Modal Logics of Topological Relations

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Abstract

We introduce a family of modal logics that are interpreted in domains consisting of regions in topological spaces, in particular the real plane. The underlying modal language has 8 operators interpreted by the RCC8(or Egenhofer-Franzosa)-relations between regions. The following results on the expressive power and computational complexity of the resulting modal systems are obtained: they are *expressively complete* for the two-variable fragment of first-order logic, and are usually *undecidable* and often not even recursively enumerable. This also holds if we interpret our language in the class of all (finite) substructures of full region spaces. If interpreted in region spaces consisting of intervals in the real line, our results significantly extend undecidability results of Halpern and Shoham in that we prove the undecidability of interval temporal logic over the class of all substructures of all full interval structures. We also analyze modal logics based on the set of RCC5-relations which are more coarse than the RCC8 relations.

1 Introduction

Reasoning about topological relations between regions in space is recognized as one of the most important and challenging research areas within Spatial Reasoning in AI and Philosophy, Spatial and Constraint Databases, and Geographical Information Systems (GISs). Research in this area can be classified according to the logical apparatus employed:

– General first-order theories of topological relations between regions are studied in AI and Philosophy [6; 27; 26], Spatial Databases [25; 30] and, from an algebraic viewpoint, in [8; 31];

– Purely existential theories formulated as constraint satisfaction systems over jointly exhaustive and mutually disjoint sets of topological relations between regions [9; 28; 15; 30; 27]

– Modal logics of space with operators interpreted by the closure and interior operator of the underlying topological space and propositions interpreted as subsets of the topological space, see e.g., [18; 5; 1].

A similar classification can be made for Temporal Reasoning: general first-order theories [3], temporal constraint systems [2; 24] and modal temporal logics like Prior's tense logics, LTL, and

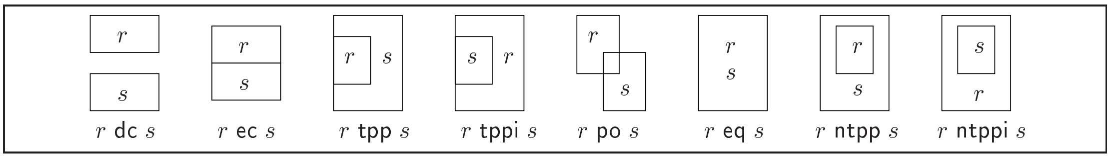

Figure 1: Examples for the RCC8 relations in the plane.

CTL [12]. However, one of the most important and influential approaches in temporal reasoning has not yet found a fully developed analogue on the spatial reasoning research agenda: Halpern and Shoham's Modal Logic of intervals [16], in which propositions are interpreted as sets of intervals (those in which they are true) and reference to other intervals is enabled by modal operators interpreted by Allen's 13 relations between intervals. Despite its bad computational behavior (undecidable, usually not even r.e.), this framework proved extremely fruitful and influential in temporal reasoning, see e.g. [32; 4; 19].

To develop an equally powerful and useful modal language for reasoning about topological relations between regions, we first have to select a set of basic relations. In the initially mentioned research areas, there seems to be consensus that the eight RCC8-relations, which are also known as "Egenhofer-Franzosa"-relations and have been independently introduced in [27] and [10], are very natural and important—both from a theoretical and a practical viewpoint, see e.g. [25; 9]. Thus, in this paper we will consider modal logics with eight modal operators interpreted by the eight RCC8-relations, and whose formulas are interpreted as sets of regions (those in which they are true). This modal framework for reasoning about regions has been suggested in an early paper by Cohn [7] and further considered in [33]. However, it proved difficult to analyze the computational behavior of such logics and, despite several efforts, to the best of our knowledge no results have been obtained so far.

To relate this approach to previous and ongoing work on first-order theories of regions [27; 26; 25; 30], it is important to observe that the modal logic we propose is a fragment of first-order logic with the eight binary RCC8-relations *and infinitely many unary predicates.* More precisely, we will show that our logic has exactly the same expressive power as the two-variable fragment of this FO logic—although the latter is exponentially more succinct. Since usual first-order theories of regions admit arbitrarily many variables but *no* unary predicates, their expressive power is incomparable to the one of our modal logics. We argue that the availability of unary predicates is essential for a wide range of application areas: in contrast to describing only purely topological properties of regions, it allows to also capture other properties such as being a country (in a GIS), a ball (for a soccer-playing robot), or a protected area (in a spatial database). In our modal logic, we can then formulate constraints such as "there are no two overlapping regions that are both countries" and "every river is connected to an ocean or a lake".

The purpose of this paper is to *introduce modal logics of topological relations in a systematic way, perform an initial investigation of their expressiveness and relationships, and analyze their computational behavior.* More precisely, this paper is organized as follows: in Section 2, we introduce region spaces, which form the semantical basis for our logics. The modal language is introduced in Section 3, and a brief analysis of its expressiveness is performed. In Section 4, we identify a number of natural logics induced by different classes of region structures, and analyze their relationship. In Section 5, we then prove the main result of this paper showing that modal logics of topological relations are usually undecidable. We also show that undecidability can not be overcome by admitting only finitely (but unboundedly) many regions. These results are strengthened in Section 6 where we prove that many logics of topological relations are even Π^1_1-hard. Finally, in Section 7 we give undecidability results for modal logics obtained from the RCC5 set of relations.

∘	dc	ec	tpp	tppi	po	ntpp	ntppi
dc	$*$	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	dc	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	dc
ec	dc,ec, po,tppi, ntppi	dc,ec, po,tpp, tppi,eq	ec,po, tpp, ntpp	dc,ec	dc,ec, po,tpp, ntpp	po, tpp, ntpp	dc
tpp	dc	dc,ec	tpp,ntpp	dc,ec, po,tpp, tppi,eq	dc,ec, po,tpp, ntpp	ntpp	dc,ec, po,tppi, ntppi
tppi	dc,ec, po,tppi, ntppi	ec,po, tppi, ntppi	po,eq, tpp, tppi	tppi,ntppi	po, tppi, ntppi	po, tpp, ntpp	ntppi
po	dc,ec, po,tppi, ntppi	dc,ec, po,tppi, ntppi	po, tpp, ntpp	dc,ec, po,tppi, ntppi	$*$	po, tpp, ntpp	dc,ec, po,tppi, ntppi
ntpp	dc	dc	ntpp	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	ntpp	$*$
ntppi	dc,ec, po,tppi, ntppi	po, tppi, ntppi	po, tppi, ntppi	ntppi	po, tppi, ntppi	po, tppi, tpp,ntpp, ntppi,eq	ntppi

Figure 2: The RCC8 composition table.

