Gödel Negation Makes Unwitnessed Consistency Crisp⋆

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Abstract. Ontology consistency has been shown to be undecidable for a wide variety of fairly inexpressive fuzzy Description Logics (DLs). In particular, for any t-norm "starting with" the Łukasiewicz t-norm, consistency of crisp ontologies (w.r.t. witnessed models) is undecidable in any fuzzy DL with conjunction, existential restrictions, and (residual) negation. In this paper we show that for any t-norm with Gödel negation, that is, any t-norm not starting with Łukasiewicz, ontology consistency for a variant of fuzzy $\mathcal{SHOI}$ is linearly reducible to crisp reasoning, and hence decidable in exponential time. Our results hold even if reasoning is not restricted to the class of witnessed models only.

1 Introduction

Fuzzy Description Logics (DLs) were introduced over a decade ago to represent and reason with vague or imprecise knowledge. Since their introduction, several variants of these logics have been studied; in fact, in addition to the constructors and kinds of axioms used, fuzzy DLs have an additional degree of liberty in the choice of the t-norm that specifies the semantics. An extensive, although slightly outdated survey of the area can be found in [18].

Very recently, it was shown that some fuzzy DLs lose the finite model property in the presence of GCIs [3]. This eventually led to a series of undecidability results [1, 2, 12, 10]. Most notably, for t-norms that "start with" the Łukasiewicz t-norm, consistency of *crisp* ontologies becomes undecidable for the inexpressive fuzzy DL $\otimes$-$\mathfrak{NEL}$, which allows only the constructors conjunction, existential restrictions and (residual) negation [10].

So far, the only known decidability results for fuzzy DLs rely on a restriction of the expressivity: either by allowing only finitely-valued semantics [6, 8], by limiting the terminological knowledge to be acyclic or unfoldable [14, 11, 5], or by using the very simple Gödel semantics [4, 21–23]. Moreover, with very few exceptions [6, 8], reasoning is usually restricted to the class of witnessed models.[1] In fact, witnessed models were introduced in [14] to correct the previous algorithms for fuzzy DL reasoning.

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1 All fuzzy logics with finitely-valued semantics have the witnessed model property.

Name	t-norm ($x \otimes y$)	t-conorm ($x \oplus y$)	residuum ($x \Rightarrow y$)
Gödel	$\min\{x, y\}$	$\max\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
product	$x \cdot y$	$x + y - x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$\min\{x + y, 1\}$	$\min\{1 - x + y, 1\}$

Table 1. The three fundamental continuous t-norms.

In this paper we show that for any t-norm with Gödel negation, ontology consistency is decidable in the very expressive fuzzy DL $\otimes$-$\mathcal{SHOI}^{-\forall}$, which is the sub-logic of $\otimes$-$\mathcal{SHOI}$ where value restrictions $\forall$ are not allowed. For these logics, ontology consistency w.r.t. general models is linearly reducible to consistency of a crisp $\mathcal{SHOI}$ ontology, and hence decidable in exponential time. In particular, this holds for the product t-norm. We emphasize that our proofs do not depend on the models being witnessed or not, hence decidability is shown for reasoning w.r.t. both, general models and witnessed models.

Since a t-norm has Gödel negation iff it does not start with the Łukasiewicz t-norm [17], this yields a full characterization of the decidability of ontology consistency (w.r.t. witnessed models) for all fuzzy DLs between $\otimes$-$\mathfrak{NEL}$ and $\otimes$-$\mathcal{SHOI}^{-\forall}$. We also provide the first decidability results w.r.t. general models for infinitely-valued, non-idempotent fuzzy DLs.

2 T-norms without Zero Divisors

Mathematical fuzzy logic [13] generalizes classical logic by allowing all the real numbers from the interval $[0, 1]$ as truth values. The interpretation of the different logical constructors depends on the choice of a triangular norm (t-norm for short). A *t-norm* is an associative, commutative, and monotone binary operator on $[0, 1]$ that has 1 as its unit element. The dual operator of a t-norm $\otimes$ is the *t-conorm* $\oplus$ defined as $x \oplus y = 1 - ((1 - x) \otimes (1 - y))$. Notice that 0 is the unit of the t-conorm, and hence

$$x \oplus y = 0 \text{ iff } x = 0 \text{ and } y = 0. \tag{1}$$

Every *continuous* t-norm $\otimes$ defines a unique residuum $\Rightarrow$ such that $x \otimes y \leq z$ iff $y \leq x \Rightarrow z$ for all $x, y, z \in [0, 1]$. It is easy to see that for all $x, y \in [0, 1]$

- $x \Rightarrow y = \sup\{z \in [0, 1] \mid x \otimes z \leq y\}$,
- $1 \Rightarrow x = x$, and
- $x \leq y$ iff $x \Rightarrow y = 1$.

