Consistency Reasoning in Lattice-Based Fuzzy Description Logics

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Abstract

Fuzzy Description Logics have been widely studied as a formalism for representing and reasoning with vague knowledge. One of the most basic reasoning tasks in (fuzzy) Description Logics is to decide whether an ontology representing a knowledge domain is consistent. Surprisingly, not much is known about the complexity of this problem for semantics based on complete De Morgan lattices. To cover this gap, in this paper we study the consistency problem for the fuzzy Description Logic $L\text{-}\mathcal{SHI}$ and its sublogics in detail.

The contribution of the paper is twofold. On the one hand, we provide a tableaux-based algorithm for deciding consistency when the underlying lattice is finite. The algorithm generalizes the one developed for classical $\mathcal{SHI}$. On the other hand, we identify decidable and undecidable classes of fuzzy Description Logics over infinite lattices. For all the decidable classes, we also provide tight complexity bounds.

Keywords: Fuzzy Description Logics, Residuated Lattices, Triangular Norms, Tableau Algorithm

1. Introduction

Description Logics (DLs) [1] are a family of knowledge representation formalisms that are widely used to model application domains. They have been successfully employed to formulate ontologies from several knowledge domains, most notably from the bio-medical sciences, where large ontologies like Galen,[1] SNOMED CT,[2] and the Gene Ontology[3] have been developed. They are also the underpinning formalism of the language OWL 2, which is the the current standard language for the Semantic Web recommended by the W3C.[4]

In DLs, knowledge is represented with the help of *concepts* (which can be understood as unary predicates) and *roles* (binary predicates) that relate the objects that belong to these concepts. Different kinds of axioms, collected in what is called an *ontology*, are used to restrict the possible interpretations of the concepts and roles. Axioms provide explicit pieces of knowledge that can be used to derive additional implicit consequences through reasoning. Among the many reasoning tasks that have been studied in these logics, *concept satisfiability* (is a given concept non-contradictory?) and *ontology consistency* (does a given ontology have a model?) are two of the most prominent. These and other reasoning problems have been studied for classical DLs, and several algorithms have been proposed and implemented [2–4]. Nowadays, many DL-based reasoners are available, which are highly optimized and perform well in practice, even for very large ontologies (for example RacerPro,[5] HermiT,[6] ELK,[7] and jcel[8]). Advanced editing tools (such as

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[1] `http://www.opengalen.org/`
[2] `http://www.ihtsdo.org/snomed-ct/`
[3] `http://www.geneontology.org`
[4] `http://www.w3.org/TR/owl2-overview/`
[5] `http://www.racer-systems.com/products/racerpro/`
[6] `http://hermit-reasoner.com/`
[7] `http://code.google.com/p/elk-reasoner/`
[8] `http://jcel.sourceforge.net/`

Protégé[9]) can aid the representation of different knowledge domains, and the development and maintenance of ontologies.

In their classical form, however, DLs are not well-suited for the representation of knowledge that is vague or imprecise in nature. Specifically in the bio-medical domain, vagueness is a characteristic that usually cannot be avoided. For example, descriptions of diseases and their symptoms, required for their diagnosis, are necessarily imprecise. Among these imprecise terms we can find "fever", "swelling", or "hypertension", just to name a few of common use. This motivates the need of knowledge representation formalisms capable of dealing with vagueness and imprecision in a well-founded manner. Fuzzy variants of description logics based on Zadeh's notion of Fuzzy Sets [5] were introduced in the nineties as a means to tackle this challenge [6, 7]. In particular, their applicability to the representation of medical knowledge and the development of medical ontologies was studied in detail in [8].

Fuzzy DLs generalize classical (or *crisp*) DLs by providing a *membership degree* semantics for their concepts and roles. Thus, it is possible to say that e.g. 130/85 belongs to the concept $\mathsf{HighBloodPressure}$ with a lower degree than, say 140/80. The membership degree of an individual to a fuzzy concept can be understood as a *weight* extending the logic with the possibility of expressing imprecision. Likewise, axioms describing the domain knowledge are equipped with a weight that gives additional flexibility in the restrictions of the membership degrees used, as described in the following section. Fuzzy extensions of OWL 2 that are based on certain fuzzy DLs have recently been proposed [9–11].

Originally, membership degrees were considered to be elements from the interval $[0, 1]$ of real numbers, but this was later generalized to lattices [12, 13], in particular allowing incomparable membership degrees. The papers [12, 13] consider a direct generalization of the fuzzy set semantics to lattices [14], where conjunction and disjunction are interpreted by the lattice operators meet and join, respectively. Following the ideas of Mathematical Fuzzy Logic [15], fuzzy DLs have been further extended to a more general lattice-based semantics that uses a *triangular norm* (t-norm) and its residuum as the interpretation functions for the logical constructors conjunction and implication, respectively. The interpretation of other constructors is also determined by this choice.

In general, the reasoning problems mentioned earlier (i.e. concept satisfiability and ontology consistency) are undecidable for these fuzzy DLs. In fact, since the interval $[0, 1]$ is also a lattice, the undecidability results for ontology consistency presented in [16] also transfer to this more general setting. Moreover, it has also been shown that concept satisfiability is undecidable even if we restrict to countable lattices that have only two limit points [17, 18]. These undecidability results have motivated a restriction of the semantics to *finite* lattices only.

Using automata-based techniques, it has been shown that the complexity of concept satisfiability in most DLs between $\mathcal{ALC}$ and $\mathcal{SHI}$ does not increase if only finitely many membership degrees are considered [19]. While automata-based algorithms are well-suited for proving tight complexity bounds, they are rarely used in practice because they require the same resources on all inputs; that is, their best-case and worst-case behaviors coincide. In classical DLs, tableaux-based algorithms have been shown to behave well in practice, despite not being optimal w.r.t. worst-case complexity. Another difference between the automata- and the tableaux-based approach is that the automata used for deciding concept satisfiability cannot be easily extended to deal with the cyclic structures that appear when ontology consistency is considered, but tableaux-based algorithms do not have this restriction.

In this paper we propose a tableaux-based algorithm for deciding the consistency problem for fuzzy DLs based on finite lattices. Our algorithm extends the algorithm for deciding consistency of (crisp) $\mathcal{SHI}$ ontologies [20] to deal with lattice-based semantics. In fact, when restricted to the lattice having only the two elements $\{0, 1\}$, our algorithm corresponds almost exactly to the one from [20] (see Section 6.3 for details). This has the advantage that many of the optimizations developed for crisp reasoning can also be adapted to the fuzzy setting. The algorithm, as presented in this paper, contains a high level of non-determinism and is not suited for an efficient implementation. However, it provides a framework for analyzing where the complexity of the logic arises, and optimizing it accordingly.

