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ the $\mathcal{E}$-connection which allows Boolean operations on links as well as applications of link operators to object variables. The following theorem is to be proved in Appendix B.2.

Theorem 19 *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be ADSs with decidable satisfiability problems. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well.*

The intuition behind the proof is similar to the basic case: we again reduce the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ to the satisfiability problem for its components. This time, however, the reduction is not so straightforward because the interaction between (complex) links has to be taken into account. For this reason, it is not enough to simply guess the 1-types and 2-types realized in a potential model together with a binary relation between them, but we have to guess a so-called pre-model which involves a relational structure *between elements* (rather than between types) and can be understood as the irregular core of an otherwise regular model. Fortunately, the size of this irregular core is at most exponential in the size of the input.

As before, a non-deterministic upper time bound for the satisfiability problem for the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is obtained by adding one exponential to the maximal time complexity of the components (cf. Appendix B.2). The following result shows that this upper bound cannot be improved, in general, since the satisfiability problem for the basic ADS $\mathcal{B}$ introduced in Section 3 is NP-complete (cf. Lemma 12).

Theorem 20 *The satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$ is* NEXPTIME*-hard, for any infinite $\mathcal{E}$.*

The proof, which can be found in Appendix C.3, is by reduction of the satisfiability problem for the modal logic **S5** $\times$ **S5** (i.e., the full binary product of modal **S5** with itself) to satisfiability in $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$. Since the ADS $\mathcal{B}$ is rather trivial, while **S5** $\times$ **S5** is known to be a variant of the two-variable fragment of first-order logic (the two-variable substitution free fragment, to be more precise) [30], this result demonstrates the considerable expressive power which the Boolean operators on links add to $\mathcal{E}$-connections.

5.3 Number restrictions on links

Another obvious need when dealing with connections is a possibility to constrain the number of objects linked by the connecting relations. For example, in the real estate agent's application we may want to say that, according to the chosen granularity of the spatial domain, the spatial extension of any house consists of precisely one point in space. Thus, the corresponding connection relation should be a partial function. The concept constructors employed in description logic to represent this kind of constraints are known as (*qualified*) *number restrictions* [38,20,42]; in modal logic they are called *graded modalities* [24,21,72]. What happens if we introduce similar constructors for links in $\mathcal{E}$-connections?

Definition 21 Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, are ADSs and that $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. Denote by

$$\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection in which the definition of i-terms, $1 \leq i \leq n$, is extended with the following clause, for every natural number r:

- if $\overline{t}_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ is a sequence of j-terms t_j, for $j \neq i$, and $k \in J$, then $\langle \leq r E_k \rangle^i (\overline{t}_i)$ and $\langle \geq r E_k \rangle^i (\overline{t}_i)$ are *i-terms.*

The semantics of the new constructors is defined as follows. Let

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle$$

be a model for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. Then

$$x \in (\langle \leq r E_j \rangle^i (\overline{t}_i))^{\mathfrak{M}} \text{ iff } \left|\{\overline{x}_i \mid (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}} \wedge x_k \in t_k^{\mathfrak{M}}\}\right| \leq r$$

and

$$x \in (\langle \geq r E_j \rangle^i (\overline{t}_i))^{\mathfrak{M}} \text{ iff } \left|\{\overline{x}_i \mid (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}} \wedge x_k \in t_k^{\mathfrak{M}}\}\right| \geq r,$$

where $\overline{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Combinations of $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ with previous extensions are denoted by the obvious names, e.g., $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ stands for the extension of basic $\mathcal{E}$-connections with both number restrictions and the Boolean operators on links.

Unfortunately, it turns out that, in general, decidability does not transfer from ADSs $\mathcal{S}_1, \ldots, \mathcal{S}_n$ to their $\mathcal{E}$-connection with number restrictions $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. As will be proved in Appendix C.2 (using two rather technical ADSs), we have:

Theorem 22 *There exist ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ with decidable satisfiability problems such that the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$ is undecidable even if $\mathcal{E}$ is a singleton.*

The intuitive reason for this 'negative' result is that number restrictions on links allow the transfer of 'counting capabilities' from one component to another. For example, in $\mathcal{C}^{\{E\}}_{Q}(\mathcal{SHIQ}^{\sharp}, \mathcal{ALCO}^{\sharp})$, we can 'export' the nominals of $\mathcal{ALCO}^\sharp$ to $\mathcal{SHIQ}^\sharp$: the assertions

$$\top_2 = \langle \leq 1E \rangle^2 (\top_1), \quad \top_2 = \langle \geq 1E \rangle^2 (\top_1), \quad \top_1 = \langle \leq 1E \rangle^1 (\top_2), \quad \top_1 = \langle \geq 1E \rangle^1 (\top_2)$$

state that E is a bijective function, and so we can use $\langle E \rangle^1 (\{a\})$, a an object variable of $\mathcal{ALCO}^\sharp$, as a nominal in $\mathcal{SHIQ}^\sharp$.

When introducing number restrictions on links, it is thus natural to confine ourselves to ADSs which, intuitively, 'cannot count.' This leads to the following definition. Given a finite set Σ of $\mathcal{L}$-assertions, we denote by $term(\Sigma)$ the set of all terms in Σ.

Definition 23 An ADS $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ is called *number tolerant* if there is a cardinal κ such that, for every $\kappa' \geq \kappa$ and every satisfiable finite set Σ of assertions, there exists a model $\mathfrak{W} \in \mathcal{M}$ satisfying Σ and such that, for each $d \in W$, there are precisely κ' elements $d' \in W$ for which

$$\{t \in term(\Sigma) \mid d \in t^{\mathfrak{W}}\} = \{t \in term(\Sigma) \mid d' \in t^{\mathfrak{W}}\}.$$

Intuitively, being number tolerant means that, if a knowledge base Σ is satisfiable, then we can find a model of Σ in which each occurring 'type' (set of terms) is satisfied a 'very large' number of times. For example, ADSs of modal logics that are invariant for the formation of disjoint unions of structures are clearly number tolerant. In contrast, ADSs with nominals cannot be number tolerant because nominals are always interpreted as singleton sets.

We now use results from [13] to obtain a straightforward proof that the ADSs for numerous description logics, in particular $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$, are number tolerant. The following notion of a *local* ADS was introduced in [13], where the transfer of decidability from local ADSs to their so-called *fusions* is proved:

Definition 24 Given a family $(\mathfrak{W}_p)_{p \in P}$ of ADMs

$$\mathfrak{W}_p = \left\langle W_p, \mathcal{V}^{\mathfrak{W}_p}, \mathcal{X}^{\mathfrak{W}_p}, \mathcal{F}^{\mathfrak{W}_p}, \mathcal{R}^{\mathfrak{W}_p} \right\rangle$$

over pairwise disjoint domains W_p, we say that

$$\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$$

is a *disjoint union* of $(\mathfrak{W}_p)_{p \in P}$ if

- $W = \bigcup_{p \in P} W_p$,
- $f^{\mathfrak{W}}(X_1, \ldots, X_{n_f}) = \bigcup\limits_{p \in P} f^{\mathfrak{W}_p}(X_1 \cap W_p, \ldots, X_{n_f} \cap W_p)$,
  for all $X_1, \ldots, X_{n_f} \subseteq W$ and all $f \in \mathcal{F}$,
- $R^{\mathfrak{W}} = \bigcup\limits_{p \in P} R^{\mathfrak{W}_p}$ for all $R \in \mathcal{R}$.

An ADS $S = (\mathcal{L}, \mathcal{M})$ is called *local* if $\mathcal{M}$ is closed under disjoint unions.

The following result is easily proved and illustrates the relationship between locality and number tolerance. For more details, consult Appendix A.

Proposition 25 *Every local ADS is number tolerant.*

It is an immediate consequence of Proposition 15 in [13] that both $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ are local. By applying Proposition 25, we thus get:

Proposition 26 *$\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ are number tolerant.*

Note, however, that locality and number tolerance are not the same. The ADS $\mathsf{S4}_u^\sharp$ is a counterexample: it is number tolerant but not local. The proof of the following proposition can be found in Appendix A.

Proposition 27 *$\mathsf{S4}_u^\sharp$ is number tolerant.*

That $\mathsf{S4}_u^\sharp$ is not local follows from the fact that it is equipped with the universal modality: if we take the disjoint union of two ADMs, then the function symbols for the universal modality 'lose' their universality.

Fortunately, number tolerance is precisely what we need in order to preserve decidability in the presence of number restrictions on links—witness the following result to be proved in Appendix B.3:

Theorem 28 *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be number-tolerant ADSs with decidable satisfiability problems. Then the satisfiability problem is also decidable for any $\mathcal{E}$-connection $\mathcal{C}_Q^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$.*

For example, the connection $\mathcal{C}_Q^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathsf{S4}_u^\sharp)$ is decidable, since both components are number tolerant.

The proof of Theorem 28 is similar to that of Theorem 11: we guess sets of 1-types and 2-types to be realized in a potential model. Additionally, for each i-type t we need to guess the number and type of witnesses for the link operators $\langle \geq rE \rangle^i(s)$ such that none of the link operators $\langle \leq rE \rangle^i(s)$ of t is violated. Similarly to the previous variants of $\mathcal{E}$-connections, we get a non-deterministic upper time bound for the satisfiability problem that is obtained by adding one exponential to the maximal time complexity of the component ADSs.

It is now a natural question to ask whether number restrictions can be combined with link operators on objects variables and/or Boolean operators on links without losing the transfer of decidability. Unfortunately, this is not the case. The proof of the following theorem is similar to the proof of Theorem 22 and can be found in Appendix C.2:

Theorem 29

(i) *There exist number tolerant ADSs* $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

(ii) *There exist number tolerant* $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

6 Connections and distributed description logics

Let us recall the knowledge base regulating relations between people in the EU from Section 4.3. We proposed to employ the $\mathcal{E}$-connection $\mathcal{C}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$: the $\mathcal{SHIQ}^\sharp$ component was used to talk about people and their relations, while the $\mathcal{ALCO}^\sharp$ component to talk about the EU countries. Apart from computational considerations, there is another important motivation for such a separation of various aspects of a large application: we may think of the components as independently maintained databases which are constantly updated, systematically linked, and import information from each other. This leads us to a discussion of distributed DLs (DDL) introduced by Borgida and Serafini [18], who observed that in some cases functional correspondences between different information systems are not enough to capture important information and provided a number of examples illustrating this point. They also stressed that, unlike other approaches relating databases, a suitable logic-based approach enlarges the possible inferences we may draw from a combined knowledge base.

In this section we show that the distributed description logics of Borgida and Serafini can be regarded as a special case of $\mathcal{E}$-connections linking a finite number of DLs. In what follows, all DLs are considered as their ADS representations.

6.1 The DDL formalism

We start with a brief, but self-contained, description of the DDL formalism. Suppose that n description logics $DL_1, \ldots, DL_n$ are given. A sequence $\mathfrak{D} = (DL_i)_{i \leq n}$ is then called a *distributed description logic* (*DDL*). We use subscripts to indicate that some concept C_i belongs to the language of the description logic DL_i. Two types of assertions—bridge rules and individual correspondence—are used to establish interconnections between the components of a DDL.

Definition 30 Let C_i and C_j be concepts from DL_i and DL_j, respectively. A *bridge rule* is an expression of the form

$$C_i \xrightarrow{\sqsubseteq} C_j \qquad \text{(into rule)}$$

or of the form

$$C_i \xrightarrow{\sqsupseteq} C_j. \qquad \text{(onto rule)}$$

Let a_i be an object name of DL_i and b_j, $b_j^1, \ldots, b_j^n$ object names of DL_j. A *partial individual correspondence* is an expression of the form

$$a_i \mapsto b_j. \qquad \text{(PIC)}$$

A *complete individual correspondence* is an expression of the form

$$a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\}. \qquad \text{(CIC)}$$

A *distributed TBox* $\mathfrak{T}$ consists of TBoxes T_i of DL_i together with a set of bridge rules. A *distributed ABox* $\mathfrak{A}$ consists of ABoxes A_i of DL_i together with a set of partial and complete individual correspondences. A *distributed knowledge base* is a pair $(\mathfrak{T}, \mathfrak{A})$.

The semantics of distributed knowledge bases is defined as follows.

Definition 31 A *distributed interpretation* $\mathfrak{I}$ of a distributed knowledge base $(\mathfrak{T}, \mathfrak{A})$ as above is a pair $(\{\mathcal{I}_i\}_{i \leq n}, \mathcal{R})$, where each $\mathcal{I}_i$ is a model for the corresponding DL_i and $\mathcal{R}$ is a function associating with every pair (i, j), $i \neq j$, a binary relation $r_{ij} \subseteq W_i \times W_j$ between the domains W_i and W_j of $\mathcal{I}_i$ and $\mathcal{I}_j$, respectively. Given a point $u \in W_i$ and a subset $U \subseteq W_i$, we set

$$r_{ij}(u) = \{v \in W_j \mid (u, v) \in r_{ij}\}, \qquad r_{ij}(U) = \bigcup_{u \in U} r_{ij}(u).$$

The *truth-relation* is standard for formulas of the component DLs. For bridge rules and individual correspondences it is defined as follows:

- $\mathfrak{I} \models C_i \xrightarrow{\sqsubseteq} C_j$ iff $r_{ij}(C_i^{\mathfrak{I}}) \subseteq C_j^{\mathfrak{I}}$;
- $\mathfrak{I} \models C_i \xrightarrow{\sqsupseteq} C_j$ iff $r_{ij}(C_i^{\mathfrak{I}}) \supseteq C_j^{\mathfrak{I}}$;
- $\mathfrak{I} \models a_i \mapsto b_j$ iff $b_j^{\mathfrak{I}} \in r_{ij}(a_i^{\mathfrak{I}})$;
- $\mathfrak{I} \models a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\}$ iff $r_{ij}(a_i^{\mathfrak{I}}) = \{(b_j^1)^{\mathfrak{I}}, \ldots, (b_j^n)^{\mathfrak{I}}\}$.

As usual, $\mathfrak{T} \models C \sqsubseteq D$ means that for every distributed interpretation $\mathfrak{I}$, if $\mathfrak{I} \models \varphi$ for all $\varphi \in \mathfrak{T}$, then $\mathfrak{I} \models C \sqsubseteq D$. The same definition applies to ABoxes $\mathfrak{A}$ and individual assertions.

It is of interest to note that, unlike $\mathcal{E}$-connections, DDLs do not provide new concept-formation operators to link the components of the DDL: both bridge rules and individual correspondences are assertions, and so atoms of knowledge bases, but not part of the concept language.

The satisfiability problem for distributed knowledge bases without complete individual correspondences (CIC) is easily reduced to the satisfiability problem for basic $\mathcal{E}$-connections. Indeed, fix a DDL $\mathfrak{D} = (DL_i)_{i \leq n}$ and associate with it the $\mathcal{E}$-connection $\mathfrak{D}^\sharp = \mathcal{C}^{\mathcal{E}}(DL_1^\sharp, \ldots, DL_n^\sharp)$, where $\mathcal{E} = \{E_{ij} \mid i, j \leq n, i \neq j\}$ consists of $n \times (n-1)$ many n-ary relations. To define a translation $\cdot^\sharp$ of $\mathfrak{D}$-assertions into $\mathfrak{D}^\sharp$-assertions, we mainly have to take care of the fact that DDL relations are binary, while $\mathcal{E}$-connection links are n-ary.

Definition 32 Suppose that $\mathfrak{K} = (\mathfrak{T}, \mathfrak{A})$ is a distributed knowledge base for $\mathfrak{D} = (DL_i)_{i \leq n}$ without complete individual correspondences. We define a translation $^\sharp$ from $\mathfrak{D}$-assertions to $\mathfrak{D}^\sharp$-assertions as follows:

- if φ is neither a bridge rule nor an individual correspondence, then $\varphi^\sharp$ is defined by translating the concepts in φ using the $^\sharp$ translation from Section 2.1;
- $(C_i \xrightarrow{\sqsubseteq} C_j)^\sharp$ is $\langle E_{ij} \rangle^j (\top_1, \ldots, C_i^\sharp, \ldots, \top_n) \sqsubseteq C_j^\sharp$;
- $(C_i \xrightarrow{\sqsupseteq} C_j)^\sharp$ is $\langle E_{ij} \rangle^j (\top_1, \ldots, C_i^\sharp, \ldots, \top_n) \sqsupseteq C_j^\sharp$;
- $(a_i \mapsto a_j)^\sharp$ is $(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) : E_{ij}$, where a_k, for $k \neq i, j$, are fresh object variables of DL_k.

Finally, we put $\mathfrak{T}^\sharp = \{\varphi^\sharp \mid \varphi \in \mathfrak{T}\}$, $\mathfrak{A}^\sharp = \{\varphi^\sharp \mid \varphi \in \mathfrak{A}\}$ and $\mathfrak{K}^\sharp = \mathfrak{T}^\sharp \cup \mathfrak{A}^\sharp$.

Note that we only need simple link assertions to translate partial individual correspondences: no application of link operators to object variables is required. The theorem below follows immediately from the definition of the translation $\cdot^\sharp$:

Theorem 33 *A distributed knowledge base $\mathfrak{K}$ for a DDL $\mathfrak{D}$ without complete individual correspondences is satisfiable iff $\mathfrak{K}^\sharp$ is satisfiable in a model of the basic $\mathcal{E}$-connection $\mathfrak{D}^\sharp$.*

Corollary 34 *The satisfiability problem for DDLs $(DL_i)_{i \leq n}$ without complete individual correspondences is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the DL_i.*

Unfortunately, complete individual correspondences cannot be translated into basic $\mathcal{E}$-connections, and Corollary 34 does not hold for arbitrary distributed description logics with knowledge bases including complete individual correspondences. To be able to deal with these as well, we introduce another extension of $\mathcal{E}$-connections.

In this section, we extend the basic $\mathcal{E}$-connections of n ADSs with an analogue of complete individual correspondences.

Definition 35 Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, are ADSs and that $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. We denote by

$$\mathcal{C}_I^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection in which the set of i-object assertions is extended with assertions of the form

$$\langle E_k \rangle^i (a_j) = B_i,$$

where $1 \leq i \leq n$, $k \in J$, B_i is a finite set of object variables of $\mathcal{L}_i$, and a_j is an object variable of $\mathcal{L}_j$, for some $j \neq i$.