2 Structures

We want to reason about models whose domains consist of regions that are related by the eight RCC8-relations dc ('disconnected'), ec ('externally connected'), tpp ('tangential proper part'), tppi ('inverse of tangential proper part'), po ('partial overlap'), eq ('equal'), ntpp ('non-tangential proper part'), and nttpi ('inverse of non-tangential proper part'). Figure 1 gives examples of the RCC8 relations in the real plane $\mathbb{R}^2$, where regions are rectangles. Different spatial ontologies give rise to different notions of regions and, therefore, different classes of models. Almost all definitions of regions provided in the literature, however, have in common that the resulting models are *region structures* $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \dots \rangle$, where W is a non-empty set (of regions) and the $\mathsf{r}^{\mathfrak{R}}$ are binary relations on W that are mutually disjoint (i.e., $\mathsf{r}^{\mathfrak{R}} \cap \mathsf{q}^{\mathfrak{R}} = \emptyset$, for $\mathsf{r} \neq \mathsf{q}$), jointly exhaustive (i.e., the union of all $\mathsf{r}^{\mathfrak{R}}$ is $W \times W$), and satisfy the following:

- eq is interpreted as the identity on W, dc, ec, and po are symmetric, and tppi and ntppi are the inverse relations of ttp and ntpp, respectively;
- the rules of the RCC8 composition table (Figure 2) are satisfied in the sense that, for any entry $\mathsf{q}_1, \dots, \mathsf{q}_k$ in row r_1 and column r_2, the first-order sentence

$$\forall x \forall y \forall z ((\mathsf{r}_1(x,y) \wedge \mathsf{r}_2(y,z)) \rightarrow (\mathsf{q}_1(x,z) \vee \cdots \vee \mathsf{q}_k(x,z))$$

  is valid ($*$ is the disjunction over all RCC8-relations).

Denote the class of all region structures by $\mathcal{RS}$. Although of definite interest as a basic class of models representing the relation between regions in space, often more restricted definitions of region structures are considered. On the one hand, one can consider further first-order conditions on region structures, say, (fragments) of the RCC-theory [27]. Another possibility is to consider only region structures that are induced by (classes of) topological spaces. Recall that a topological space is a pair $\mathfrak{T} = (U, \mathbb{I})$, where U is a set and $\mathbb{I}$ is an *interior operator* on U, i.e., for all $s, t \subseteq U$, we have

$$\mathbb{I}(U) = U, \quad \mathbb{I}(s) \subseteq s, \quad \mathbb{I}(s) \cap \mathbb{I}(t) = \mathbb{I}(s \cap t), \quad \text{and} \quad \mathbb{I}\mathbb{I}(s) = \mathbb{I}(s).$$

The closure $\mathbb{C}(s)$ of s is then $\mathbb{C}(s) = U - (\mathbb{I}(U - s))$. Of particular interest are n-dimensional Euclidean spaces $\mathbb{R}^n$ based on Cartesian products of the real line with the standard topology induced by the Euclidean metric.

Depending on the application domain, different definitions of regions in topological spaces have been introduced. A rather general notion identifies regions with non-empty, *regular closed* sets, i.e. non-empty subsets $s \subseteq U$ such that $\mathbb{C}\mathbb{I}(s) = s$. We write $\mathfrak{T}_{\mathsf{reg}}$ to denote the set of non-empty, regular closed subsets of the topological space $\mathfrak{T}$. Various more restrictive definitions of regions are important in the Euclidean spaces $\mathbb{R}^n$, e.g.,

- the set $\mathbb{R}^n_{\mathsf{conv}}$ of non-empty convex regular closed subsets of $\mathbb{R}^n$;
- the set $\mathbb{R}^n_{\mathsf{rect}}$ of closed hyper-rectangular subsets of $\mathbb{R}^n$, i.e., regions of the form $\prod_{i=1}^n C_i$, where $C_1, \ldots, C_n$ are non-singleton closed intervals in $\mathbb{R}$.

In both cases we allow unbounded regions, in particular $\mathbb{R}^n$. However, we should note that the technical results proved in this paper also hold if we consider bounded regions, only.

Given a topological space $\mathfrak{T}$ and a set of regions $U_{\mathfrak{T}}$ in $\mathfrak{T}$ as introduced above, we obtain a region structure $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}}) = \langle U_{\mathfrak{T}}, \mathsf{dc}^{\mathfrak{T}}, \ldots \rangle$ by putting:

$$
\begin{array}{rcl}
(s,t) \in \mathsf{dc}^{\mathfrak{T}} & \text{iff} & s \cap t = \emptyset \\
(s,t) \in \mathsf{ec}^{\mathfrak{T}} & \text{iff} & \mathbb{I}(s) \cap \mathbb{I}(t) = \emptyset \ \wedge \ s \cap t \neq \emptyset \\
(s,t) \in \mathsf{po}^{\mathfrak{T}} & \text{iff} & \mathbb{I}(s) \cap \mathbb{I}(t) \neq \emptyset \ \wedge \ s \setminus t \neq \emptyset \ \wedge t \setminus s \neq \emptyset \\
(s,t) \in \mathsf{eq}^{\mathfrak{T}} & \text{iff} & s = t \\
(s,t) \in \mathsf{tpp}^{\mathfrak{T}} & \text{iff} & s \cap \overline{t} = \emptyset \ \wedge \ s \cap \overline{\mathbb{I}(t)} \neq \emptyset \\
(s,t) \in \mathsf{ntpp}^{\mathfrak{T}} & \text{iff} & s \cap \overline{\mathbb{I}(t)} = \emptyset \\
(s,t) \in \mathsf{tppi}^{\mathfrak{T}} & \text{iff} & (t,s) \in \mathsf{tpp} \\
(s,t) \in \mathsf{ntppi}^{\mathfrak{T}} & \text{iff} & (t,s) \in \mathsf{ntpp}
\end{array}
$$

$\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})$ is called the region structure *induced* by $(\mathfrak{T}, U_{\mathfrak{T}})$. It is easy (but tedious) to verify that the conditions of region structures are satisfied. We set $\mathcal{TOP} = \{\mathfrak{R}(\mathfrak{T}, \mathfrak{T}_{\mathsf{reg}}) \mid \mathfrak{T} \text{ topological space } \}$.

3 Languages

The modal language $\mathcal{L}_{\mathsf{RCC8}}$ extends propositional logic with countably many variables $p_1, p_2, \ldots$ and the Boolean connectives $\wedge$ and $\neg$ by means of the modal operators [dc], [ec], etc. (one for each RCC8 relation). A *region model* $\mathfrak{M} = \langle \mathfrak{R}, p_1^{\mathfrak{M}}, p_2^{\mathfrak{M}}, \ldots \rangle$ for $\mathcal{L}_{\mathsf{RCC8}}$ consists of a region structure $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \ldots \rangle$ and the interpretation $p_i^{\mathfrak{M}}$ of the variables of $\mathcal{L}_{\mathsf{RCC8}}$ as subsets of W. A formula φ is either true at $s \in W$ (written $\mathfrak{M}, s \models \varphi$) or false at s (written $\mathfrak{M}, s \not\models \varphi$), the inductive definition being as follows:

1. If φ is a prop. variable, then $\mathfrak{M}, s \models \varphi$ iff $s \in \varphi^{\mathfrak{M}}$.
2. $\mathfrak{M}, s \models \neg\varphi$ iff $\mathfrak{M}, s \not\models \varphi$.
3. $\mathfrak{M}, s \models \varphi_1 \wedge \varphi_2$ iff $s \models \varphi_1$ and $s \models \varphi_2$.
4. $\mathfrak{M}, s \models [\mathsf{r}]\varphi$ iff, for all $t \in W$, $(s,t) \in \mathsf{r}^{\mathfrak{R}}$ implies $\mathfrak{M}, t \models \varphi$.