Based on the residuum, one can define the unary *precomplement* $\ominus x = x \Rightarrow 0$. Three important continuous t-norms are the Gödel, product and Łukasiewicz t-norms shown in Table 1, together with their t-conorms and residua. These are *fundamental* in the sense that every continuous t-norm can be described as an ordinal sum of copies of these t-norms [20].

In this paper, we are interested in t-norms that do not have zero divisors. An element $x \in (0,1)$ is called a *zero divisor* for $\otimes$ if there is a $z \in (0,1)$ such that $x \otimes z = 0$. Of the three fundamental continuous t-norms, only the Łukasiewicz t-norm has zero divisors. In fact, every element of the interval $(0,1)$ is a zero divisor for this t-norm. The Gödel and product t-norms are just two elements of the uncountable class of continuous t-norms without zero divisors.

Proposition 1. *For any t-norm $\otimes$ without zero divisors and every $x \in [0,1]$,*

1. $x \Rightarrow y = 0$ *iff* $x > 0$ *and* $y = 0$, *and*
2. $\ominus x = 0$ *iff* $x > 0$.

Proof. For the first claim, we prove only the *if* direction, since the other direction is known to hold for every t-norm [17]. Assume that $x > 0$ and $y = 0$. Then $x \Rightarrow y = x \Rightarrow 0 = \sup\{z \mid z \otimes x = 0\}$. Since $\otimes$ has no zero divisors, $z \otimes x > 0$ for all $z > 0$. Therefore $\{z \mid z \otimes x = 0\} = \{0\}$ and thus $x \Rightarrow y = 0$. The second statement follows from the first one since $\ominus x = x \Rightarrow 0$. □

In particular, this implies that if the t-norm $\otimes$ does not have zero divisors, then its precomplement is the so-called *Gödel negation*, i.e. for every $x \in [0,1]$,

$$\ominus x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{otherwise.} \end{cases}$$

It can be shown that the converse also holds: if the precomplement is the Gödel negation, then the t-norm has no zero divisors.

We now define the function $\mathbb{1}$ that maps fuzzy truth values to crisp truth values by setting, for all $x \in [0,1]$,

$$\mathbb{1}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

For a t-norm without zero divisors it follows from Proposition 1 that $\mathbb{1}(x) = \ominus\ominus x$ for all $x \in [0,1]$. This function is compatible with negation, the t-norm, the corresponding t-conorm, implication and suprema.

Lemma 2. *Let $\otimes$ be a t-norm without zero divisors. For all $x, y \in [0,1]$ and all non-empty sets $X \subseteq [0,1]$ it holds that*

1. $\mathbb{1}(\ominus x) = \ominus \mathbb{1}(x)$,
2. $\mathbb{1}(x \otimes y) = \mathbb{1}(x) \otimes \mathbb{1}(y)$,
3. $\mathbb{1}(x \oplus y) = \mathbb{1}(x) \oplus \mathbb{1}(y)$,
4. $\mathbb{1}(x \Rightarrow y) = \mathbb{1}(x) \Rightarrow \mathbb{1}(y)$, *and*
5. $\mathbb{1}\left(\sup\{x \mid x \in X\}\right) = \sup\{\mathbb{1}(x) \mid x \in X\}$.

Proof. It holds that $\mathbb{1}(\ominus x) = \ominus\ominus\ominus x = \ominus\mathbb{1}(x)$ which proves 1. Since $\otimes$ does not have zero divisors, $x \otimes y = 0$ iff $x = 0$ or $y = 0$. This shows that

$$\mathbb{1}(x \otimes y) = 0 \text{ iff } \mathbb{1}(x) \otimes \mathbb{1}(y) = 0. \tag{2}$$

Both $\mathbb{1}(x \otimes y)$ and $\mathbb{1}(x) \otimes \mathbb{1}(y)$ can only have the values 0 or 1. Hence, (2) proves the second statement. Following similar arguments we obtain from (1) that $\mathbb{1}(x \oplus y) = 0$ holds iff $\mathbb{1}(x) \oplus \mathbb{1}(y) = 0$, thus proving 3. We use Proposition 1 to prove 4:

$$\mathbb{1}(x \Rightarrow y) = \begin{cases} 1 & \text{iff } x = 0 \text{ or } y > 0 \\ 0 & \text{iff } x > 0 \text{ and } y = 0 \end{cases} = \begin{cases} 1 & \text{iff } \mathbb{1}(x) = 0 \text{ or } \mathbb{1}(y) = 1 \\ 0 & \text{iff } \mathbb{1}(x) = 1 \text{ and } \mathbb{1}(y) = 0 \end{cases}$$
$$= \mathbb{1}(x) \Rightarrow \mathbb{1}(y).$$

To prove 5, observe that $\sup X = 0$ iff $X = \{0\}$. Thus,

$$\mathbb{1}\big(\sup X\big) = 0 \Leftrightarrow \sup X = 0 \Leftrightarrow X = \{0\}$$
$$\Leftrightarrow \{\mathbb{1}(x) \mid x \in X\} = \{0\} \Leftrightarrow \sup\{\mathbb{1}(x) \mid x \in X\} = 0.$$

□

Notice that in general $\mathbb{1}$ is not compatible with the infimum. Consider for example the set $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\inf X = 0$ and hence $\mathbb{1}(\inf X) = 0$, but $\inf\{\mathbb{1}(\frac{1}{n}) \mid n \in \mathbb{N}\} = \inf\{1\} = 1$.