The truth-relation for the new assertions is defined as follows. Given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle,$$

we put

$$\begin{aligned} &\mathfrak{M} \models \langle E_k \rangle^i (a_j) = B_i \qquad \text{iff} \\ &\qquad \{x_i \in W_i \mid \underset{l \neq i,j}{\exists} x_l \in W_l (x_1, \ldots, a_j^{\mathfrak{M}}, \ldots, x_n) \in E_k^{\mathfrak{M}}\} = \{b_i^{\mathfrak{M}} \mid b_i \in B_i\}. \end{aligned}$$

The assertion $\langle E_k \rangle^i (a_j) = B_i$ can be expressed in the basic $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ if all its components have nominals: if $B_i = \{b_i^1, \ldots, b_i^r\}$ then $\langle E_k \rangle^i (a_j) = B_i$ is equivalent to

$$\langle E_k \rangle^i (\top_1, \ldots, \{a_j\}, \ldots, \top_n) = \{b_i^1\} \sqcup \cdots \sqcup \{b_i^r\}.$$

Therefore, as a consequence of Theorem 19 we obtain:

Theorem 36 *Suppose that $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are ADSs with decidable satisfiability problems and that each of them has nominals. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}_{OBI}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well.*

Moreover, there exists a connection to number restrictions on links: if we consider the connection of two ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\langle E_k \rangle^1 (a) = B$, where a is an object variable of $\mathcal{S}_2$ and $B = \{b_1^1, \ldots, b_1^r\}$ is a set of object variables of $\mathcal{S}_1$, is equivalent to the set of object assertions

$$\{(b_1^1, a) : E_k, \ldots, (b_1^r, a) : E_k,\ a : (\leq r E_k)^2 \top_1\}$$

if we adopt the *unique name assumption* (*UNA*), i.e., assume that $(b_k^i)^{\mathfrak{M}} \neq (b_k^j)^{\mathfrak{M}}$ for any distinct b_k^i and b_k^j and any model $\mathfrak{M}$. It should be clear that this assumption can be made without loss of generality: reasoning without UNA can be reduced to reasoning with UNA by first 'guessing' an equivalence relation on the set of object names of each $\mathcal{S}_i$, then choosing a representative of each equivalence class, and finally replacing each object name with the representative of its class. We thus obtain from Theorem 28:

Theorem 37 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be number tolerant ADSs with decidable satisfiability problems. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{QI}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Let us now transfer these results to distributed description logics. Obviously, the translation $\cdot^\sharp$ can be extended to a map from distributed knowledge bases which possibly contain CICs into the set of assertions of the corresponding $\mathcal{E}$-connection by taking

- $(a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\})^\sharp = \langle E_{ij} \rangle^j (a_i) = \{b_j^1, \ldots, b_j^n\}$.

We then obtain the following transfer results for DDLs:

Corollary 38
(i) *The satisfiability problem for DDLs $\mathfrak{D} = (DL_i)_{i \leq n}$ is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the DL_i, and all of them have nominals.*

(ii) *The satisfiability problem for distributed description logics $\mathfrak{D} = (DL_1, DL_2)$ is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the DL_i, and both of them are number tolerant.*

Although we were able to identify some natural cases in which decidability transfers from $\mathcal{S}_1, \mathcal{S}_2$ to $\mathcal{C}^{\mathcal{E}}_I(\mathcal{S}_1, \mathcal{S}_2)$, the transfer of decidability fails in general. The proof of the following theorem is similar to the proofs of Theorems 22 and 29 and can be found in Appendix C.2:

Theorem 39
(i) *There exist ADS $\mathcal{S}_1$ and $\mathcal{S}_2$ with decidable satisfiability problems such that the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_I(\mathcal{S}_1, \mathcal{S}_2)$ is undecidable even if $\mathcal{E}$ is a singleton.*

(ii) *There exist number tolerant $\mathcal{S}_1, \mathcal{S}_2$ with decidable satisfiability problems such that the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{IB}(\mathcal{S}_1, \mathcal{S}_2)$ is undecidable even if $\mathcal{E}$ is a singleton.*

(iii) *There exist number tolerant $\mathcal{S}_1, \mathcal{S}_2$ with decidable satisfiability problems such that the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{IO}(\mathcal{S}_1, \mathcal{S}_2)$ is undecidable even if $\mathcal{E}$ is a singleton.*

7 Link constraints

Yet another interesting way of increasing the expressive power of $\mathcal{E}$-connections is by imposing various kinds of first-order constraints. Suppose, for example, that we want to extend the geographical knowledge bases considered in Section 4 with the following information:

- the spatial extension of the capital of every country is included in the spatial extension of that country, and that
- the EU will never contract.

Unfortunately, the $\mathcal{E}$-connections $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^{\sharp}, \mathsf{S4}_u^{\sharp})$ and $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^{\sharp}, \mathsf{S4}_u, \mathsf{PTL}^{\sharp})$ provide no means to do this. Of course, the conditions above can easily be expressed in the language of first-order logic:

$$(1) \qquad \forall x \forall y \forall z \ \Big(x \ \textit{capital-of} \ y \to (xEz \to yEz)\Big),$$

$$(2) \qquad \forall x \forall y \forall z \ \Big(y < z \to (E(EU, x, y) \to E(EU, x, z))\Big),$$

where *capital-of* is an $\mathcal{ALCO}^{\sharp}$ relation, $<$ is the precedence relation of the flow of time $\langle \mathbb{N}, < \rangle$, and the link E denotes spatial extension. Thus it would be interesting to find out what kinds of first-order constraints are 'harmless' from the computational point of view.

A general investigation of this question seems to be rather complex and is out of the scope of this paper. Here we only consider constraints of the form (1) and (2) above. Note that (1) and (2) have the same structure in the sense that they enforce a new E-link between the models under certain conditions. We show that under some weak conditions constraints of this form do not ruin the transfer of decidability. We begin by introducing link constraints formally.

Definition 40 Suppose that we are given $n \geq 2$ ADSs $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, R is a binary relation symbol of $\mathcal{L}_1$, $\overline{a} = a_3, \ldots, a_n$ are object variables in $\mathcal{L}_3, \ldots, \mathcal{L}_n$, respectively, and $E \in \mathcal{E}$. Then the formula

$$\forall x \forall y \forall z \ \Big(xRy \to (E(x, z, \overline{a}) \to E(y, z, \overline{a}))\Big)$$

is called a *link constraint* for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$.

We say that the binary relation R of $\mathcal{L}_1$ is *describable* in $\mathcal{S}_1$ if there exists a term t_R in $\mathcal{L}_1$ such that, for every model $\mathfrak{M} \in \mathcal{M}_1$ with domain W, every $x \in W$ and every $X \subseteq W$, we have

$$x \in t_R^{\mathfrak{M}}(X) \quad \text{iff} \quad \forall y \in W \ (xR^{\mathfrak{M}}y \to y \in X).$$

A link constraint with describable R is called a *describable link constraint.*

Clearly, the relations R and $<$ in link constraints (1) and (2) above are describable by the $\mathcal{ALC}^\sharp$- and $\mathsf{PTL}^\sharp$-terms corresponding to the 'box operators' $\forall R.C$ and $\Box_F p$, respectively. In what follows, we only consider those link constraints that are describable.

Definition 41 Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, are abstract description systems and $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. We denote by

$$\mathcal{C}^{\mathcal{E}}_{LO}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection in which the set of link assertions is extended with describable link constraints and the link operators can be applied to object variables. The truth-relation for the new $\mathcal{E}$-connection is defined in the obvious way; in particular, satisfiability of link constraints is defined via the standard first-order reading of these constraints.

The following transfer theorem can be proved by appropriately extending the proof of Theorem 16. Details can be found in Appendix B.1.

Theorem 42 *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be ADSs with decidable satisfiability problems. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{LO}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well.*

As already noted, a further investigation of first-order constraints on links is beyond the limits of this paper. As to the link constraints of the form above, we conjecture that by dropping the describability condition one destroys the (general) transfer of decidability. The combination of link constraints with other variants of $\mathcal{E}$-connections and the computational properties of different kinds of first-order constraints are left for future work.

8 Comparison with other combination methodologies

We now briefly compare $\mathcal{E}$-connections with three other combination methodologies which are relevant for knowledge representation and reasoning.

8.1 Multi-dimensional systems

The formation of multi-dimensional systems out of one-dimensional ones is probably the most frequently employed methodology of combining knowledge representation and reasoning formalisms. Given n languages $L_1, \ldots, L_n$ interpreted in domains $D_1, \ldots, D_n$, we take the union L of the L_i and interpret it in the Cartesian product $D_1 \times \cdots \times D_n$ consisting of all n-tuples $(d_1, \ldots, d_n)$,

$d_i \in D_i$. (The combined language L contains no new constructors as compared with the original languages L_i.) Typical examples of such multi-dimensional formalisms are:

- *temporal-epistemic logics* for reasoning about multi-agent systems—these are based on the Cartesian product of a flow of time and a set of possible states of a system (see [23,35] and references therein),
- *first-order modal and temporal logics* based on the Cartesian product of a set of possible worlds or moments of time and a domain of first-order individuals [27,30],
- *spatio-temporal logics* based on the Cartesian product of a flow of time and a model of space (see, e.g., [77,78]),
- *modal and temporal description logics* based on the Cartesian product of a set of possible worlds and a description logic domain [51,12,14,17,6,75,76].

The main difference between multi-dimensional systems and $\mathcal{E}$-connections is the range of the quantifiers: while the former quantify (at least implicitly) over the set of n-tuples, in $\mathcal{E}$-connections we can quantify only over one-dimensional objects which form a component of a link. This seems to be the main reason why, as we show in this paper, $\mathcal{E}$-connections exhibit a much more robust computational behavior than multi-dimensional combinations (see, e.g., [30] and references therein). In the multi-dimensional setting, even the two-dimensional combination of simple, say, NP-complete logics, can be highly undecidable [68]. In contrast to $\mathcal{E}$-connections, no general transfer results are available for multi-dimensional combinations (their algorithmic behavior is governed by rather subtle features of the component logics, so that the concept of abstract description systems is 'too abstract' to be useful in this context). On the contrary, it has been recently proved that three-dimensional products of standard unimodal logics (and even the two-dimensional products of CTL^* with standard unimodal logics) are usually undecidable [36,37]. In this respect, $\mathcal{E}$-connections do not 'feel' the number of combined formalisms.

8.2 Independent fusions and Gabbay's fibring methodology

Another way of combining formalisms without adding new constructors to the union of the languages is known as the formation of *independent fusions* or *joins* [46,25,74,13,68]. In this case, it is assumed that the component languages L_i actually speak about the *same domain* D. In other words, the expressive capabilities of the L_i are combined by the independent fusion in order to reason about the same objects, yet viewed from different perspectives. As in the case of multi-dimensional systems, no new constructors are added.

A typical example of an independent fusion is the standard multi-modal *epi-*

stemic logic modeling knowledge of $n > 1$ agents [33], where we simply join n epistemic logics for a *single* agent. Sometimes *temporal epistemic logics* degenerate to fusions of temporal and epistemic logics [35].

Independent fusions have also been suggested in the context of *description logics* [13], where constructors of different DLs may be required to represent knowledge about certain domains. Note that putting the constructors of different DLs together to form a new DL often results in an undecidable logic, even if the components are decidable.[6] It has been shown in [13] that independent fusions form a more robust (but, of course, less expressive) way of combining the constructors of different DLs than multi-dimensional combination.

In contrast to $\mathcal{E}$-connections, independent fusions behave 'badly' if the class of models is not closed under the formation of disjoint unions (the corresponding ADS is not local), for instance, when nominals or negations of roles are present [13] or when we combine logics of time and space—while linear orders are natural models of time, their disjoint unions are certainly not.

Gabbay's [28] *fibring methodology* is a generalization of independent fusions: when constructing the fibring of two formalisms L_1 and L_2, their models are not matched, but combined by a so-called *fibring function* F which associates with any element of the domain D_i of a model M_i for L_i a model $M_{\bar{i}}$ of the other formalism. The truth-values of formulas at point w are computed inductively: the Boolean operators are treated as usual, and the inductive step for a given constructor of L_i depends on whether w is a member of a model M_i for L_i—in which case it is computed as in M_i—or a member of a model $M_{\bar{i}}$ for the other logic $L_{\bar{i}}$, in which case the truth-value is computed in the model $F(w)$ for L_i. In contrast to $\mathcal{E}$-connections and similarly to multi-dimensional systems and independent fusions, the fibring formalisms do not add any new constructors to the combined languages, but are based on their unions. Also, in contrast to $\mathcal{E}$-connections, the atoms of the component languages are supposed to be identical. Finally, because of the guarded quantification in $\mathcal{E}$-connections in 'any direction' of a link relation, the interaction between the fibred components is much weaker than the interaction between the $\mathcal{E}$-connected ones.

8.3 Description logics with concrete domains

As demonstrated in Section 4.1, $\mathcal{E}$-connections can be used to connect a description logic, such as $\mathcal{ALC}$, with another logic, such as $\mathcal{MS}$, which is evalu-

[6] As an example, consider the DLs $\mathcal{ALCF}$ (introduced on Page 19) and $\mathcal{ALC}^{+,\circ,\sqcup}$ (extending $\mathcal{ALC}$ with transitive closure, composition, and union of roles). For both DLs, the subsumption of concept descriptions is known to be decidable [39,64,7]. However, the subsumption problem for their *union* $\mathcal{ALCF}^{+,\circ,\sqcup}$ is undecidable [8].

ated in a *single* model, say, a map of London. This idea—to fix a single model in one of the combined formalisms—also underlies the extension of description logics with so-called *concrete domains*: since 'classical' description logics represent knowledge at a rather abstract logical level, concrete domains have been proposed to cope with applications that require predefined predicates or temporal and spatial dimensions [10,56]. Examples of concrete domains include the natural numbers equipped with predicates like $=_{17}$, $<$, and $+$ [11,55], Allen's interval algebra [1,53], and the RCC-8 calculus discussed in Section 2.2 [32]. However, the expressive power provided by concrete domains is largely orthogonal to the expressive power of $\mathcal{E}$-connections. First, in DLs with concrete domains, the coupling of the two formalisms is 'one way,' i.e., we can only talk about the concrete domain in the description logic, but not vice versa. Clearly, $\mathcal{E}$-connections are 'two way' in this sense. Second, in DLs with concrete domains, the description logic is equipped with operators which allow us to make statements about relations (of arbitrary arity) between 'concrete elements.' In contrast, $\mathcal{E}$-connections allow us only to express that formulas (i.e., *unary* predicates) are satisfied by domain elements of other components. It should also be noted that the addition of a concrete domain to a DL is a rather sensitive operation as far as the preservation of computational properties is concerned: even 'weak' DLs combined with rather 'weak' concrete domains can become undecidable, see, e.g., [11,32,54]. In fact, except for a result in [13] which treats extremely inexpressive concrete domains, no general decidability transfer results for the extension of description logics with concrete domains are known. Indeed, investigating the computational properties of DLs with concrete domains is a cumbersome task which involves the development of new and specialized techniques, consult, e.g., the survey [56].

9 Discussion

In this paper, we have developed a new methodology of combining knowledge representation and reasoning formalisms. The key idea of the methodology is to keep the domains of the combined formalisms disjoint and to introduce 'link relations' which keep track of existing correspondences between objects in different domains. Typical link relations are as follows:

- 'x is in the spatial extension of y,'
- 'x belongs to the lifespan of y,'
- object x in information system IS_1 'corresponds' to object y in information system IS_2.

The new methodology is introduced within the framework of *abstract description systems* in order to provide coverage of a wide range of KR&R formalisms such as description logics, temporal logics, modal logics of space, epistemic logics, etc. The resulting combinations are called *$\mathcal{E}$-connections.*

The main technical result of the paper is a number of theorems which show that the formation of various kinds of $\mathcal{E}$-connections is computationally robust, even if we allow expressive link operators such as qualified number restrictions and Boolean combinations of link relations. On the other hand, our complexity and undecidability results show that this nice computational behavior of $\mathcal{E}$-connections does not come for free. As we have argued in the introduction, the design of 'practical' reasoning systems for $\mathcal{E}$-connections cannot be specified at this level of generality, but depends on the features of the combined formalisms. The message of the present investigation is, however, that the chances of *$\mathcal{E}$-connections* to be reasonably efficient on practical examples are as high as those of standard description or temporal logics.

Although we have considered in-depth various extensions of the basic $\mathcal{E}$-connections, a number of interesting problems remain open. Here are some of them:

- Starting from the theoretical results obtained in this paper, develop 'practical' decision procedures for interesting $\mathcal{E}$-connections like, for instance, $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^{\sharp}, \mathcal{ALCO}^{\sharp})$, $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^{\sharp}, \mathcal{MS}^{\sharp})$, or $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^{\sharp}, \mathsf{S4}_u^{\sharp}, \mathsf{PTL}^{\sharp})$. In all these cases, efficient decision procedures for the components have been implemented. Is it possible to devise decision procedures for the $\mathcal{E}$-connections which are modular and integrate known decision procedures for the components without substantial modifications? Compare the performance of implemented algorithms for the $\mathcal{E}$-connections with the performance of decision procedures for their components.
- Consider more general first-order constraints for the link relations and classify them according to their algorithmic behavior. This can also lead to a deeper analysis of the structural properties of ADSs because more subtle conditions than describability are required for decidability transfer results covering larger classes of first-order constraints.
- Introduce elements of 'fuzziness' to link relations between different domains in order to reflect the fact that spatial extensions or 'corresponding' objects in distributed databases can be often specified only approximately. It would, therefore, be of interest to allow link operators stating, for example, that 'the probability that y belongs to the spatial extension of x is not more than 75%.'
- The embedding of the product logic $\mathbf{S5} \times \mathbf{S5}$ into the $\mathcal{E}$-connection with the Booleans $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$ provides the first evidence that there might be an interesting and useful hierarchy of formalisms between the 'weak' basic $\mathcal{E}$-connections and multi-dimensional formalisms. For example, we can take the closure of the set of link relations $\mathcal{E}$ not only under the Booleans, but

also under the operations $\langle R \rangle E$ and $[R]E$ defined by taking

$$(\langle R \rangle E)^{\mathfrak{M}} = \{\langle x, y\rangle \mid \exists z\, (\langle x, z\rangle \in R^{\mathfrak{M}} \wedge \langle z, y\rangle \in E^{\mathfrak{M}})\},$$
$$([R]E)^{\mathfrak{M}} = \{\langle x, y\rangle \mid \forall z\, (\langle x, z\rangle \in R^{\mathfrak{M}} \rightarrow \langle z, y\rangle \in E^{\mathfrak{M}})\},$$

for every binary relation symbol R of the first component of a binary $\mathcal{E}$-connection $\mathcal{C}$ (and similarly for the binary relations of the second component). Using these new constructors, we can easily 'simulate' most of multi-dimensional formalisms. Useful and interesting intermediate formalisms could be obtained by restricting applications of the Boolean operators to links.

A Properties of ADSs

This section proves Propositions 25, 8, 27, and 10. For the reader's convenience we formulate these propositions once again.

Proposition 25. *Every local ADS is number tolerant.*

Proof. Suppose that an ADS $(\mathcal{L}, \mathcal{M})$ is local. Let κ be any infinite cardinal such that, for every finite satisfiable Σ, there exists a model $\mathfrak{W} \in \mathcal{M}$ of cardinality $\leq \kappa$ which satisfies Σ. The supremum of all the minimal cardinals needed to satisfy each Σ will do, for instance. We show that κ is as required. Suppose that $\kappa' \geq \kappa$ and that Σ is satisfiable. Take any model

$$\mathfrak{W}_0 = \left\langle W_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, \mathcal{F}^{\mathfrak{W}_0}, \mathcal{R}^{\mathfrak{W}_0} \right\rangle$$

from $\mathcal{M}$ which satisfies Σ and is of cardinality $\leq \kappa$. Now take the disjoint union $\mathfrak{W}$ of κ' isomorphic copies $\mathfrak{W}_i$, $i < \kappa'$, of $\mathfrak{W}_0$ in which

– $x^{\mathfrak{W}} = \bigcup_{i<\kappa'} x^{\mathfrak{W}_i}$, for $x \in \mathcal{V}$;
– $a^{\mathfrak{W}} = a^{\mathfrak{W}_0}$, for $a \in \mathcal{X}$.