We use the usual abbreviations: $\varphi \to \psi$ for $\neg\varphi \vee \psi$ and $\langle \mathsf{r} \rangle \varphi$ for $\neg[\mathsf{r}]\neg\varphi$.

The discussion of the expressive power of our logic starts with three simple examples. First, the useful *universal box* $\Box_u \varphi$ can obviously be expressed as $\bigwedge_{\mathsf{r} \in \mathsf{RCC8}} [\mathsf{r}]\varphi$. Second, we can express that a formula φ holds in precisely one region (is *a nominal*) by

$$\mathsf{nom}(\varphi) = \Diamond_u(\varphi \wedge \bigwedge_{\mathsf{r} \in \mathsf{RCC8} \setminus \{\mathsf{eq}\}} [\mathsf{r}]\neg\varphi),$$

where $\Diamond_u \varphi = \neg\Box_u\neg\varphi$. The definability of nominals means, in particular, that we can express RCC8-constraints [28] in our language: just observe that constraints $(x \mathsf{\ r\ } y)$, where r is an RCC8-relation, correspond to the assertion $(p_x \wedge \langle \mathsf{r} \rangle p_y) \wedge \mathsf{nom}(p_x) \wedge \mathsf{nom}(p_y)$. Another main advantage of having nominals is that we can introduce names for regions; e.g., the formulas $\mathsf{nom}(Elbe)$ and $\mathsf{nom}(Dresden)$ state that "*Elbe*" (the name of a river) and "*Dresden*" each apply to exactly one region. Third, it is useful to define operators [pp] and [ppi] as abbreviations:

$$[\mathsf{pp}]\varphi = [\mathsf{tpp}]\varphi \wedge [\mathsf{nttp}]\varphi \qquad [\mathsf{ppi}]\varphi = [\mathsf{tppi}]\varphi \wedge [\mathsf{nttpi}]\varphi.$$

As in the temporal case [16] and following Cohn [7], we can classify propositions φ according to whether

- they are homogeneous, i.e. they hold continuously throughout subregions: $\Box_u(\varphi \rightarrow [\mathsf{pp}]\varphi)$.
- they are anti-homogeneous, i.e. they hold only in regions whose interiors are mutually disjoint: $\Box_u(\varphi \rightarrow ([\mathsf{pp}]\neg\varphi \wedge [\mathsf{po}]\neg\varphi)$

Instances of anti-homogeneous propositions are "*river*" and "*city*", while "*occupied-by-water*" is homogeneous. The following are some example statements in our logic (neglecting for simplicity the existence of sea harbors):

$$\begin{array}{l}
\Box_u(\textit{harbor-city} \leftrightarrow (\textit{city} \wedge \langle \mathsf{ntppi} \rangle(\textit{harbor} \wedge \langle \mathsf{ec} \rangle \textit{river}))) \\
\Box_u(\textit{Dresden} \rightarrow \textit{harbor-city}) \\
\Box_u(\textit{Elbe} \rightarrow \textit{river}) \\
\Box_u(\textit{Dresden} \rightarrow (\langle \mathsf{po} \rangle \textit{Elbe} \wedge [\mathsf{po}](\textit{river} \rightarrow \textit{Elbe}))) \\
\Box_u(\textit{Dresden} \rightarrow [\mathsf{ppi}]\neg \textit{river})
\end{array}$$

From these formulas, it follows that Dresden has a harbor that is related via ec to the river Elbe.

We now relate the expressive power of the modal language $\mathcal{L}_{\mathsf{RCC8}}$ to the expressive power of first-order languages over region structures. Since spatial first-order theories are usually formulated in first-order languages equivalent to $\mathcal{FO}_{\mathsf{RCC8}}$ with eight binary relations for the RCC8 relations and *no* unary predicates [25; 26; 30; 27], we cannot reduce $\mathcal{L}_{\mathsf{RCC8}}$ to such languages. A formal proof is provided by the observation that $\mathcal{FO}_{\mathsf{RCC8}}$ is decidable over the region space consisting of rectangles in $\mathbb{R}^2$ (in fact it is reducible to the decidable first order theory of $\langle \mathbb{R}, < \rangle$), while in Section 5 we show that $\mathcal{L}_{\mathsf{RCC8}}$ is not even r.e. over that space. Thus, the proper first-order language to compare $\mathcal{L}_{\mathsf{RCC8}}$ with is the monadic extension $\mathcal{FO}^m_{\mathsf{RCC8}}$ of $\mathcal{FO}_{\mathsf{RCC8}}$ that is obtained by adding unary predicates $p_1, p_2, \ldots$. By well-known results from modal correspondence theory, any modal formula φ can be polynomially translated into an equivalent formula φ^* of $\mathcal{FO}^m_{\mathsf{RCC8}}$ with only two variables such that, for any region model $\mathfrak{M}$ and any region s, $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M} \models \varphi^*[s]$. More surprisingly, the converse holds as well: this follows from recent results of [20] since the RCC8 relations are mutually exclusive and jointly exhaustive.

Theorem 1. *For every $\mathcal{FO}^m_{\mathsf{RCC8}}$-formula $\varphi(x)$ with free variable x that uses only two variables, one can effectively construct a $\mathcal{L}_{\mathsf{RCC8}}$-formula φ^* of length at most exponential in the length of $\varphi(x)$ such that, for every region model $\mathfrak{M}$ and any region s, $\mathfrak{M}, s \models \varphi^*$ iff $\mathfrak{M} \models \varphi^*[s]$.*

A proof sketch can be found in [21]. There, we also argue that, due to a result of Etessami, Vardi, and Wilke [11], there exist properties that can be formulated exponentially more succinct in the two-variable fragment of $\mathcal{FO}^m_{\mathsf{RCC8}}$ than in $\mathcal{L}_{\mathsf{RCC8}}$.

4 Logics

In this section, we analyze the impact of choosing different underlying classes of region structures. As discussed in Section 2, the most important such classes are induced by topological spaces.

$$
\begin{array}{ccccc}
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R},\mathbb{R}_{\mathsf{rect}}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R},\mathbb{R}_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^2,\mathbb{R}^2_{\mathsf{rect}}) & \supset & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^2,\mathbb{R}^2_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^3,\mathbb{R}^3_{\mathsf{rect}}) & \supset & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^3,\mathbb{R}^3_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n,\mathbb{R}^n_{\mathsf{reg}}) \\
\cup & & \cup & & \cup \\
L_{\mathsf{RCC8}}(\mathcal{RS}) & = & L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP}) & = & L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n,\mathbb{R}^n_{\mathsf{reg}}) \\
\cap & & & & \\
L_{\mathsf{RCC8}}(\mathcal{TOP}) & & & & \\
\cap & & & & \\
L_{\mathsf{RCC8}}(\mathbb{R}^n,\mathbb{R}^n_{\mathsf{reg}}) & & & &
\end{array}
$$

Figure 3: Inclusions between logics.

