Quantitative Temporal Logics: PSPACE and below

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Abstract

Often, the addition of metric operators to qualitative temporal logics leads to an increase of the complexity of satisfiability by at least one exponential. In this paper, we exhibit a number of metric extensions of qualitative temporal logics of the real line that do not lead to an increase in computational complexity. We show that the language obtained by extending since/until logic of the real line with the operators 'sometime within n time units', n coded in binary, is PSPACE-complete even without the finite variability assumption. Without qualitative temporal operators the complexity of this language turns out to depend on whether binary or unary coding of parameters is assumed: it is still PSPACE-hard under binary coding but in NP under unary coding.

1 Introduction

The extension of qualitative temporal logics (TLs) with metric operators has been studied for almost fifteen years [5, 4, 8]. Of particular interest are metric extensions of temporal logics of the real line, since the resulting quantitative TLs are an important tool for the specification and verification of real-time systems [2]. Unfortunately, moving from qualitative to quantitative logics is often accompanied by an increase in computational complexity of the satisfiability problem. The most important example witnessing this effect is the PSPACE-complete since/until logic of the real line [7], whose extension with a metric operator 'sometime in at least n but not more than m time units' (n and m coded in binary) becomes EXPSPACE-complete if the case $n = m$ is not admitted, and even undecidable if it is [1, 3, 5].

It is well known that the complexity of the metric temporal logic obtained by this extension can be reduced to PSPACE again by further restricting the values of n and m, e.g., by enforcing that $n = 0$ [1]. However, in contrast to the EXPSPACE-completeness and undecidability results above,

this improvement has only been proven under the *finite variability assumption (FVA)* which states that no propositional variable changes its truth-value infinitely many times in any finite interval. While the FVA is a natural condition for various computer science applications, we believe that there are at least two reasons to consider also the non-FVA case: first, qualitative temporal logic originated in philosophy and mathematics to study time itself, rather than the behaviour of systems with discrete state changes as considered in computer science. If quantitative TL is used for the former purpose, the FVA is less convincing than in computer science applications. Second, even in computer science reasoning without the FVA can be fruitfully employed: assume that a formula φ of a quantitative TL describes the specification of a real-time system. Further assume that φ has been found to be unsatisfiable under FVA, indicating that the described specification is not realizable. If an additional satisfiability check *without* FVA is made revealing that dropping FVA regains satisfiability of φ, then the user obtains additional information on the source of the unrealizability of her specification: namely that it enforces an infinite number of state changes in a finite interval.

The purpose of this paper is to *investigate metric temporal logics of the real line that are at most* PSPACE-*complete*. More precisely, we prove three results. Our first result is that extending since/until logic of the real line with metric operators 'sometime in at most n time units', n coded in binary, is PSPACE-complete even without FVA. (Note that the logic without FVA is more general than with FVA in the sense that the latter logic can be polynomially reduced to the former [5].) To show this result, we propose a new method for polynomially reducing satisfiability in metric TLs whose numerical parameters are coded in binary to satisfiability in the same logic with numbers coded in unary. The essence of the reduction is to introduce new propositional variables that serve as the bits of a binary counter measuring distances. For the metric TL mentioned above, we obtain a PSpace upper bound since Hirshfeld and Rabinovich have shown that QTL, i.e., the same logic with num-

bers coded in unary, is PSPACE-complete without FVA [5]. In the technical report accompanying this paper, we show that our proof method can be used for other logics as well. For instance, metric extensions of (discrete-time) LTL and the branching time logic CTL are shown to be in PSPACE and EXPTIME, respectively [6].

Our second result concerns a sharpening of PSPACE lower bounds for metric temporal logics of the real line. In the current literature, such logics usually contain qualititative since/until logic as a proper fragment, and thus trivially inherit PSPACE-hardness [2, 5, 7]. We consider metric TLs with weaker qualitative operators and show that PSPACE-hardness can already be observed in the following three cases: (i) a future diamond and a future operator 'sometime in at most n time units', n coded in unary; (ii) *only* the future operator 'sometime in at most n time units', n coded in binary (i.e., no qualitative operators at all); (iii) only a metric version of the until operator for the interval $[0, 1]$.

As a third result, we explore the transition from NP to PSPACE. In particular, we show that the quantitative TL with only the metric operator 'sometime within n time units', n coded in unary, is NP-complete. This result extends the result of [10] that satisfiability of the qualitative TL with operators 'eventually in the future' and 'eventually in the past' over the real line is decidable in NP. When compared with result (ii) above, it also shows that the complexity of metric TLs without qualitative operators depends on the coding of numbers. To establish the NP upper bound, we show that satisfiability of a formula φ can be decided by first 'guessing' a system of rational linear inequalities, and checking whether this system has a solution over the real (or, equivalently, rational) numbers.

2 Preliminaries

We introduce the metric temporal language QTL of [5], which is closely related to the language MITL of [1]. Fix a countably infinite supply $p_0, p_1, \ldots$ of propositional variables. A QTL-formula is built according to the syntax rule

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi\,\mathcal{S}\,\psi \mid \varphi\,\mathcal{U}\,\psi \mid \varphi\mathcal{S}^I\psi \mid \varphi\mathcal{U}^I\psi$$

with p ranging over the propositional variables and I ranging over intervals of the forms $(0, n)$, $(0, n]$, $[0, n)$, and $[0, n]$, where $n > 0$ is a natural number. The Boolean operators $\top$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as abbreviations in the usual way. Moreover, we introduce additional future modalities as abbreviations: $\diamondsuit_F^I\varphi = \top\,\mathcal{U}^I\,\varphi$, $\square_F^I\varphi = \neg\diamondsuit_F^I\neg\varphi$, $\diamondsuit_F\varphi = \top\,\mathcal{U}\,\varphi$, and $\square_F\varphi = \neg\diamondsuit_F\neg\varphi$. Their past counterparts are defined analogously and have a subscript P.