[9] `http://protege.stanford.edu/`

As mentioned before, there are known examples of relatively simple countable lattices for which reasoning becomes undecidable. However, the causes of undecidability in lattice-based fuzzy DLs are far from being well-understood. For the class of continuous t-norms over the interval $[0, 1]$, decidability has been almost fully characterized. In a nutshell, ontology consistency is decidable if (i) the t-norm is idempotent; that is, the Gödel t-norm [21, 22], or (ii) the t-norm has no zero divisors and the involutive negation operator is disallowed [23]. With very few exceptions, all other cases are known to be undecidable [16, 24]. For the case of general lattices, we show that the conditions (ii) also imply decidability of the problem, but for t-norms with zero divisors, even if finitely many, the distinction is not clear. We show that there are infinitely many t-norms that have exactly one zero divisor for which the problem is undecidable, but also infinitely many for which it is decidable. The construction we present can be easily generalized to any finite number of zero divisors.

The paper is structured as follows. First we introduce the relevant notions of lattice theory and fuzzy DLs that will be used throughout the paper. Then we study the case of infinite lattices, describing families of t-norms for which the problem is (un-)decidable. In Section 4 we present our tableau algorithm. We close with an overview of some related work, and possibilities of future work.

Parts of this paper, describing the tableau algorithm, have appeared in a shorter version in [18]. This paper extends and improves the results from [18] and additionally provides a more detailed analysis of the complexity of ontology consistency over infinite lattices. To increase readability, most of the technical proofs of this paper have been moved to the appendix.

2. Preliminaries

We start with a short introduction to residuated lattices, which provide the base for the semantics of the fuzzy DL L-$\mathcal{SHI}$, described later in this section. For a more comprehensive view on these lattices, in particular in connection with mathematical fuzzy logic, we refer the reader to [25–27].

2.1. Residuated Lattices

A *lattice* is a triple $(L, \vee, \wedge)$, consisting of a *carrier set* L and two idempotent, associative, and commutative binary operators $\vee$ (*join*) and $\wedge$ (*meet*) on L that satisfy the absorption laws

$$\ell_1 \vee (\ell_1 \wedge \ell_2) = \ell_1 = \ell_1 \wedge (\ell_1 \vee \ell_2)$$

for all $\ell_1, \ell_2 \in L$. These operations induce a partial order $\leq$ on L where $\ell_1 \leq \ell_2$ iff $\ell_1 \wedge \ell_2 = \ell_1$. As usual, we will write $\ell_1 < \ell_2$ if $\ell_1 \leq \ell_2$ and $\ell_1 \neq \ell_2$. A subset $T \subseteq L$ is called an *antichain (in L)* if there are no two elements $\ell_1, \ell_2 \in T$ with $\ell_1 < \ell_2$. Whenever it is clear from the context, we will use the carrier set L to refer to the lattice $(L, \vee, \wedge)$.

The lattice L is *distributive* if the operators $\vee$ and $\wedge$ distribute over each other, *finite* if L is finite, and *bounded* if it has a *minimum* and a *maximum* element, denoted as $\mathbf{0}$ and $\mathbf{1}$, respectively. It is *complete* if joins and meets of arbitrary subsets $T \subseteq L$, i.e. $\bigvee_{\ell \in T} \ell$ and $\bigwedge_{\ell \in T} \ell$, respectively, exist. Clearly, every finite lattice is also complete, and every complete lattice is bounded with

$$\mathbf{0} = \bigwedge_{\ell \in L} \ell \qquad \text{and} \qquad \mathbf{1} = \bigvee_{\ell \in L} \ell.$$

A complete lattice L is called *completely distributive* if infinite joins and meets distributive over each other, i.e. for all families of lattice elements $(\ell_{s,t})_{s \in S, t \in T}$, we have

$$\bigwedge_{s \in S} \bigvee_{t \in T} \ell_{s,t} = \bigvee_{f \in T^S} \bigwedge_{s \in S} \ell_{s,f(s)}.$$

A *De Morgan lattice* is a bounded distributive lattice L extended with an involutive and anti-monotonic unary operation $\sim$, called *(De Morgan) negation*, satisfying the De Morgan laws $\sim(\ell_1 \vee \ell_2) = \sim\ell_1 \wedge \sim\ell_2$ and $\sim(\ell_1 \wedge \ell_2) = \sim\ell_1 \vee \sim\ell_2$ for all $\ell_1, \ell_2 \in L$.

Table 1: The three fundamental continuous t-norms over $[0,1]$.

Name	t-norm ($\ell \otimes m$)	t-conorm ($\ell \oplus m$)	residuum ($\ell \Rightarrow m$)
Gödel	$\min\{\ell, m\}$	$\max\{\ell, m\}$	$\begin{cases} 1 & \text{if } \ell \leq m \\ m & \text{otherwise} \end{cases}$
Product	$\ell \cdot m$	$\ell + m - \ell \cdot m$	$\begin{cases} 1 & \text{if } \ell \leq m \\ m/\ell & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{\ell + m - 1, 0\}$	$\min\{\ell + m, 1\}$	$\min\{1 - \ell + m, 1\}$

A very important notion in the area of mathematical fuzzy logic is that of a triangular norm, or t-norm for short. We define this for arbitrary lattices, although in the literature the term is usually only used when talking about the real interval $[0,1]$ or finite chains [26–28].

Definition 2.1. Given a lattice L, a *(generalized) t-norm* is an associative and commutative binary operator on L that is monotonic and has $\mathbf{1}$ as its unit. A *residuated lattice* is a lattice L extended with a t-norm $\otimes$ and a binary operator $\Rightarrow$ (called the *residuum*) such that for every $\ell_1, \ell_2, \ell_3 \in L$ it holds that $\ell_1 \otimes \ell_2 \leq \ell_3$ iff $\ell_2 \leq \ell_1 \Rightarrow \ell_3$. The *residual negation* $\ominus \ell$ is defined with the help of the residuum as $\ominus \ell := \ell \Rightarrow \mathbf{0}$.

It should be noted that what we call a residuated lattice corresponds to commutative, distributive, integral, zero-bounded FL-algebras from [27]. We chose to call these residuated lattices to keep the relation with mathematical fuzzy logic explicit.