By cardinal arithmetic, the size of $\mathfrak{W}$ is κ', and it is not difficult to show that $\mathfrak{W}$ satisfies all of the conditions we need. □

Propositions 8 and 27. $\mathsf{S4}_u^\sharp$ *is number tolerant. The satisfiability problem for* $\mathsf{S4}_u^\sharp$ *is PSPACE-complete.*

Proof. We first show that the satisfiability problem for $\mathsf{S4}_u^\sharp$ is PSPACE-complete. PSPACE-hardness follows from PSPACE-hardness of the satisfiability problem for $\mathsf{S4}$ [50]. We establish the corresponding upper bound by means of a reduction to the satisfiability problem for $\mathsf{S4}_u$ enriched with *nom-*

inals in topological models,[7] which is known to be PSPACE-complete [4]. Namely, given a set Γ of $\mathsf{S4}_u^\sharp$-assertions we define an $\mathsf{S4}_u$-formula φ_Γ as the conjunction of all formulas in the set

$$\{\boxed{\forall}(\varphi_1 \to \varphi_2) \mid (\varphi_1^\sharp \sqsubseteq \varphi_2^\sharp) \in \Gamma\} \cup \{\boxed{\forall}(\{a\} \to \psi) \mid (a : \psi^\sharp) \in \Gamma\}.$$

Obviously, φ_Γ is satisfiable in some topological model iff Γ is satisfiable, which gives us the required PSPACE-upper bound. Reductions of this type are known as 'internalizations' of TBoxes by means of the universal box [64].

To prove that $\mathsf{S4}_u^\sharp$ is number tolerant, we show that $\aleph_0$ is the required cardinal number. Suppose that $\kappa' \geq \aleph_0$ and that Σ is satisfiable. Let

$$\mathfrak{W}_0 = \left\langle T_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, f_{\boldsymbol{I}}^{\mathfrak{W}_0}, f_{\boldsymbol{C}}^{\mathfrak{W}_0}, f_{\boxed{\forall}}^{\mathfrak{W}_0} \right\rangle$$

be a countable model satisfying Σ. Take the disjoint union $\mathfrak{W}'$ of κ' isomorphic copies $\mathfrak{W}_i'$, $i < \kappa'$, of the reduct

$$\mathfrak{W}_0' = \left\langle T_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, f_{\boldsymbol{I}}^{\mathfrak{W}_0}, f_{\boldsymbol{C}}^{\mathfrak{W}_0} \right\rangle$$

of $\mathfrak{W}_0$ in which

- $x^{\mathfrak{W}'} = \bigcup_{i<\kappa'} x^{\mathfrak{W}_0}$, for $x \in \mathcal{V}$;
- $a^{\mathfrak{W}'} = a^{\mathfrak{W}_0}$, for $a \in \mathcal{X}$.

Now we extend $\mathfrak{W}'$ to a model $\mathfrak{W}$ of the required signature by setting

$$f_{\boxed{\forall}}^{\mathfrak{W}}(Y) = \begin{cases} \emptyset & \text{if } Y \neq \bigcup\limits_{i<\kappa'} T_i, \\ \bigcup\limits_{i<\kappa'} T_i & \text{if } Y = \bigcup\limits_{i<\kappa'} T_i, \end{cases}$$

for every subset Y of $\bigcup_{i<\kappa'} T_i$. It is readily seen that the constructed ADM $\mathfrak{W}$ is as required. $\square$

Proposition 10. $\mathsf{PTL}^\sharp$ *has nominals. The satisfiability problem for* $\mathsf{PTL}^\sharp$ *is PSPACE-complete.*

Proof. It it proved in [67] that the satisfiability problem for PTL is PSPACE-complete. As we have already seen above, the nominals and the binary relation $<$ can be simulated in PTL. Observe that the universal box $\boxed{\forall}\varphi$ can be expressed as well, by using the formula $\Box_F\varphi \wedge \varphi \wedge \Box_P\varphi$. Therefore, we can employ the same internalization reduction as in the proof of Proposition 8 to show that the satisfiability problem for $\mathsf{PTL}^\sharp$ is PSPACE-complete. $\square$

[7] Nominals $\{a\}$ are interpreted as singleton sets of topological spaces.

B Decidability results

This section establishes decidability results for $\mathcal{E}$-connections of abstract description systems. Before we actually start proving these results, let us introduce some notation that will be used in all of the proofs in this section. To make presentation as clear as possible, throughout the appendix we confine ourselves to $\mathcal{E}$-connections of only two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$. In this case it will be convenient to write $\overline{1}$ for 2 and $\overline{2}$ for 1. Let Γ be a finite set of assertions of some $\mathcal{E}$-connection of $\mathcal{S}_1$ and $\mathcal{S}_2$ (possibly allowing link operators on object variables and/or Boolean combinations of link relations). Then we use the following notation:

- We write $ob_i(\Gamma)$ to denote the set of object variables from $\mathcal{L}_i$ which occur in Γ, for $i = 1, 2$.
- We write $\mathcal{X}_i(\Gamma)$ to denote the set of object variables

$$\mathcal{X}_i \setminus (ob_i(\Gamma) \cup (\mathcal{X}_i)_{\mathcal{G}_i}),$$

  where $\mathcal{G}_i$ is the set of function symbols of $\mathcal{L}_i$ which occur in Γ and $(\mathcal{X}_i)_{\mathcal{G}_i}$ is the set of object variables supplied by the closure condition of Definition 3 (ii), for $i = 1, 2$.
- In each of the decidability proofs, we will use $\mathsf{cl}_i(\Gamma)$, $i = 1, 2$, to refer to some finite closure of the set of i-terms occurring in Γ. Since different closures are required in different proofs, we do not fix the exact details here.
- We assume that, for every i-term t of the form $\langle F \rangle^i (s)$ occurring in $\mathsf{cl}_i(\Gamma)$ (where s is an $\bar{i}$-term or an object name of $\mathcal{L}_{\bar{i}}$, $i = 1, 2$, and F is a link symbol or a Boolean combination of such symbols), there exists a set variable x_t of $\mathcal{L}_i$ not occurring in Γ. Given an i-term t, denote by $sur_i(t)$—the *surrogate of* t—the term which results from t by replacing all subterms t' of the form $\langle F \rangle^i (s)$ that are not within the scope of another term $\langle G \rangle^i (s)$ with $x_{t'}$. Clearly, $sur_i(t)$ belongs to the language $\mathcal{L}_i$.
- The *i-consistency set* $\mathfrak{C}_i(\Gamma)$ is defined as the set $\{t_c \mid c \subseteq \mathsf{cl}_i(\Gamma)\}$, where

$$t_c = \bigwedge\{\chi \mid \chi \in c\} \wedge \bigwedge\{\neg\chi \mid \chi \in \mathsf{cl}_i(\Gamma) \setminus c\}.$$

  Sometimes we will identify $t \in \mathfrak{C}_i(\Gamma)$ with the set of its conjuncts. Then $s \in t$ means that s is a conjunct of t.
- Recall that by $\top_i$ we denote $x_i \vee \neg x_i$, where x_i is a set variable from $\mathcal{L}_i$.

B.1 Basic $\mathcal{E}$-connections, link operators on object variables, and link constraints

This section proves Theorems 11, 16, and 42. More precisely, we start by proving Theorem 16. Since Theorem 16 clearly implies Theorem 11, a separate proof for the latter is omitted. As was said above, we confine ourselves to $\mathcal{E}$-connections of only *two* ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$. Moreover, for simplicity we assume that $\mathcal{E}$ contains only a *single* link symbol E. Thus, our first aim is to prove the following variant of Theorem 16.

Theorem 43 *Suppose the satisfiability problems for the ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable. Then the satisfiability problem for the $\{E\}$-connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

The reader should be able to extend the proofs to n-ary $\mathcal{E}$-connections with multiple link relations without any difficulty. Having proved Theorem 43, we then extend it to take into account link constraints, thus obtaining a proof of Theorem 42. Here is the simplified variant of this theorem:

Theorem 44 *Suppose the satisfiability problems for the ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable. Then the satisfiability problem for the $\{E\}$-connection $\mathcal{C}_{LO}^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Observe that, since we restrict ourselves to the connection of only two ADSs, the additional function symbols $\langle E \rangle^1$ and $\langle E \rangle^2$ of the connection are unary. Since the connections treated in this section allow the application of link operators to object variables, we do not explicitly treat link assertions of the form $(a_1, a_2) : E$. Clearly, such a link assertion can be replaced with the equivalent object assertion $a_1 : \langle E \rangle^1 (a_2)$.

Proof of Theorem 43

Fix two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$ with decidable satisfiability problems, and let Γ be a finite set of assertions of the $\{E\}$-connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$. To define the closure $\mathsf{cl}_i(\Gamma)$ of i-terms occurring in Γ, we first introduce the abbreviation

$$o_i(\Gamma) = \{\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a) \mid a \in ob_i(\Gamma)\},$$

for $i = 1, 2$. The set $o_i(\Gamma)$ contains i-terms that must be present in the closure $\mathsf{cl}_i(\Gamma)$ in order to ensure a proper treatment of link operators applied to object variables. Note that, given a model $\mathfrak{M}$ of the $\{E\}$-connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$,

$$(\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a))^{\mathfrak{M}} = \{x \in W_i \mid \exists y \in W_{\bar{i}} \left((a, y) \notin E^{\mathfrak{M}} \wedge (x, y) \in E^{\mathfrak{M}}\right)\},$$

and so $a^{\mathfrak{M}} \notin (\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a))^{\mathfrak{M}}$.

We now define $\mathsf{cl}_i(\Gamma)$, $i = 1, 2$, to be the closure under negation of the set of i-terms which occur in $\Gamma \cup o_i(\Gamma)$. Without loss of generality we can identify $\neg\neg t$ with t. Thus, $\mathsf{cl}_i(\Gamma)$ is finite.

The following theorem is the core component in the proof of Theorem 43: it provides us with a criterion of satisfiability of sets of $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions Γ which almost immediately implies decidability of the satisfiability problem for $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$.

Theorem 45 *Let Γ be a $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base. Then Γ is satisfiable iff there exist* (i) *subsets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$,* (ii) *a relation $e \subseteq \Delta_1 \times \Delta_2$,* (iii) *functions σ_1 from $ob_1(\Gamma)$ into Δ_1 and σ_2 from $ob_2(\Gamma)$ into Δ_2 such that, for $i = 1, 2$, the following conditions are satisfied:*

(1) *for any $a \in ob_i(\Gamma)$, we have $\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a) \notin \sigma_i(a)$,*
(2) *the union Γ_i of*
  – $\{sur_i(\bigvee \Delta_i) = \top_i\}$,
  – $\{a_t : sur_i(t) \mid t \in \Delta_i\}$,
  – $\{a : sur_i(\sigma_i(a)) \mid a \in ob_i(\Gamma)\}$,
  – $\{sur_i(t_1) \sqsubseteq sur_i(t_2) \mid t_1 \sqsubseteq t_2 \in \Gamma$ *is an i-term assertion*$\}$,
  – $\{R_j(a_1, \dots, a_{m_j}) \mid R_j(a_1, \dots, a_{m_j}) \in \Gamma$ *is an i-object assertion*$\}$,
  – $\{(a : sur_i(t)) \mid (a : t) \in \Gamma$ *is an i-object assertion*$\}$

  is $\mathcal{S}_i$-satisfiable, where $a_t \in \mathcal{X}_i(\Gamma)$ is a fresh object variable for each $t \in \Delta_i$,
(3) *for all $t \in \Delta_1$ and $\langle E \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$ with s a 2-term, we have $\langle E \rangle^1 (s) \in t$ iff there exists $t' \in \Delta_2$ with $(t, t') \in e$ and $s \in t'$,*
(4) *for all $t \in \Delta_2$ and $\langle E \rangle^2 (s) \in \mathsf{cl}_2(\Gamma)$ with s a 1-term, we have $\langle E \rangle^2 (s) \in t$ iff there exists $t' \in \Delta_1$ with $(t', t) \in e$ and $s \in t'$,*
(5) *for all $t \in \Delta_1$ and $\langle E \rangle^1 (a) \in \mathsf{cl}_1(\Gamma)$ with $a \in ob_2(\Gamma)$, we have $\langle E \rangle^1 (a) \in t$ iff $(t, \sigma_2(a)) \in e$,*
(6) *for all $t \in \Delta_2$ and $\langle E \rangle^2 (a) \in \mathsf{cl}_2(\Gamma)$ with $a \in ob_1(\Gamma)$, we have $\langle E \rangle^2 (a) \in t$ iff $(\sigma_1(a), t) \in e$.*

Proof. ($\Rightarrow$) Suppose Γ is $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-satisfiable and $\mathfrak{M} = ((\mathfrak{W}_1, \mathfrak{W}_2), E^{\mathfrak{M}})$ is a model of Γ, with W_1 being the domain of $\mathfrak{W}_1$ and W_2 being the domain of $\mathfrak{W}_2$. For $i = 1, 2$ and each $d \in W_i$, define

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}.$$

Then set $\Delta_i = \{t(d) \mid d \in W_i\}$ for $i = 1, 2$ and define $e \subseteq \Delta_1 \times \Delta_2$ by putting $(t, t') \in e$ iff there exist $d_1 \in W_1$ and $d_2 \in W_2$ such that $t = t(d_1)$, $t' = t(d_2)$, and $(d_1, d_2) \in E^{\mathfrak{M}}$. Finally, for $i = 1, 2$ and each $a \in ob_i(\Gamma)$, define

$$\sigma_i(a) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid a^{\mathfrak{M}} \in s^{\mathfrak{M}}\} = t(a^{\mathfrak{M}}) \in \Delta_i.$$

It remains to check that Δ_1, Δ_2, e, σ_1, and σ_2 satisfy conditions (1)–(6).

(1) Suppose that there is an $a \in ob_i(\Gamma)$ such that $\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a) \in \sigma_i(a)$. Then, by the definition of σ_i, $a^{\mathfrak{M}} \in (\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a))^{\mathfrak{M}}$, which is impossible.

(2) We have to show that the Γ_i are $\mathcal{S}_i$-satisfiable. The models

$$\mathfrak{W}_i = \left\langle W_i, \mathcal{V}_i^{\mathfrak{W}_i}, \mathcal{X}_i^{\mathfrak{W}_i}, \mathcal{R}_i^{\mathfrak{W}_i}, \mathcal{F}_i^{\mathfrak{W}_i} \right\rangle$$

are almost as required: we just have to give appropriate values to the fresh set variables x_t (which result from taking surrogates) and the fresh object names a_t from $\mathcal{X}_i(\Gamma)$. To this end, put

$$x_s^{\mathfrak{W}_i'} = s^{\mathfrak{M}}$$

for every term $s \in \mathsf{cl}_i(\Gamma)$ of the form $\langle E \rangle^i (s')$ and $x^{\mathfrak{W}_i'} = x^{\mathfrak{W}_i}$ for the remaining variables. For every $t \in \Delta_i$, choose $a_t^{\mathfrak{W}_i'}$ such that

$$a_t^{\mathfrak{W}_i'} \in t^{\mathfrak{M}}$$

and set $a^{\mathfrak{W}_i'} = a^{\mathfrak{W}_i}$ for the remaining object names. Note that

$$\mathfrak{W}_i' = \left\langle W_i, \mathcal{V}_i^{\mathfrak{W}_i'}, \mathcal{X}_i^{\mathfrak{W}_i'}, \mathcal{R}_i^{\mathfrak{W}_i}, \mathcal{F}_i^{\mathfrak{W}_i'} \right\rangle \in \mathcal{M}_i$$

for some interpretation $\mathcal{F}_i^{\mathfrak{W}_i'}$ of the function symbols in $\mathcal{F}_i$ such that $f^{\mathfrak{W}_i} = f^{\mathfrak{W}_i'}$ for all function symbols f of Γ (due to the closure condition for the class $\mathcal{M}_i$ formulated in Definition 3). Using induction on the term structure of s, it is straightforward to show that

$$d \in (sur_i(s))^{\mathfrak{W}_i'} \quad \text{iff} \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$ and $s \in \mathsf{cl}_i(\Gamma)$. By considering the construction of Γ_i, it is readily checked that this implies $\mathfrak{W}_i' \models \Gamma_i$. Hence Γ_i is $(\mathcal{L}_i, \mathcal{M}_i)$-satisfiable.

(3) Let $t \in \Delta_1$ and $\langle E \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$ with s a 2-term. Since $t \in \Delta_1$, there is a $d \in W_1$ such that $t(d) = t$. First assume that $\langle E \rangle^1 (s) \in t$. By definition, this means that there exists a $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s^{\mathfrak{M}}$. This, in turn, clearly yields $s \in t(d')$ and $(t, t(d')) \in e$, as required. Now assume that $(t, t') \in e$ and $s \in t'$. Then there exist $d \in W_1$ and $d' \in W_2$ such that $t = t(d)$, $t' = t(d')$, and $(d, d') \in E^{\mathfrak{M}}$. We have $d' \in s^{\mathfrak{W}}$, and so $d \in (\langle E \rangle^1 (s))^{\mathfrak{M}}$, from which $\langle E \rangle^1 (s) \in t$, as required.

(4) is proved similarly to (3).

(5) Let $t \in \Delta_1$ and $\langle E \rangle^1 (a) \in \mathsf{cl}_1(\Gamma)$ with $a \in ob_2(\Gamma)$. Since $t \in \Delta_1$, there is a $d \in W_1$ such that $t(d) = t$. First assume $\langle E \rangle^1 (a) \in t$. By definition, we then have $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$. Hence $(t, t(a^{\mathfrak{M}})) \in e$, i.e., $(t, \sigma_2(a)) \in e$, as required. Conversely, suppose $(t, \sigma_2(a)) \in e$. condition (1) yields $\langle E \rangle^2 \neg \langle E \rangle^1 (a) \notin \sigma_2(a)$. By condition (4), we have $\neg \langle E \rangle^1 (a) \notin t$. Hence $\langle E \rangle^1 (a) \in t$, as required.

(6) is proved similarly to (5).

($\Leftarrow$) Conversely, suppose that Δ_1, Δ_2, e, σ_1, and σ_2 satisfy the conditions of the theorem. By (2), there exist a model $\mathfrak{W}_1 \in \mathcal{M}_1$ of Γ_1 and a model $\mathfrak{W}_2 \in \mathcal{M}_2$ of Γ_2. For $i = 1, 2$, let $\mathfrak{W}_i$ be based on the domain W_i. For each $d \in W_i$, we set

$$t(d) = \bigwedge\{t \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(t))^{\mathfrak{W}_i}\} \in \mathfrak{C}_i(\Gamma).$$

Now define the extension $E^{\mathfrak{M}} \subseteq W_1 \times W_2$ of the link symbol E by taking:

$$E^{\mathfrak{M}} = \{(d, d') \mid (t(d), t(d')) \in e\}.$$

In the following, we prove that $\mathfrak{M} = (\mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}})$ is a model of Γ. Using the construction of the Γ_i, it is readily checked that it suffices to show that

$$d \in (sur_i(s))^{\mathfrak{W}_i} \quad \text{iff} \quad d \in s^{\mathfrak{M}} \tag{$*$}$$

for $i = 1, 2$, all $d \in W_i$, and all $s \in \mathsf{cl}_i(\Gamma)$.