Note that formulas with rational numbers as parameters of metric operators $\mathcal{S}^I$ and $\mathcal{U}^I$ can be translated (in polynomial time) to satisfiability equivalent formulas contain-

ing only the natural numbers by multiplying with the least common denominator of all (rational) parameters.

Formulas of QTL are interpreted on the real line. A *model* $\mathfrak{M} = \langle\mathbb{R}, \mathfrak{V}\rangle$ is a pair consisting of the real numbers and a valuation $\mathfrak{V}$ mapping every propositional variable p to a set $\mathfrak{V}(p) \subseteq \mathbb{R}$. The satisfaction relation '$\models$' is defined inductively as follows, where we write $w + I$ to denote the set $\{w + x \mid x \in I\}$ for each time point $w \in \mathbb{R}$ and interval I of one of the above forms; $w - I$ is defined analogously.

$\mathfrak{M}, w \models p$	iff	$w \in \mathfrak{V}(p)$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi\,\mathcal{U}\,\psi$	iff	there exists $u > w$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $w < v < u$
$\mathfrak{M}, w \models \varphi\,\mathcal{S}\,\psi$	iff	there exists $u < w$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $u < v < w$
$\mathfrak{M}, w \models \varphi\,\mathcal{U}^I\,\psi$	iff	there exists $u \in w + I$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $w < v < u$
$\mathfrak{M}, w \models \varphi\,\mathcal{S}^I\,\psi$	iff	there exists $u \in w - I$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $u < v < w$

We will also write $w \models_{\mathfrak{V}} \varphi$ for $\langle\mathbb{R}, \mathfrak{V}\rangle, w \models \varphi$. A QTL-formula φ is satisfiable if there exists a model $\mathfrak{M}$ and a $w \in \mathbb{R}$ such that $\mathfrak{M}, w \models \varphi$. It is satisfiable under the *finite variability assumption (FVA)* if it is satisfiable in a model in which no propositional variable changes its truth-value infinitely many times in any finite interval.

Our presentation of QTL deviates from that of [5], where only the metric operators $\diamondsuit_F^{(0,1)}$ and $\diamondsuit_P^{(0,1)}$ are admitted. If the numerical parameters of the metric operators are coded in unary, there exists an easy polynomial translation from Hirshfeld and Rabinovich's version of QTL to ours and vice versa. However, in this paper we also consider binary coding of numbers. If we want to emphasize this fact, we shall write QTL^b instead of QTL, and likewise, QTL^u will denote unary coding of numbers.

We close this section with a discussion of the relation between temporal logics with and without FVA. The satisfiability of a formula φ of a temporal logic with operators since and until and with FVA can be polynomially reduced to satisfiability in the same logic without FVA; see [5]: Let $\delta = \bigwedge\limits_{p \text{ used in } \varphi} (\square_P\delta_p \wedge \delta_p \wedge \square_F\delta_p)$ where

$$\delta_p = \big((p\,\mathcal{U}\,p) \vee (\neg p\,\mathcal{U}\,\neg p)\big)$$
$$\wedge\big((p\,\mathcal{S}\,p) \vee (\neg p\,\mathcal{S}\,\neg p)\big).$$

Then it is not hard to verify that φ is satisfiable with FVA iff $\varphi \wedge \delta$ is without FVA. Note that the length of $\varphi \wedge \delta$ is polynomial in the length of φ. To the best of our knowledge there is no polynomial reduction of this type for the language without since or until.

3 QTL^b is PSPACE-complete without FVA

The purpose of this section is to prove that QTL^b-satisfiability without FVA is decidable in PSPACE. Note that, due to polynomial reducibility of QTL^b with FVA to this logic, the upper complexity bound carries over. This result is already known for QTL^u without FVA [5] and QTL^u with FVA [1]. We first show that our result indeed improves upon the existing ones by proving that QTL^b is exponentially more succinct than QTL^u.

Theorem 1. *Let ψ be a QTL-formula with numbers coded in unary that is equivalent to $\square_F^{[0,n]} p$. Then ψ has length at least n.*

Proof. Suppose by contradiction that there exists a QTL-formula ψ with numbers coded in unary such that ψ is equivalent to $\square_F^{[0,n]} p$, for some $n \geq 1$, and the length of ψ is strictly smaller than n. We may assume that ψ contains no other propositional letters than p: otherwise, just replace them with $\top$. Then, for $n \geq 1$, set $\mathfrak{V}_n(p) := [-n, n]$ and $\mathfrak{M}_n := \langle \mathbb{R}, \mathfrak{V}_n \rangle$. Then $\mathfrak{M}_n, 0 \models \square_F^{[0,n]} p$. Therefore, $\mathfrak{M}_n, 0 \models \psi$. Now, it is straightforward to prove the following by induction: for every subformula χ of ψ of length $\leq k$ and all $x \geq k$ such that $n - x \geq -n + k$:

$$\mathfrak{M}, (n-k) \models \chi \text{ iff } \mathfrak{M}_n, (n-x) \models \chi.$$

Since the length of ψ is smaller than n, it follows that, in $\mathfrak{M}_n$, the points 0 and 1 satisfy the same subformulas of ψ. In particular, $\mathfrak{M}_n, 1 \models \psi$. We have derived a contradiction since $\mathfrak{M}_n, 1 \not\models \square_F^{[0,n]} p$. ❏

We now establish the first result of this paper.