A formula φ is *valid* in a class of regions structures $\mathcal{S}$ if it is true in all points of all models based on region structures from $\mathcal{S}$. We use $L_{\mathsf{RCC8}}(\mathcal{S})$ to denote the logic of the class $\mathcal{S}$, i.e. the set of all $\mathcal{L}_{\mathsf{RCC8}}$-formulas valid in $\mathcal{S}$. If $\mathcal{S} = \{\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})\}$ for some topological space $\mathfrak{T}$ with regions $U_{\mathfrak{T}}$, then we write $L_{\mathsf{RCC8}}(\mathfrak{T}, U_{\mathfrak{T}})$ instead of $L_{\mathsf{RCC8}}(\mathcal{S})$.

The basic logic we consider is $L_{\mathsf{RCC8}}(\mathcal{RS})$, the logic of all region structures. On arbitrary topological spaces, we investigate $L_{\mathsf{RCC8}}(\mathcal{TOP})$, the logic of all region structures induced by topological spaces in which regions are non-empty regular closed sets. On $\mathbb{R}^n$, $n \geq 1$, we investigate the family of logics $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$, where $\mathbb{R}^n_{\mathsf{reg}} \supseteq U_n \supseteq \mathbb{R}^n_{\mathsf{rect}}$. In particular, we may have $U_n = \mathbb{R}^n_{\mathsf{conv}}$.

In many applications, it does not seem natural to enforce the presence of *all* regions with some characteristics (say, non-empty and regular closed) in every model. Instead, one could include only those regions that are "relevant" for the application. Thus, given a class $\mathcal{S}$ of region structures, we are interested in the classes $\mathsf{S}(\mathcal{S})$ of all substructures of structures in $\mathcal{S}$. Then we write $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{S})$ as abbreviation of $L_{\mathsf{RCC8}}(\mathsf{S}(\mathcal{S}))$. Going one step further, one could even postulate that the set of relevant regions is finite (but unbounded). Thus we use $\mathsf{S}_{\mathsf{fin}}(\mathcal{S})$ to denote all finite substructures of structures in $\mathcal{S}$ and write $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{S})$ for $L_{\mathsf{RCC8}}(\mathsf{S}_{\mathsf{fin}}(\mathcal{S}))$.

It is natural to ask for the relationship between the logics just introduced. We start with two examples: first, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$ (and any other logic of spaces with finitely many regions) differs from $L_{\mathsf{RCC8}}(\mathcal{RS})$, $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and the logics $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ since

$$[\mathsf{pp}]([\mathsf{pp}]p \to p) \to [\mathsf{pp}]p$$

is valid in $\mathcal{S}_{\mathsf{fin}}(\mathcal{RS})$ (it states that there does not exist an infinite ascending pp-chain). Second, the logic $L_{\mathsf{RCC8}}(\mathcal{RS})$ differs from $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and the logics $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ since

$$\Diamond_u(p \land \langle \mathsf{dc} \rangle q) \to \Diamond_u(\langle \mathsf{ppi} \rangle p \land \langle \mathsf{ppi} \rangle q)$$

is not valid in $\mathcal{RS}$ (it states that any two disconnected regions are proper parts of a region).

These and some more relationships are summarized in Figure 3. Perhaps most interesting is the fact that $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$ and $L_{\mathsf{RCC8}}(\mathcal{RS})$ can be regarded as logics of topological spaces, and even of $\mathbb{R}^n$:

Theorem 2. *For $n > 0$:*
(i) $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$
(ii) $L_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.

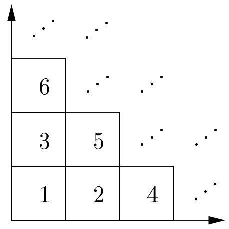

Figure 4: Enumerating tile positions.

A proof of the theorem and a justification of the inclusions in Figure 3 can be found in [21].

5 Undecidability

In this section, we prove the undecidability of the logics introduced previously. The results are summarized by the following theorem. We start with establishing a quite general undecidability result.

Theorem 3. *Let* $\mathfrak{R}(\mathbb{R}^n, U) \in \mathcal{S} \subseteq \mathcal{RS}$ *with* $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$*, for some* $n > 0$*. Then* $L_{\mathsf{RCC8}}(\mathcal{S})$ *is undecidable. Thus the logics* $L_{\mathsf{RCC8}}(\mathcal{S})$ *and* $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{S})$ *are undecidable, for* $\mathcal{S}$ *one of* $\mathcal{RS}$, $\mathcal{TOP}$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$, *and* $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$*, with* $n > 0$*.*

The proof is by reduction of the domino problem that requires tiling of the first quadrant of the plane to the satisfiability problem.

Definition 4. Let $\mathcal{D} = (T, H, V)$ be a *domino system*, where T is a finite set of *tile types* and $H, V \subseteq T \times T$ represent the horizontal and vertical matching conditions. We say that $\mathcal{D}$ *tiles the first quadrant of the plane* iff there exists a mapping $\tau : \mathbb{N}^2 \to T$ such that, for all $(x, y) \in \mathbb{N}^2$:

- if $\tau(x, y) = t$ and $\tau(x + 1, y) = t'$, then $(t, t') \in H$
- if $\tau(x, y) = t$ and $\tau(x, y + 1) = t'$, then $(t, t') \in V$

Such a mapping τ is called a *solution* for $\mathcal{D}$. ◇

For reducing this domino problem to satisfiability in L_{RCC8} logics, we fix an enumeration of all the tile positions in the first quadrant of the plane as indicated in Figure 4. The function λ takes positive integers to $\mathbb{N} \times \mathbb{N}$-positions, i.e. $\lambda(1) = (0, 0)$, $\lambda(2) = (1, 0)$, $\lambda(3) = (1, 1)$, etc.

Our proof strategy is inspired by [23; 29]. Let $\mathcal{D} = (T, H, V)$ be a domino system. In the reduction, we use the following propositional letters:

- for each tile type $t \in T$, a letter p_t;
- propositional letters a, b, and c that are used to mark certain, important regions;
- propositional letters wall and floor that are used to identify regions corresponding to tiles with positions from the sets $\{0\} \times \mathbb{N}$ and $\mathbb{N} \times \{0\}$, respectively.

The reduction formula $\varphi_{\mathcal{D}}$ is defined as

$$a \wedge b \wedge \mathsf{wall} \wedge \mathsf{floor} \wedge [\mathsf{ntppi}]\neg a \wedge \Box_u \chi,$$

where χ is the conjunction of a number of formulas. We list these formulas together with some intuitive explanations:

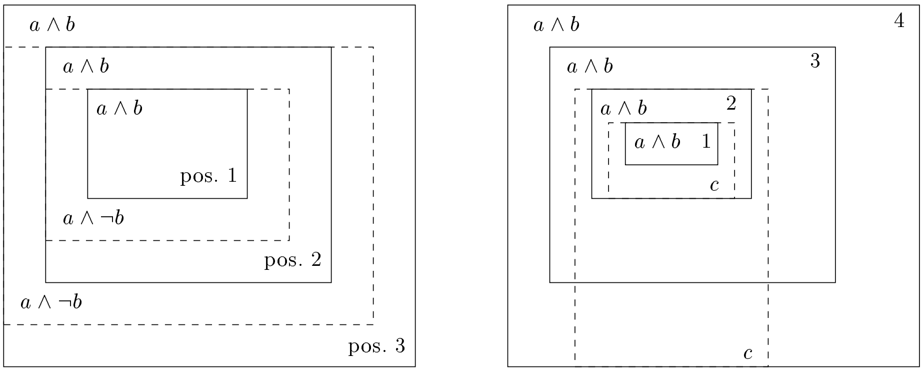

Figure 5: Left: a discrete ordering in the plane Right: the "going right" regions.