The proof of this claim is by induction on the term structure of s, simultaneously for $i = 1, 2$. For set variables, the claim is an immediate consequence of the definition of $\mathfrak{M}$. The cases of the Boolean operators and the function symbols of $\mathcal{L}_i$, $i = 1, 2$, are trivial. Thus, it remains to consider the cases

(a) $s = \langle E\rangle^i (s')$ with s' an $\bar{\imath}$-term and
(b) $s = \langle E\rangle^i (a)$ with $a \in ob_{\bar{\imath}}(\Gamma)$.

We assume $i = 1$, since the case $i = 2$ is dual.

(a) $s = \langle E\rangle^1 (s')$ with s' a 2-term. Let $d \in (sur_1(\langle E\rangle^1 s'))^{\mathfrak{W}_1}$. Then we have $\langle E\rangle^1 (s') \in t(d)$. Since $\mathfrak{W}_1$ is a model of Γ_1,

$$\mathfrak{W}_1 \models sur_1(\bigvee \Delta_1) = \top_1.$$

Thus $t(d) \in \Delta_1$. By condition (3), we find a $t' \in \Delta_2$ with $(t(d), t') \in e$ and $s' \in t'$. By the definition of Γ_2, we have

$$\mathfrak{W}_2 \models a_{t'} : sur_2(t'),$$

and so there is a $d' \in W_2$ such that $t' = t(d')$. Hence we have $(d, d') \in E^{\mathfrak{M}}$ by the definition of $E^{\mathfrak{M}}$. From $s' \in t'$, we obtain $d' \in (sur_2(s'))^{\mathfrak{M}_2}$, and therefore the induction hypothesis yields $d' \in s'^{\mathfrak{M}}$. Thus, $d \in (\langle E\rangle^1 (s'))^{\mathfrak{M}}$ by definition.

Conversely, suppose $d \in (\langle E\rangle^1 (s'))^{\mathfrak{M}}$. We find $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s^{\mathfrak{M}}$. By the induction hypothesis, $d' \in (sur_2(s'))^{\mathfrak{W}_2}$ and so $s' \in t(d')$. The definition of $E^{\mathfrak{M}}$ together with $(d, d') \in E^{\mathfrak{M}}$ yields $(t(d), t(d')) \in e$. Finally, by (3), we obtain $\langle E\rangle^1 (s') \in t(d)$ which implies $d \in (sur_1(\langle E\rangle^1 s'))^{\mathfrak{W}_1}$.

(b) $s = \langle E \rangle^1(a)$ with $a \in ob_2(\Gamma)$. Let $d \in (sur_1(\langle E \rangle^1(a)))^{\mathfrak{W}_1}$. This implies $\langle E \rangle^1(a) \in t(d)$. As in the previous case, we have $t(d) \in \Delta_1$. By condition (5), we thus obtain $(t(d), \sigma_2(a)) \in e$. Also, as in the previous case, we know that

$$\mathfrak{W}_2 \models a : sur_2(\sigma_2(a)).$$

Hence $(t(d), \sigma_2(a)) \in e$ and the definition of $E^{\mathfrak{M}}$ yields $(d, a^{\mathfrak{W}_2}) \in E^{\mathfrak{M}}$, which implies $d \in (\langle E \rangle^1(a))^{\mathfrak{M}}$.

Conversely, suppose $d \in (\langle E \rangle^1(a))^{\mathfrak{M}}$. Then $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by definition, and so $(t(d), t(a^{\mathfrak{M}})) \in e$ by the definition of $E^{\mathfrak{M}}$. We have $t(a^{\mathfrak{M}}) = \sigma_2(a)$, and therefore $(t(d), \sigma_2(a)) \in e$. Together with condition (5), this yields $\langle E \rangle^1(a) \in t(d)$ which clearly implies $d \in (sur_1(\langle E \rangle^1(a)))^{\mathfrak{W}_1}$. □

Theorem 43 follows from Theorem 45. Indeed, since the sets $\mathfrak{C}_i(\Gamma)$ are finite, Theorem 45 provides us with a decision procedure for the connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$ if decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$ are known. To decide whether a set Γ of $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$, a relation $e \subseteq \Delta_1 \times \Delta_2$, and functions $\sigma_i : ob_i(\Gamma) \to \Delta_i$, $i = 1, 2$, and then check whether they satisfy the conditions listed in the formulation of the theorem.

To estimate the complexity of the obtained decision procedure, note that the cardinality of the sets $\mathfrak{C}_i(\Gamma)$ is exponential in the size of Γ. Thus, the same holds for the sets Δ_1 and Δ_2 and for the constructed sets of assertions Γ_1 and Γ_2 which are passed to decision procedures for $\mathcal{S}_i$-satisfiability. This means that the time complexity of the obtained decision procedure for $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-satisfiability is one exponential higher than the time complexity of the original decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$-satisfiability. Moreover, the combined decision procedure is non-deterministic: if, for example, $\mathcal{S}_1$ and $\mathcal{S}_2$-satisfiability are in EXPTIME, then our algorithm yields a 2-NEXPTIME decision procedure for $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-satisfiability.

Proof of Theorem 44

We now extend Theorem 45 and its proof to take into account constraints, thus obtaining a proof of Theorem 44. Let Φ be a finite set of link constraints talking only about the link relation E such that the relations $R_1, \ldots, R_k$ occurring in Φ are describable in $\mathcal{S}_1$. Observe that no vectors of object variables $\overline{a}$ appear in the constraints, as we are concerned with the connection of only two ADSs. We make the following modifications of Theorem 45 and the notions it uses:

(1) We redefine the closure $\mathsf{cl}_1(\Gamma)$ as follows (but keep the definition of $\mathsf{cl}_2(\Gamma)$): let Θ_0 denote the closure under negation of the set of 1-terms occurring

in Γ and $o_1(\Gamma)$. Then set

$$\Theta_1 = \Theta_0 \cup \{\langle E\rangle^1 (s) \mid s = \neg \langle E\rangle^2 (s') \in \mathsf{cl}_2(\Gamma) \text{ or } s = \neg \langle E\rangle^2 (a) \in \mathsf{cl}_2(\Gamma)\},$$

$$\Theta_2 = \Theta_1 \cup \bigcup_{1\leq j\leq k} \{t_{R_j}(s) \mid s = \langle E\rangle^1 (s') \in \Theta_1 \text{ or } s = \langle E\rangle^1 (a) \in \Theta_1\},$$

where, for $1 \leq j \leq k$, t_{R_j} is the $\mathcal{L}_1$-term describing the relation R_j (cf. the definition of 'describable' in Section 7). Finally, define $\mathsf{cl}_1(\Gamma)$ to be the closure of Θ_2 under subformulas and negation; again we identify $\neg\neg t$ with t, so that the closure is finite.

(2) We add the following to the definition of Γ_1 in condition (2) of Theorem 45 (but leave Γ_2 unchanged):

$$\{sur_1(\langle E\rangle^1 (s)) \sqsubseteq t_R(sur_1(\langle E\rangle^1 (s))) \mid \langle E\rangle^1 (s) \in \mathsf{cl}_1(\Gamma)\}$$

$$\{sur_1(\langle E\rangle^1 (a)) \sqsubseteq t_R(sur_1(\langle E\rangle^1 (a))) \mid \langle E\rangle^1 (a) \in \mathsf{cl}_1(\Gamma)\}.$$

The proof of the theorem remains largely unchanged. Only in the 'if' direction, the definition of the link relation $E^{\mathfrak{M}}$ is modified: we set

$$E_0^{\mathfrak{M}} = \{(d, d') \mid (t(d), t(d')) \in e\};$$

$$E_{n+1}^{\mathfrak{M}} = E_n^{\mathfrak{M}} \cup \{(d, d') \mid \exists d'' \ (d'', d) \in R_j^{\mathfrak{W}_1} \text{ with } 1 \leq j \leq k \text{ and } (d'', d') \in E_n^{\mathfrak{M}}\};$$

$$E^{\mathfrak{M}} = \bigcup_{n\geq 0} E_n^{\mathfrak{M}}.$$

It is easy to see that $E^{\mathfrak{M}}$ satisfies all of the constraints in Φ. Since the definition of $E^{\mathfrak{M}}$ has changed, we need to adapt the proof of $(*)$ on Page 51. The cases of the Boolean operators, the function symbols of $\mathcal{L}_1$ and $\mathcal{L}_2$, and the 'only if' directions of the link operators remain unchanged. However, the 'if' directions of the link operators have to be extended. Let us start with proving the following auxiliary lemma:

Lemma 46 *Let s and s' be, respectively, a 1- and a 2-term, a an object variable of $\mathcal{L}_1$, and a' an object variable of $\mathcal{L}_2$ with $\{\langle E\rangle^1 (s'), \langle E\rangle^1 (a')\} \subseteq \mathsf{cl}_1(\Gamma)$ and $\{\langle E\rangle^2 (s), \langle E\rangle^2 (a)\} \subseteq \mathsf{cl}_2(\Gamma)$. If $(d, d') \in E^{\mathfrak{M}}$, then the following holds:*

(i) *$s' \in t(d')$ implies $\langle E\rangle^1 (s') \in t(d)$;*
(ii) *$s \in t(d)$ implies $\langle E\rangle^2 (s) \in t(d')$;*
(iii) *${a'}^{\mathfrak{M}} = d'$ implies $\langle E\rangle^1 (a') \in t(d)$;*
(iv) *$a^{\mathfrak{M}} = d$ implies $\langle E\rangle^2 (a) \in t(d')$.*

Proof. The proof is by induction on n. Let $n = 0$. Then $(d, d') \in E_0^{\mathfrak{M}}$ implies $(t(d), t(d')) \in e$. Thus, (i) is an immediate consequence of condition (3), (ii) is an immediate consequence of (4), (iii) of (5), and (iv) of (6).

Let $n > 0$. Then $(d, d') \in E_n^{\mathfrak{M}}$ implies that either $(d, d') \in E_{n-1}^{\mathfrak{M}}$ or there exists a d'' such that $(d'', d) \in R_j^{\mathfrak{W}_1}$ for some j with $1 \leq j \leq k$ and $(d'', d') \in E_{n-1}^{\mathfrak{M}}$. In the former case, (i)–(iv) follow by the induction hypotheses. Let us consider the latter one.

(i) Let $s' \in t(d')$. By the induction hypotheses and since $(d'', d') \in E_{n-1}^{\mathfrak{M}}$, we have $\langle E \rangle^1 (s') \in t(d'')$ and so $d'' \in sur_1(\langle E \rangle^1 (s'))^{\mathfrak{W}_1}$. Due to the new components of Γ_1 and the fact that $(d'', d) \in R_j^{\mathfrak{W}_1}$, we then have $d \in sur_1(\langle E \rangle^1 (s'))^{\mathfrak{W}_1}$, which yields $\langle E \rangle^1 (s') \in t(d)$, as required.

(ii) Assume by contraposition that $\neg \langle E \rangle^2 (s) \in t(d')$. By induction hypotheses (and since we extended the closure $\mathsf{cl}_1(\Gamma)$), we obtain $\langle E \rangle^1 \neg \langle E \rangle^2 (s) \in t(d'')$ using (i), and thus $d'' \in sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s))^{\mathfrak{W}_1}$. Due to the new components of Γ_1, this yields $d'' \in t_R(sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s)))^{\mathfrak{W}_1}$. Because $(d'', d) \in R_j^{\mathfrak{W}_1}$, we have $d \in sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s))^{\mathfrak{W}_1}$, and hence $\langle E \rangle^1 \neg \langle E \rangle^2 (s) \in t(d)$. By conditions (3) and (4), we then have $s \notin t(d)$, which had to be shown.

Finally, (iii) is proved analogously to (i) and (iv) is proved analogously to (ii); details are left to the reader. $\square$

We can now adapt the 'if' directions in the proof of (a) and (b) of $(*)$. As before, we restrict ourselves to the case $i = 1$.

(a) $s = \langle E \rangle^1 (s')$ with s' a 2-term. Suppose $d \in (\langle E \rangle^1 (s'))^{\mathfrak{M}}$. We find $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s'^{\mathfrak{M}}$. By the induction hypothesis, $d' \in (sur_2(s'))^{\mathfrak{W}_2}$ and so $s' \in t(d')$. As $(d, d') \in E^{\mathfrak{M}}$, part (i) of Lemma 46 yields $\langle E \rangle^1 (s') \in t(d)$, which implies $d \in (sur_1(\langle E \rangle^1 (s')))^{\mathfrak{W}_1}$.

(b) $s = \langle E \rangle^1 (a)$ with $a \in ob_2(\Gamma)$. Let $d \in (\langle E \rangle^1 (a))^{\mathfrak{M}}$. Then $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by definition and thus $\langle E \rangle^1 (a) \in t(d)$ by part (iii) of Lemma 46. This obviously implies $d \in (sur_1(\langle E \rangle^1 (a)))^{\mathfrak{W}_1}$, as required.

The case $i = 2$ is similar and uses parts (ii) and (iv) of Lemma 46 instead of parts (i) and (iii).

B.2 Boolean operators on link relations

In this section, we prove that decidability of ADSs transfers to their $\mathcal{E}$-connection even if Boolean operators may be applied to link relations and the link operators may be used on object variables, i.e., we prove Theorem 19. As before, we confine ourselves to considering $\mathcal{E}$-connections of only two ADSs. In contrast to the previous section, however, we admit an arbitrary number of link relations, since otherwise the Boolean operators on link relations cannot deploy their full power. Under these restrictions, Theorem 19 reads as follows:

Theorem 47 *Suppose that the satisfiability problems for ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Let us fix two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$ with decidable satisfiability problems and a set of link symbols $\mathcal{E}$. Let Γ be a finite set of assertions of the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$. We start by defining some notions:

- In contrast to the previous section, $\mathsf{cl}_i(\Gamma)$ (for $i = 1, 2$) simply denotes the closure under negation of the set of i-terms occurring in Γ. As before, we identify $\neg\neg t$ with t, and so $\mathsf{cl}_i(\Gamma)$ is finite.
- By *rel*(Γ) we denote the set of link symbols used in Γ. A *link type for* Γ is a set $T \subseteq \mathit{rel}(\Gamma)$. We use $\mathfrak{T}(\Gamma)$ to denote the set of all link types for Γ. If we interpret the symbols of *rel*(Γ) as propositional variables, then a link type T for Γ can clearly be viewed as a propositional logic interpretation. Thus we can write $T \models F$ for a link type T and a link F if T is a model of F.
- For $t \in \mathfrak{C}_i(\Gamma)$, $t' \in \mathfrak{C}_{\bar{i}}(\Gamma)$, and T a link type for Γ, we write $t \leadsto^T t'$ if the following conditions are satisfied:

(1) for all $\neg \langle F \rangle^i (s) \in t$ with s $\bar{i}$-term and $T \models F$, we have $s \notin t'$;

(2) for all $\neg \langle F \rangle^{\bar{i}} (s) \in t'$ with s i-term and $T \models F$, we have $s \notin t$.

- Let S_1, S_2, and S_3 be sets. We call a total function

$$f : (S_1 \times S_2) \cup (S_2 \times S_1) \to S_3$$

  a *symmetric function from* S_1, S_2 *to* S_3 if for all $(x_1, x_2) \in S_1 \times S_2$ we have

$$f(x_1, x_2) = f(x_2, x_1).$$

We assume without loss of generality that $\mathcal{S}_1$ and $\mathcal{S}_2$ support assertions of the form $a = a'$ and $a \neq a'$, where a and a' are object names. An assertion $a = a'$ ($a \neq a'$) is satisfied by a model $\mathfrak{W}$ iff $a^{\mathfrak{W}} = a'^{\mathfrak{W}}$ ($a^{\mathfrak{W}} \neq a'^{\mathfrak{W}}$). It should be clear that reasoning with such assertions can be reduced to reasoning without them: first perform appropriate substitutions of object names to eliminate all assertions of the form $a = a'$. Then introduce a fresh set variable x from the respective language for every assertion of the form $a \neq a'$ and replace $a \neq a'$ with $\{a : x, a' : \neg x\}$. As in the previous section, we assume that link assertions $(a_1, a_2) : E$ are replaced by the equivalent object assertion $a_1 : \langle E \rangle^1 (a_2)$.

Our aim is to formulate a criterion of satisfiability of sets of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions Γ similar to Theorem 45, from which we will derive decidability of the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$. However, in the presence of the Boolean operators on link relations, things are somewhat more complicated. To see why this is the case, consider the ($\Leftarrow$) direction of the proof of Theorem 45 in which we 'connect' the models for the sets Γ_1 and Γ_2 to a model for Γ. Whenever an element $d \in W_i$ should satisfy a term $\langle E \rangle^i(s)$, then Properties (3) to (6) ensure that there is a $t \in \Delta_{\bar{i}}$ such that (i) $s \in t$ and (ii) $s' \notin t$ for all $\langle E\rangle^i(s')$ that d should not satisfy. Moreover, $\Gamma_{\bar{i}}$ ensures that t is 'realized' at least once in $\mathfrak{W}_{\bar{i}}$, and thus we can connect d to an appropriate witness via the relation E. This simple strategy does not work with Boolean operators on link relations: since the element $d \in W_i$ may need a witness for the term s for many complex link relations $E_1, \ldots, E_k$ that are mutually exclusive (the simplest case is a an atomic link relation and its negation), it does not suffice to ensure that there is only one appropriate $t \in \Delta_{\bar{i}}$ that is realized only once in $\mathfrak{W}_{\bar{i}}$. The requirement of having enough witnesses for each term is in conflict with the fact that the involved ADSs may not allow certain terms to be realized an arbitrary number of times.

Our solution is to view models of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ as having a core of complex structure which is 'surrounded' by a shell of more regular structure. Intuitively, the core provides a 'sufficient' number of witnesses required for the model construction: witness requirements inside the core are satisfied inside the core, and witness requirements of elements outside the core (whose existence may be enforced by the class of models of the involved ADMs) are also satisfied inside the core.

In what follows, pre-models are used to describe the core part of models.

Definition 48 Let $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$. A *pre-model for* Δ_1, Δ_2 is a structure

$$\langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle \,,$$

where

- P_1 and P_2 are disjoint sets,
- t_i is a surjective function mapping each $p \in P_i$ to an element of Δ_i,
- e is a symmetric function from P_1, P_2 to $\mathfrak{T}(\Gamma)$,
- and σ_i is a function mapping each $a \in ob_i(\Gamma)$ to an element of P_i

such that, for $i \in \{1, 2\}$, the following conditions are satisfied:

(1) for all $p \in P_i$, if $\langle F\rangle^i(s) \in t_i(p)$, then there is a $p' \in P_{\bar{i}}$ such that $e(p, p') \models F$ and $s \in t_{\bar{i}}(p')$;
(2) for all $p \in P_i$, if $\langle F\rangle^i(a) \in t_i(p)$, then $e(p, \sigma_{\bar{i}}(a)) \models F$;
(3) for all $p \in P_i$ and $p' \in P_{\bar{i}}$, we have $t_i(p) \rightsquigarrow^{e(p,p')} t_{\bar{i}}(p')$;
(4) for all $p \in P_i$, if $\neg\langle F\rangle^i(a) \in t_i(p)$, then $e(p, \sigma_{\bar{i}}(a)) \not\models F$.

We are now in a position to formulate a satisfiability criterion for sets of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions.