Theorem 2. *Satisfiability in QTL with numbers coded in binary is PSPACE-complete without FVA.*

Since (qualitative) since/until logic on the real line is PSPACE-hard [7], it suffices to prove the upper bound. For simplicity, we prove the upper bound for the future fragment of QTL, i.e., we omit past operators. The proofs are easily extended to the general case. Within the future fragment, we consider only the metric operators $\Diamond_F^{(0,1)}$, $\Diamond_F^{(0,1]}$, $\Diamond_F^{[0,1)}$, and $\Diamond_F^{[0,n]}$. This can be done w.l.o.g. due to the following observations:

First, satisfiability in QTL^b can be reduced to satisfiability in QTL^b without the metric operators $\psi_1 \mathcal{U}^I \psi_2$: to decide satisfiability of a QTL^b-formula φ, introduce a new propositional variable p_{ψ_2} for every ψ_2 which occurs in a subformula of the form $\psi_1 \mathcal{U}^I \psi_2$ of φ. Inductively define a translation on QTL^b-formulas such that, for any subformula χ of φ, χ^p to denotes the result of replacing all outermost subformulas $\psi_1 \mathcal{U}^I \psi_2$ of χ by $\psi_1^p \mathcal{U} p_{\psi_2} \wedge \Diamond_F^I p_{\psi_2}$. Set $\square_F^+ \psi = \psi \wedge \square_F \psi$. Then φ is satisfiable iff

$$\varphi^p \wedge \square_F^+ \Big[\bigwedge_{\psi_1 \mathcal{U}^I \psi_2 \in sub(\varphi)} (p_{\psi_2} \leftrightarrow \psi_2^p) \Big]$$

is satisfiable. Note that the length of the latter formula is polynomial in the length of φ. Second, for any interval I of the form $(0, n)$, $(0, n]$, or $[0, n)$ with $n > 1$, $\Diamond_F^I \varphi$ is equivalent to $\Diamond_F^J \Diamond_F^{[0,n-1]} \varphi$, where J is obtained from I by replacing the upper interval bound n by 1.

In the following, we reduce satisfiability of QTL^b-formulas to the satisfiability of QTL^1-formula, i.e., QTL-formulas in which all upper interval bounds have value 1. As the coding of numbers is not an issue in the latter logic, we obtain a PSPACE upper bound from the result of [5] that QTL^u-satisfiability in models without FVA is decidable in PSPACE.

Let φ be a QTL-formula meeting the restrictions laid out above. Let k be the greatest number occurring as a parameter to a metric operator in φ, $n_c = \lceil \log_2(k + 2) \rceil$, and $\chi_1, \ldots, \chi_\ell$ the subformulas of φ that occur as an argument to a metric operator of the form $\Diamond_F^{[0,n]}$ with $n > 1$. We reserve, for $1 \leq i \leq \ell$, fresh propositional variables x_i, y_i, and $c_{n_c-1}^i, \ldots, c_0^i$ that do not occur in φ. The sequences $c_{n_c-1}^i, \ldots, c_0^i$ of propositional variables will be used to implement binary counters, one for each χ_i. Intuitively, these counters measure the distance to the "nearest" future occurrence of the formula χ_i, rounded up to the next largest natural number. A counter value greater than or equal to $k + 1$ is a special case indicating that the nearest occurrence is too far away to be of any relevance. The variables x_i and y_i will serve as markers with the following meaning: x_i holds in a point iff there is a natural number n such that χ_i holds at distance n, but not in between; similarly, y_i holds iff there is a natural number n such that χ_i does not hold at any distance up to (and including) n, but χ_i holds at future points that converge from the right to the future point with distance n. In the following, we call the structure imposed on the real line by the markers x_i and y_i the (one-dimensional) 'grid'.

To implement the counters, we introduce auxiliary formulas. For $1 \leq i \leq \ell$, let

- $(C_i = m)$ be a formula saying that, at the current point, the value of the i-th counter is m, for $0 \leq m < 2^{n_c}$. There are exponentially many such formulas, but we will use only polynomially many of them in the reduction.

- $(C_i \leq m)$ is a formula saying that, at the current point, the value of the i-th counter does not exceed m, for $0 \leq m < 2^{n_c}$.

- $\bigcirc\varphi := \neg(x_i \vee y_i)\,\mathcal{U}((x_i \vee y_i) \wedge \varphi)$ says that, at the next grid point, φ is satisfied.

To deal with effects of convergence, it is convenient to introduce an additional abbreviation. The formula $rc(\psi) := \neg(\neg\psi\,\mathcal{U}\,\top) \wedge \neg\psi$ describes convergence of ψ-points from the right against a point where ψ does not hold. We now inductively define a translation of QTL^b-formulas to QTL^1-formulas:

$$
\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(\psi_1\,\mathcal{U}\,\psi_2)^* &:= \psi_1^*\,\mathcal{U}\,\psi_2^* \\
(\Diamond_F^I\psi)^* &:= \Diamond_F^I\psi^* \\
(\Diamond_F^{[0,n]}\chi_i)^* &:= (C_i \leq n-1) \vee ((C_i = n) \wedge \neg y_i)
\end{aligned}
$$

Here, I ranges over intervals $(0,1]$, $(0,1)$, and $[0,1)$. It remains to enforce the existence of the grid and the behavior of the counters as described above. This is done with the following auxiliary formulas, for $1 \leq i \leq \ell$:

$$
\vartheta_1^i := (C_i = 0) \leftrightarrow \left(\chi_i^* \vee rc(\chi_i^*)\right)
$$

$$
\begin{aligned}
\vartheta_2^i := x_i \leftrightarrow \Big[&\chi_i^* \vee \big(\square_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \\
&\wedge \Diamond_F^{(0,1]}x_i \wedge \Diamond_F\chi_i^*\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_3^i := y_i \leftrightarrow \Big[&rc(\chi_i^*) \vee \big(\square_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \\
&\wedge \Diamond_F^{(0,1]}y_i \wedge \Diamond_F rc(\chi_i^*)\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_4^i := &\neg(C_i = 0) \wedge \Diamond_F^{(0,1]}(x_i \vee y_i) \to \\
&\Big(\bigvee_{t=0..n_c-1}\big(c_t^i \wedge \bigcirc\neg c_t^i \wedge \bigwedge_{\ell=0..t-1}(\neg c_\ell^i \wedge \bigcirc c_\ell^i) \\
&\wedge \bigwedge_{\ell=t+1..n_c-1}(c_\ell^i \leftrightarrow \bigcirc c_\ell^i)\big) \vee \bigwedge_{\ell=0..n_c-1}(c_\ell^i \wedge \bigcirc c_\ell^i)\Big)
\end{aligned}
$$

$$
\vartheta_5^i := \neg\Diamond_F^{(0,1)}(x_i \vee y_i) \to (C_i = 2^{n_c}-1)
$$

Intuitively, ϑ_1^i initializes the counter, ϑ_2^i and ϑ_3^i ensure that the grid points x_i and y_i behave as described above, ϑ_4^i increments the counter when travelling to the left, and ϑ_5^i ensures that, when travelling left, the counter stays in maximal value after the last occurrence of χ_i^*. Let ϑ^i be the conjunction of ϑ_1^i to ϑ_5^i. The following is proved in [6] and finishes the reduction.