A simple consequence of Definition 2.1 is that for every $\ell_1, \ell_2 \in L$ it holds that (i) $\mathbf{1} \Rightarrow \ell_1 = \ell_1$, and (ii) $\ell_1 \leq \ell_2$ iff $\ell_1 \Rightarrow \ell_2 = \mathbf{1}$. For a t-norm $\otimes$ over a complete lattice L, there is a binary operator $\Rightarrow$ that satisfies the residuation property w.r.t. $\otimes$ iff the t-norm is *join-preserving* [27], i.e. for all $\ell \in L$ and $T \subseteq L$ we have

$$\ell \otimes (\bigvee_{\ell' \in T} \ell') = \bigvee_{\ell' \in T} (\ell \otimes \ell').$$

In this case, $\Rightarrow$ is unique and can be computed as $\ell_1 \Rightarrow \ell_2 = \bigvee\{m \mid \ell_1 \otimes m \leq \ell_2\}$ for all $\ell_1, \ell_2 \in L$. Using this result, we will often characterize a complete residuated lattice through its t-norm, without explicitly mentioning its residuum. If L is a completely distributive lattice, then it can always be extended to a residuated lattice with the meet operator $\ell_1 \wedge \ell_2$ as its t-norm. This t-norm is often called the *Gödel t-norm.*

In a residuated De Morgan lattice L, the *t-conorm* $\oplus$ is defined as $\ell_1 \oplus \ell_2 := {\sim}({\sim}\,\ell_1 \otimes {\sim}\,\ell_2)$. From the De Morgan laws, it follows that the t-conorm of the Gödel t-norm is the join operator $\ell_1 \vee \ell_2$. Note that, by monotonicity of $\otimes$ and anitonicity of $\sim$, the value $\ell_1 \otimes \ell_2$ is always smaller than or equal to $\ell_1 \wedge \ell_2$ and $\ell_1 \oplus \ell_2$ is always greater than or equal to $\ell_1 \vee \ell_2$. In other words, the Gödel t-norms is the largest possible t-norm over a given lattice L.

Most work on fuzzy logic is focused on the chain $L = [0,1]$ of all real numbers between 0 and 1, with the usual ordering. In this setting, a t-norm is join-preserving iff it is left-continuous as a function from $[0,1] \times [0,1]$ to $[0,1]$ with the standard topology. If we restrict to continuous t-norms (left- and right-continuous), there are three basic t-norms, which are presented in Table 1. All continuous t-norms over $[0,1]$ can be constructed from those three as follows. Let $((a_i, b_i))_{i \in I}$ be a (possibly infinite) family of disjoint open subintervals of $[0,1]$ and $(\otimes_i)_{i \in I}$ be a family of continuous t-norms over $[0,1]$ over the same index set I. Then the *ordinal sum* of the t-norms $\otimes_i$ is defined as the t-norm $\otimes$, where, for all $\ell_1, \ell_2 \in [0,1]$,

$$\ell_1 \otimes \ell_2 = \begin{cases} a_i + (b_i - a_i)\left(\frac{\ell_1 - a_i}{b_i - a_i} \otimes_i \frac{\ell_2 - a_i}{b_i - a_i}\right) & \text{if } \ell_1, \ell_2 \in [a_i, b_i] \text{ for some } i \in I, \\ \min\{\ell_1, \ell_2\} & \text{otherwise.} \end{cases}$$

The ordinal sum of a family of continuous t-norms is itself a continuous t-norm, with the residuum given by

$$\ell_1 \Rightarrow \ell_2 = \begin{cases} 1 & \text{if } \ell_1 \le \ell_2, \\ a_i + (b_i - a_i)\left(\frac{\ell_1 - a_i}{b_i - a_i} \Rightarrow_i \frac{\ell_2 - a_i}{b_i - a_i}\right) & \text{if } a_i \le \ell_2 < \ell_1 \le b_i \text{ for some } i \in I, \\ \ell_2 & \text{otherwise,} \end{cases}$$

where $\Rightarrow_i$ is the residuum of $\otimes_i$, for each $i \in I$. Intuitively, this means that the t-norm $\otimes$ and its residuum "behave like" $\otimes_i$ and its residuum in each of the intervals $[a_i, b_i]$, and like the Gödel t-norm and residuum everywhere else.

Theorem 2.2 ([29]). *Every continuous t-norm over $[0,1]$ is isomorphic to the ordinal sum of copies of the Łukasiewicz and product t-norms.*

Motivated by this representation as an ordinal sum, we say that a continuous t-norm $\otimes$ *starts with the Łukasiewicz t-norm* if in its representation as ordinal sum there is an $i \in I$ such that $a_i = 0$ and $\otimes_i$ is isomorphic to the Łukasiewicz t-norm. Another important notion is that of zero divisors. An element $\ell > \mathbf{0}$ of a residuated lattice L is called a *zero divisor* if there exists an $\ell' \in L$ such that $\ell' > \mathbf{0}$ and $\ell \otimes \ell' = \mathbf{0}$. For every lattice without zero divisors, the residual negation is *crisp*, i.e. it is always either $\mathbf{0}$ or $\mathbf{1}$.

Proposition 2.3 ([27]). *Let L be a complete residuated De Morgan lattice without zero divisors. For all $\ell_1, \ell_2 \in L$, the following two statements hold:*

a) $\ell_1 \Rightarrow \ell_2 = \mathbf{0}$ iff $\ell_1 > \mathbf{0}$ and $\ell_2 = \mathbf{0}$, and

b) $\ominus \ell_1 = \begin{cases} \mathbf{0} & \text{if } \ell_1 > \mathbf{0}, \\ \mathbf{1} & \text{otherwise.} \end{cases}$

Of the three fundamental continuous t-norms over $[0,1]$ from Table 1, only the Łukasiewicz t-norm has zero divisors. In fact, every element ℓ in the interval $(0,1)$ is a zero divisor for this t-norm since $\ominus \ell = 1 - \ell > 0$ and $\ell \otimes (1-\ell) = \max\{\ell + 1 - \ell - 1, 0\} = 0$. Moreover, a continuous t-norm over $[0,1]$ can only have zero divisors if it starts with the Łukasiewicz t-norm.

Lemma 2.4 ([28]). *A continuous t-norm over $[0,1]$ has zero divisors iff it starts with the Łukasiewicz t-norm.*

We are interested in fuzzy logics where the set of membership degrees is not restricted to be a total order, nor does it need to be infinite.

Example 2.5. Consider the simple finite lattice L_4 with the elements f, u, i, and t shown in Figure 1.[10] We can use it to collate information from different sources, e.g. medical textbooks or doctors. Most books only have information on a specific domain and tell us nothing about other topics. Furthermore, a doctor might disagree with a textbook on a certain issue because it contradicts her experience or she knows about the latest clinical trials.

We can use the truth degree t to indicate that at least one source supports a given statement, and no other source refutes it. Similarly, the value f means that the statement is known to be false by some sources, and not known to be true by any other. Finally, u says that no information on the truth or falsity of the statement is available, while i is used when two or more sources disagree. For example, we might say that a patient has high blood pressure to degree i if the examination by two doctors led to different diagnoses.

Partial orders are useful in situations where incomparable membership degrees need to be modeled, for instance disagreement (i) and no information (u). The incomparability of these degrees reflects the fact

[10] This lattice has often been used for paraconsistent reasoning about incomplete and contradictory knowledge [30, 31].