Theorem 49 *Let Γ be a $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base. Then Γ is satisfiable iff there exist subsets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$, and a pre-model*

$$\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$$

for Δ_1, Δ_2 such that, for $i \in \{1,2\}$, the following conditions are satisfied:

(i) $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$, *where* $\delta = \max(|ob_1(\Gamma)|, |ob_2(\Gamma)|, |\mathsf{cl}_1(\Gamma)|, |\mathsf{cl}_2(\Gamma)|)$,

(ii) *the union Γ_i of the sets*
- $\{sur_i(\bigvee \Delta_i) = \top_i\}$
- $\{a_p : sur_i(t_i(p)) \mid p \in P_i\}$
- $\{a_p = a \mid \sigma_i(a) = p\}$
- $\{a_p \neq a_{p'} \mid p, p' \in P_i$ *and* $p \neq p'\}$
- $\{sur_i(t_1) \sqsubseteq sur_i(t_2) \mid t_1 \sqsubseteq t_2 \in \Gamma$ *an i-term assertion*$\}$
- $\{R_j(a_1, \ldots, a_{m_j}) \mid R_j(a_1, \ldots, a_{m_j}) \in \Gamma$ *an i-object assertion*$\}$
- $\{(a : sur_i(t)) \mid (a : t) \in \Gamma$ *an i-object assertion*$\}$

is $\mathcal{S}_i$-satisfiable for $i \in \{1,2\}$, where a_p is a fresh object name from $\mathcal{X}_i(\Gamma)$ for each $p \in P_i$.

Proof. ($\Rightarrow$) Let $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, (E_i^{\mathfrak{M}})_{i \leq k} \right\rangle$ be a model for Γ, where $\mathfrak{W}_1$ has domain W_1 and $\mathfrak{W}_2$ has domain W_2. We use $\mathfrak{M}$ to choose sets Δ_1 and Δ_2 and define a pre-model $\mathfrak{P}$ satisfying the conditions given in the theorem: for $i \in \{1,2\}$ and $d \in W_i$, put

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}.$$

Further, for $d \in W_1$ and $d' \in W_2$, define their link type $ct(d,d')$ as

$$ct(d,d') = \{E \in rel(\Gamma) \mid (d,d') \in E^{\mathfrak{M}}\} \in \mathfrak{T}(\Gamma).$$

Then set

$$\Delta_i = \{t(d) \mid d \in W_i\}.$$

The construction of $\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$ requires a bit more effort. We proceed in several steps:

1. Choose a set $L_1 \subseteq W_1$ such that the following conditions are satisfied:

(a) for $t \in \Delta_1$ and $\Sigma_t = \{d \in W_1 \mid t(d) = t$ and $a^{\mathfrak{M}} \neq d$ for all $a \in ob_1(\Gamma)\}$ we let

$$\{d \in W_1 \mid t(d) = t\} \subseteq L_1,$$

if $|\Sigma_t| = |\mathsf{cl}_2(\Gamma)|$, and, otherwise, choose a set

$$\Sigma' \subseteq \Sigma_t \text{ with } |\Sigma'| = |\mathsf{cl}_2(\Gamma)| \text{ and let } \Sigma' \subseteq L_1;$$

(b) for all $a \in ob_1(\Gamma)$, we have $a^{\mathfrak{M}} \in L_1$;

(c) $|L_1| \leq |\Delta_1| \cdot |\mathsf{cl}_2(\Gamma)| + |ob_1(\Gamma)|$.

It is easy to see that such a set exists.

2. Choose a set $R_1 \subseteq W_2$ satisfying the following conditions:

(a) for each $t \in \Delta_2$, there is a $d \in R_1$ such that $t(d) = t$;

(b) for all $a \in ob_2(\Gamma)$, we have $a^{\mathfrak{M}} \in R_1$;
(c) for each $d \in L_1$ and $\langle F \rangle^1 (s) \in t(d)$, there exists a $d' \in R_1$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;
(d) $|R_1| \leq |L_1| \cdot |\mathsf{cl}_1(\Gamma)| + |\Delta_2| + |ob_2(\Gamma)|$.

Such a set exists since property (2.c) can clearly be satisfied by choosing at most $|L_1| \cdot |\mathsf{cl}_1(\Gamma)|$ elements of W_2 for R_1.

3. Choose a set $L_2 \subseteq W_1$ such that the following conditions are satisfied:

(a) $L_1 \cap L_2 = \emptyset$;
(b) for each $d \in R_1$ and $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in L_1 \cup L_2$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;
(c) $|L_2| \leq |R_1| \cdot |\mathsf{cl}_2(\Gamma)|$.

4. Choose a set $R_2 \subseteq W_2$ such that the following conditions are satisfied:

(a) $R_1 \cap R_2 = \emptyset$;
(b) for each $d \in L_2$ and $\langle F \rangle^1 (s) \in t(d)$, there exists a $d' \in R_1 \cup R_2$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;
(c) $|R_2| \leq |L_2| \cdot |\mathsf{cl}_1(\Gamma)|$.

5. Choose a function K from $L_1 \times R_2$ to $\mathfrak{T}(\Gamma)$ such that the following conditions are satisfied:

(a) for each $d \in R_2$ and each $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in L_1$ such that $K(d', d) \models F$ and $s \in t(d')$;
(b) for each $d \in R_2$ and $\langle F \rangle^2 (a) \in t(d)$, we have $K(a^{\mathfrak{M}}, d) \models F$;
(c) for all $(d, d') \in L_1 \times R_2$, we have $d \rightsquigarrow^{K(d,d')} d'$;
(d) for each $d \in R_2$ and $\neg \langle F \rangle^2 (a) \in t(d)$, we have $K(a^{\mathfrak{M}}, d) \not\models F$.

Let us show that such a function does exist. First, fix for each $d \in R_2$ a subset $\tau(d) \subseteq W_1$ of cardinality $\leq |\mathsf{cl}_2(\Gamma)|$ such that, for each $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in \tau(d)$ such that $(d', d) \in F^{\mathfrak{M}}$ and $s \in t(d')$. Due to properties (1.a) and (1.b) of L_1, we can find a map

$$\pi : \bigcup_{d \in R_2} \tau(d) \to L_1$$

whose restriction to $\tau(d)$ is injective for each $d \in R_2$ and such that, for all d' in the domain of π, we have

(i) $t(d') = t(\pi(d'))$,
(ii) $d' = a^{\mathfrak{M}}$ for some $a \in ob_1(\Gamma)$ implies $d' = \pi(d')$, and
(iii) $d' \neq a^{\mathfrak{M}}$ for all $a \in ob_1(\Gamma)$ implies $\pi(d') \neq a^{\mathfrak{M}}$ for all $a \in ob_1(\Gamma)$.

We now define K in three steps:

(1) for each $a \in ob_1(\Gamma)$ and $d \in R_2$, set $K(a^{\mathfrak{M}}, d) = ct(a^{\mathfrak{M}}, d)$;
(2) for each $d \in R_2$ and $d' \in \tau(d)$, set $K(\pi(d'), d) = ct(d', d)$;
(3) for each $d \in L_1$ and each $d' \in R_2$ such that $K(d, d')$ is undefined, we set $K(d, d') = ct(d, d')$.

Due to properties (ii) and (iii) of π, K is well-defined. It is straightforward to verify that K satisfies properties (5.a) to (5.d).

6. We now define the pre-model $\mathfrak{P}$ as follows:

(1) Set $P_1 = L_1 \cup L_2$ and $P_2 = R_1 \cup R_2$.
(2) For $i = 1, 2$, set $t_i(d) = t(d)$ for all $d \in P_i$. In view of property (1.a) of L_1 and property (3.a) of L_2, it is clear that the t_i are surjective.
(3) Let $d \in P_1$ and $d' \in P_2$. If $d \notin L_1$ or $d' \notin R_2$, then set $e(d, d') = e(d', d) = ct(d, d')$. If $d \in L_1$ and $d' \in R_2$, then set $e(d, d') = e(d', d) = K(d, d')$.
(4) For $i = 1, 2$ and $a \in ob_i(\Gamma)$, set $\sigma_i(a) = a^{\mathfrak{M}}$ (we do not 'leave' P_1 and P_2 due to property (1.b) of L_1 and property (3.b) of L_2).

A lengthy but easy computation yields the upper bound $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$ for the size of the sets P_i. Next, we show that $\mathfrak{P}$ is indeed a pre-model, i.e., that it satisfies properties (1)–(4) from Definition 48

(1) Let $d \in L_1$ and $\langle F \rangle^1 (s) \in t_1(d)$. Since $t_1(d) = t(d)$ by the definition of $\mathfrak{P}$, property (2.c) of R_1 yields a $d' \in R_1$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$. By the definition of $\mathfrak{P}$, we have $e(d, d') = ct(d, d')$ and $t_2(d') = t(d')$. Thus, $e(d, d') \models F$ and $s \in t_2(d')$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (s) \in t_2(d)$, we may use an analogous argument employing property (3.b) of L_2 instead of property (2.c) of R_1. Similarly, in the case $d \in L_2$ we may use property (4.b) of R_2.

Now let $d \in R_2$ and $\langle F \rangle^2 (s) \in t_2(d)$. By property (5.a) of K, there exists a $d' \in L_1$ such that $K(d', d) \models F$ and $s \in t(d')$. By the definition of $\mathfrak{P}$, we have $e(d', d) = K(d', d)$ and $t_1(d') = t(d')$. Thus, $e(d, d') \models F$ and $s \in t_1(d')$.

(2) Let $d \in L_1 \cup L_2$ and $\langle F \rangle^1 (a) \in t_1(d)$. By property (2.b) of R_1, we have $a^{\mathfrak{M}} \in R_1$. Moreover, by the definition of $\mathfrak{P}$, we have $t_1(d) = t(d)$. Thus, $\langle F \rangle^1 (a) \in t(d)$ which implies $ct(d, a^{\mathfrak{M}}) \models F$. Since $e(d, a^{\mathfrak{M}}) = ct(d, a^{\mathfrak{M}})$ and $\sigma_2(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_2(a)) \models F$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (a) \in t_2(d)$, we may use an analogous argument employing property (1.b) of L_1 instead of property (2.b) of R_1.

Now let $d \in R_2$ and $\langle F \rangle^2 (a) \in t_2(d)$. By property (1.b) of L_1, we have $a^{\mathfrak{M}} \in L_1$.

By property (5.b) of K, we have $K(a^{\mathfrak{M}}, d) \models F$. Since $e(a^{\mathfrak{M}}, d) = K(a^{\mathfrak{M}}, d)$ and $\sigma_1(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(\sigma_1(a), d) = e(d, \sigma_1(a)) \models F$, as required.

(3) As the definition of $\rightsquigarrow$ is symmetric, it suffices to show $t_1(d_1) \rightsquigarrow^{e(d_1,d_2)} t_2(d_2)$ for all $d_1 \in P_1$ and $d_2 \in P_2$. First, let $d_1 \in P_1$ and $d_2 \in R_1$. The definition of $\mathfrak{P}$ implies $e(d_1, d_2) = ct(d_1, d_2)$. By the definition of $\rightsquigarrow$, we need to show two properties:

- Let $\neg \langle F \rangle^1 (s) \in t_1(d_1)$ and $e(d_1, d_2) \models F$. Since $t_1(d_1) = t(d_1)$, we have $\neg \langle F \rangle^1 (s) \in t(d_1)$. Since $e(d_1, d_2) = ct(d_1, d_2)$ and $e(d_1, d_2) \models F$, we obtain $s \notin t(d_2)$. Now $t(d_2) = t_2(d_2)$ implies $s \notin t_2(d_2)$, as required.
- The case of $\neg \langle F \rangle^2 (s) \in t_2(d_2)$ and $e(d_1, d_2) \models F$ is considered analogously.

Now let $d_1 \in P_1$ and $d_2 \in R_2$. The definition of $\mathfrak{P}$ implies $e(d_1, d_2) = K(d_1, d_2)$. Again we need to show two properties:

- Let $\neg \langle F \rangle^1 (s) \in t_1(d_1)$ and $e(d_1, d_2) \models F$. Since $t_1(d_1) = t(d_1)$, we have $\neg \langle F \rangle^1 (s) \in t(d_1)$. Since $e(d_1, d_2) = K(d_1, d_2)$, we obtain $s \notin t(d_2)$ by property (5.c) of K. Now $t(d_2) = t_2(d_2)$ implies $s \notin t_2(d_2)$ as required.
- The case of $\neg \langle F \rangle^2 s \in t_2(d_2)$ and $e(d_1, d_2) \models F$ is considered analogously.

(4) Let $d \in L_1 \cup L_2$ and $\neg \langle F \rangle^1 (a) \in t_1(d)$. By property (2.b) of R_1, $a^{\mathfrak{M}} \in R_1$. Moreover, by the definition of $\mathfrak{P}$ we have $t_1(d) = t(d)$. Thus $\neg \langle F \rangle^1 (a) \in t(d)$, which implies $ct(d, a^{\mathfrak{M}}) \not\models F$. Since $e(d, a^{\mathfrak{M}}) = ct(d, a^{\mathfrak{M}})$ and $\sigma_2(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_2(a)) \not\models F$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (a) \in t_2(d)$, we may use an analogous argument employing property (1.b) of L_1 instead of property (2.b) of R_1.

Now let $d \in R_2$ and $\neg \langle F \rangle^2 (a) \in t_2(d)$. By property (1.b) of L_1, we have $a^{\mathfrak{M}} \in L_1$. By property (5.d) of K, $K(d, a^{\mathfrak{M}}) \not\models F$. Since $e(d, a^{\mathfrak{M}}) = K(d, a^{\mathfrak{M}})$ and $\sigma_1(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_1(a)) \not\models F$ as required.

To complete the proof of the 'only if' direction, it remains to show that the sets Γ_i are $\mathcal{S}_i$ satisfiable, which is done as in Theorem 45 by additionally setting $(a_p)^{\mathfrak{M}'_i} = p$ for all $p \in P_i$.

($\Leftarrow$) Suppose that Δ_1, Δ_2, and $\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$ satisfying the conditions of the theorem are given. We construct a model satisfying Γ. To this end, take models $\mathfrak{W}_i \in \mathcal{M}_i$ with domain W_i satisfying Γ_i, for $i = 1, 2$. Let, for $d \in W_i$,

$$t(d) = \bigwedge \{ s \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(s))^{\mathfrak{W}_i} \}.$$

By the definition of Γ_i, we clearly have $t(d) \in \Delta_i$ for each $d \in W_i$. Now fix an element $\rho(d) \in P_i$ for each $d \in W_i$ such that $t(d) = t_i(\rho(d))$ and $d = a_p^{\mathfrak{W}_i}$ implies $\rho(d) = p$, for all $p \in P_i$. This is possible, since the functions t_i of $\mathfrak{P}$ are surjective and $t(a_p^{\mathfrak{W}_i}) = t_i(p)$ by the definition of Γ_i. Let $rel(\Gamma) = \{E_1, \ldots, E_k\}$. For $1 \leq j \leq k$, we define the extension $E_j^{\mathfrak{M}}$ of a link relation E_j by setting

$$d\, E_j^{\mathfrak{M}}\, d' \quad \text{iff} \quad E_i \in e(\rho(d), \rho(d')).$$

The proof of the following claim is straightforward and left to the reader:

(♣) For all links F, $d_1 \in W_1$, and $d_2 \in W_2$,

$$(d_1, d_2) \in F^{\mathfrak{M}} \quad \text{iff} \quad e(\rho(d_1), \rho(d_2)) \models F.$$

We now show that $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, (E_j^{\mathfrak{M}})_{j \leq k} \right\rangle$ is a model for Γ. It clearly suffices to prove that

$$d \in sur_i(s)^{\mathfrak{W}_i} \quad \text{iff} \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$, $s \in \mathsf{cl}_i(\Gamma)$, and $i \in \{1, 2\}$, which can be done by simultaneous structural induction. We only consider the interesting cases, i.e., (i) $t = \langle F \rangle^i (s')$ and (ii) $t = \langle F \rangle^i (a)$, for $i = 1$ (the case $i = 2$ is symmetric).

(i) Assume $t = \langle F \rangle^1 (s')$. Let $d \in sur_1(\langle F \rangle^1 s')^{\mathfrak{W}_1}$. This implies $\langle F \rangle^1 (s') \in t(d)$, and so $\langle F \rangle^1 (s') \in t_1(\rho(d))$. By property (1) of pre-models, there exists a $p \in P_2$ such that $e(\rho(d), p) \models F$ and $s' \in t_2(p)$. By the choice of ρ, we have $\rho(a_p^{\mathfrak{W}_2}) = p$. Since $e(\rho(d), p) \models F$, we thus obtain $(d, a_p^{\mathfrak{W}_2}) \in F^{\mathfrak{M}}$ from (♣). Moreover, $s' \in t_2(p)$ and $\rho(a_p^{\mathfrak{W}_2}) = p$ yield $s' \in t(a_p^{\mathfrak{W}_2})$, and hence $a_p^{\mathfrak{W}_2} \in sur_2(s')^{\mathfrak{W}_2}$, from which we obtain $a_p^{\mathfrak{W}_2} \in s'^{\mathfrak{M}}$ by the induction hypotheses. To sum up, $d \in (\langle F \rangle^1 (s'))^{\mathfrak{M}}$.

For the 'if' direction, we show the contrapositive. Let $d \notin sur_1(\langle F \rangle^1 (s'))^{\mathfrak{W}_1}$. We need to prove that $d \notin (\langle F \rangle^1 (s'))^{\mathfrak{M}}$. Fix a $d' \in W_2$ such that $(d, d') \in F^{\mathfrak{M}}$. By (♣), we have $e(\rho(d), \rho(d')) \models F$, and $d \notin sur_1(\langle F \rangle^1 (s'))^{\mathfrak{W}_i}$ yields $\neg \langle F \rangle^1 (s') \in t(d)$ and $\neg \langle F \rangle^1 (s') \in t_1(\rho(d))$. Thus, we have $s' \notin t_2(\rho(d'))$ by property (3) of pre-models and the definition of $\rightsquigarrow$. This clearly yields $s' \notin t(d')$ and thus $d' \notin sur_2(s')^{\mathfrak{W}_2}$, which implies $d' \notin s'^{\mathfrak{M}}$ by the induction hypotheses. Since this holds independently of the choice of d', we obtain $d \notin (\langle F \rangle^1 (s'))^{\mathfrak{M}}$, as required.

(ii) Let $t = \langle F \rangle^1 (a)$ and $d \in sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$. This implies $\langle F \rangle^1 (a) \in t(d)$ and so $\langle F \rangle^1 (a) \in t_1(\rho(d))$. By property (2) of pre-models, $e(\rho(d), \sigma_2(a)) \models F$. By the construction of Γ_2, there is a $p \in P_2$ such that $p = \sigma_2(a)$ and $a_p^{\mathfrak{W}_2} = a^{\mathfrak{W}_2}$. By the choice of ρ, we then have $\rho(a_p^{\mathfrak{W}_2}) = \sigma_2(a)$. Since $e(\rho(d), \sigma_2(a)) \models F$, we thus obtain $(d, a_p^{\mathfrak{W}_2}) \in F^{\mathfrak{M}}$ from (♣). Hence, $d \in (\langle F \rangle^1 (a))^{\mathfrak{M}}$.