Lemma 3. φ *is satisfiable iff* $\square_F(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell) \wedge \varphi^*$ *is satisfiable.*

4   From NP to PSPACE

Qualitative since/until logic on the real line is PSPACE-complete, and thus not computationally simpler than QTL^b. However, several natural fragments are only NP-complete, an important example being the qualitative TL with temporal operators 'eventually in the future' and 'eventually in the past' [10]. In this section, we explore the transition from NP to PSPACE for fragments of *quantitative* logics of the real line, i.e., for QTL and its fragments. We start with determining several weak, but still PSPACE-hard fragments of QTL. Observe that two of the fragments are purely quantitative, i.e., they do not admit qualitative temporal operators at all.

Theorem 4. *Satisfiability (with and without FVA) is* PSPACE-*hard for the fragments of QTL whose only temporal operators are:*

(i) $\Diamond_F$ *and* $\Diamond_F^{[0,n]}$ *with* $n > 0$ *coded in unary;*

(ii) $\Diamond_F^{[0,n]}$ *with* $n > 0$ *coded in binary;*

(iii) $\mathcal{U}^{[0,1]}$.

Point (i) is proved by reducing satisfiability in LTL (i.e., qualitative temporal logic of the natural numbers) with the only temporal operators $\bigcirc$ and $\Diamond_F$ (which is PSPACE-hard [9]) to satisfiability in the logic defined in (i). To do so, the main task is to represent the discrete natural numbers on the real line by alternating intervals making a true and intervals making $\neg a$ true, with the former representing the time points of discrete time. This structure is enforced such that the length of the a-intervals is between 2 and 3 (including 2), the length of the $\neg a$-intervals is between 7 and 8 (including 7), and the length of an a-interval together with the subsequent $\neg a$ interval is between 9 and 10. This is done by the formula $\vartheta = \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3$:

$$
\begin{aligned}
\vartheta_1 &= \square_F^{[0,2]}a, \\
\vartheta_2 &= \square_F(a \to \Diamond_F^{[0,3]}\square_F^{[0,7]}\neg a), \\
\vartheta_3 &= \square_F(a \to \Diamond_F^{[0,10]}\square_F^{[0,2]}a).
\end{aligned}
$$

For the reduction, inductively define a translation $(\cdot)^*$ as follows:

$$
\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(\bigcirc\psi)^* &:= \Diamond_F^{[0,3]}\big(\square_F^{[0,7]}\neg a \wedge \Diamond_F^{[0,8]}(\psi^* \wedge a)\big) \\
(\Diamond_F\psi)^* &:= \Diamond_F(\neg a \wedge \Diamond_F(\psi^* \wedge a))
\end{aligned}
$$

Additionally, a formula ϑ' is needed to take care of uniformity, i.e., to make sure that the same propositional variables

hold in all points of an interval that makes a true:

$$\vartheta' = \Box_F \bigwedge_{p \text{ used in } \varphi} \left[\left(p \wedge a \to \Box_F^{[0,3]}(a \to p) \right) \right.$$
$$\left. \wedge \left(\neg p \wedge a \to \Box_F^{[0,3]}(a \to \neg p) \right) \right].$$

Now, φ is satisfiable over the natural numbers iff $\varphi^* \wedge \vartheta \wedge \vartheta' \wedge a$ is satisfiable over the real numbers with FVA iff it is satisfiable over the real numbers without FVA.

A similar reduction can be used to prove (ii). Notice that satisfiability in LTL is already PSPACE-hard if the natural numbers are replaced by a finite strict linear order (an initial segment of the natural numbers). Moreover, any formula φ which is satisfiable in a finite strict linear order is also satisfiable in a finite strict linear order of length not exceeding $2^{|\varphi|}$. Based on this observation, using the operator $\Diamond_F^{[0,n]}$, $n > 0$ coded in binary, instead of $\Diamond_F$, we can reduce satisfiability of an LTL-formula φ in such a finite strict linear order to satisfiability over the real line (with and without FVA).

Finally, (iii) can be proved by reducing satisfiability over the real line in the qualitative fragment of QTL with the only temporal operator $\mathcal{U}$ (which is PSPACE-hard) to satisfiability of formulas with the operator $\mathcal{U}^{[0,1]}$ over the interval $(0, 1)$; details can be found in [6].

We now exhibit a purely quantitative temporal logic of the real line for which satisfiability is NP-complete: the fragment of QTL with only the quantitative diamond and numbers coded in unary, with and without FVA. This logic may appear rather weak since it does not allow to make statements about *all* time points. Still, it is useful for reasoning about the behaviour of systems up to a previously fixed time point. Note that our NP-completeness result shows that Points (i) and (ii) of Theorem 4 are optimal in the following sense: in Point (i) we cannot drop $\Diamond_F$, and in Point (ii) we cannot switch to unary coding.

Theorem 5. *In the fragment of QTL with temporal operators $\Diamond_F^I$ and $\Diamond_P^I$, I of the form $(0, n)$, $[0, n)$, $[0, n]$, or $(0, n]$, and $n > 0$ coded in unary, satisfiability is decidable in* NP, *both, with and without FVA.*

The lower bound is immediate from propositional logic and thus we only have to prove the upper bound. Since numbers are coded in unary, we may restrict our attention to temporal operators whose upper interval bound is 1. In the proof, we only consider the temporal operator $\Diamond_F^{[0,1]}$. An extension to past operators and open intervals is straightforward.