3 The Fuzzy DL $\otimes$-$\mathcal{SHOI}^{-\forall}$

A fuzzy description logic usually inherits its syntax from the underlying crisp description logic. We consider the constructors of $\mathcal{SHOI}$ with the addition of $\to$, which in the crisp case can be expressed by $\sqcup$ and $\neg$.

Definition 3 (syntax). *Let* $\mathsf{N_C}$, $\mathsf{N_R}$, *and* $\mathsf{N_I}$, *be disjoint sets of* concept, role, *and* individual names, *respectively, and* $\mathsf{N_R^+} \subseteq \mathsf{N_R}$ *be a set of* transitive role names. *The set of* (complex) roles *is* $\mathsf{N_R} \cup \{r^- \mid r \in \mathsf{N_R}\}$. $\otimes$-$\mathcal{SHOI}$ (complex) concepts *are constructed by the following syntax rule:*

$$C ::= A \mid \top \mid \bot \mid \{a\} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid C \to C \mid \exists s.C \mid \forall s.C,$$

where A is a concept name, a is an individual name, and s is a complex role.

The *inverse* of a complex role s (denoted by $\overline{s}$) is s^- if $s \in \mathsf{N_R}$ and r if $s = r^-$. A role s is *transitive* if either s or $\overline{s}$ belongs to $\mathsf{N_R^+}$.

Given a continuous t-norm $\otimes$, concepts in the fuzzy DL $\otimes$-$\mathcal{SHOI}$ are interpreted by functions specifying the membership degree of each domain element to the concept. The interpretation of the constructors is based on the t-norm $\otimes$ and the induced operators $\oplus$, $\Rightarrow$, and $\ominus$.

Definition 4 (semantics). *An* interpretation *is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the* domain $\Delta^{\mathcal{I}}$ *is a non-empty set and* $\cdot^{\mathcal{I}}$ *is a function that assigns to every concept name A a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, to every individual name a an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, and to every role name r a function $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$ such that $r^{\mathcal{I}}(x,y) \otimes r^{\mathcal{I}}(y,z) \leq r^{\mathcal{I}}(x,z)$ holds for all $x,y,z \in \Delta^{\mathcal{I}}$ if $r \in \mathsf{N}_\mathsf{R}^+$. The function $\cdot^{\mathcal{I}}$ is extended to complex roles and concepts as follows for every $x,y \in \Delta^{\mathcal{I}}$,*

- $(r^-)^{\mathcal{I}}(x,y) = r^{\mathcal{I}}(y,x)$,
- $\top^{\mathcal{I}}(x) = 1, \quad \bot^{\mathcal{I}}(x) = 0$,
- $\{a\}^{\mathcal{I}}(x) = 1$ *if* $a^{\mathcal{I}} = x$ *and* 0 *otherwise,*
- $(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$,
- $(C_1 \sqcap C_2)^{\mathcal{I}}(x) = C_1^{\mathcal{I}}(x) \otimes C_2^{\mathcal{I}}(x), \quad (C_1 \sqcup C_2)^{\mathcal{I}}(x) = C_1^{\mathcal{I}}(x) \oplus C_2^{\mathcal{I}}(x)$,
- $(C_1 \to C_2)^{\mathcal{I}}(x) = C_1^{\mathcal{I}}(x) \Rightarrow C_2^{\mathcal{I}}(x)$,
- $(\exists s.C)^{\mathcal{I}}(x) = \sup_{z \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,z) \otimes C^{\mathcal{I}}(z)$, *and*
- $(\forall s.C)^{\mathcal{I}}(x) = \inf_{z \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,z) \Rightarrow C^{\mathcal{I}}(z)$.

$\mathcal{I}$ is finite *if its domain $\Delta^{\mathcal{I}}$ is finite, and* crisp *if $A^{\mathcal{I}}(x), r^{\mathcal{I}}(x,y) \in \{0,1\}$ for all $A \in \mathsf{N}_\mathsf{C}$, $r \in \mathsf{N}_\mathsf{R}$, and $x,y \in \Delta^{\mathcal{I}}$.*

Recall from the previous section that $\neg$ is interpreted by the Gödel negation iff the t-norm $\otimes$ does not have zero divisors. In particular, $(\neg C)^{\mathcal{I}}(x) \in \{0,1\}$ holds for every concept C, interpretation $\mathcal{I}$, and $x \in \Delta^{\mathcal{I}}$, i.e. the value of $\neg C$ is always crisp.