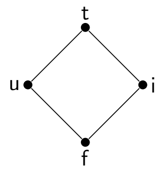

Figure 1: The residuated De Morgan lattice L_4 with $\sim\mathsf{u} = \mathsf{u}$ and $\sim\mathsf{i} = \mathsf{i}$.

that none of them represents a higher degree of truth than the other. On the other hand, using discrete structures it is possible to model the notion of a *next* membership degree, which does not exist in the $[0, 1]$ continuum. For example, in the lattice L_4 there is exactly one membership degree that is greater than u, namely t.

Notice that any t-norm on the lattice L_4 will have zero divisors. In fact, if we consider the Gödel t-norm, then $\mathsf{i} \otimes \mathsf{u} = \mathsf{f}$, and hence both i and u are zero divisors.[11] In that case, it holds that $\mathsf{i} \Rightarrow \mathsf{f} = \mathsf{u}$ and $\mathsf{u} \Rightarrow \mathsf{f} = \mathsf{i}$. It is also important to point out that for general lattices no representation theorem analogous to Theorem 2.2, nor a characterization of t-norms with zero divisors in the style of Lemma 2.4, is known.

For the rest of this paper, L will denote a complete residuated De Morgan lattice with the t-norm $\otimes$ and the residuum $\Rightarrow$, unless explicitly stated otherwise.

2.2. Lattice-Based Fuzzy Description Logics

The fuzzy DL L-$\mathcal{SHI}$ is a generalization of the expressive crisp DL $\mathcal{SHI}$ that uses the elements of the lattice L as truth values, instead of just the Boolean *true* and *false*. The syntax of L-$\mathcal{SHI}$ is the same as in the classical DL $\mathcal{SHI}$ with the addition of the constructor $\to$ for the implication, which is expressible by $\neg$ and $\sqcup$ in the crisp case, but not in the fuzzy case in general.

Definition 2.6 (syntax of L-$\mathcal{SHI}$). Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be pairwise disjoint sets of *concept-*, *role-*, and *individual names*, respectively, and $\mathsf{N_{tR}} \subseteq \mathsf{N_R}$ a set of *transitive role names*. The set of *(complex) roles* is $\mathsf{N_R} \cup \{r^- \mid r \in \mathsf{N_R}\}$. The set of *(complex) concepts* is the smallest set containing all concept names $A \in \mathsf{N_C}$ such that if C, D are concept names and s is a (complex) role, then the *top concept* $\top$, the *bottom concept* $\bot$, the *conjunction* $C \sqcap D$, the *disjunction* $C \sqcup D$, the *implication* $C \to D$, the *negation* $\neg C$, the *existential restriction* $\exists s.C$, and the *value restriction* $\forall s.C$ are also concepts. The *inverse* of a complex role s (denoted by $\overline{s}$) is s^- if $s \in \mathsf{N_R}$ and r if $s = r^-$. A complex role s is *transitive* if either s or $\overline{s}$ belongs to $\mathsf{N_{tR}}$.

The semantics of this logic is based on interpretation functions that specify, for every individual x of an interpretation domain and concept C, the degree of membership for x to belong to C, as motivated by the notion of L-fuzzy sets [14].

Definition 2.7 (semantics of L-$\mathcal{SHI}$). An *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty *domain*, and $\cdot^\mathcal{I}$ is a function that assigns to every individual name a an element $a^\mathcal{I} \in \Delta^\mathcal{I}$, to every concept name A a function $A^\mathcal{I} \colon \Delta^\mathcal{I} \to L$, and to every role name r a function $r^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to L$, where $r^\mathcal{I}(x, y) \otimes r^\mathcal{I}(y, z) \leq r^\mathcal{I}(x, z)$ holds for all $r \in \mathsf{N_{tR}}$ and $x, y, z \in \Delta^\mathcal{I}$.

The function $\cdot^\mathcal{I}$ is extended to complex L-$\mathcal{SHI}$ roles and concepts as follows. For all $x, y \in \Delta^\mathcal{I}$ and $r \in \mathsf{N_R}$, we define $(r^-)^\mathcal{I}(x, y) := r^\mathcal{I}(y, x)$. For every $x \in \Delta^\mathcal{I}$,

- $\top^\mathcal{I}(x) := \mathbf{1}$,
- $\bot^\mathcal{I}(x) := \mathbf{0}$,
- $(C \sqcap D)^\mathcal{I}(x) := C^\mathcal{I}(x) \otimes D^\mathcal{I}(x)$,
- $(C \sqcup D)^\mathcal{I}(x) := C^\mathcal{I}(x) \oplus D^\mathcal{I}(x)$,

[11] Recall that the Gödel t-norm is the largest t-norm.

- $(C \to D)^{\mathcal{I}}(x) := C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$,
- $(\neg C)^{\mathcal{I}}(x) := {\sim} C^{\mathcal{I}}(x)$,
- $(\exists s.C)^{\mathcal{I}}(x) := \bigvee_{y\in\Delta^{\mathcal{I}}} \big(s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\big)$,
- $(\forall s.C)^{\mathcal{I}}(x) := \bigwedge_{y\in\Delta^{\mathcal{I}}} \big(s^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\big)$.

It is important to note that the constructor $\neg$ is interpreted by the De Morgan negation $\sim$, and not by the residual negation $\ominus$. The semantics of the existential and value restrictions is the direct application of the semantics of quantification of fuzzy first-order logic [15, 26] to L-fuzzy DLs. Notice that, unlike in crisp $\mathcal{SHI}$, existential and value restrictions are not dual to each other, i.e. in general, $(\neg\exists s.C)^{\mathcal{I}}(x) = (\forall s.\neg C)^{\mathcal{I}}(x)$ does not hold. Likewise, the implication constructor $\to$ cannot be expressed in terms of the negation $\neg$ and disjunction $\sqcup$. This is shown in the following example.

Example 2.8. Consider the lattice L_4 from Figure 1, with the Gödel t-norm, and let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation such that $\Delta^{\mathcal{I}} = \{\delta, \gamma\}$ and whose interpretation function satisfies

$$\begin{aligned} A^{\mathcal{I}}(\delta) &= \mathsf{u}, & A^{\mathcal{I}}(\gamma) &= \mathsf{i}, \\ B^{\mathcal{I}}(\delta) &= \mathsf{i}, & & \\ r^{\mathcal{I}}(\delta,\delta) &= \mathsf{u}, & r^{\mathcal{I}}(\delta,\gamma) &= \mathsf{t}. \end{aligned}$$

Under this interpretation, it follows that $(A \to B)^{\mathcal{I}}(\delta) = A^{\mathcal{I}}(\delta) \Rightarrow B^{\mathcal{I}}(\delta) = \mathsf{u} \Rightarrow \mathsf{i} = \mathsf{i}$, but $(\neg A)^{\mathcal{I}}(\delta) = \mathsf{u}$ and hence $(\neg A \sqcup B)^{\mathcal{I}}(\delta) = \mathsf{u} \vee \mathsf{i} = \mathsf{t}$.