1. ensure that the regions $\{s \in W \mid \mathfrak{M}, s \models a\}$ are ordered by the relation pp (i.e. the union of tpp and ntpp):
$$a \rightarrow [\mathsf{dc}]\neg a \wedge [\mathsf{ec}]\neg a \wedge [\mathsf{po}]\neg a \tag{1}$$

2. enforce that the regions $\{s \mid \mathcal{M}, s \models a \wedge b\}$ are *discretely* ordered by ntpp. These regions will correspond to positions of the grid. In order to ensure discreteness, we use sequence of alternating $a \wedge b$ and $a \wedge \neg b$ regions as shown in the left part of Figure 5.
$$a \wedge b \rightarrow \langle \mathsf{tpp} \rangle (a \wedge \neg b) \tag{2}$$
$$a \wedge \neg b \rightarrow \langle \mathsf{tpp} \rangle (a \wedge b) \tag{3}$$
$$a \wedge \neg b \rightarrow [\mathsf{tpp}](a \rightarrow b) \tag{4}$$
$$a \wedge b \rightarrow [\mathsf{tpp}](a \rightarrow \neg b) \tag{5}$$
If we are at an $a \wedge b$ region, we can access the region corresponding to the next grid position (w.r.t. the fixed ordering) and to the previous grid position using
$$\begin{aligned} \Diamond^+(\varphi) &= \langle \mathsf{tpp} \rangle (a \wedge \neg b \wedge \langle \mathsf{tpp} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^-(\varphi) &= \langle \mathsf{tppi} \rangle (a \wedge \neg b \wedge \langle \mathsf{tppi} \rangle (a \wedge b \wedge \varphi)). \end{aligned}$$

3. we need a way to "go right" in the grid. To this end, we introduce additional regions satisfying c as displayed in the right part of Figure 5. For example, Grid cell 2 in the figure is right of Grid cell 1, and Grid cell 4 is right of Grid cell 2.
$$a \wedge b \rightarrow \langle \mathsf{tpp} \rangle c \tag{6}$$
$$c \rightarrow \langle \mathsf{tpp} \rangle (a \wedge b) \tag{7}$$
$$c \rightarrow [\mathsf{dc}]\neg c \wedge [\mathsf{ec}]\neg c \wedge [\mathsf{po}]\neg c \wedge [\mathsf{tpp}]\neg c \wedge [\mathsf{tppi}]\neg c \tag{8}$$
We can go to the right and upper element with
$$\begin{aligned} \Diamond^R(\varphi) &= \langle \mathsf{tpp} \rangle (c \wedge \langle \mathsf{tpp} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^U(\varphi) &= \Diamond^R(\Diamond^+(\varphi)). \end{aligned}$$
Similarly, we can go to the left and down:
$$\begin{aligned} \Diamond^L(\varphi) &= \langle \mathsf{tppi} \rangle (c \wedge \langle \mathsf{tppi} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^D(\varphi) &= \Diamond^L(\Diamond^-(\varphi)). \end{aligned}$$

Considering Formulas (6) to (8), it can be checked that going to the right is a monotone and injective total function (see [21]).

4. axiomatize the behavior of tiles on the floor and on the wall to enforce that our "going to the right" relation brings us to the expected position:

$$[\mathsf{ntppi}]\neg a \vee (\neg(\mathsf{floor} \wedge \mathsf{wall})) \tag{9}$$
$$\mathsf{wall} \rightarrow \Diamond^{+}\mathsf{floor} \tag{10}$$
$$\mathsf{wall} \rightarrow \Diamond^{U}(\mathsf{wall}) \tag{11}$$
$$[\mathsf{ntppi}]\neg a \vee (\mathsf{wall} \rightarrow \Diamond^{D}(\mathsf{wall})) \tag{12}$$
$$\Diamond^{R}(\neg\mathsf{wall}) \tag{13}$$
$$\neg\mathsf{wall} \rightarrow \Diamond^{L}\top \tag{14}$$

5. finally, we enforce the tiling:

$$\bigwedge_{t,t \in T} \neg(p_t \wedge p_{t'}) \tag{15}$$
$$\bigvee_{(t,t') \in H} p_t \wedge \Diamond^{R} p_{t'} \tag{16}$$
$$\bigvee_{(t,t') \in V} p_t \wedge \Diamond^{U} p_{t'} \tag{17}$$

The main strength of our reduction is that it requires only very limited prerequisites. Indeed, we will show that satisfiability of $\varphi_{\mathcal{D}}$ in *any* region model implies that $\mathcal{D}$ has a solution. Thus, to prove undecidability of some logic $L_{\mathsf{RCC8}}(\mathcal{S})$, it suffices to show that $\varphi_{\mathcal{D}}$ is satisfiable in $\mathcal{S}$ if $\mathcal{D}$ has a solution. This can be done for each region space $\mathfrak{R}(\mathbb{R}^n, U)$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$ and $n > 0$:

Lemma 5. *Let $\mathcal{D}$ be a domino system. Then:*

(i) if the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model, then the domino system $\mathcal{D}$ has a solution;

(ii) if the domino system $\mathcal{D}$ has a solution, then the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n > 0$ and each U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$.

Theorem 3 is an immediate consequence of this lemma, which is proved in [21].

Although Theorem 3 covers the case of substructure logics, it does not cover the case of finite region spaces, including finite substructures. By using a different variant of the domino problem, however, we can also establish a quite general undecidability result for this case.

Theorem 6. *If $\mathsf{S}_{\mathsf{fin}}(\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})) \subseteq \mathcal{S} \subseteq \mathsf{S}_{\mathsf{fin}}(\mathcal{RS})$ for some $n \geq 1$, then $L_{\mathsf{RCC8}}(\mathcal{S})$ is undecidable. Thus, the following logics are undecidable for each $n \geq 1$:*
$L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$, and $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$.

The employed variant of the domino problem is as follows: for $k \in \mathbb{N}$, the *k-triangle* is the set $\{(i, j) \mid i + j \leq k\} \subseteq \mathbb{N}^2$. The task of the new domino problem is to tile an arbitrary k-triangle, $k \in \mathbb{N}$, such that the position $(0, 0)$ is occupied with a distinguished tile s_0 and some position is occupied with a distinguished tile f_0. A proof that this domino problem is undecidable and the details of the proof of Theorem 6 can be found in [21].