Knowledge is encoded using axioms, which restrict the class of interpretations that are considered and specify a degree to which the restrictions should hold.

Definition 5 (axioms). *A* $\otimes$-$\mathcal{SHOI}$-axiom *is either an* assertion *of the form $\langle a{:}C,\ \ell\rangle$ or $\langle (a,b){:}s,\ \ell\rangle$, a* GCI *of the form $\langle C \sqsubseteq D,\ \ell\rangle$, or a* role inclusion *of the form $\langle s \sqsubseteq t,\ \ell\rangle$, where C and D are $\otimes$-$\mathcal{SHOI}$-concepts, $a,b \in \mathsf{N}_\mathsf{I}$, s,t are complex roles, and $\ell \in (0,1]$. An axiom is called* crisp *if $\ell = 1$.*

An interpretation $\mathcal{I}$ satisfies *an assertion $\langle a{:}C,\ \ell\rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \ell$ and an assertion $\langle (a,b){:}s,\ \ell\rangle$ if $s^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \ell$. It* satisfies *the GCI $\langle C \sqsubseteq D,\ \ell\rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ holds for all $x \in \Delta^{\mathcal{I}}$. It* satisfies *a role inclusion $\langle s \sqsubseteq t,\ \ell\rangle$ if $s^{\mathcal{I}}(x,y) \Rightarrow t^{\mathcal{I}}(x,y) \geq \ell$ holds for all $x,y \in \Delta^{\mathcal{I}}$.*

A $\otimes$-$\mathcal{SHOI}$-ontology $(\mathcal{A}, \mathcal{T}, \mathcal{R})$ *is defined by a finite set $\mathcal{A}$ of assertions* (ABox)*, a finite set $\mathcal{T}$ of GCIs* (TBox)*, and a finite set $\mathcal{R}$ of role inclusions* (RBox)*. It is* crisp *if every axiom in $\mathcal{A}$, $\mathcal{T}$, and $\mathcal{R}$ is crisp. An interpretation $\mathcal{I}$ is a* model *of this ontology if it satisfies all its axioms.*

We consider also the logic $\otimes$-$\mathcal{SHOI}^{-\forall}$, which restricts $\otimes$-$\mathcal{SHOI}$ by disallowing the constructor $\forall$. $\otimes$-$\mathcal{SHOI}^{-\forall}$-concepts, axioms and ontologies are defined in the obvious way. Notice that, contrary to the crisp case, value- and existential-restrictions are not dual. In fact, we will show in Section 4 that for every t-norm $\otimes$ without zero divisors $\otimes$-$\mathcal{SHOI}$ is strictly more expressive than $\otimes$-$\mathcal{SHOI}^{-\forall}$.

Several reasoning problems are of interest in the area of fuzzy DLs. Here we focus only on deciding whether a $\otimes$-$\mathcal{SHOI}$ (or $\otimes$-$\mathcal{SHOI}^{-\forall}$) ontology is *consistent*; that is, whether it has a model. We will show that, if the t-norm

$\otimes$ has no zero divisors, then consistency in $\otimes$-$\mathcal{SHOI}^{-\forall}$ is effectively the same problem as consistency in crisp $\mathcal{SHOI}$. Moreover, the precise values appearing in the axioms in the ontology are then irrelevant. The same is not true, however, for consistency in $\otimes$-$\mathcal{SHOI}$.

Recall that the semantics of the quantifiers require the computation of a supremum or infimum of the membership degrees of a possibly infinite set of elements of the domain. In the fuzzy DL community it is customary to restrict reasoning to a special kind of models, called witnessed models [14].

Definition 6 (witnessed). *An interpretation $\mathcal{I}$ is called* witnessed *if for every $x \in \Delta^{\mathcal{I}}$, every role s and every concept C there are $y_1, y_2 \in \Delta^{\mathcal{I}}$ such that*

$$(\exists s.C)^{\mathcal{I}}(x) = s^{\mathcal{I}}(x, y_1) \otimes C^{\mathcal{I}}(y_1), \qquad (\forall s.C)^{\mathcal{I}}(x) = s^{\mathcal{I}}(x, y_2) \Rightarrow C^{\mathcal{I}}(y_2).$$

In particular, if an interpretation $\mathcal{I}$ is crisp or finite, then it is also witnessed. Witnessed models were introduced to simplify the reasoning tasks. In fact, although this concept was only formalized in [14], the earlier reasoning algorithms for fuzzy DLs semantics based on the Gödel t-norm (e.g. [23]) implicitly used only witnessed models. We show that consistency of $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontologies can be decided in exponential time, without restricting to witnessed models.