Let φ be a formula whose satisfiability is to be decided. We introduce some convenient abbreviations: m_φ denotes the nesting depth of operators $\Diamond_F^I$ in φ (henceforth *diamond depth*), $n_\varphi = 2|\varphi|^3 + |\varphi|^2$, and $r_\varphi = |\varphi| \times n_\varphi$. Denote by $cl(\varphi)$ the closure of the set of subformulas of φ under single negation. A *type* t *for* φ is a subset of $cl(\varphi)$ such that

(i) $\neg\psi \in t$ iff $\psi \notin t$ for each $\neg\psi \in cl(\varphi)$, and (ii) $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$ for each $\psi_1 \wedge \psi_2 \in cl(\varphi)$. For a model $\langle \mathbb{R}, \mathfrak{V} \rangle$ and $w \in \mathbb{R}$, set

$$t(w) = \{\psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \psi\},$$
$$t^{<}(w) = \{\Diamond_F^I \psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \Diamond_F^I \psi\}.$$

Notice that $t(w)$ is a type for φ. First, we devise an algorithm for satisfiability without FVA. To begin with, we show that satisfiability of φ implies satisfiability of φ in a 'homogeneous' model. In particular, in such models the number of realized types is polynomial in the length of φ.

Lemma 6. *Let φ be satisfiable without FVA. Then there is a sequence $x_0, \ldots, x_{n_\varphi}$ in $\mathbb{R}$ such that $0 = x_0 < x_1 < \cdots < x_{n_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that $\langle \mathbb{R}, \mathfrak{V} \rangle, 0 \models \varphi$ and*

- $|\{t(w) \mid 0 \leq w \leq m_\varphi\}| \leq r_\varphi$;

- *for every n with $0 \leq n < n_\varphi$ and each type t for φ, the set $\{w \in \mathbb{R} \mid x_n < w < x_{n+1}$ and $w \models_{\mathfrak{V}} t\}$ is either empty or dense in the interval (x_n, x_{n+1}).*

Proof. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with $\mathfrak{M}, 0 \models \varphi$. By the semantics, we clearly have the following:

(∗) for any $\Diamond_F^I \psi \in sub(\varphi)$, the set $\{w \in \mathbb{R} \mid 0 \leq w \leq m_\varphi$ and $w \models_{\mathfrak{V}'} \Diamond_F^I \psi\}$ is a union of intervals of length at least 1 and at most two intervals of length smaller than 1.

The two possibly shorter intervals are the one starting at 0 and the one ending at m_φ. Using (∗), we can show that there is a sequence $y_0, \ldots, y_k$ in $\mathbb{R}$ for some $k \leq 2|\varphi|^2 + |\varphi|$ such that

- $0 = y_0 < \cdots < y_k = m_\varphi$ and

- $t^{<}(w) = t^{<}(w')$ whenever $y_i < w < w' < y_{i+1}$ for any $i < k$.

To see this, take a formula $\Diamond_F^I \psi \in sub(\varphi)$. The *toggle points* for $\Diamond_F^I \psi$ in the interval $[0, m_\varphi]$ are those points x such that either (i) there is a $y > x$ such that the truth value of $\Diamond_F^I \psi$ at x is different from the truth value of $\Diamond_F^I \psi$ at all points z with $x < z < y$ or (ii) there is a $y < x$ such that the truth value of $\Diamond_F^I \psi$ at x is different from the truth value of $\Diamond_F^I \psi$ at all points z with $y < z < x$. By (∗), there are at most $2 \cdot m_\varphi + 1 < 2 \cdot |\varphi| + 1$ toggle points for each formula $\Diamond_F^I \psi$, and thus at most $2|\varphi|^2 + |\varphi|$ toggle points altogether. These points form the required sequence $y_0, \ldots, y_k$.

We convert this sequence into the desired sequence $x_0, \ldots, x_{n_\varphi}$ by arranging the elements of the set

$$\{y_0, \ldots, y_k\} \cup \bigcup_{\substack{i < k \\ 1 \leq j < m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\}$$

in ascending order according to '<', possibly introducing (arbitrary) intermediate points to obtain a sequence of length $n_\varphi + 1$.

To obtain a valuation $\mathfrak{V}$ as required by the lemma, fix a set T_i of types for each $i < n_\varphi$ as follows: for each $\Diamond_F^I \psi \in sub(\varphi)$, choose a $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ if such a w exists. Then, T_i is the set of types $t(w)$ of all points w chosen in this way. Clearly $|T_i| \leq |\varphi|$. For each $i < n_\varphi$, take a collection $(X_t^i)_{t \in T_i}$, of subsets of (x_i, x_{i+1}) which form a partitioning of (x_i, x_{i+1}) such that each X_t^i is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) := (\mathfrak{V}'(p) \cap \{x_0, \ldots, x_{n_\varphi}\}) \cup \bigcup_{i < n_\varphi, t \in T_i} \{X_t^i \mid p \in t\}.$$

Let t_i, $i \leq n_\varphi$, be the type $\{\psi \in sub(\varphi) \mid x_i \models_{\mathfrak{V}'} \psi\}$ for φ realized in point x_i of the original model $\mathfrak{M}$. To show that $\mathfrak{V}$ is as required, it is sufficient to show that, for each $k \leq m_\varphi$, each $\psi \in sub(\varphi)$ with diamond depth bounded by k, and each $w \in [0, m_\varphi - k]$, we have

$w \models_{\mathfrak{V}} \psi$ iff   there is an $i \leq n_\varphi$ such that

    (a) $w = x_i$ and $\psi \in t_i$, or

    (b) $w \in X_t^i$ and $\psi \in t$ for some $t \in T_i$.

Proof. Let k, ψ, and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\Diamond_F^{[0,1]}$.