For the 'if' direction, we show the contrapositive. Let $d \notin sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$. We need to prove that $d \notin (\langle F \rangle^1 (a))^{\mathfrak{M}}$. Fix a $d' \in W_2$ such that $(d, d') \in F^{\mathfrak{M}}$. By the claim, we have $e(\rho(d), \rho(d')) \models F$. Moreover, $d \notin sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$ yields $\neg \langle F \rangle^1 (a) \in t(d)$ and $\neg \langle F \rangle^1 (a) \in t_1(\rho(d))$. Thus, $e(\rho(d), \sigma_2(a)) \not\models F$, i.e., $\rho(d') \neq \sigma_2(a)$, by property (4) of pre-models, and so $d' \neq a^{\mathfrak{W}_2}$ by the definition of Γ_2 and the choice of ρ. Thus $d' \neq a^{\mathfrak{M}}$. Since this holds independently of the choice of d', we obtain $d \notin (\langle F \rangle^1 (a))^{\mathfrak{M}}$, as required. $\Box$

Similarly to the previous section, Theorem 49 almost immediately provides us with a decision procedure for the connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ if decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$ are known: since the sets $\mathfrak{C}_i(\Gamma)$ are finite and $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$, to decide whether a set Γ of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we may 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$ and a pre-model $\mathfrak{P}$, and then check whether they satisfy the conditions listed in the formulation of the theorem.

The time complexity of the obtained decision procedure is the same as in the case without Boolean operators on link relations (see the previous section): it is one exponential higher than the complexity of the original decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$-satisfiability. It should be noted that the combined decision procedure is non-deterministic.

B.3 Qualified number restrictions

Now we prove Theorem 28, which states that decidability of ADSs transfers to their $\mathcal{E}$-connection even if we allow qualified number restrictions on link relations. Note that, by Theorem 29, we have to disallow Boolean operators on link relations and the use of link operators on object variables in order to avoid undecidability. As in the previous sections, we restrict ourselves to two ADSs and a single link relation E. For simplicity, we will therefore write number restrictions as $\langle \geq r \rangle^i (s)$ and $\langle \leq r \rangle^i (s)$, thus omitting the link symbol E.

Here is the variant of Theorem 28 obtained by the two restrictions:

Theorem 50 *Suppose that the satisfiability problems for the ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable and both $\mathcal{S}_1$ and $\mathcal{S}_2$ are number tolerant. Then the satisfiability problem for the $\{E\}$-connection $\mathcal{C}^{\{E\}}_Q(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Fix two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$ with decidable satisfiability problems. Note that for any model $\mathfrak{M}$ of $\mathcal{C}^{\{E\}}_Q(\mathcal{S}_1, \mathcal{S}_2)$ and any i-term s of $\mathcal{S}_i$ $(i = 1, 2)$ we have

$$\begin{aligned} (\langle \leq r \rangle^{\bar{i}} (s))^{\mathfrak{M}} &= (\neg \langle \geq r+1 \rangle^{\bar{i}} (s))^{\mathfrak{M}} \text{ for all } r \in \mathbb{N}, \text{ and} \\ (\langle E \rangle^{\bar{i}} (s))^{\mathfrak{M}} &= (\langle \geq 1 \rangle^{\bar{i}} (s))^{\mathfrak{M}}. \end{aligned}$$

Therefore, without loss of generality we may assume that we do not have terms of the form $\langle \leq r \rangle^{\bar{i}} (s)$ and $\langle E \rangle^{\bar{i}} (s)$. Let us fix some notational conventions:

- As in the previous section, we use $\mathsf{cl}_i(\Gamma)$, $i = 1, 2$, to denote the closure under negation of the set of i-terms occurring in Γ. Without loss of generality we can identify $\neg\neg t$ with t and thus $\mathsf{cl}_i(\Gamma)$ is finite.

- For an i-term t, we define a *surrogate* $sur_i(t)$ as described at the beginning of Section B, but now replacing subterms s of the form $\langle \geq r \rangle^i (s')$ with surrogate variables x_s.
- For $i \in \{1, 2\}$, we use $deg_i(\Gamma)$ to denote the maximum number r such that $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, for some term s.
- Given domain elements $d \in W_i$ and $d' \in W_{\bar{i}}$ (or object variables a of $\mathcal{S}_i$ and b of $\mathcal{S}_{\bar{i}}$) we use the expression $[d, d']$ (or $[a, b]$) to denote the pair (d, d') (respectively, (a, b)), if $i = 1$, and the pair (d', d) (or (b, a)), if $i = 2$.

As observed in Section B.2, without loss of generality we may assume that the ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ support assertions of the form $a = a'$ and $a \neq a'$, where a and a' are object names. Note that, since we do not allow the application of link operators on object variables, we cannot replace link assertions with object assertions as in the previous sections. Hence, we will treat link assertions $(a, b) : E$ explicitly in the proof.

We can now reduce satisfiability for the connection $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$ to satisfiability for the components $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem 51 *Let Γ be a $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base, where the $\mathcal{S}_i$ are number tolerant. Then Γ is satisfiable iff there are sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$ and equivalence relations $\sim_1 \subseteq ob_1(\Gamma) \times ob_1(\Gamma)$ and $\sim_2 \subseteq ob_2(\Gamma) \times ob_2(\Gamma)$ such that, for $i \in \{1, 2\}$, the following conditions are satisfied:*

(1) *For each $t \in \Delta_i$, there exists a set $\mathcal{W}_t = \{(Z_1, \gamma_1), \ldots, (Z_{k(t)}, \gamma_{k(t)})\}$, where $Z_j \subseteq \Delta_{\bar{i}}$ and the γ_j are functions from Z_j to $\{1, \ldots, deg_i(\Gamma)\}$ such that, for each $(Z_j, \gamma_j) \in \mathcal{W}_t$, we have the following:*
   (a) *For each term $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, we have*
$$\langle \geq r \rangle^i (s) \in t \quad \textit{iff} \quad \sum_{\{t' \in Z_j \mid s \in t'\}} \gamma_j(t') \geq r;$$
   (b) *for each $t' \in Z_j$, there exists $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t \in Z$.*

(2) *For each equivalence class C of $\sim_i$, there exist a type $t_C \in \Delta_i$, a set of types $Z_C \subseteq \Delta_{\bar{i}}$, and a function $\gamma_C : Z_C \to \{1, \ldots, deg_i(\Gamma)\}$ such that*
   (a) *for each term $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, we have*
$$\langle \geq r \rangle^i (s) \in t_C \quad \textit{iff} \quad \sum_{\{t' \in Z_C \mid s \in t'\}} \gamma_C(t') + |\{C' \in conn_\Gamma(C) \mid s \in t_{C'}\}| \geq r,$$
   where the set $conn_\Gamma(C)$ contains precisely those equivalence classes C' of $\sim_{\bar{i}}$ for which $[a, b] : E \in \Gamma$, for some $a \in C$ and $b \in C'$;
   (b) *for each $t' \in Z_C$, there is $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t_C \in Z$.*

(3) *The union Γ_i of the sets*
   – $\{sur_i(\bigvee \Delta_i) = \top_i\}$
   – $\{a_t : sur_i(t) \mid t \in \Delta_i\}$

– $\{a = a' \mid a \sim_i a'\}$
– $\{a \neq a' \mid a \not\sim_i a'\}$
– $\{a : sur_i(t_{[a]_i}) \mid a \in ob_i(\Gamma)\}$
– $\{sur_i(s_1) \sqsubseteq sur_i(s_2) \mid (s_1 \sqsubseteq s_2) \in \Gamma$ *an i-term assertion*$\}$
– $\{R_j(a_1, \ldots, a_{m_j}) \mid R_j(a_1, \ldots, a_{m_j}) \in \Gamma$ *an i-object assertion*$\}$
– $\{a : sur_i(s) \mid (a : s) \in \Gamma$ *an i-object assertion*$\}$

is $\mathcal{S}_i$-satisfiable, where $[a]_i$ denotes the equivalence class of a with respect to $\sim_i$ and a_t is a fresh object name from $\mathcal{X}_i(\Gamma)$ for each $t \in \Delta_i$.

Proof. ($\Rightarrow$) Let $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}} \right\rangle$ be a model for Γ, where $\mathfrak{W}_1$ is based on the domain W_1 and $\mathfrak{W}_2$ is based on the domain W_2. We use $\mathfrak{M}$ to choose sets Δ_1 and Δ_2 and equivalence relations $\sim_1$ and $\sim_2$ satisfying the conditions given in the formulation of the theorem.

We start with some preliminaries. A domain element $d \in W_i$ is called *anonymous* if $d \neq a^{\mathfrak{M}}$ for all $a \in ob_i(\Gamma)$. For $i \in \{1, 2\}$, $d \in W_i$, and $t' \in \mathfrak{C}_{\bar{i}}(\Gamma)$, define the abbreviations

$$\begin{aligned} t(d) &= \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}; \\ R(d) &= \{d' \in W_{\bar{i}} \mid [d, d'] \in E^{\mathfrak{M}}\}; \\ P(d) &= \{t(d') \mid d' \in R(d)\}; \\ P_A(d) &= \{t(d') \mid d' \in R(d) \text{ is anonymous}\}; \\ c(d, t') &= \min\{deg_i(\Gamma), \left|\{d' \in R(d) \mid t(d') = t'\}\right|\}; \\ c_A(d, t') &= \min\{deg_i(\Gamma), \left|\{d' \in R(d) \mid t(d') = t' \text{ and } d' \text{ is anonymous}\}\right|\}. \end{aligned}$$

Then we set

- $\Delta_i = \{t(d) \mid d \in W_i\}$;
- $\sim_i = \{(a, b) \in ob_i(\Gamma) \times ob_i(\Gamma) \mid a^{\mathfrak{M}} = b^{\mathfrak{M}}\}$;
- $\mathcal{W}_t = \{(P(d), \gamma_d) \mid d \in W_i \text{ and } t(d) = t\}$ for each $t \in \Delta_i$, where

$$\gamma_d = \{t' \mapsto c(d, t') \mid t' \in P(d)\};$$

- $t_C = t(a^{\mathfrak{M}})$, with $a \in C$, for each equivalence class C of $\sim_i$;
- $Z_C = P_A(a^{\mathfrak{M}})$, with $a \in C$, for each equivalence class C of $\sim_i$;
- $\gamma_C = \{t' \mapsto c_A(a^{\mathfrak{M}}, t') \mid t' \in P_A(a^{\mathfrak{M}})\}$, with $a \in C$, for each equivalence class C of $\sim_i$.

Note that t_C, Z_C, and γ_C are well-defined by the definition of the relations $\sim_i$. It remains to show that these definitions satisfy conditions (1)–(3) from the formulation of the theorem. We only do this for $i = 1$, since the case $i = 2$ is symmetric.

1. Fix terms $\langle \geq r \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$, $t \in \Delta_1$, and a pair $(Z, \gamma) \in \mathcal{W}_t$. Then there is

a $d \in W_1$ such that $t(d) = t$, $Z = P(d)$, and $\gamma = \gamma_d$. Let

$$\Sigma_d^s = \{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}.$$

By definition we have

$$\langle \geq r \rangle^1 (s) \in t \quad \text{iff} \quad d \in (\langle \geq r \rangle^1 (s))^{\mathfrak{M}} \quad \text{iff} \quad |\Sigma_d^s| \geq r.$$

By the definition of $P(d)$ and γ_d,

$$\sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') = |\{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}|$$

if for all $t' \in Z$ with $s \in t'$ we have $|\{d' \in R(d) \mid t(d') = t'\}| < deg_i(\Gamma)$, and

$$\sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') \geq deg_2(\Gamma) \geq r$$

otherwise. The latter case implies $|\Sigma_d^s| \geq deg_2(\Gamma) \geq r$. We thus obtain

$$|\Sigma_d^s| \geq r \quad \text{iff} \quad \sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') \geq r,$$

which gives (1.a).

To prove (1.b), let $t' \in Z$. Then there exists a $d' \in W_2$ such that $(d, d') \in E^{\mathfrak{M}}$ and $t(d') = t'$. It is readily checked that $(P(d'), \gamma_{d'}) \in \mathcal{W}_{t'}$ is as required, i.e., $t \in P(d')$.

2. Fix an equivalence class C of $\sim_1$, an $a \in C$ and a term $\langle \geq r \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$. Let

$$\Sigma_a^s = \{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}.$$

As above, we have by definition that $\langle \geq r \rangle^1 (s) \in t_C$ iff $|\Sigma_a^s| \geq r$ and, moreover,

$$\begin{aligned} |\Sigma_a^s| = {} & |\{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}, s \in t(d'), \text{ and } d' \text{ anonymous}\}| \; + \\ & |\{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}, s \in t(d'), \text{ and } d' \text{ not anonymous}\}|. \end{aligned}$$

By the definition of P_A, c_A, $\sim_1$, Z_C, and γ_C, the sum

$$\sum_{\{t' \in Z_C \mid s \in t'\}} \gamma_C(t')$$

is equal to the former component of $|\Sigma_a^s|$ or is at least $deg_2(\Gamma)$. Further, by the definition of $\sim_1$ and t_C, the second component is equal to

$$|\{C' \in conn_\Gamma(C) \mid s \in t_{C'}\}|.$$

Thus, as in the proof of (1.a), we obtain

$$|\Sigma_a^s| \geq r \quad \text{iff} \quad \sum_{\{t' \in Z_C \mid s \in t'\}} \gamma_C(t') + |\{C' \in conn_\Gamma(C) \mid s \in t_{C'}\}| \geq r$$

which gives (2.a).

To prove (2.b), let $t' \in Z_C$. Then there is a $d' \in W_2$ such that $(a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}$ and $t(d') = t'$. It is readily checked that $(P(d'), \gamma_{d'}) \in \mathcal{W}_{t'}$ is as required, i.e., $t_C \in P(d')$.

3. Take the model $\mathfrak{W}_1$ and extend it as follows:

- for each surrogate variable x_s occurring in Γ_1 with s of the form $\langle \geq r \rangle^i (s')$, set $x_s^{\mathfrak{W}_1} = s^{\mathfrak{W}}$;
- for each newly introduced object name a_t (with $t \in \Delta_1$), set $a_t^{\mathfrak{W}_1}$ to some element of $t^{\mathfrak{W}}$.

Note that the resulting model $\mathfrak{W}_1'$ can be found in the set of models $\mathcal{M}_1$ by the closure conditions that are required to hold for $\mathcal{M}_1$. It is easy to prove by induction that, for all $d \in W_1$ and $s \in \mathsf{cl}_1(\Gamma)$, we have $d \in sur_1(s)^{\mathfrak{W}_1'}$ iff $d \in s^{\mathfrak{M}}$; details are left to the reader. Using this fact, in turn, it is straightforward to verify that $\mathfrak{W}_1'$ is a model of Γ_1.

($\Leftarrow$) Suppose that there exist Δ_1, Δ_2, $\sim_1$, and $\sim_2$ satisfying the conditions of the theorem. Hence, there also exist sets $\mathcal{W}_t$, for $t \in \Delta_i$, and types t_C, sets of types Z_C, and functions γ_C, for equivalence classes C of $\sim_i$, satisfying conditions (1.a), (1.b) and (2.a), (2.b). Our aim is to construct a model satisfying Γ. For each ADS $\mathcal{S}_i$, $i = 1, 2$, let κ_i denote the cardinal number for $\mathcal{S}_i$ from the definition of 'number tolerance.' Take an infinite cardinal κ such that $\kappa \geq \kappa_i$, for $i = 1, 2$, and models $\mathfrak{W}_i \in \mathcal{M}_i$ with domains W_i satisfying Γ_i, for $i = 1, 2$. Let, for $d \in W_i$,

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(s))^{\mathfrak{W}_i}\}.$$

By the definition of the Γ_i, we clearly have $t(d) \in \Delta_i$ for each $d \in W_i$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are number tolerant and $t \in \Delta_i$ implies the existence of some $d \in W_i$ such that $t(d) = t$ by the definition of the Γ_i, by the choice of κ we may assume that

$$\Big|\{d \in W_i \mid t(d) = t\}\Big| = \kappa \text{ for each } t \in \Delta_i. \qquad (*)$$

Again, a domain element $d \in W_i$ is called *anonymous* if $d \neq a^{\mathfrak{M}}$ for all $a \in ob_i(\Gamma)$. We now show that there exists a relation $E^{\mathfrak{M}} \subseteq W_1 \times W_2$ satisfying the following conditions:

(I) For all $a \in ob_1(\Gamma)$ and $b \in ob_2(\Gamma)$, we have

$$(a^{\mathfrak{W}_1}, b^{\mathfrak{W}_2}) \in E^{\mathfrak{M}} \quad \text{iff there are} \quad a' \in [a]_1, b' \in [b]_2 \text{ such that } (a', b') : E \in \Gamma.$$

(II) For all $i \in \{1, 2\}$ and $a \in ob_i(\Gamma)$, we have
- $[a^{\mathfrak{W}_i}, d'] \in E^{\mathfrak{M}}$ implies $t(d') \in Z_{[a]_i}$;
- for each $t \in Z_{[a]_i}$,

$$\gamma_{[a]_i}(t) = \Big|\{d' \in W_{\bar{i}} \mid [a^{\mathfrak{W}_i}, d'] \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\}\Big|.$$

(III) For all $i \in \{1, 2\}$ and $d \in W_i$, there exists a $(Z, \gamma) \in \mathcal{W}_{t(d)}$ such that
- $[d, d'] \in E^{\mathfrak{M}}$ implies $t(d') \in Z$;
- for each $t \in Z$,

$$\gamma(t) = \Big|\{d' \in W_{\bar{i}} \mid [d, d'] \in E^{\mathfrak{M}} \text{ and } t(d') = t\}\Big|.$$

Since there are only finitely many types $t \in \Delta_i$ and each t is of the form $t(d)$ for some $d \in W_i$, we have $|W_i| = |\Delta_i| \cdot \kappa = \kappa$. Hence, we can assume that the sets W_i are ordered by $<_i$ such that $(\kappa, \in)$ is order-isomorphic to $(W_i, <_i)$ (i.e., $<_i$ is a well-ordering on W_i such that no $<_i$-initial subset of W_i is of cardinality κ). We construct the relation $E^{\mathfrak{M}}$ by transfinite induction as

$$E^{\mathfrak{M}} = \bigcup_{\alpha<\kappa} E^{\mathfrak{M}}_\alpha,$$

and simultaneously define (partial) functions π_i^α, $\alpha < \kappa$, $i = 1, 2$, that take anonymous domain elements $d \in W_i$ to elements of $\mathcal{W}_{t(d)}$. We start with $\alpha = 0, 1$:

- Set $E^{\mathfrak{M}}_0 = \{(a^{\mathfrak{W}_1}, b^{\mathfrak{W}_2}) \mid (a, b) : E \in \Gamma\}$ and $\pi_1^0 = \pi_2^0 = \emptyset$.
- For all $i \in \{1, 2\}$, $a \in ob_i(\Gamma)$, $t \in Z_{[a]_i}$, and j, $1 \leq j \leq \gamma_{[a]_i}(t)$, choose an anonymous element $d_{a,t,j} \in W_{\bar{i}}$ with $t(d_{a,t,j}) = t$ such that $(a, t, j) \neq (a', t', j')$ implies $d_{a,t,j} \neq d_{a',t',j'}$—this is possible since $Z_{[a]_i} \subseteq \Delta_{\bar{i}}$ and in view of $(*)$. Then set, for each a, t, j as above, $\pi_i^1(d_{a,t,j})$ to some $(Z, \gamma) \in \mathcal{W}_t$ such that $t_{[a]} \in Z$, which exists by property (2.b). Further, set

$$E^{\mathfrak{M}}_1 = E^{\mathfrak{M}}_0 \cup \bigcup_{i\in\{1,2\}} \bigcup_{a\in ob_i(\Gamma)} \bigcup_{t\in Z_{[a]_i}} \bigcup_{1\leq j\leq\gamma_{[a]_i}(t)} \{[a^{\mathfrak{W}_i}, d_{a,t,j}]\}.$$

- Suppose that $\alpha < \kappa$ is the minimal ordinal for which $E^{\mathfrak{M}}_\alpha$ is not yet defined. If α is a limit ordinal, then set

$$E^{\mathfrak{M}}_\alpha = \bigcup_{\beta<\alpha} E^{\mathfrak{M}}_\beta \quad \text{and} \quad \pi_i^\alpha = \bigcup_{\beta<\alpha} \pi_i^\beta \text{ for } i = 1, 2.$$

  Now suppose that $\alpha = \alpha' + 1$. Let β be the largest limit ordinal which is smaller than α, or 0 if no such limit ordinal exists. If $\alpha = \beta + 2n$ for some natural number n, set $i = 1$. Otherwise set $i = 2$. Choose the $<_i$-minimal domain element $d \in W_i$ such that

(i) $\pi_i^{\alpha'}(d)$ is undefined, or

(ii) $\pi_i^{\alpha'}(d) = (Z, \gamma)$ and there is a $t' \in Z$ such that

$$\left|\{d' \in W_{\bar{i}} \mid [d, d'] \in E_{\alpha'}^{\mathfrak{M}} \text{ and } t(d') = t'\}\right| < \gamma(t').$$

In case (i), set

$$E_\alpha^{\mathfrak{M}} = E_{\alpha'}^{\mathfrak{M}}, \quad \pi_i^\alpha = \pi_i^{\alpha'} \cup \{(d, (Z, \gamma))\}, \quad \pi_{\bar{i}}^\alpha = \pi_{\bar{i}}^{\alpha'},$$

where (Z, γ) is an element of $\mathcal{W}_{t(d)}$. In case (ii), we do the following: choose an anonymous element $d' \in W_{\bar{i}}$ with $t(d') = t'$ and $[d, d'] \notin E_{\alpha'}^{\mathfrak{M}}$ such that $\pi_{\bar{i}}^{\alpha'}(d')$ is undefined—this is possible since $Z \subseteq \Delta_{\bar{i}}$ and by $(*)$. Then set

$$E_\alpha^{\mathfrak{M}} = E_{\alpha'}^{\mathfrak{M}} \cup \{[d, d']\}, \quad \pi_i^\alpha = \pi_i^{\alpha'}, \quad \pi_{\bar{i}}^\alpha = \pi_{\bar{i}}^{\alpha'} \cup \{(d', (Z, \gamma))\},$$

for some $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t(d) \in Z$, which is possible by property (1.b).