4 The Crisp Model Property

The existing undecidability results for fuzzy DLs all rely heavily on the fact that one can design ontologies that allow only models with infinitely many truth values. We shall see that for t-norms without zero divisors one cannot construct such an ontology in $\otimes$-$\mathcal{SHOI}^{-\forall}$. It is even true that all consistent $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontologies have a crisp model; that is, using at most two truth values.

Definition 7. *A fuzzy DL $\mathcal{L}$ has the* crisp model property *if every consistent $\mathcal{L}$-ontology has a crisp model.*

For the rest of this paper we assume that $\otimes$ is a continuous t-norm that does not have zero divisors, and hence has the properties described in Section 2. In particular, Lemma 2 allows us to construct a crisp interpretation from a fuzzy interpretation by simply applying the function $\mathbb{1}$.

Let $\mathcal{I}$ be a fuzzy interpretation for the concept names $\mathsf{N_C}$ and role names $\mathsf{N_R}$. We construct the interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$, where $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$ and for all concept names $A \in \mathsf{N_C}$, all role names $r \in \mathsf{N_R}$, and all $x, y \in \Delta^{\mathcal{I}}$,

$$A^{\mathcal{J}}(x) := \mathbb{1}\left(A^{\mathcal{I}}(x)\right) \text{ and } r^{\mathcal{J}}(x, y) := \mathbb{1}\left(r^{\mathcal{I}}(x, y)\right).$$

To show that $\mathcal{J}$ is a valid interpretation, we first verify the transitivity condition for all $r \in \mathsf{N_R^+}$ and all $x, y, z \in \Delta^{\mathcal{J}}$. From Lemma 2, we obtain

$$r^{\mathcal{J}}(x, y) \otimes r^{\mathcal{J}}(y, z) = \mathbb{1}\big(r^{\mathcal{I}}(x, y)\big) \otimes \mathbb{1}\big(r^{\mathcal{I}}(y, z)\big) = \mathbb{1}\left(r^{\mathcal{I}}(x, y) \otimes r^{\mathcal{I}}(y, z)\right).$$

Since $\mathcal{I}$ satisfies the transitivity condition and $\mathbb{1}$ is monotonic, we have

$$\mathbb{1}\left(r^{\mathcal{I}}(x,y)\otimes r^{\mathcal{I}}(y,z)\right)\leq \mathbb{1}\big(r^{\mathcal{I}}(x,z)\big)=r^{\mathcal{J}}(x,z),$$

and thus $r^{\mathcal{J}}(x,y)\otimes r^{\mathcal{J}}(y,z)\leq r^{\mathcal{J}}(x,z)$.

Lemma 8. *For all complex roles s and $x,y\in\Delta^{\mathcal{I}}$, $s^{\mathcal{J}}(x,y)=\mathbb{1}(s^{\mathcal{I}}(x,y))$.*

Proof. If s is a role name, this follows directly from the definition of $\mathcal{J}$. If $s=r^-$ for some $r\in\mathsf{N_R}$, then $s^{\mathcal{J}}(x,y)=r^{\mathcal{J}}(y,x)=\mathbb{1}(r^{\mathcal{I}}(y,x))=\mathbb{1}(s^{\mathcal{I}}(x,y))$. □

The interpretation $\mathcal{J}$ preserves the compatibility of $\mathbb{1}$ with all the constructors of $\otimes$-$\mathcal{SHOI}^{-\forall}$.

Lemma 9. *For every $\otimes$-$\mathcal{SHOI}^{-\forall}$-concept C and $x\in\Delta^{\mathcal{I}}$, $C^{\mathcal{J}}(x)=\mathbb{1}\left(C^{\mathcal{I}}(x)\right)$.*

Proof. We use induction over the structure of C. The claim holds trivially for $C=\bot$ and $C=\top$. For $C=A\in\mathsf{N_C}$ it follows immediately from the definition of $\mathcal{J}$. It also holds for $C=\{a\}$, $a\in\mathsf{N_I}$, because $\{a\}^{\mathcal{I}}(x)$ can only take the values 0 or 1 for all $x\in\Delta^{\mathcal{I}}$.

Assume now that the concepts D and E satisfy $D^{\mathcal{J}}(x)=\mathbb{1}(D^{\mathcal{I}}(x))$ and $E^{\mathcal{J}}(x)=\mathbb{1}(E^{\mathcal{I}}(x))$ for all $x\in\Delta^{\mathcal{I}}$. In the case where $C=D\sqcap E$, Lemma 2 yields that for all $x\in\Delta^{\mathcal{I}}$

$$\begin{aligned}C^{\mathcal{J}}(x)=D^{\mathcal{J}}(x)\otimes E^{\mathcal{J}}(x)&=\mathbb{1}\big(D^{\mathcal{I}}(x)\big)\otimes\mathbb{1}\big(E^{\mathcal{I}}(x)\big)\\&=\mathbb{1}\big(D^{\mathcal{I}}(x)\otimes E^{\mathcal{I}}(x)\big)=\mathbb{1}\big(C^{\mathcal{I}}(x)\big).\end{aligned}$$

Likewise, the compatibility of $\mathbb{1}$ with the t-conorm, the residuum, and the negation entails the result for the cases $C=D\sqcup E$, $C=D\to E$, and $C=\neg D$.