6 Axiomatizability

In this section, we show that many of the introduced logics are Π^1_1-hard, thus *highly* undecidable and not even recursively enumerable. We start with some easy "positive" results and then prove a general "negative" result. First, we remind the reader of the following consequence of the translation of $\mathcal{L}_{\mathsf{RCC8}}$ into $\mathcal{FO}^m_{\mathsf{RCC8}}$:

Proposition 7. *If a class $\mathcal{S}$ of region structures is characterized by a finite set of axioms from $\mathcal{FO}_{\mathsf{RCC8}}$, then $L_{\mathsf{RCC8}}(\mathcal{S})$ is recursively axiomatizable.*

Recall that $\mathcal{RS}$ was defined by first-order axioms. Hence, $L_{\mathsf{RCC8}}(\mathcal{RS})$ and any $L_{\mathsf{RCC8}}(\mathcal{S})$ with $\mathcal{S}$ a first-order definable subclass of $\mathcal{S}$ are recursively enumerable. Actually, using general results on modal logics with names [14] and the fact that $\mathcal{RS}$ is axiomatized by universal first-order sentences, it is not difficult to provide a finitary axiomatization of $L_{\mathsf{RCC8}}(\mathcal{RS})$ using non-standard rules. By Theorem 2, we obtain axiomatizations for $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP})$ and every $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $n > 0$.

We now establish a non-axiomatizability result that applies to many logics $L_{\mathsf{RCC8}}(\mathcal{S})$ whose class of region spaces $\mathcal{S}$ is induced by a class of topological spaces:

Theorem 8. *The following logics are Π^1_1-hard: $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ with $U_n \in \{\mathbb{R}^n_{\mathsf{reg}}, \mathbb{R}^n_{\mathsf{conv}}, \mathbb{R}^n_{\mathsf{rect}}\}$ and $n \geq 1$.*

To prove this result, the domino problem of Definition 4 is modified by requiring that, in solutions, a distinguished tile $t_0 \in T$ occurs infinitely often in the first column of the grid. It has been shown in [17] that this variant of the domino problem is Σ^1_1-hard. Since we reduce it to satisfiability, this yields a Π^1_1-hardness bound for validity.

As a first step toward reducing this stronger variant of the domino problem, we extend $\varphi_{\mathcal{D}}$ with the following conjunct:

$$\Box_u \Big(\langle \mathsf{ntpp} \rangle (a \wedge b \wedge \mathsf{wall} \wedge p_{t_0}) \wedge [\mathsf{ntpp}] \big((a \wedge b \wedge \mathsf{wall} \wedge p_{t_0}) \rightarrow \langle \mathsf{ntpp} \rangle (a \wedge b \wedge \mathsf{wall} \wedge p_{t_0}) \big) \Big) \quad (18)$$

However, this is not yet sufficient: in models of $\varphi_{\mathcal{D}}$, we can have not only one discrete ordering of $a \wedge b$ regions, but rather many "stacked" such orderings. Due to this effect, the above formula does not enforce that the main ordering (there is only one for which we can ensure a proper "going to the right relation") has infinitely many occurrences of t_0.

It is thus obvious that we have to prevent stacked orderings. This is done by enforcing that there is only one "limit region", i.e. only one region approached by an infinite sequence of a-regions in the limit. We add the following formula to $\varphi_{\mathcal{D}}$:

$$\Box_u \big([\mathsf{tppi}] \langle \mathsf{po} \rangle a \rightarrow (\neg a \wedge [\mathsf{tpp}] \neg a \wedge [\mathsf{ntpp}] \neg a) \big) \quad (19)$$

Let $\varphi'_{\mathcal{D}}$ be the resulting extension of $\varphi_{\mathcal{D}}$. The classes of region spaces to which the extended reduction applies is more restricted than for the original one. We adopt the following property:

Definition 9 (Closed under infinite unions). Suppose that $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots \rangle$ is a region space. Then $\mathfrak{R}$ is called *closed under infinite unions* if $\mathfrak{R} = \mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})$ is a region space induced by a topological space $\mathfrak{T}$, and, additionally, $\mathfrak{R}$ satisfies the following property: for any sequence $r_1, r_2, \ldots \in W$ such that $r_1 \ \mathsf{ntpp}\ r_2 \ \mathsf{ntpp}\ r_3 \ \cdots$, we have $\mathbb{CI}(\bigcup_{i \in \omega} r_i)) \in W$. ◇

We can now formulate the first part of correctness for the extended reduction. The proofs of this and the following lemma can be found in [21].

Lemma 10. *Let $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}}) = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots \rangle$ be a region space that is closed under infinite unions such that all regions in $U_{\mathfrak{T}}$ are regular closed. Then the formula $\varphi'_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}$ only if the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall.*

For the second part of correctness, we again consider region spaces $\mathfrak{R}(\mathbb{R}^n, U)$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$.

Lemma 11. *If the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall, then the formula $\varphi'_\mathcal{D}$ is satisfiable in a region model based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n \geq 1$ and each U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U \subseteq \mathbb{R}^n_{\mathsf{reg}}$.*

Note that the region spaces $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$ and $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$ are closed under infinite unions. Since $\mathbb{R}^n_{\mathsf{rect}} \subseteq \mathbb{R}^n_{\mathsf{conv}} \subseteq \mathbb{R}^n_{\mathsf{reg}}$, Lemmas 10 and 11 immediately yield Theorem 8.

It is worth noting that there are a number of interesting region spaces to which this proof method does not apply. Interesting examples are the region space based on simply connected regions in $\mathbb{R}^2$ [30] and the space of polygons in $\mathbb{R}^2$ [26]. Since these spaces are not closed under infinite unions, the above proof does not show the non-axiomatizability of the induced logics. We conjecture, however, that slight modifications of the proof introduced here can be used to prove their Π^1_1-hardness as well.

Finally, we consider the recursive enumerability of logics of finite region spaces: obviously, undecidability of a logic of finite region spaces implies that it is not recursively enumerable if it is based on a class of region structures $\mathsf{S}_{\mathsf{fin}}(\mathcal{S})$ with $\mathcal{S}$ first-order definable (since we can enumerate all finite models). Thus, Theorems 6 and Theorem 2 give us the following:

Corollary 12. *The following logics are not r.e., for each $n \geq 1$: $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.*

7 The RCC5 set of Relations

For several applications, the RCC8 relations are weakened into a set of only 5 relations called RCC5 (or medium resolution topological relations) [15; 8]. This is done by keeping the relation eq and po but coarsening (1) the tpp and ntpp relations into a new "proper-part of" relation pp; (2) the tppi and ntppi relations into a new "has proper-part" relation ppi; and (3) the dc and ec relations into a new disjointness relation dr. The modal language $\mathcal{L}_{\mathsf{RCC5}}$ for reasoning about RCC8-style region structures $\mathfrak{R} = \langle W, \mathsf{ec}^\mathfrak{R}, \ldots \rangle$ thus extends propositional logic with the operators [r], where r ranges over the five RCC5-relations. They are interpreted by the relations $\mathsf{eq}^\mathfrak{R}$, $\mathsf{po}^\mathfrak{R}$, and

- $\mathsf{dr}^\mathfrak{R} = \mathsf{dc}^\mathfrak{R} \cup \mathsf{ec}^\mathfrak{R}$;
- $\mathsf{pp}^\mathfrak{R} = \mathsf{tpp}^\mathfrak{R} \cup \mathsf{nttp}^\mathfrak{R}$;
- $\mathsf{ppi}^\mathfrak{R} = \mathsf{tppi}^\mathfrak{R} \cup \mathsf{nttpi}^\mathfrak{R}$.