It is not hard to verify that the relation $E^{\mathfrak{M}} = \bigcup_{\alpha < \kappa} E_\alpha^{\mathfrak{M}}$ constructed in this way indeed satisfies Properties (I)–(III).

We now show that $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}} \right\rangle$ is a model for Γ. Since $(a, b) : E \in \Gamma$ implies $(a^{\mathfrak{M}}, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by property (I) of $E^{\mathfrak{M}}$, it clearly suffices to show that

$$d \in sur_i(s)^{\mathfrak{W}_i} \quad \text{iff} \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$, $s \in \mathsf{cl}_i(\Gamma)$, and $i \in \{1, 2\}$, which can be done by simultaneous structural induction. The case of set variables and the Boolean cases are trivial, so we only consider the case $s = \langle \geq r \rangle^i (s')$ and $i = 1$.

Let $s = \langle \geq r \rangle^1 (s')$ for s' a 2-term. First assume that $d \in sur_1(s)^{\mathfrak{W}_1}$, i.e., $s \in t(d)$, and consider the case where d is not anonymous, i.e., there exists an $a \in ob_1(\Gamma)$ such that $a^{\mathfrak{M}} = d$. By condition (2.a), we then have

$$r \leq \sum_{\{t \in Z_{[a]_1} \mid s' \in t\}} \gamma_{[a]_1}(t) + |\{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\}|.$$

By the definitions of Γ_2 and of $\mathfrak{M}$, we have $b^{\mathfrak{M}} = b'^{\mathfrak{M}}$ if and only if $b \sim_2 b'$ for all $b, b' \in ob_2(\Gamma)$. Thus, property (I) of $E^{\mathfrak{M}}$ and the definition of Γ_2 yield

$$\begin{aligned}
&\left|\{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}},\ s' \in t(d'),\ \text{and } d' \text{ not anonymous}\}\right| = \\
&\left|\{[b]_2 \mid s' \in t_{[b]_2} \text{ and } (a', b') : E \in \Gamma \text{ for some } a' \in [a]_1, b' \in [b]_2\}\right| = \\
&\left|\{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\}\right|.
\end{aligned}$$

By property (II) of $E^{\mathfrak{M}}$, we have for each $t \in Z_{[a]_1}$:

$$\gamma_{[a]_1}(t) = \left|\{d' \in W_2 \mid (a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\}\right|.$$

Moreover, $(a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}$ implies $t(d') \in Z_{[a]_1}$. This yields

$$\Big|\{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\}\Big| \geq r.$$

Since, by the induction hypotheses, $s' \in t(d')$ iff $d' \in s'^{\mathfrak{M}}$, this yields $d \in s^{\mathfrak{M}}$, as required.

Now assume that $d \in sur_1(s)^{\mathfrak{W}_1}$ and d is anonymous. Then $s \in t(d)$, property (1.a), and property (III) of $E^{\mathfrak{M}}$ yield

$$\Big|\{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\}\Big| \geq r,$$

which is equivalent to $d \in s^{\mathfrak{M}}$, and we are done.

Conversely, assume that $d \in s^{\mathfrak{M}}$. By definition and the induction hypotheses, we have that

$$|\Sigma_d^{s'}| \geq r, \qquad \text{where} \qquad \Sigma_d^{s'} = \{d' \in W_2 \mid (d, d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\}.$$

Assume first that $d = a^{\mathfrak{M}}$ for some $a \in ob_1(\Gamma)$. Clearly, for each $d' \in \Sigma_d^{s'}$ that is not anonymous, i.e., $b^{\mathfrak{M}} = d'$ for some $b \in ob_2(\Gamma)$, there are $a' \in [a]_1$ and $b' \in [b]_2$ such that $(a', b') : E \in \Gamma$ and $s' \in t_{[b']_2}$, by condition (I) and the definition of Γ_2. By condition (II) of $E^{\mathfrak{M}}$ we further have that for all $d' \in \Sigma_d^{s'}$, $s' \in t(d') \in Z_{[a]_1}$ and for any $t \in Z_{[a]_1}$,

$$\gamma_{[a]_1}(t) = \Big|\{d' \in W_2 \mid (a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\}\Big|.$$

Hence

$$r \leq |\Sigma_d^{s'}| = \sum_{\{t \in Z_{[a]_1} \mid s' \in t\}} \gamma_{[a]_1}(t) + \Big|\{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\}\Big|.$$

By condition (2.a), $s \in t_{[a]_1}$. Since, by the definition of Γ_1, we have $t(d) = t_{[a]_1}$, this yields $d \in sur_1(s)^{\mathfrak{W}_1}$, as required.

Assume now that d is anonymous. By condition (III) of $E^{\mathfrak{M}}$ there exists $(Z, \gamma) \in \mathcal{W}_{t(d)}$ such that $s' \in t(d') \in Z$ for all $d' \in \Sigma_d^{s'}$. As above we obtain

$$r \leq |\Sigma_d^{s'}| = \sum_{\{t \in Z \mid s' \in t\}} \gamma(t),$$

and so $s \in t(d)$ by condition (1.a), which completes the proof. $\square$

Assuming that there exist decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$, it is now easy to use Theorem 51 to derive a decision procedure for the connection $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$. Since the sets $\mathfrak{C}_i(\Gamma)$ are finite, to decide whether a set Γ of $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we may 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$, equivalence relations $\sim_1$ and $\sim_2$, sets $\mathcal{W}_t$ for each $t \in \Delta_1 \cup \Delta_2$, and types t_C, sets Z_C, and functions γ_C for each equivalence class C of $\sim_1$ and $\sim_2$, and then check whether they satisfy the conditions listed in the formulation of the theorem.

The time complexity of the obtained decision procedure is the same as in the previous two sections (see Section B.1 for details): it is one exponential higher than the complexity of the original decision procedures for $\mathcal{S}_1$- and $\mathcal{S}_2$-satisfiability. Moreover, the decision procedure for the connection is non-deterministic.

C Undecidability and lower bounds

In this section, we prove the undecidability results and lower bounds for the computational complexity. In Section C.1, we consider A-satisfiability and show that, for this reasoning problem, decidability of the component ADSs does not always transfer to their $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2)$. In Section C.2, we prove that, for some more powerful types of $\mathcal{E}$-connections, even the decidability transfer for the *satisfiability* problem fails. An example of such a connection type is $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \mathcal{S}_2)$, which allows both the Boolean operations on link relations and qualified number restrictions. Finally, in Section C.3 we prove that the basic connection $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$ is EXPTIME-hard, while its extension with the Boolean operators on links is already NEXPTIME-hard.

C.1 Undecidability of A-satisfiability

Our aim is to prove the following:

Theorems 14 and 17. *Let $\mathcal{E}$ be a non-empty set of links. Then A-satisfiability is undecidable for*

(1) *the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^{\sharp}, \mathcal{ALCO}^{\sharp})$;*
(2) *the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{\mathcal{O}}(\mathcal{ALCF}^{\sharp}, \mathcal{S})$, for any ADS $\mathcal{S}$.*

Recall that $\mathcal{ALCO}$ is the extension of the basic DL $\mathcal{ALC}$ with nominals, whereas $\mathcal{ALCF}$ extends $\mathcal{ALC}$ with functional roles and the feature agreement/disagreement constructors introduced on Page 19. The proof below depends on the possibility of applying link operators to object variables (or nominals) of the connection's second component. We therefore either have to allow such applications explicitly as in (2) or, alternatively, equip the second ADS with nominals as in (1).

Proof. Since the proofs of (1) and (2) are similar, we concentrate on (1). As noted above, the satisfiability problem for ABoxes relative to TBox axioms in $\mathcal{ALCF}$ is undecidable. For simplicity, however, we will consider the concept satisfiability problem relative to TBox axioms which is formulated as follows: given an $\mathcal{ALCF}$-concept C and a set Γ of $\mathcal{ALCF}$ TBox assertions of the form $D \sqsubseteq D'$, does there exist a model $\mathcal{I}$ for Γ such that $C^{\mathcal{I}} \neq \emptyset$? As shown in [8], this problem is undecidable for $\mathcal{ALCF}$. To prove (1), we reduce this problem to the A-satisfiability problem for the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$.

Let C be an $\mathcal{ALCF}$-concept and Γ a set of $\mathcal{ALCF}$ TBox assertions. We use $\mathcal{R}$ to denote the set of roles occurring in C or Γ, and $[E]^i D$ as an abbreviation for $\neg \langle E \rangle^i \neg D$. Let a be an object variable of $\mathcal{ALCF}^\sharp$ and b an object variable of $\mathcal{ALCO}^\sharp$. Define the following set of $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$-object assertions:

$$\begin{aligned}\Gamma^* = \; & \{a : C^\sharp \wedge \langle E \rangle^1 \{b\}\} \\ \cup \; & \{b : [E]^2 (D^\sharp \to D'^\sharp) \mid D \sqsubseteq D' \in \Gamma\} \\ \cup \; & \{b : [E]^2 f_{\forall R}(\langle E \rangle^1 \{b\}) \mid R \in \mathcal{R}\},\end{aligned}$$

where E is some link from $\mathcal{E}$. We show that

$$C \text{ is satisfiable relative to } \Gamma \text{ in } \mathcal{ALCF} \quad \text{iff} \quad \Gamma^* \text{ is A-satisfiable in } \mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp).$$

($\Rightarrow$) Suppose that $\{a : C\} \cup \Gamma$ is satisfiable relative to Γ. Due to the correspondence between $\mathcal{ALCF}$ and the ADS $\mathcal{ALCF}^\sharp$, there is an $\mathcal{ALCF}^\sharp$-model $\mathfrak{W}_1$ of $\{a : C\} \cup \Gamma$ with domain Δ_1. Define a model $\mathfrak{M}$ for $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$ by taking an arbitrary $\mathcal{ALCO}^\sharp$-model $\mathfrak{W}_2$ with domain Δ_2 and putting $E^{\mathfrak{M}} = \Delta_1 \times \Delta_2$. It is easily checked that $\mathfrak{M} \models \Gamma^*$.

($\Leftarrow$) Suppose $\mathfrak{M} \models \Gamma^*$ for a $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$-model $\mathfrak{M} = (\mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}})$. Let Δ be the domain of $\mathfrak{W}_1$. Denote by Δ' the minimal subset of Δ containing $a^{\mathfrak{M}}$ and satisfying the following closure condition for all $d, d' \in \Delta$:

$$\text{if } (d, d') \in S^{\mathfrak{M}} \text{ for some } d \in \Delta' \text{ and } S \in \mathcal{R}, \text{ then } d' \in \Delta'.$$

Let $\mathfrak{W}_1'$ be the substructure of $\mathfrak{W}_1$ induced by Δ'. Since it is straightforward to prove that $\mathcal{ALCF}^\sharp$ is invariant under taking generated substructures, we have $a^{\mathfrak{W}_1'} \in C^{\mathfrak{W}_1'}$. To show that $\mathfrak{W}_1'$ satisfies Γ, it obviously suffices to prove that, for every assertion $D \sqsubseteq D' \in \Gamma$, we have $(D^\sharp)^{\mathfrak{W}} \cap \Delta' \subseteq (D'^\sharp)^{\mathfrak{W}} \cap \Delta'$. To this end, note that $d \in (D^\sharp \to D'^\sharp)^{\mathfrak{M}}$ whenever $(d, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ due to the third component of Γ^*. Hence, it is sufficient to prove that, for all $d \in \Delta'$, we have $(d, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$. This, however, is an easy consequence of the facts that $(a^{\mathfrak{M}}, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ and $\mathfrak{M}$ satisfies the third component of Γ^*. $\square$

The undecidability proofs in this section use a new reasoning problem: singleton satisfiability of terms. For an ADS $\mathcal{S} = (\mathcal{L}, \mathcal{M})$, we call an $\mathcal{L}$-term t *singleton satisfiable* if there exists $\mathfrak{M} \in \mathcal{M}$ such that $|t^{\mathfrak{M}}| = 1$. As the following lemma shows, there exist ADSs that are number tolerant (cf. Definition 23) and have decidable satisfiability problems, but for which singleton satisfiability is, nevertheless, undecidable:

Lemma 52 *There exist number tolerant ADSs with decidable satisfiability problems for which singleton satisfiability is undecidable. In particular, there exist number tolerant ADSs with decidable satisfiability problems whose extensions with nominals have undecidable satisfiability problems.*

Proof. Consider the ADS $\mathcal{ALC}^\sharp = (\mathcal{L}, \mathcal{M})$ corresponding to the description logic $\mathcal{ALC}$. It follows from, e.g., Theorem 13.15 of [19] that there exists an uncountable set $\mathcal{K} = \{\mathcal{S}_i \mid i \in I\}$ of ADSs $\mathcal{S}_i = (\mathcal{L}, \mathcal{M}_i)$ such that $\mathcal{M}_i \subseteq \mathcal{M}$ for $i \in I$ and

(1) for all $i \in I$ and any $\mathcal{L}$-term t, satisfiability of $a : t$ in $\mathcal{S}_i$ implies singleton satisfiability of t in $\mathcal{S}_i$;
(2) for all $i, j \in I$ with $i \neq j$, there exists a constant term t (i.e., a term composed using the Booleans and function symbols from the symbol $\top$) such that $a : t$ is $\mathcal{S}_i$-satisfiable and not $\mathcal{S}_j$-satisfiable or vice versa.

By property (2), $i \neq j$ implies that the set of constant terms satisfiable in $\mathcal{S}_i$ is not identical to the set of constant terms satisfiable in $\mathcal{S}_j$. Since there exist only countably many algorithms (i.e., Turing machines), the fact that $\mathcal{K}$ is uncountable implies that there exists an $i_0 \in I$ such that satisfiability of constant terms in $\mathcal{S}_{i_0}$ is undecidable. Since for any satisfiable $a : t$ the term t is singleton satisfiable by (1), it is undecidable whether a constant term is singleton satisfiable in $\mathcal{S}_{i_0}$.

Let $\mathcal{M}'$ denote those members of $\mathcal{M}$ which are disjoint unions of at least $\aleph_0$ isomorphic copies of some model in $\mathcal{M}$. By $\mathcal{M}'' \subseteq \mathcal{M}$ we denote the closure of $\mathcal{M}'$ under disjoint unions and arbitrary re-interpretations of object and set variables. The important properties of $\mathcal{M}''$ are as follows:

(a) A knowledge base Γ is satisfiable in $(\mathcal{L}, \mathcal{M}'')$ iff it is satisfiable in $\mathcal{ALC}^\sharp$.
(b) If $a : t$ is satisfied in some $\mathfrak{M} \in \mathcal{M}''$ and t is a constant term, then $|t^{\mathfrak{M}}| \geq \aleph_0$. Hence no satisfiable constant term t is singleton satisfiable in $\mathcal{M}''$.

That property (a) holds should be clear. Property (b) follows from the fact that the extension of constant terms does not depend on the interpretation of set or object variables. Now set $\mathcal{N} = \mathcal{M}_{i_0} \cup \mathcal{M}''$ and $\mathcal{S} = (\mathcal{L}, \mathcal{N})$. We claim that $\mathcal{S}$ is as required. Obviously, $\mathcal{S}$ is number tolerant and singleton satisfiability is undecidable. It remains to observe that the satisfiability problem for $\mathcal{S}$ coincides with the satisfiability problem for $\mathcal{ALC}^\sharp$, which is decidable.

The extension of $\mathcal{S}$ by means of nominals has the undecidable satisfiability problem, since $\{a\} = t$ is satisfiable iff t is singleton satisfiable, for any term t.
□

Apart from ADSs for which singleton satisfiability is undecidable, there exists one more ADS that will play an important role in this section:

Definition 53 The ADS $\mathcal{B}_1 = (\mathcal{L}_\mathcal{B}, \mathcal{M}_{\mathcal{B}_1})$ is defined as follows:

- $\mathcal{L}_\mathcal{B}$ is, as defined above already, the ADL without function symbols (apart from the Booleans) and relation symbols;
- $\mathcal{M}_{\mathcal{B}_1}$ consists of all ADMs of the signature of $\mathcal{L}_\mathcal{B}$ based on a singleton domain.

It is obviously trivial to decide satisfiability in $\mathcal{B}_1$. Note also that $\mathcal{B}_1$ is not number tolerant.