For $C=\exists s.D$, where s is a complex role and D is a concept description satisfying $D^{\mathcal{J}}(x)=\mathbb{1}(D^{\mathcal{I}}(x))$ for all $x\in\Delta^{\mathcal{I}}$, we obtain

$$\begin{aligned}\mathbb{1}\big(C^{\mathcal{I}}(x)\big)=\mathbb{1}\big((\exists s.D)^{\mathcal{I}}(x)\big)&=\mathbb{1}\big(\sup_{y\in\Delta^{\mathcal{I}}}\left\{s^{\mathcal{I}}(x,y)\otimes D^{\mathcal{I}}(y)\right\}\big)\\&=\sup_{y\in\Delta^{\mathcal{I}}}\left\{\mathbb{1}\big(s^{\mathcal{I}}(x,y)\big)\otimes\mathbb{1}\big(D^{\mathcal{I}}(y)\big)\right\}\end{aligned}\tag{3}$$

because $\mathbb{1}$ is compatible with the supremum and the t-norm. Lemma 8 yields

$$\sup_{y\in\Delta^{\mathcal{I}}}\left\{\mathbb{1}(r^{\mathcal{I}}(x,y))\otimes\mathbb{1}(D^{\mathcal{I}}(y))\right\}=\sup_{y\in\Delta^{\mathcal{I}}}\left\{r^{\mathcal{J}}(x,y)\otimes D^{\mathcal{J}}(y)\right\}=(\exists r.D)^{\mathcal{J}}(x).\tag{4}$$

Equations (3) and (4) prove $\mathbb{1}(C^{\mathcal{I}}(x))=C^{\mathcal{J}}(x)$ for $C=\exists r.D$. □

We can use this lemma to show that the crisp interpretation $\mathcal{J}$ satisfies all the axioms that are satisfied by $\mathcal{I}$.

Lemma 10. *Let $\mathcal{O}=(\mathcal{A},\mathcal{T},\mathcal{R})$ be a $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology. If $\mathcal{I}$ is a model of $\mathcal{O}$, then $\mathcal{J}$ is also a model of $\mathcal{O}$.*

Proof. We prove that $\mathcal{J}$ satisfies all assertions, GCIs, and role inclusions from $\mathcal{O}$. Let $\langle a{:}C,\ \ell\rangle$, $\ell \in (0,1]$, be a concept assertion from $\mathcal{A}$. Since the assertion is satisfied by $\mathcal{I}$, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \ell > 0$ holds. Lemma 9 yields $C^{\mathcal{J}}(a^{\mathcal{J}}) = 1 \geq \ell$. The same argument can be used for role assertions.

Let now $\langle C \sqsubseteq D,\ \ell\rangle$ be a GCI in $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies the GCI, we get $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell > 0$. By Lemmata 2 and 9, we obtain

$$C^{\mathcal{J}}(x) \Rightarrow D^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x)) \Rightarrow \mathbb{1}(D^{\mathcal{I}}(x)) = \mathbb{1}(C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)) = 1 \geq \ell,$$

and thus $\mathcal{J}$ satisfies the GCI $\langle C \sqsubseteq D,\ \ell\rangle$. A similar argument, using Lemma 8 instead of Lemma 9, shows that $\mathcal{J}$ satisfies all role inclusions in $\mathcal{R}$. □

The previous results show that by applying $\mathbb{1}$ to the truth degrees we obtain a crisp model $\mathcal{J}$ from any fuzzy model $\mathcal{I}$ of a $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology $\mathcal{O}$.

Theorem 11. *If $\otimes$ is a t-norm without zero divisors, then $\otimes$-$\mathcal{SHOI}^{-\forall}$ has the crisp model property.*

A trivial consequence of this theorem is that every consistent $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology has also a witnessed model, since every crisp model is also crisp.

Corollary 12. *If $\otimes$ is a t-norm without zero divisors, then $\otimes$-$\mathcal{SHOI}^{-\forall}$ has the witnessed model property.*

In the next section we will use this result from Theorem 11 to show that $\otimes$-$\mathcal{SHOI}^{-\forall}$ ontology consistency can be decided in exponential time, by testing consistency of a (crisp) $\mathcal{SHOI}$ ontology. But first, we show that value restrictions destroy the crisp model property, even if only crisp axioms are used.