"$\Rightarrow$": Suppose $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$. Then there is a $w' \in w + [0,1]$ such that $w' \models_{\mathfrak{V}} \psi$ by the semantics. Distinguish four cases:

- $w = x_i$ for some $i \leq n_\varphi$ and $w' = x_j$ for some $j \geq i$. By (a), the induction hypothesis yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. From $x_j - x_i \leq 1$ follows $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Hence $\Diamond_F^{[0,1]} \psi \in t_i$.

- $w = x_i$ for some $i \leq n_\varphi$ and $w' \in X_t^j$ for some j with $i \leq j < n_\varphi$ and $t \in T_j$. The induction hypothesis in (b) yields $\psi \in t$. Then, by definition of T_j, there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. Note that there is an ℓ with $i < \ell \leq n_\varphi$ such that $x_\ell = x_i + 1$. But then $x_{j+1} \leq x_\ell$; otherwise $w' \in X_t^j$ contradicts $w' \in w + [0,1]$. Now, from $w'' - w < 1$ follows $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Hence $\Diamond_F^{[0,1]} \psi \in t_i$.

- $w \in X_t^i$ for some $i < n_\varphi$ and $t \in T_i$, and $w' = x_j$ for some $j > i$. By (a), the induction hypothesis yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. Now, from $x_j - w \leq 1$ follows $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. But then, by definition of

the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that, for all $w'' \in (x_i, x_{i+1})$, $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Therefore, $\Diamond_F^{[0,1]} \psi \in t'$ for any $t' \in T_i$. Hence $\Diamond_F^{[0,1]} \psi \in t$.

- $w \in X_t^i$ for some $i < n_\varphi$ and $t \in T_i$, and $w' \in X_t^j$ for some j with $i \leq j < n_\varphi$ and $t' \in T_j$. This case is similar to the previous cases and left to the reader, see [6].

"$\Leftarrow$": Let $i \leq n_\varphi$ such that

(a) $w = x_i$ and $\Diamond_F^{[0,1]} \psi \in t_i$. Then $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. There is a $w' \in x_i + [0,1]$ such that $w' \models_{\mathfrak{V}'} \psi$. Distinguish two cases:

- $w' = x_j$ for some j with $i \leq j \leq n_\varphi$. Then $\psi \in t_j$. The induction hypothesis in (a) yields $w' \models_{\mathfrak{V}} \psi$. From $w' - x_i \leq 1$ follows $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

- $w' \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. By definition of T_j, there is a $t \in T_j$ such that $\psi \in t$. The induction hypothesis in (b) yields $w'' \models_{\mathfrak{V}} \psi$ for any $w'' \in X_t^j$. Since X_t^j is dense in (x_j, x_{j+1}), there is such a w'' such that $w'' \leq w'$. Then $w'' - x_i \leq 1$. Hence, $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

(b) $w \in X_t^i$ and $\Diamond_F^{[0,1]} \psi \in t$ for some $t \in T_i$. This case is similar to the previous one and left to the reader.

❏

Lemma 6 suggests the following idea for deciding in non-deterministic polynomial time whether a formula φ is satisfiable: guess a (polynomially bounded) set of types for φ to be realized in a homogeneous model, a sequence $v_0, \ldots, v_{n_\varphi}$ of variables, and construct a system of linear inequalities whose solution in $\mathbb{R}$ determines a sequence of points $x_0, \ldots, x_{n_\varphi}$ from which we can build a homogeneous model realizing the guessed types. More precisely, to decide the satisfiability of φ, we non-deterministically choose

- a set T of types for φ such that $|T| \leq r_\varphi$;

- a type $t_i \in T$ such that $\varphi \in t_0$, for every $i \leq n_\varphi$;

- a non-empty set of types $T_i \subseteq T$, for every $i < n_\varphi$.

Intuitively, the type t_i is to be realized at point x_i, and the types in T_i are those types realized in the interval (x_i, x_{i+1}). Then, we take variables $v_0, \ldots, v_{n_\varphi}$ and check whether the system of inequalities given in Figure 1 has a solution in $\mathbb{R}$. The Inequalities 2 to 9 are only added if $i < n_\varphi$. To understand the inequalities (in particular 4 and 5), note that the point x_i described by variable v_i is not intended to realize the *whole* type t_i, but only those elements of t_i whose diamond depth is at most $\lfloor m_\varphi - x_i \rfloor$. Similarly, points from

$$(1)\ 0 = v_0 < v_1 < \cdots < v_{n_\varphi} = m_\varphi$$

(2) $v_j - v_i > 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t_j$

(3) $v_j - v_i \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t$ for some $t \in T_j$

(4) $m_\varphi - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$, but there is no $j \geq i$ such that $\psi \in t_j$ or $\psi \in t$ for a $t \in T_j$

(5) $m_\varphi - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, there is no $j > i$ such that $\psi \in t_j$,
                    and there is no $j \geq i$ such that $\psi \in t'$ for some $t' \in T_j$

(6) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t_j$ and,
                    for every j' with $i \leq j' < j$, $\psi \notin t$ for any $t \in T_{j'}$

(7) $v_j - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t$ for some $t \in T_j$ and
                    there is no j' with $i \leq j' \leq j$ such that $\psi \in t_{j'}$

(8) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, $\psi \notin t'$ for any $t' \in T_i$, and $j > i$ is minimal
                    such that $\psi \in t_j$ or $\psi \in t'$ for some $t' \in T_j$

(9) $v_j - v_{i+1} \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, and ($j \geq i$ and $\psi \in t'$ for some $t' \in T_j$) or
               ($j > i$ and $\psi \in t_j$)

Figure 1. The system of inequalities.

(x_i, x_{i+1}) described by a type $t \in T_i$ realize only elements of t whose diamond depth is at most $\lfloor m_\varphi - x_i \rfloor$; cf. the structural induction in the proof of Lemma 6.