Additionally, it follows that $(\exists r.A)^{\mathcal{I}}(\delta) = (r^{\mathcal{I}}(\delta,\delta) \wedge A^{\mathcal{I}}(\delta)) \vee (r^{\mathcal{I}}(\delta,\gamma) \wedge A^{\mathcal{I}}(\gamma)) = (\mathsf{u}\wedge\mathsf{u}) \vee (\mathsf{t}\wedge\mathsf{i}) = \mathsf{u}\vee\mathsf{i} = \mathsf{t}$, but $(\forall r.\neg A)^{\mathcal{I}}(\delta) = (\mathsf{u} \Rightarrow \mathsf{u}) \wedge (\mathsf{t} \Rightarrow \mathsf{i}) = \mathsf{t} \wedge \mathsf{i} = \mathsf{i}$.

The axioms of the fuzzy DL L-$\mathcal{SHI}$ are similar to those of crisp $\mathcal{SHI}$, but have an associated lattice value, which expresses the degree to which the restriction must be satisfied.

Definition 2.9 (axioms). An *assertion* is either a *concept assertion* of the form $\langle a{:}C \rhd \ell\rangle$ or a *role assertion* of the form $\langle (a,b){:}s \rhd \ell\rangle$, where C is a concept, s is a complex role, a, b are individual names, $\ell \in L$, and $\rhd \in \{=, \geq\}$. If $\rhd$ is $=$, then it is called an *equality assertion.* A *general concept inclusion (GCI)* is of the form $\langle C \sqsubseteq D \geq \ell\rangle$, where C, D are concepts, and $\ell \in L$. A *role inclusion* is of the form $s \sqsubseteq s'$, where s and s' are complex roles.

An *ontology* $(\mathcal{A}, \mathcal{T}, \mathcal{R})$ consists of a finite set $\mathcal{A}$ of assertions (*ABox*), a finite set $\mathcal{T}$ of GCIs (*TBox*), and a finite set $\mathcal{R}$ of role inclusions (*RBox*). The ABox $\mathcal{A}$ is called *local* if there is an individual $a \in \mathsf{N_I}$ such that all assertions in $\mathcal{A}$ are of the form $\langle a{:}C = \ell\rangle$, for some concept C and $\ell \in L$.

An interpretation $\mathcal{I}$ *satisfies* the assertion $\langle a{:}C \rhd \ell\rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \ell$ and the assertion $\langle (a,b){:}s \rhd \ell\rangle$ if $s^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \ell$. It satisfies the GCI $\langle C \sqsubseteq D \geq \ell\rangle$ if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ holds for every $x \in \Delta^{\mathcal{I}}$. It satisfies the role inclusion $s \sqsubseteq s'$ if for all $x, y \in \Delta^{\mathcal{I}}$ we have $s^{\mathcal{I}}(x,y) \leq s'^{\mathcal{I}}(x,y)$.

$\mathcal{I}$ is a *model* of the ontology $(\mathcal{A},\mathcal{T},\mathcal{R})$ if it satisfies all axioms in $\mathcal{A}$, $\mathcal{T}$, and $\mathcal{R}$.

Given an RBox $\mathcal{R}$, the *role hierarchy* $\sqsubseteq_{\mathcal{R}}$ on the set of complex roles is the reflexive and transitive closure of the relation

$$\{(s,s') \mid s \sqsubseteq s' \in \mathcal{R} \text{ or } \overline{s} \sqsubseteq \overline{s'} \in \mathcal{R}\}.$$

Using reachability algorithms, the role hierarchy can be computed in polynomial time in the size of $\mathcal{R}$. An RBox $\mathcal{R}$ is called *acyclic* if it contains no cycles of the form $s \sqsubseteq_{\mathcal{R}} s'$, $s' \sqsubseteq_{\mathcal{R}} s$ for two roles $s \neq s'$.

There are several sublogics of L-$\mathcal{SHI}$ which we will study in more detail:

$\mathbf{2}$-$\mathcal{SHI}$ We consider the restriction of our logic to the two-element sublattice $\mathbf{2}$ of L over the set $\{\mathbf{0}, \mathbf{1}\}$. The resulting logic $\mathbf{2}$-$\mathcal{SHI}$ is a syntactic variant of the crisp DL $\mathcal{SHI}$, where sets $X \subseteq \Delta^{\mathcal{I}}$ are instead viewed as characteristic functions $X\colon \Delta^{\mathcal{I}} \to \{\mathbf{0}, \mathbf{1}\}$. Since in this setting t-norm, t-conorm, and negation behave just like classical conjunction, disjunction, and negation, respectively, and the concepts $C \to D$ and $(\neg C) \sqcup D$ have the same semantics, these two logics are indeed equally expressive. However, it is sometimes more convenient to use **2**-$\mathcal{SHI}$ as a fuzzy sublogic of L-$\mathcal{SHI}$ instead of $\mathcal{SHI}$, to keep a consistent representation.

L-$\mathcal{SHI}^-$ We also consider the sublogic L-$\mathcal{SHI}^-$ of L-$\mathcal{SHI}$ in which the negation constructor $\neg$ and equality assertions $\langle a\!:\!C = \ell\rangle$ and $\langle (a,b)\!:\!r = \ell\rangle$ are not allowed. Notice that in this logic it is still possible to express the residual negation using the concept $C \to \bot$.

L-$\mathcal{ALC}$ The fuzzy DL L-$\mathcal{ALC}$ is the sublogic of L-$\mathcal{SHI}$ where no role inclusions, transitive roles, or inverse roles are allowed. The sublogic L-$\mathfrak{NEL}$ of L-$\mathcal{ALC}$ only allows the constructors conjunction, existential restriction, and top, and the special constructor $C \to \bot$ for the residual negation. Finally, L-$\mathfrak{IELU}$ extends L-$\mathfrak{NEL}$ by allowing disjunctions and arbitrary implications.

Recall that the semantics of the quantifiers require the computation of a join or meet of the membership degrees of a possibly infinite set of elements of the domain. To obtain effective decision procedures, reasoning is usually restricted to a special kind of models, called witnessed models [15].

Definition 2.10 (witnessed model). Let $n \in \mathbb{N}$. A model $\mathcal{I}$ of an ontology $\mathcal{O}$ is *n-witnessed* if for every $x \in \Delta^{\mathcal{I}}$, every role s and every concept C there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that

$$(\exists s.C)^{\mathcal{I}}(x) = \bigvee_{i=1}^{n} \left(s^{\mathcal{I}}(x, x_i) \otimes C^{\mathcal{I}}(x_i)\right) \text{ and}$$
$$(\forall s.C)^{\mathcal{I}}(x) = \bigwedge_{i=1}^{n} \left(s^{\mathcal{I}}(x, y_i) \Rightarrow C^{\mathcal{I}}(y_i)\right).$$

In particular, if $n = 1$, the joins and meets from the semantics of $\exists s.C$ and $\forall s.C$ are maxima and minima, respectively, and we say that $\mathcal{I}$ is *witnessed*.