Example 13. Consider the $\otimes$-$\mathcal{SHOI}$-ontology

$$\mathcal{O} = \{\langle \top \sqsubseteq \neg\neg A,\ 1\rangle, \langle a{:}\neg\forall r.A,\ 1\rangle\}.$$

The interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \mathbb{N}$ (the set of all natural numbers), $a^{\mathcal{I}} = 1$, $A^{\mathcal{I}}(n) = 1/(n+1)$, $r^{\mathcal{I}}(1,n) = 1$, and $r^{\mathcal{I}}(n',n) = 0$ for all $n, n' \in \mathbb{N}$ with $n' > 1$ is a model of $\mathcal{O}$, and hence $\mathcal{O}$ is consistent.

Let now $\mathcal{J}$ be a crisp interpretation satisfying the first axiom in $\mathcal{O}$. Then, $A^{\mathcal{J}}(x) = 1$ for all $x \in \Delta^{\mathcal{J}}$. This implies that

$$\begin{aligned}(\forall r.A)^{\mathcal{J}}(a^{\mathcal{J}}) &= \inf_{y \in \Delta^{\mathcal{J}}} r^{\mathcal{J}}(a^{\mathcal{J}}, y) \Rightarrow A^{\mathcal{J}}(y)\\ &= \inf_{y \in \Delta^{\mathcal{J}}} r^{\mathcal{J}}(a^{\mathcal{J}}, y) \Rightarrow 1\\ &= 1.\end{aligned}$$

And thus, $(\neg\forall r.A)^{\mathcal{J}}(a^{\mathcal{J}}) = 0$, violating the second axiom. This means that $\mathcal{O}$ has no crisp model.

The example shows that no fuzzy DL with the constructor $\forall$ and Gödel negation[2] has the crisp model property. A similar example in [14] demonstrates that no fuzzy DL with the constructors $\exists$ and $\forall$ and Gödel negation has the witnessed model property.

Theorem 14. *For any continuous t-norm $\otimes$ and any fuzzy DL $\otimes$-$\mathcal{L}$ having the constructors $\top$, $\neg$, and $\forall$, $\otimes$-$\mathcal{L}$ does not have the crisp model property.*

In particular, this means that $\otimes$-$\mathcal{SHOI}$ does not have the crisp model property and is strictly more expressive than $\otimes$-$\mathcal{SHOI}^{-\forall}$.

Corollary 15. *If $\otimes$ is a t-norm without zero divisors, then $\otimes$-$\mathcal{SHOI}$ is strictly more expressive than $\otimes$-$\mathcal{SHOI}^{-\forall}$.*

5 Deciding Consistency

For a given $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology $\mathcal{O}$, we define $\mathrm{crisp}(\mathcal{O})$ to be the crisp $\mathcal{SHOI}$-ontology that is obtained from $\mathcal{O}$ by replacing all the truth values appearing in the axioms by 1. For example, for the ontology

$$\mathcal{O} = \big\{\langle a{:}C,\ 0.2\rangle, \langle (a,b){:}r,\ 0.8\rangle, \langle C \sqsubseteq D,\ 0.5\rangle, \langle r \sqsubseteq s,\ 0.1\rangle\big\}$$

we obtain

$$\mathrm{crisp}(\mathcal{O}) = \big\{\langle a{:}C,\ 1\rangle, \langle (a,b){:}r,\ 1\rangle, \langle C \sqsubseteq D,\ 1\rangle, \langle r \sqsubseteq s,\ 1\rangle\big\}.$$

Lemma 16. *Let $\mathcal{O}$ be a $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology and $\mathcal{I}$ be a crisp interpretation. Then $\mathcal{I}$ is a model of $\mathcal{O}$ iff it is a model of* $\mathrm{crisp}(\mathcal{O})$.

Proof. Assume that $\mathrm{crisp}(\mathcal{O})$ has a model $\mathcal{I}$. Let $\langle C \sqsubseteq D,\ \ell\rangle$, $\ell > 0$, be an axiom from $\mathcal{O}$. Since $\mathcal{I}$ is a model of $\mathrm{crisp}(\mathcal{O})$, it must satisfy $\langle C \sqsubseteq D,\ 1\rangle$; that is, $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq 1 \geq \ell$ holds for all $x \in \Delta^{\mathcal{I}}$. Thus $\mathcal{I}$ satisfies $\langle C \sqsubseteq D,\ \ell\rangle$. The proof that $\mathcal{I}$ satisfies assertions and role inclusions is analogous. Hence $\mathcal{I}$ is also a model of $\mathcal{O}$.