The algorithm runs in non-deterministic polynomial time and returns 'φ is satisfiable' if there is a solution to this system of inequalities, and 'φ is not satisfiable' otherwise. By considering the contrapositive, it is easily seen that φ is unsatisfiable if the algorithm answers 'no': if φ has a model, then by Lemma 6 it also has a homogeneous model, and this model suggests a choice of types such that the corresponding system of inequalities is satisfiable. Conversely, if the algorithm returns 'yes', we can construct a homogeneous model:

Lemma 7. *If the algorithm returns 'φ is satisfiable', then φ is satisfiable.*

Proof. Suppose there are types t_i, $i \leq n_\varphi$, and sets of types T_i, $i < n_\varphi$, such that there is a solution $x_0, \ldots, x_{n_\varphi}$ for the corresponding system of inequalities. For $i < n_\varphi$, take a partitioning $(X_t^i)_{t \in T_i}$ of (x_i, x_{i+1}) such that each X_t^i is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by putting, for every propositional variable p,

$$\mathfrak{V}(p) := \bigcup_{i \leq n_\varphi} \left(\{x_i \mid p \in t_i\} \cup \bigcup_{i < n_\varphi, t \in T_i} \{X_t^i \mid p \in t\} \right).$$

In [6], the following is proved: for all $k \leq m_\varphi$, all $\psi \in sub(\varphi)$ with diamond depth bounded by k, and all $w \in [0, m_\varphi - k]$, we have

$w \models_\mathfrak{V} \psi$ iff   there is an $i \leq n_\varphi$ such that
                           (a) $w = x_i$ and $\psi \in t_i$, or
                           (b) $w \in X_t^i$ and $\psi \in t$ for some $t \in T_i$.

It is an immediate consequence that $0 \models_\mathfrak{V} \varphi$.   ❏

We now come to the proof of Theorem 5 with FVA. Again, the first step is to show that if φ is satisfiable under FVA, then it is satisfiable in a homogeneous model (this time with FVA) in which only polynomially many types are realized:

Lemma 8. *Suppose φ is satisfiable with FVA. Then there exists a sequence $z_0, \ldots, z_{r_\varphi}$ in $\mathbb{R}$ such that $0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that $\langle \mathbb{R}, \mathfrak{V} \rangle, 0 \models \varphi$ and*

- *$|\{t(w) \mid 0 \leq w \leq m_\varphi\}| \leq r_\varphi$;*

- *for all n with $0 \leq n < r_\varphi$, all $\psi \in sub(\varphi)$, and all $z_n < w < w' < z_{n+1}$, $w \models_\mathfrak{V} \psi$ iff $w' \models_\mathfrak{V} \psi$.*

Proof. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with FVA satisfying φ in 0. First, construct a sequence $0 = y_0 < y_1 < \cdots < y_k = m_\varphi$, $k \leq 2|\varphi|^2 + |\varphi|$, as in Lemma 6. Then the sequence $x_0, \ldots, x_{n_\varphi}$ is obtained by arranging the elements

of the set

$$\{y_0, \ldots, y_k\} \ \cup \ \bigcup_{\substack{i<k \\ 1\le j<m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\}$$

$$\cup \ \bigcup_{\substack{i\le k \\ 1\le j<m_\varphi}} \{y_i - j \mid y_i - j > 0\}$$

in ascending order according to $<$ (where we possibly have to add new x_i to obtain a sequence of length $n_\varphi + 1$). Let

$$\sigma = \min\{x_{i+1} - x_i \mid 0 \le i < n_\varphi\},$$

and set, for $i < n_\varphi$, $\sigma_i = \frac{1}{|\varphi|^{i+1}} \times \sigma$. The sequence

$$0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$$

is obtained by adding to the sequence $x_0, \ldots, x_{n_\varphi}$ the points

$$y_i^j = x_i + \frac{j}{|\varphi|} \times \sigma_i,$$

for all $i < n_\varphi$ and $j \le |\varphi|$. For $i < n_\varphi$, denote by t^{-i} the type t which is realized in some interval of the form (x_i, y). Note that such an interval exists since we are in a model with FVA. Also, denote by t^{+i} the type which is realized in some interval of the form (y, x_{i+1}). Now, for $i < n_\varphi$, take for any $\Diamond_F^I \psi \in sub(\varphi)$ such that there exists $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ such a type $t(w)$ and denote the collection of selected types plus the types t^{-i} and t^{+i} by T_i. Notice that $|T_i| \le |\varphi|$. Let $t_0^i, \ldots, t_{|\varphi|-1}^i$ be an ordering of the types in T_i such that $t_0^i = t^{-i}$ (if T_i has cardinality $< |\varphi|$, then take some t from T_i more than once in this ordering.) Define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) = \ \{x_i \mid i \le n_\varphi, x_i \models_{\mathfrak{V}'} p\}$$

$$\cup \ \bigcup_{i<n_\varphi, j<|\varphi|} \{(y_i^j, y_i^{j+1}] \mid p \in t_j^i\}$$

$$\cup \ \bigcup_{i<n_\varphi} \{(y_i^{|\varphi|}, x_{i+1}) \mid p \in t^{+i}\}.$$

We show that $\mathfrak{V}$ is as required. To this end, it is sufficient to show by induction that, for each $k \le m_\varphi$, every $\psi \in sub(\varphi)$ in which the number of nestings of $\Diamond_F^{[0,1]}$ does not exceed k, and all $w \in [0, m_\varphi - k]$:

$$w \models_{\mathfrak{V}} \psi \Leftrightarrow \text{ there is an } i \le n_\varphi \text{ such that}$$

(a) $w = x_i$ and $x_i \models_{\mathfrak{V}'} \psi$, or

(b) $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\psi \in t_\ell^i$ for

some $\ell < |\varphi|$, or

(c) $w \in (y_i^{|\varphi|}, x_{i+1})$ and $\psi \in t^{+i}$.

Proof. Let k, ψ, and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\Diamond_F^{[0,1]}$.