Given a class $\mathcal{S}$ of region structures, we denote by $L_{\mathsf{RCC5}}(\mathcal{S})$ the set of $\mathcal{L}_{\mathsf{RCC5}}$-formulas which are valid in all members of $\mathcal{S}$. The sets $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{S})$ and $L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{S})$ are defined analogously to the RCC8 case.

A number of results from our investigation of $\mathcal{L}_{\mathsf{RCC8}}$ have obvious analogues for $\mathcal{L}_{\mathsf{RCC5}}$: First, we can characterize the logics $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{TOP})$ by means of a composition table: denote by $\mathcal{RS}^5$ the class of all structures $\mathfrak{R} = \langle W, \mathsf{dr}^\mathfrak{R}, \mathsf{eq}^\mathfrak{R}, \mathsf{pp}^\mathfrak{R}, \mathsf{ppi}^\mathfrak{R}, \mathsf{po}^\mathfrak{R} \rangle$, where W is non-empty and the $\mathsf{r}^\mathfrak{R}$ are mutually exclusive and jointly exhaustive binary relations on W such that (1) eq is interpreted as the identity relation on W, (2) po and dr are symmetric, (3) pp is the inverse of ppi and (4) the rules of the RCC5-composition table (Figure 6) are valid. Second, it is possible to prove an analogue of Theorem 2, i.e. that, for $n \geq 1$, we have

$$\text{(i) } L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{RS}^5) = L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{TOP}) = L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$$
$$\text{(ii) } L_{\mathsf{RCC5}}(\mathcal{RS}^5) = L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP}) = L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}).$$

∘	dr	po	pp	ppi
dr	$*$	dr,po,pp	dr,po,pp	dr
po	dr,po,ppi,	$*$	po,pp	dr,po,ppi
pp	dr	dr,po,pp	pp	$*$
ppi	dr,po,ppi,	po,pp	eq,po,pp,ppi	ppi

Figure 6: The RCC5 composition table.

Third, on region models, $\mathcal{L}_{\mathsf{RCC5}}$ has the same expressive power as the two-variable fragment of $\mathcal{FL}^m_{\mathsf{RCC5}}$, i.e. the first-order language with the five binary RCC5-relation symbols and infinitely many unary predicates.

We now investigate the computational properties of logics based on $\mathcal{L}_{\mathsf{RCC5}}$. Analogously to the RCC8 case, the most natural logics are undecidable. Still, our RCC5 undecidability result is less powerful than the one for RCC8. More precisely, we have to restrict ourselves to region structures with certain properties: denote by $\mathcal{RS}^{\exists}$ the class of all region structures $\mathfrak{R} = \langle W, \mathsf{ec}^{\mathfrak{R}}, \dots \rangle$ such that, for any set $S \subseteq W$ of cardinality two or three, there exists a unique region $\mathsf{Sup}(S)$ such that

- $s \;\mathsf{eq}\; \mathsf{Sup}(S)$ or $s \;\mathsf{pp}\; \mathsf{Sup}(S)$ for each $s \in S$;
- for every region $t \in W$ with $s \;\mathsf{pp}\; t$ for each $s \in S$, we have $\mathsf{Sup}(S) \;\mathsf{eq}\; t$ or $\mathsf{Sup}(S) \;\mathsf{pp}\; t$,
- for every region $t \in W$ with $t \;\mathsf{dr}\; s$ for each $s \in S$, we have $t \;\mathsf{dr}\; \mathsf{Sup}(S)$.

It is easy to verify that $\mathcal{TOP} \subseteq \mathcal{RS}^{\exists}$ and $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{RS}^{\exists}$ for each $n > 0$.

Theorem 13. *Suppose $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{S} \subseteq \mathcal{RS}^{\exists}$, for some $n \geq 1$. Then $L_{\mathsf{RCC5}}(\mathcal{S})$ is undecidable. Thus, the following logics are undecidable, for each $n \geq 1$: $L_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.*

The proof is by reduction of the satisfiability problem for the undecidable modal logic $\mathsf{S5}^3$ (see [22] for the original proof in an algebraic setting. We use the modal notation of [13]). Due to space limitations, we refer the reader to [13] or to [21] for a formal definition of $\mathsf{S5}^3$, and just recall here that the domain of $\mathsf{S5}^3$ is a product $W_1 \times W_2 \times W_3$, and that there are three modal operators for referring to triples that are identical to the current one, but for one component.

With every $\mathsf{S5}^3$-formula φ, we associate a $\mathcal{L}_{\mathsf{RCC5}}$-formula

$$\Box_u \chi \wedge d \wedge \varphi^{\sharp} \qquad (*)$$

such that φ is $\mathsf{S5}^3$-satisfiable iff $\Box_u \chi \wedge d \wedge \varphi^{\sharp}$ is satisfiable in a model from $\mathcal{S}$. In $(*)$, χ is the conjunction of the following formulas:

1. Each sets W_i of $\mathsf{S5}^3$-models is simulated by the set $\{r \in W \mid \mathfrak{M}, r \models a_i\}$. Thus, we introduce fresh variables a_i, $i = 1, 2, 3$, and state

$$a_i \to \bigwedge_{j=1,2,3} ([\mathsf{pp}]\neg a_j \wedge [\mathsf{ppi}]\neg a_j \wedge [\mathsf{po}]\neg a_j) \qquad (20)$$

$$a_1 \to \neg a_2, a_2 \to \neg a_3, a_2 \to \neg a_3, \qquad (21)$$

$$\bigwedge_{i=1,2,3} \Diamond_u a_i \qquad (22)$$

2. the set $W_1 \times W_2 \times W_3$ is simulated by a fresh variable d, so we add

$$d \leftrightarrow (\bigwedge_{i=1,2,3} \langle \mathsf{ppi} \rangle a_i) \wedge \neg \langle \mathsf{ppi} \rangle (\bigwedge_{i=1,2,3} \langle \mathsf{ppi} \rangle a_i) \tag{23}$$

3. the sets $W_i \times W_j$, $1 \leq i < j \leq 3$ are simulated by fresh variables d_{ij}, so we add

$$d_{ij} \leftrightarrow (\bigwedge_{k=i,j} \langle \mathsf{ppi} \rangle a_k) \wedge \neg \langle \mathsf{ppi} \rangle (\bigwedge_{k=i,j} \langle \mathsf{ppi} \rangle a_k). \tag{24}$$

Now, we define $\varphi^\sharp$ inductively by

$$\begin{array}{rcl} p_i^\sharp & := & p_i \\ (\neg\varphi)^\sharp & := & d \wedge \neg\varphi^\sharp \\ (\varphi \wedge \psi)^\sharp & := & \varphi^\sharp \wedge \psi^\sharp \\ (\Diamond_1 \varphi)^\sharp & := & \langle \mathsf{ppi} \rangle (d_{23} \wedge \langle \mathsf{pp} \rangle (d \wedge \varphi^\sharp)) \\ (\Diamond_2 \varphi)^\sharp & := & \langle \mathsf{ppi} \rangle (d_{13} \wedge \langle \mathsf{pp} \rangle (d \wedge \varphi^\sharp)) \\ (\Diamond^3 \varphi)^\sharp & := & \langle \mathsf{ppi} \rangle (d_{12} \wedge \langle \mathsf{pp} \rangle (d \wedge \varphi^\sharp)) \end{array}$$

The following Lemma immediately yields Theorem 13 and is proved in [21].