We are now in a position to prove the undecidability results. We start with $\mathcal{E}$-connections that allow qualified number restrictions, but do *not* require number tolerance: together with Lemma 52, the following lemma implies Theorem 22:

Lemma 54 *Let $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ be an ADS for which singleton satisfiability is undecidable and let $\mathcal{E}$ be a non-empty set of link symbols. Then the satisfiability problem for $\mathcal{C}^\mathcal{E}_Q(\mathcal{S}, \mathcal{B}_1)$ is undecidable.*

Proof. We prove the lemma by reducing singleton satisfiability in $\mathcal{S}$ to satisfiability in $\mathcal{C}^\mathcal{E}_Q(\mathcal{S}, \mathcal{B}_1)$: it is readily checked that an $\mathcal{L}$-term t is singleton satisfiable if and only if the set of $\mathcal{C}^\mathcal{E}_Q(\mathcal{S}, \mathcal{B}_1)$-assertions (consisting of a 1-assertion and a 2-assertion)

$$\{t \sqsubseteq \langle E \rangle^1 (\top_2)\,, \quad \top_2 \sqsubseteq \langle = 1\, E \rangle^2 (t)\}$$

is satisfiable, where E is a link relation from $\mathcal{E}$ and $\langle = 1\, E \rangle^i (t)$ is an abbreviation for $\langle \leq 1\, E \rangle^i (t) \wedge \langle \geq 1\, E \rangle^i (t)$. □

The proofs of the other undecidability results are similar to the proof of Lemma 54. Therefore, we give only the set of reduction assertions which varies with the type of $\mathcal{E}$-connection under consideration. Again together with Lemma 52, the following lemma yields Theorem 29 (i) and (ii), which deal with $\mathcal{E}$-connections of number-tolerant ADSs allowing (i) both qualified number restrictions and the Boolean operators on link relations, or (ii) both qualified number restrictions and the application of link operators to object variables.

Lemma 55
(i) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any ADS* $\mathcal{S}_2$*.*

(ii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any ADS* $\mathcal{S}_2$*.*

Proof. The proof of (i) is analogous to the proof of Lemma 54: we use the following set, which consists only of a single 2-assertion:

$$\{b : (\langle E \rangle^2 (t) \wedge \langle {=}\, 1\, E \rangle^2 (\top_1) \wedge \neg \langle \neg E \rangle^2 (t))\}$$

The proof of (ii) is similar to the proof of Lemma 54, using the following set of assertions (one 1-assertion and one 2-assertion):

$$\{t \sqsubseteq \langle E \rangle^1 (b)\ , \quad b : \langle {=}\, 1\, E \rangle^2 (t)\}$$

Here b is an object variable from $\mathcal{L}_2$. □

The following lemma takes care of Theorem 39. This theorem is concerned with $\mathcal{E}$-connections that provide for CIC (complete individual correspondence, see Section 6.2) assertions.

Lemma 56
(i) *Let* $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{I}(\mathcal{S}, \mathcal{B}_1)$ *is undecidable.*

(ii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any ADS* $\mathcal{S}_2$*.*

(iii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any ADS* $\mathcal{S}_2$*.*

Proof. (i) is similar to the proof of Lemma 54: we use the following set of assertions (three 1-assertions):

$$\{a : t\ , \quad t \sqsubseteq \langle E \rangle^1 (\top_2)\ , \quad \langle E \rangle^1 b = \{a\}\}$$

Here, the last assertion is a CIC assertion, a is an object variable from $\mathcal{L}_1$, and b is an object variable from $\mathcal{L}_{\mathcal{B}}$.

(ii) is analogous to the proof of Lemma 54; it uses the following set of assertions (two 1-assertions and one 2-assertion):

$$\{a : t\ ,\quad \langle E\rangle^1 (b) = \{a\}\ ,\quad b : \neg \langle \neg E\rangle^2 (t)\}$$

Here, a is an object variable from $\mathcal{L}_1$ and b is an object variable from $\mathcal{L}_2$.

(iii) is similar to the proof of Lemma 54; it uses the following set of assertions (three 1-assertions):

$$\{a : t\ ,\quad t \sqsubseteq \langle E\rangle^1 (b)\ ,\quad \langle E\rangle^1 (b) = \{a\}\}.$$

Again, a is an object variable from $\mathcal{L}_1$ and b is such a variable from $\mathcal{L}_2$. □

C.3 Lower bounds

In this section we give proofs of Theorem 13 and Theorem 20.

Theorem 13. *The satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$ is EXPTIME-hard for any infinite set $\mathcal{E}$ of links.*

Proof. We reduce the EXPTIME-complete satisfiability problem for $\mathcal{ALC}$-concepts relative to TBoxes [64] to the satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$. To this end, select for any role name $R \in \mathcal{R}$ of $\mathcal{ALC}$ two links E_1^R and E_2^R, set

$$\mathcal{E} = \{E_1^R, E_2^R \mid R \in \mathcal{R}\},$$

and associate with any concept name A_i of $\mathcal{ALC}$ a set variable X^{A_i} of the first component of $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$. Now define a translation $.^\dagger$ by taking

$$\begin{aligned} A_i^\dagger &= X^{A_i} & (C_1 \wedge C_2)^\dagger &= C_1^\dagger \wedge C_2^\dagger \\ (\neg C)^\dagger &= \neg C^\dagger & (\exists R.C)^\dagger &= \left\langle E_1^R\right\rangle^1 \left(\left\langle E_2^R\right\rangle^2 (C^\dagger)\right) \\ (C_1 \sqsubseteq C_2)^\dagger &= C_1^\dagger \sqsubseteq C_2^\dagger & (a : C)^\dagger &= a : C^\dagger \end{aligned}$$

We claim that for every set Γ of $\mathcal{ALC}$-assertions and the corresponding set $\Gamma^\dagger = \{\varphi^\dagger \mid \varphi \in \Gamma\}$ of $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$ assertions,

$$\Gamma \text{ is } \mathcal{ALC}\text{-satisfiable} \quad \text{iff} \quad \Gamma^\dagger \text{ is } \mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})\text{-satisfiable.} \qquad (\spadesuit)$$

For assume that Γ is satisfied in an $\mathcal{ALC}$-model

$$\mathcal{I} = \left\langle \Delta, A_1^{\mathcal{I}}, \ldots, R_1^{\mathcal{I}}, \ldots, a_1^{\mathcal{I}}, \ldots \right\rangle.$$

Define a model

$$\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{(E_1^R)^{\mathfrak{M}}, (E_2^R)^{\mathfrak{M}}\}_{R \in \mathcal{R}} \right\rangle,$$

where $\mathfrak{M}_1 = \left\langle \Delta, (X^{A_1})^{\mathfrak{M}_1}, \ldots, a_1^{\mathfrak{M}_1}, \ldots \right\rangle$ with $(X^{A_i})^{\mathfrak{M}_1} = A_i^{\mathcal{I}}$ and $a_i^{\mathfrak{M}_1} = a_i^{\mathcal{I}}$, $\mathfrak{M}_2$ is some arbitrary ADM for $\mathcal{B}$ with domain Δ, and

$$(E_1^R)^{\mathfrak{M}} = \{(x,x) \mid x \in \Delta\}, \quad (E_2^R)^{\mathfrak{M}} = \{(x,y) \mid (y,x) \in R^{\mathcal{I}}\}.$$

Clearly, it suffices to show that, for any concept C of $\mathcal{ALC}$,

$$C^{\mathcal{I}} = (C^{\dagger})^{\mathfrak{M}}.$$

The proof is by induction on the construction of C. We consider the case $C = \exists R.D$, leaving the remaining ones to the reader:

$$\begin{aligned}
\left(\left\langle E_1^R \right\rangle^1 (\left\langle E_2^R \right\rangle^2 (D^{\dagger}))\right)^{\mathfrak{M}} &= \{x \mid \exists y.\ y \in \left(\left\langle E_2^R \right\rangle^2 (D^{\dagger})\right)^{\mathfrak{M}} \wedge (x,y) \in (E_1^R)^{\mathfrak{M}}\} \\
&= \{x \mid x \in \left(\left\langle E_2^R \right\rangle^2 (D^{\dagger})\right)^{\mathfrak{M}}\} \\
&= \{x \mid \exists y \in (D^{\dagger})^{\mathfrak{M}}\ (x,y) \in R^{\mathcal{I}}\} \\
&= (\exists R.D)^{\mathcal{I}}.
\end{aligned}$$

Conversely, suppose $\Gamma^{\dagger}$ is satisfied in a model $\mathfrak{M}$ of $\mathcal{C}^{\mathcal{E}}(\mathcal{B},\mathcal{B})$. Define a model $\mathcal{I}$ of $\mathcal{ALC}$ by taking $A_i^{\mathcal{I}} = (X^{A_i})^{\mathfrak{M}_1}$, $a_i^{\mathcal{I}} = a_i^{\mathfrak{M}_1}$, and

$$R^{\mathcal{I}} = \{(x,y) \mid \exists z \in \Delta_2.\ (x,z) \in (E_1^R)^{\mathfrak{M}} \wedge (y,z) \in (E_2^R)^{\mathfrak{M}}\}.$$

Again, it suffices to show that, for any concept C of $\mathcal{ALC}$, $C^{\mathcal{I}} = (C^{\dagger})^{\mathfrak{M}}$. We consider only the case $C = \exists R.D$ of the inductive proof:

$$\begin{aligned}
(\exists R.D)^{\mathcal{I}} &= \{x \in \Delta_1 \mid \exists y \in D^{\mathcal{I}}\ (x,y) \in R^{\mathcal{I}}\} \\
&= \{x \in \Delta_1 \mid \exists y \in (D^{\dagger})^{\mathfrak{M}} \exists z \in \Delta_2.\ (x,z) \in (E_1^R)^{\mathfrak{M}} \wedge (y,z) \in (E_2^R)^{\mathfrak{M}}\} \\
&= \{x \in \Delta_1 \mid \exists z \in \Delta_2.\ z \in \left(\left\langle E_2^R \right\rangle^2 (D^{\dagger})\right)^{\mathfrak{M}} \wedge (x,z) \in (E_1^R)^{\mathfrak{M}}\} \\
&= \left(\left\langle E_1^R \right\rangle^1 (\left\langle E_2^R \right\rangle^2 (D^{\dagger}))\right)^{\mathfrak{M}}.
\end{aligned}$$

This completes the proof. □

Let us now prove Theorem 20. To this end, we are going to reduce the NEXPTIME-complete satisfiability problem for the modal logic $\mathbf{S5} \times \mathbf{S5}$ [57] to the satisfiability problem for $\mathcal{C}_B^{\mathcal{E}}(\mathcal{B},\mathcal{B})$.

Theorem 20. *The satisfiability problem for $\mathcal{C}_B^{\mathcal{E}}(\mathcal{B},\mathcal{B})$ is* NEXPTIME*-hard, for any infinite $\mathcal{E}$.*

Proof. Recall that $\mathbf{S5} \times \mathbf{S5}$–formulas are composed from propositional variables $p_1, \ldots$ by means of the Boolean operators and the modal operators $\Box_1$ and $\Box_2$. $\mathbf{S5} \times \mathbf{S5}$-models $\mathfrak{N} = \langle W_1 \times W_2, \mathfrak{V} \rangle$ consist of the Cartesian product of two non-empty sets W_1 and W_2 and a valuation $\mathfrak{V}$ which maps any propositional variable to a subset of $W_1 \times W_2$. The extension $\varphi^{\mathfrak{N}}$ of an $\mathbf{S5} \times \mathbf{S5}$-formula

φ in $\mathfrak{N}$ is computed inductively as follows:

$$
\begin{aligned}
p_i^{\mathfrak{N}} &= \mathfrak{V}(p_i), \quad (\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}}, \quad (\neg\psi)^{\mathfrak{N}} = (W_1 \times W_2) \setminus \psi^{\mathfrak{N}}, \\
(\Box_1\psi)^{\mathfrak{N}} &= \{(w_1, w_2) \mid \forall v \in W_1\ (v, w_2) \in \psi^{\mathfrak{N}}\}, \\
(\Box_2\psi)^{\mathfrak{N}} &= \{(w_1, w_2) \mid \forall v \in W_2\ (w_1, v) \in \psi^{\mathfrak{N}}\}.
\end{aligned}
$$

A formula φ is $\mathbf{S5} \times \mathbf{S5}$-satisfiable if there exists an $\mathbf{S5} \times \mathbf{S5}$-model in which φ has a non-empty extension.

Suppose now that φ is an $\mathbf{S5} \times \mathbf{S5}$-formula. Denote by $sub(\varphi)$ the set of all subformulas of φ. For any $\psi \in sub(\varphi)$, take a link $E_\psi \in \mathcal{E}$ and let the $\mathcal{C}_B^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$-knowledge base Γ consist of:

(1) $E_{\psi_1 \wedge \psi_2} = E_{\psi_1} \wedge E_{\psi_2}$, for $\psi_1 \wedge \psi_2 \in sub(\varphi)$,
(2) $E_{\neg\psi} = \neg E_\psi$, for $\neg\psi \in sub(\varphi)$;
(3) $\langle \neg E_\psi \rangle^2 (\top_1) = [E_{\Box_1\psi}]^2(\bot_1)$, $[E_{\Box_1\psi}]^2(\bot_1) = \langle \neg E_{\Box_1\psi} \rangle^2 (\top_1)$, for $\Box_1\psi \in sub(\varphi)$;
(4) $\langle \neg E_\psi \rangle^1 (\top_2) = [E_{\Box_2\psi}]^1(\bot_2)$, $[E_{\Box_2\psi}]^1(\bot_2) = \langle \neg E_{\Box_2\psi} \rangle^1 (\top_2)$, for $\Box_2\psi \in sub(\varphi)$.

It was shown in Section 5.2 that such equations can be added to the vocabulary when working with connections allowing the Boolean closure of links. More precisely, an equation of the form $F = G$ is a shorthand for the conjunction of the two link inclusions $F \sqsubseteq G$ and $G \sqsubseteq F$. We now claim that

$$
\varphi \text{ is } \mathbf{S5} \times \mathbf{S5}\text{-satisfiable} \quad \text{iff} \quad \Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\} \text{ is satisfiable in } \mathcal{C}_B^{\mathcal{E}}(\mathcal{B}, \mathcal{B}), \quad (\clubsuit)
$$

where a is an object name of the first component of $\mathcal{C}_B^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$.

To prove ($\clubsuit$), assume first that φ is satisfied in $\mathfrak{N} = \langle W_1 \times W_2, \mathfrak{V} \rangle$. We construct a model $\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{E_\psi^{\mathfrak{M}}\}_{\psi \in sub(\varphi)} \right\rangle$ that satisfies $\Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\}$. Let $\mathfrak{M}_2$ be any model for $\mathcal{B}$ with domain W_2. By assumption, $\varphi^{\mathfrak{N}} \neq \emptyset$, so we can pick some $(u, v) \in \varphi^{\mathfrak{N}}$ and choose $\mathfrak{M}_1$ to be any model for $\mathcal{B}$ with domain W_1, where $a^{\mathfrak{M}_1} = u$. Finally, we can define $E_\psi^{\mathfrak{M}} = \psi^{\mathfrak{N}} \subseteq W_1 \times W_2$, for every $\psi \in sub(\varphi)$. By construction, $\mathfrak{M} \models a : \langle E_\varphi \rangle^1 (\top_2)$, so it suffices to show that equations (1)–(4) hold in $\mathfrak{M}$, which can be done by structural induction.

If $\psi_1 \wedge \psi_2 \in sub(\varphi)$, then

$$
E_{\psi_1 \wedge \psi_2}^{\mathfrak{M}} = (\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}} = E_{\psi_1}^{\mathfrak{M}} \cap E_{\psi_2}^{\mathfrak{M}}.
$$

Equation (2) is shown in the same way. To prove (3), notice that the following equivalences hold:

$$\begin{aligned} v \in (\langle \neg E_\psi \rangle^2 (\top_1))^{\mathfrak{M}} \quad \text{iff} \quad \exists u\ (u,v) \notin E_\psi^{\mathfrak{M}} \quad \text{iff} \quad \exists u\ (u,v) \notin \psi^{\mathfrak{N}} \\ \text{iff} \quad \forall u\ (u,v) \notin (\Box_1 \psi)^{\mathfrak{N}} \quad \text{iff} \quad \forall u\ (u,v) \notin E_{\Box_1\psi}^{\mathfrak{M}} \quad \text{iff} \quad v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \end{aligned}$$

and

$$\begin{aligned} v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \quad \text{iff} \quad \forall u\ (u,v) \notin (\Box_1\psi)^{\mathfrak{N}} \quad \text{iff} \quad \exists u\ (u,v) \notin (\Box_1\psi)^{\mathfrak{N}} \\ \text{iff} \quad \exists u\ (u,v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \quad \text{iff} \quad v \in (\langle \neg E_{\Box_1\psi} \rangle^2 (\top_1))^{\mathfrak{M}} \end{aligned}$$

The equations in (4) are proved in exactly the same way.

Conversely, assume that $\Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\}$ is satisfied in a model $\mathfrak{M}$, where $\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{E_\psi^{\mathfrak{M}}\}_{\psi \in sub(\varphi)} \right\rangle$ is based on the domains W_1 and W_2. We define a model $\mathfrak{N}$ for **S5** $\times$ **S5** based on the domain $W_1 \times W_2$ by letting $p_i^{\mathfrak{N}} = E_{p_i}^{\mathfrak{M}}$ for $p_i \in sub(\varphi)$ and arbitrary otherwise. It can now be shown by induction that, for all $\psi \in sub(\varphi)$,

$$E_\psi^{\mathfrak{M}} = \psi^{\mathfrak{N}}. \qquad (\heartsuit)$$

The base case, $\psi = p_i$, follows from the definition of $\mathfrak{N}$. If $\psi = \psi_1 \wedge \psi_2$, then $(\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}} = E_{\psi_1}^{\mathfrak{M}} \cap E_{\psi_2}^{\mathfrak{M}} = E_{\psi_1 \wedge \psi_2}^{\mathfrak{M}}$ by (1). The case of $\psi = \neg\chi$ is shown in the same way using (2). The case of $\psi = \Box_1\chi$ is shown using (3) as follows:

$$\begin{aligned} (u,v) \notin (\Box_1\psi)^{\mathfrak{N}} &\text{ iff } \exists \hat{u} \in W_1\ (\hat{u}, v) \notin \psi^{\mathfrak{N}} \\ &\text{ iff } \exists \hat{u} \in W_1\ (\hat{u}, v) \notin E_\psi^{\mathfrak{M}} \text{ (by induction)} \\ &\text{ iff } v \in (\langle \neg E_\psi \rangle^2 (\top_1))^{\mathfrak{M}} \\ &\text{ iff } v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \text{ (by (3.1))} \\ &\text{ iff } \forall \hat{u} \in W_1\ (\hat{u}, v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \\ &\Rightarrow (u,v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \end{aligned}$$

and

$$\begin{aligned} (u,v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} &\Rightarrow \exists \hat{u} \in W_1\ (\hat{u}, v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \\ &\text{ iff } v \in (\langle \neg E_{\Box_1\psi} \rangle^2 (\top_1))^{\mathfrak{M}} \\ &\text{ iff } v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \text{ (by (3.2))} \\ &\text{ iff } (u,v) \notin (\Box_1\psi)^{\mathfrak{N}} \text{ (from above).} \end{aligned}$$

The case of $\psi = \Box_2\chi$ is treated in the same way. This shows ($\heartsuit$).

As $\mathfrak{M} \models a : \langle E_\varphi \rangle^1 (\top_2)$, there is a $v \in W_2$ such that $(a^{\mathfrak{M}}, v) \in E_\varphi^{\mathfrak{M}} = \varphi^{\mathfrak{N}} \neq \emptyset$. It follows that φ is satisfied in $\mathfrak{N}$, which proves ($\clubsuit$). $\Box$

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