"$\Rightarrow$": Suppose $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$. Then there is a $w' \in w + [0, 1]$ such that $w' \models_{\mathfrak{V}} \psi$. Similarly to the previous proof distinguish four cases. Here only the most intricate case is considered; the other cases are left to the reader and can also be found in [6]:

- $w \in (x_i, x_{i+1})$ for some $i < n_\varphi$, and $w' \in (x_j, x_{j+1})$ for some $j \ge i$. If $w' \in (y_j^\ell, y_j^{\ell+1}]$ for some $\ell < |\varphi|$, then the induction hypothesis in (b) yields $\psi \in t_\ell^j$. Otherwise, i.e., if $w' \in (y_j^{|\varphi|}, x_{j+1})$, $\psi \in t^{+j}$ by the induction hypothesis in (c). Since $t_\ell^j, t^{+j} \in T_j$, it follows by definition of T_j that there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. Note that there is an $i' > i + 1$ such that $x_{i'} = x_{i+1} + 1$. But then $x_{j+1} \le x_{i'}$; otherwise $x_j \ge x_{i'}$ and thus $w' - w > 1$ contradicting $w' \in w + [0, 1]$. Thus there is a $v \in (x_i, x_{i+1})$ such that $w'' - v \le 1$. By the semantics, $v \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Then it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $v' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$ for all $v' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]} \psi \in t'$ for any $t' \in T_i$. Hence $\Diamond_F^{[0,1]} \psi \in t_\ell^i$ if $w \in (y_i^\ell, y_i^{\ell+1}]$ for some $\ell < |\varphi|$, and $\Diamond_F^{[0,1]} \psi \in t^{+i}$ if $w \in (y_i^{|\varphi|}, x_{i+1})$.

"$\Leftarrow$": Here only Case (b) is considered; the cases (a) and (c) are left to the reader and can also be found in [6]. Let $i \le n_\varphi$ such that (b) $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\Diamond_F^{[0,1]} \psi \in t_\ell^i$ for some $\ell < |\varphi|$. By definition of T_i, there is a $w' \in (x_i, x_{i+1})$ such that $w' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Then it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$ for any $w'' \in (x_i, x_{i+1})$. In particular, $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Then $v \models_{\mathfrak{V}'} \psi$ for some $v \in w + [0, 1]$ by the semantics. Distinguish three cases:

- $v = x_j$ for some $j > i$. The induction hypothesis in (a) yields $v \models_{\mathfrak{V}} \psi$. Since $v - w \le 1$, it follows by the semantics that $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

- $v \in (x_i, x_{i+1})$. By definition of T_i, there is a $t \in T_i$ such that $\psi \in t$. Distinguish two subcases: First, suppose that $\psi \in t_{\ell'}^i$ for some $\ell' \ge \ell$, or $\psi \in t^{+i}$. The induction hypothesis in (b) or (c) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^{\ell'}, y_j^{\ell'+1}]$, or all $v' \in (y_i^{|\varphi|}, x_{i+1})$, respectively. Then there is such a v' such that $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.
  Second, suppose there is no $\ell' \ge \ell$ such that $\psi \in t_{\ell'}^i$, and $\psi \notin t^{+i}$. Note that this implies $\ell > 0$. Since $\psi \notin t^{+i}$, there is an interval of the form (y, x_{i+1}) such that $y' \not\models_{\mathfrak{V}'} \psi$ for all $y' \in (y, x_{i+1})$. Take such a y'. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$, it follows by definition of

$x_0, \ldots, x_{n_\varphi}$ that $y' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$. Then there is a $v' \in y' + [0,1]$ such that $v' \models_{\mathfrak{V}'} \psi$ and $v' \geq x_{i+1}$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $v' \in (x_j, x_{j+1})$ where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction. Therefore $j = i'$, i.e., $x_j = x_i + 1$. By definition of T_j, there is an $\ell' < |\varphi|$ such that $t_{\ell'}^j \in T_j$ and $\psi \in t_{\ell'}^j$. Then the induction hypothesis in (b) yields $v'' \models_{\mathfrak{V}} \psi$ for all $v'' \in (y_j^{\ell'}, y_j^{\ell'+1}]$. Take such a v''. Since $\ell > 0$ and $\sigma_j \leq \frac{\sigma_i}{|\varphi|}$ by definition of σ_j, it holds that $y_i^\ell + 1 \geq x_j + \sigma_j$. Then $y_j^{\ell'+1} - y_i^\ell < 1$ and thus $v'' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

- $v \in (x_j, x_{j+1})$ for some $j > i$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction.
Distinguish three subcases:

  - $\ell = 0$ and $w' \models_{\mathfrak{V}'} \psi$ for some w' with $x_i < w' \leq x_j$. Then it is easy to see that there is a $v'' \geq w$ such that $v'' \models_{\mathfrak{V}} \psi$ and $v'' - w \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

  - $\ell = 0$ and $w' \not\models_{\mathfrak{V}'} \psi$ for all w' with $x_i < w' \leq x_j$. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$, it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$ for all w'' with $x_i < w'' < w$. Take such a w''. Then there is a $v'' \in w'' + [0,1]$ such that $v'' \models_{\mathfrak{V}'} \psi$ and $v'' > x_j$. This implies that $\psi \in t^{-j} = t_0^j$. Then the induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^0, y_j^1]$. Clearly, there is such a v' such that $w - v' \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

  - $1 \leq \ell < |\varphi|$. By definition of T_j, there is an $\ell' < |\varphi|$ such that $t_{\ell'}^j \in T_j$ and $\psi \in t_{\ell'}^j$. The induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^{\ell'}, y_j^{\ell'+1}]$. Take such a v'. Since $\sigma_j \leq \frac{\sigma_i}{|\varphi|}$ by definition of σ_j, it holds that $y_i^\ell + 1 \geq x_j + \sigma_j$ and thus $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

❏

Using Lemma 8, one can now modify the decision procedure for satisfiability without FVA to obtain a decision procedure running in nondeterministic polynomial time for satisfiability with FVA. The crucial step is to determine a set of rational linear inequalities which represent the truth conditions in models of the form described in Lemma 8. We leave this exercise to the reader.

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