Lemma 14. *Suppose $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{S} \subseteq \mathcal{RS}^\exists$, for some $n \geq 1$. Then an $\mathsf{S5}^3$-formula φ is satisfiable in an $\mathsf{S5}^3$-model iff $\Box_u \chi \wedge d \wedge \varphi^\sharp$ is satisfiable in $\mathcal{S}$.*

8 Conclusion

Several open questions for future research remain. The main challenge is to exhibit a decidable and still useful variant of the logics proposed in this paper. Perhaps the most interesting candidate is $L_{\mathsf{RCC5}}(\mathcal{RS})$, which coincides with the substructure logics $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, and to which the proof of Theorem 13 does not apply. Other candidates could be obtained by modifying the set of relations, e.g. giving up some of them. It has for example been argued that dropping po still results in a useful formalism for geographic applications. Finally, it as an open problem whether $L_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ are recursively enumerable.

Let us also relate (some special cases of) our results to Halpern and Shoham's results on interval temporal logic [16]: Theorems 3, 8, and 6 apply to logics induced by the region space $\mathfrak{R}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, which is clearly an interval structure. Interestingly, on this interval structure our results are stronger than those of Halpern and Shoham in two respects: first, we only need the RCC8 relations, which can be viewed as a "coarsening" of the Allen interval relations used by Halpern and Shoham. Second and more interestingly, by Theorem 3 we have also proved undecidability of the *substructure logic* $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, which is a natural but much weaker variant of the full (interval temporal) logic $L_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, and not captured by Halpern and Shoham's undecidability proof.

References

[1] M. Aiello and J. van Benthem. A modal walk through space. *Journal of Applied Non-Classical logic*, pages 319 – 363, 2002.

[2] J. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11), 1983.

[3] J. F. Allen. Towards a general theory of action and time. *Artificial Intelligence*, 23(2):123–154, 1984.

[4] A. Artale and E. Franconi. A temporal description logic for reasoning about actions and plans. *Journal of Artificial Intelligence Research (JAIR)*, 9:463–506, 1998.

[5] B. Bennett. Modal logics for qualitative spatial reasoning. *Bulletin of the Interest Group in Pure and Applied Logic (IGPL)*, 4(1):23–45, 1996.

[6] B. Clarke. Individuals and points. *Notre Dame Journal of Formal Logic*, 26:61–75, 1985.

[7] A. G. Cohn. Modal and non modal qualitative spatial logics. In F. D. Anger, H. M. Guesgen, and J. van Benthem, editors, *Proc. of the IJCAI-93 Workshop on Spatial and Temporal Reasoning*, 1993.

[8] I. Düntsch, H. Wang, and S. McCloskey. A relation algebraic approach to the Region Connection Calculus. *Theoretical Computer Science*, 255:63–83, 2001.

[9] M. Egenhofer. Deriving the composition of binary topological relations. *Journal of Visual Languages and Computing*, 5 (2):133–149, 1994.

[10] M. Egenhofer and R. Franzosa. Point-set topological spatial relations. *International Journal of Geographical Information Systems*, 5 (2):161–174, 1991.

[11] K. Etessami, M. Y. Vardi, and Th. Wilke. First-order logic with two variables and unary temporal logic. *Inform. and Computation*, 179(2):279–295, 2002.

[12] D. M. Gabbay, I. M. Hodkinson, and M. A. Reynolds, editors. *Temporal Logic: Mathematical Foundations and Computational Aspects, Volume 1.* Oxford University Press, Logic Guides 28, 1994.

[13] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional Modal Logics: Theory and Applications.* Elsevier, 2003.

[14] G. Gargov and V. Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22:607–636, 1993.

[15] M. Grigni, D. Papadias, and C. Papadimitriou. Topological inference. In *Proc. of the 14th Int. Joint Conf. on AI (IJCAI)*, pages 901–906. Morgan Kaufmann, 1995.

[16] J. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, 1991.

[17] D. Harel. Recurring dominoes: Making the highly undecidable highly understandable. *Annals of Discrete Mathematics*, 24:51–72, 1985.

[18] J. M. Kinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, pages 141–191, 1944.

[19] C. Lutz. Combining interval-based temporal reasoning with general tboxes. *Artificial Intelligence*, 2003. to appear.

[20] C. Lutz, U. Sattler, and F. Wolter. Modal logic and the two-variable fragment. In *Computer Science Logic*, number 2142 in LNCS, pages 247–261. Springer-Verlag, 2001.

[21] C. Lutz and F. Wolter. Modal logics of topological relations. LTCS-Report 04-05, Technical University Dresden, 2004. Available from http://lat.inf.tu-dresden.de/research/reports.html.

[22] R. Maddux. The equational theory of ca_3 is undecidable. *Journal of Symbolic Logic*, 45:311 – 315, 1980.

[23] M. Marx and M. Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9(6), 1999.

[24] B. Nebel and H. Bürckert. Reasoning about temporal relations: A maximal tractable subclass of allen's interval algebra. *Journal of the ACM*, 42(1):43–66, 1995.

[25] C. Papadimitriou, D. Suciu, and V. Vianu. Topological queries in spatial databases. *Journal of Computer and System Sciences*, 58:29–53, 1999.

[26] I. Pratt and D. Schoop. A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, 27:621–658, 1998.

[27] D. Randell, Z. Cui, and A. Cohn. A spatial logic based on regions and connection. In *Proc. 3rd Int. Conf. on KR&R*, pages 165–176, San Mateo, 1992. Morgan Kaufmann.

[28] J. Renz and B. Nebel. On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the region connection calculus. *Artificial Intelligence*, 108(1-2):69–123, 1999.

[29] Reynolds and Zakharyaschev. On the products of linear modal logics. *Journal of Logic and Computation*, 11:909–931, 2001.

[30] M. Schaefer and D. Stefankovic. Decidability of string graphs. In *Proc. of the thirty-third annual ACM symposium on Theory of Computing*, pages 241–246. ACM Press, 2001.

[31] J. G. Stell. Boolean connection algebras: A new approach to the region-connection calculus. *Artificial Intelligence*, 122:111–136, 2000.

[32] Y. Venema. Expressiveness and completeness of an interval tense logic. *Notre Dame Journal of Formal Logic*, 31(4):529–547, 1990.

[33] M. Wessel. Obstacles on the way to qualitative spatial reasoning with description logics: Some undecidability results. In *Proc. of the Int. Workshop in Description Logics 2001 (DL2001)*, number 49 in CEUR-WS (http://ceur-ws.org/), pages 96–105, 2001.