We now generalize the reasoning problems for $\mathcal{SHI}$ to the fuzzy semantics of L-$\mathcal{SHI}$.

Definition 2.11 (decision problems). Let $\mathcal{O}$ be an ontology, C, D be two concepts, $a \in \mathsf{N_I}$, and $\ell \in L$.

- $\mathcal{O}$ is *consistent* if it has a (witnessed) model.
- C is *strongly ℓ-satisfiable* if there is a (witnessed) model $\mathcal{I}$ of $\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$ with $C^{\mathcal{I}}(x) \geq \ell$.
- The individual a is an *ℓ-instance* of C if $\langle a\!:\!C \geq \ell\rangle$ is satisfied by all (witnessed) models of $\mathcal{O}$.
- C is *ℓ-subsumed* by D if $\langle C \sqsubseteq D \geq \ell\rangle$ is satisfied by all (witnessed) models of $\mathcal{O}$.

In the following example we show how the different membership degrees can be useful for representing knowledge containing vague concepts.

Example 2.12. It is known that coffee drinkers and salt consumers tend to have a higher blood pressure. On the other hand, bradycardia is highly correlated with a lower blood pressure. However, some sources may disagree on the validity of these statements, which can be expressed through the TBox

$$\{\langle \mathsf{CoffeeDrinker} \sqsubseteq \mathsf{HighBloodPressure} \geq \mathsf{i}\rangle,\ \langle \mathsf{SaltConsumer} \sqsubseteq \mathsf{HighBloodPressure} \geq \mathsf{i}\rangle,$$
$$\langle \mathsf{Bradycardia} \sqsubseteq \neg\mathsf{HighBloodPressure} \geq \mathsf{i}\rangle\},$$

over the lattice L_4 from Example 2.5. The degree i in these axioms expresses that the relation between the causes and $\mathsf{HighBloodPressure}$ is not absolute. Consider the patients ana, who is a coffee drinker, and bob, a salt consumer with bradycardia, as expressed by the ABox

$$\{\langle \mathsf{ana}\!:\!\mathsf{CoffeeDrinker} = \mathsf{t}\rangle,\ \langle \mathsf{bob}\!:\!\mathsf{SaltConsumer} \sqcap \mathsf{Bradycardia} = \mathsf{t}\rangle\}.$$

We can deduce that both patients are an i-instance of HighBloodPressure, but only bob is an i-instance of ¬HighBloodPressure. Notice that if we changed all the degrees from the GCIs to the value t, then bob would have to be a t-instance of both HighBloodPressure and ¬HighBloodPressure, which means that the ontology is inconsistent.

We have shown in [17, 18] that satisfiability and consistency in L-$\mathcal{SHI}$ is undecidable in general, even if we restrict the semantics to countable total orders. However, there is no characterization of the properties under which reasoning w.r.t. an infinite lattice is undecidable. In the next section, we will study the decidability of the consistency problem for particular classes of infinite lattices. In particular, we provide a non-trivial infinite family of t-norms for which ontology consistency is decidable. A class of undecidable t-norms is also described.

Afterwards, we will provide reasoning procedures for the case that the underlying lattice is finite. There, we focus first on a version of the consistency problem where the ABox is required to be a local ABox; we call this problem *local consistency*. We show in Section 5 that local consistency can be used for solving other reasoning problems in L-$\mathcal{SHI}$ if L is finite.

3. Consistency over Infinite Lattices

In this section, we restrict our considerations to the logic L-$\mathcal{SHI}^-$ that does not allow the involutive negation operator and equality assertions. We show some undecidability results, which obviously also transfer to the logic L-$\mathcal{SHI}$ with the involutive negation. We also characterize some cases where the problem is decidable; these decidability results do not necessarily transfer to full L-$\mathcal{SHI}$.

For continuous t-norms over the standard chain $[0,1]$, the decidability status of L-$\mathcal{SHI}^-$ is well understood. In fact, it has been shown that if the t-norm has no zero divisors, then consistency can be reduced in linear time to consistency of crisp ontologies [23]. On the other hand, for any t-norm having (infinitely many) zero divisors,[12] ontology consistency has been shown to be undecidable [16], even if we restrict expressivity to the logic L-$\mathfrak{NEL}$, allowing only existential restrictions, conjunctions, and residual negations, and all axioms hold with degree 1. Based on these results, we identify a class of t-norms for which the problem is decidable, and another class in which the problem becomes undecidable.

First we show that in the absence of zero divisors ontology consistency is linearly reducible to crisp consistency, regardless of the shape of the lattice, similarly to the result in [23]. For any L-$\mathcal{SHI}^-$-ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$, we define a $\mathbf{2}$-$\mathcal{SHI}^-$-ontology $\mathcal{O}' = (\mathcal{A}', \mathcal{T}', \mathcal{R})$ of size linear in the size of $\mathcal{O}$ that is consistent in $\mathbf{2}$-$\mathcal{SHI}^-$ iff $\mathcal{O}$ is consistent in L-$\mathcal{SHI}^-$. Since consistency in $\mathbf{2}$-$\mathcal{SHI}^-$ is decidable in ExpTime [32], this shows that consistency in L-$\mathcal{SHI}^-$ is also in ExpTime. We set

$$\mathcal{A}' := \{\langle \alpha \geq \mathbf{1} \rangle \mid \langle \alpha \geq \ell \rangle \in \mathcal{A}, \ell > \mathbf{0}\} \text{ and}$$
$$\mathcal{T}' := \{\langle C \sqsubseteq D \geq \mathbf{1} \rangle \mid \langle C \sqsubseteq D \geq \ell \rangle \in \mathcal{T}, \ell > \mathbf{0}\}.$$

The proof of the equivalence of the two consistency problems can be found in the appendix.

Theorem 3.1. *If L has no zero divisors, then consistency in L-$\mathcal{SHI}^-$ is decidable in* ExpTime.

The question is now what happens with t-norms that have zero divisors. We show that, in general, ontology consistency becomes undecidable, even if only one zero divisor exists. In fact, we provide a stronger result: we identify an infinite family of infinite residuated chains that have exactly one zero divisor, and for which ontology consistency is undecidable.