For the other direction, assume that $\mathcal{I}$ satisfies $\langle C \sqsubseteq D,\ \ell\rangle$ with $\ell > 0$. As $\mathcal{I}$ is a crisp interpretation it holds that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \in \{0,1\}$ for all $x \in \Delta^{\mathcal{I}}$. Together with $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell > 0$ we obtain $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) = 1$. Thus, $\mathcal{I}$ satisfies the GCI $\langle C \sqsubseteq D,\ 1\rangle$. The same argument can be used for role inclusions and assertions. Thus, $\mathcal{I}$ is also a model of $\mathrm{crisp}(\mathcal{O})$. $\square$

In particular, a $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology $\mathcal{O}$ has a crisp model iff $\mathrm{crisp}(\mathcal{O})$ has a crisp model. Together with Theorem 11, this shows that a $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontology $\mathcal{O}$ is consistent iff $\mathrm{crisp}(\mathcal{O})$ has a crisp model. Therefore, one can use any reasoning procedure for crisp $\mathcal{SHOI}$ to decide consistency of $\otimes$-$\mathcal{SHOI}^{-\forall}$-ontologies. Reasoning in crisp $\mathcal{SHOI}$ is known to be ExpTime-complete [15]. Recall that under crisp semantics, value restrictions can be expressed by negation and existential restrictions, and hence, crisp $\mathcal{SHOI}$ is equivalent to crisp $\mathcal{SHOI}^{-\forall}$.

[2] Recall that a fuzzy DL has Gödel negation iff its semantics is based on a t-norm without zero divisors.

Corollary 17. *Deciding consistency in* $\otimes$-$\mathcal{SHOI}^{-\forall}$ *is* ExpTime*-complete.*

Lemma 16 and Theorem 11 still hold when we restrict the semantics to the slightly less expressive logics $\otimes$-$\mathcal{SHO}^{-\forall}$, which does not allow for inverse roles, or $\otimes$-$\mathcal{SI}^{-\forall}$ which does not allow for nominals and role hierarchies. The crisp DLs $\mathcal{SHO}$ and $\mathcal{SI}$ are known to have the finite model property [16, 19], and $\otimes$-$\mathcal{SI}^{-\forall}$ and $\otimes$-$\mathcal{SHO}^{-\forall}$ inherit the finite model property from their crisp counterparts.

Theorem 18. *The logics* $\otimes$-$\mathcal{SHO}^{-\forall}$ *and* $\otimes$-$\mathcal{SI}^{-\forall}$ *and their sublogics have the finite model property.*

6 Conclusions

In this paper we have described a family of expressive fuzzy DLs for which ontology consistency is decidable. More precisely, we have shown that if $\otimes$ is a t-norm without zero divisors, consistency of $\otimes$-$\mathcal{SHOI}^{-\forall}$ ontologies is ExpTime-complete, and hence as hard as consistency of (crisp) $\mathcal{SHOI}$ ontologies. Our construction shows that the fuzzy values appearing in $\otimes$-$\mathcal{SHOI}^{-\forall}$ ontologies are irrelevant for consistency: a $\otimes$-$\mathcal{SHOI}^{-\forall}$ ontology $\mathcal{O}$ has a (fuzzy) model iff its crisp variant crisp($\mathcal{O}$), where the degrees of all the axioms in $\mathcal{O}$ are changed to 1, has a crisp model. This implies that $\otimes$-$\mathcal{SHOI}^{-\forall}$ has the crisp model property, and hence also the witnessed model property. If the constructor $\forall$ is also allowed, hence obtaining the logic $\otimes$-$\mathcal{SHOI}$, then these properties do not hold anymore.

For other reasoning problems such as entailment and subsumption it is unknown whether they are decidable in $\otimes$-$\mathcal{SHOI}^{-\forall}$. In [7] it is shown for $\otimes$-$\mathcal{SHOI}$ with witnessed models that subsumption and entailment, as well as computing the best subsumption and entailment degrees, cannot be reduced to crisp reasoning by simply mapping all nonzero truth degrees to 1. We conjecture that this is also the case in $\otimes$-$\mathcal{SHOI}^{-\forall}$.

It has recently been shown that if the t-norm $\otimes$ has zero divisors, then consistency of crisp ontologies in the very inexpressive fuzzy DL $\otimes$-$\mathfrak{NEL}$ w.r.t. witnessed models [10] and w.r.t. general models [9] is undecidable.[3] Combining these results, we obtain a characterization of the decidability of consistency w.r.t. witnessed and general models for all fuzzy DLs between $\otimes$-$\mathfrak{NEL}$ and $\otimes$-$\mathcal{SHOI}^{-\forall}$: it is decidable (in ExpTime) iff $\otimes$ has no zero divisors.

In future work, we plan to study reasoning problems in fuzzy DLs allowing for value restrictions without the restriction to witnessed models. In this direction it is worth looking at the decidability results for $\otimes$-$\mathcal{ALC}$ with product t-norm w.r.t. *quasi-witnessed* models [11].

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[3] $\otimes$-$\mathfrak{NEL}$ is the sublogic of $\otimes$-$\mathcal{SHOI}^{-\forall}$ that allows only the constructors $\top$, $\neg$ and $\exists$.

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