Definition 3.2 (L_∞). Let $\otimes$ be a continuous t-norm on the interval $[0,1]$. The De Morgan lattice L_∞ is given by $L_\infty := [0,1] \cup \{-\infty, -2, 2, \infty\}$ with the usual ordering and De Morgan negation $\sim\ell = 1 - \ell$ for $\ell \in [0,1]$ and $\sim\ell = -\ell$ for $\ell \in \{-\infty, -2, 2, \infty\}$.

[12] If an element ℓ is a zero divisor, then every $\ell', \mathbf{0} < \ell' \leq \ell$ is also a zero divisor. Thus, a t-norm over $[0,1]$ has a zero divisor iff it has infinitely many zero divisors.

The t-norm $\otimes_\infty$ over the lattice L_∞ is defined as follows for every $\ell_1, \ell_2 \in L_\infty$:

$$\ell_1 \otimes_\infty \ell_2 := \begin{cases} \ell_1 \otimes \ell_2 & \text{if } \ell_1, \ell_2 \in [0,1] \\ 1 & \text{if } \ell_1 = \ell_2 = 2 \\ -\infty & \text{if } \ell_1 = \ell_2 = -2 \\ \min\{\ell_1, \ell_2\} & \text{otherwise.} \end{cases}$$

A simple consequence of the continuity of $\otimes$ is that $\otimes_\infty$ is join-preserving, and hence has a unique residuum $\Rightarrow_\infty$ such that $\ell_1 \Rightarrow_\infty \ell_2 = \infty$ if $\ell_1 \leq \ell_2$, and for all $\ell_1, \ell_2 \in L_\infty$ such that $\ell_1 > \ell_2$,

$$\ell_1 \Rightarrow_\infty \ell_2 = \begin{cases} \ell_1 \Rightarrow \ell_2 & \text{if } \ell_1, \ell_2 \in [0,1] \\ 2 & \text{if } \ell_1 = 2, \ell_2 = 1 \\ -2 & \text{if } \ell_1 = -2, \ell_2 = -\infty \\ \ell_2 & \text{otherwise.} \end{cases}$$

The t-conorm $\oplus_\infty$ defined by $\otimes_\infty$ is given, for every $\ell_1, \ell_2 \in L_\infty$ by

$$\ell_1 \oplus_\infty \ell_2 = \begin{cases} \ell_1 \oplus \ell_2 & \text{if } \ell_1, \ell_2 \in [0,1] \\ \infty & \text{if } \ell_1 = \ell_2 = 2 \\ 0 & \text{if } \ell_1 = \ell_2 = -2 \\ \max\{\ell_1, \ell_2\} & \text{otherwise.} \end{cases}$$

Notice, moreover, that -2 is the only zero divisor w.r.t. the t-norm $\otimes_\infty$. We now show that if the t-norm $\otimes$ has zero divisors, then ontology consistency in the fuzzy DL L_∞-$\mathcal{IELU}$ is undecidable.

Theorem 3.3. *Let $\otimes$ be a continuous t-norm over $[0,1]$ that starts with the Łukasiewicz t-norm. Then consistency of L_∞-$\mathcal{IELU}$ ontologies is undecidable.*

We prove Theorem 3.3 by a reduction from ontology consistency in $\otimes$-$\mathfrak{NEL}$. It has been previously shown that for any continuous t-norm $\otimes$ over the interval $[0,1]$ that starts with the Łukasiewicz t-norm, ontology consistency is undecidable for the inexpressive DL $\otimes$-$\mathfrak{NEL}$ even if all axioms are crisp; that is, they hold with degree 1 [16].

For a given crisp $\otimes$-$\mathfrak{NEL}$ ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \emptyset)$, we build an L_∞-$\mathcal{IELU}$ ontology $\mathcal{O}_\infty$ that preserves the semantics of $\mathcal{O}$. Let Bot be a concept name not appearing in $\mathcal{O}$. We first recursively define the function ρ that maps $\mathfrak{NEL}$ concepts to $\mathcal{IELU}$ concepts as follows. We set $\rho(\top) := \top$ and, for every concept name A, $\rho(A) := A$. If C and D are two $\mathfrak{NEL}$ concepts, then

- $\rho(C \sqcap D) := \rho(C) \sqcap \rho(D)$,
- $\rho(\exists r.C) := \exists r.\rho(C)$, and
- $\rho(C \to \bot) := \rho(C) \to \mathsf{Bot}$.

The ontology $\mathcal{O}_\infty := (\mathcal{A}_\infty, \mathcal{T}_\infty, \emptyset)$ is then given by

$$\begin{aligned} \mathcal{A}_\infty \;&:=\; \{\langle a : \rho(C) \geq 1\rangle \mid \langle a : C \geq 1\rangle \in \mathcal{O}\} \cup \\ & \{\langle (a,b) : s \geq 1\rangle \mid \langle (a,b) : s \geq 1\rangle \in \mathcal{O}\} \\ \mathcal{T}_\infty \;&:=\; \{\langle \rho(C) \sqsubseteq \rho(D) \geq 1\rangle \mid \langle C \sqsubseteq D \geq 1\rangle \in \mathcal{O}\} \cup \\ & \{\langle \top \sqsubseteq \mathsf{Nil} \geq -2\rangle, \langle \top \sqsubseteq (\mathsf{Nil} \sqcap \mathsf{Nil}) \to \bot \geq \infty\rangle\} \cup \qquad (1) \\ & \{\langle \mathsf{Bot} \sqsubseteq \mathsf{Nil} \sqcup \mathsf{Nil} \geq \infty\rangle, \langle \mathsf{Nil} \sqcup \mathsf{Nil} \sqsubseteq \mathsf{Bot} \geq \infty\rangle\}, \qquad (2) \end{aligned}$$

where Nil is a new concept name not appearing in $\mathcal{O}$ and different from Bot.

We comment first on the last axioms of $\mathcal{O}_\infty$ appearing in the lines (1) and (2) of the definition of $\mathcal{T}_\infty$. The axioms from (1) restrict the concept name Nil to be interpreted as the constant -2 in every model of $\mathcal{O}_\infty$, as described next. The first axiom requires the interpretation of Nil to be always greater or equal to -2. The second axiom expresses that for every model $\mathcal{I}$ and every $x \in \Delta^\mathcal{I}$ it holds that $\mathsf{Nil}^\mathcal{I}(x) \otimes_\infty \mathsf{Nil}^\mathcal{I}(x) \leq \bot^\mathcal{I}(x) = -\infty$. Thus, together these two axioms restrict every model of $\mathcal{O}_\infty$ to interpret the concept name Nil as the constant -2. Consider now the axioms in (2). These axioms state that

$$\mathsf{Bot}^\mathcal{I}(x) = \mathsf{Nil}^\mathcal{I}(x) \oplus_\infty \mathsf{Nil}^\mathcal{I}(x) = -2 \oplus_\infty -2 = 0$$

for every model $\mathcal{I}$ and every $x \in \Delta^\mathcal{I}$. The idea behind this restriction is that Bot will be used to simulate the bottom concept $\bot$ from the original ontology $\mathcal{O}$, as suggested by the transformation ρ. We now show that $\mathcal{O}$ is consistent iff $\mathcal{O}_\infty$ is consistent.

Let $\mathcal{I}$ be a model of $\mathcal{O}$, and let $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be the interpretation where $\Delta^\mathcal{J} := \Delta^\mathcal{I}$, for every role name r and individual name a we have $r^\mathcal{J} := r^\mathcal{I}$ and $a^\mathcal{J} := a^\mathcal{I}$, and for every $x \in \Delta^\mathcal{J}$ and concept name A,

$$A^\mathcal{J}(x) := \begin{cases} 0 & \text{if } A = \mathsf{Bot} \\ -2 & \text{if } A = \mathsf{Nil} \\ A^\mathcal{I}(x) & \text{otherwise.} \end{cases}$$

Clearly, as $\mathsf{Bot}^\mathcal{J}(x) = 0$ and $\mathsf{Nil}^\mathcal{J}(x) = -2$ for every $x \in \Delta^\mathcal{J}$, $\mathcal{J}$ satisfies the axioms from (1) and (2). In fact, $\mathcal{J}$ is a model of $\mathcal{O}_\infty$. The proof of the following lemma can be found in the appendix.

Lemma 3.4. *$\mathcal{J}$ is a model of $\mathcal{O}_\infty$.*

This shows that if $\mathcal{O}$ is consistent, then $\mathcal{O}_\infty$ is consistent too. For the converse direction, let now $\mathcal{I}$ be a model of $\mathcal{O}_\infty$. The $[0,1]$-interpretation $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ uses the same domain as $\mathcal{I}$; that is, $\Delta^\mathcal{J} := \Delta^\mathcal{I}$, for every individual name a we have $a^\mathcal{J} := a^\mathcal{I}$, and for every role name r, every concept name A and $x, y \in \Delta^\mathcal{I}$,

$$r^\mathcal{J}(x,y) := \begin{cases} 0 & \text{if } r^\mathcal{I}(x,y) \leq 0 \\ 1 & \text{if } r^\mathcal{I}(x,y) \geq 1 \\ r^\mathcal{I}(x,y) & \text{otherwise,} \end{cases} \qquad A^\mathcal{J}(x) := \begin{cases} 0 & \text{if } A^\mathcal{I}(x) \leq 0 \\ 1 & \text{if } A^\mathcal{I}(x) \geq 1 \\ A^\mathcal{I}(x) & \text{otherwise.} \end{cases}$$

The interpretation $\mathcal{J}$ can be seen as an approximation of $\mathcal{I}$ to the interval $[0,1]$ by mapping all values outside this interval to the closest element. Again, the proof that this constitutes a model of $\mathcal{O}$ can be found in the appendix.

Lemma 3.5. *$\mathcal{J}$ is a model of $\mathcal{O}$.*

As this lemma shows, the $\otimes$-$\mathfrak{NEL}$-ontology $\mathcal{O}$ is satisfiable if $\mathcal{O}_\infty$ is satisfiable. Together with Lemma 3.4, we obtain that $\mathcal{O}$ is satisfiable if and only if $\mathcal{O}_\infty$ is. Since consistency of $\otimes$-$\mathfrak{NEL}$-ontologies is undecidable for any t-norm starting with the Łukasiewicz t-norm, this shows undecidability of L_∞-$\mathfrak{IELU}$ ontologies, finishing the proof of Theorem 3.3.

This might suggest that a similar dichotomy as for $[0,1]$ holds for infinite lattices: ontology consistency is decidable if and only if the underlying t-norm has no zero divisors. However, as we show next, this is not the case. Complementing the undecidability result from Theorem 3.3, we will show that if $\otimes$ has no zero divisors, then ontology consistency in L_∞-$\mathcal{SHI}$ is decidable in exponential time, even though L_∞ has a zero divisor, namely the element -2. The idea for proving this is similar to the one used in Theorem 3.1, but more cases need to be distinguished.

Consider the sublattice $\mathbf{4}$ of L_∞ containing only the four elements $\mathbf{4} := \{-\infty, -2, 0, \infty\}$, and all the operations restricted to only this subset. Notice that this sublattice is closed under $\otimes_\infty$, $\oplus_\infty$ and $\Rightarrow_\infty$. We also define the function $\mathbb{4} : L_\infty \to \mathbf{4}$, where

$$\mathbb{4}(\ell) := \begin{cases} \ell & \text{if } \ell \leq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Given an L_∞-$\mathcal{SHI}^-$ ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$, we construct the **4**-$\mathcal{SHI}^-$ ontology $\mathcal{O}' = (\mathcal{A}', \mathcal{T}', \mathcal{R})$ where

$$\begin{aligned} \mathcal{A}' &:= \{\langle \alpha \geq \mathbb{4}(\ell)\rangle \mid \langle \alpha \geq \ell\rangle \in \mathcal{A}\}, \text{ and} \\ \mathcal{T}' &:= \{\langle C \sqsubseteq D \geq \mathbb{4}(\ell)\rangle \mid \langle C \sqsubseteq D \geq \ell\rangle \in \mathcal{T}\}. \end{aligned}$$

The size of $\mathcal{O}'$ is linear in the size of $\mathcal{O}$. We show in the appendix that it is consistent in **4**-$\mathcal{SHI}^-$ iff $\mathcal{O}$ is consistent in L_∞-$\mathcal{SHI}^-$.

Lemma 3.6. *Let $\mathcal{O}$ be an L_∞-$\mathcal{SHI}^-$ ontology, and $\mathcal{O}'$ constructed as described above. $\mathcal{O}$ is consistent in L_∞-$\mathcal{SHI}^-$ iff $\mathcal{O}'$ is consistent in* **4**-$\mathcal{SHI}^-$*.*

Decidability in ExpTime then follows from the results of Section 5.1.

Theorem 3.7. *If $\otimes$ has no zero divisors, then consistency in L_∞-$\mathcal{SHI}^-$ is decidable in* ExpTime*.*

The constructed lattice L_∞ has exactly one zero divisor, regardless of which continuous t-norm $\otimes$ it is based upon. If we include additional values $\pm 3, \pm 4, \ldots \pm (n+1)$, it is possible to extend the t-norm $\otimes_\infty$ in such a way that it has exactly n zero divisors, simply by setting $\ell \otimes_\infty \ell = -\infty$ for every $\ell \leq -2$. Arguments analogous to the ones used in Theorems 3.3 and 3.7 can be used to prove that for any natural number n there is an infinite family of residuated lattices with exactly n zero divisors for which ontology consistency is undecidable, and another infinite family for which this problem is decidable in exponential time. In other words, the decidability of ontology consistency in L-$\mathcal{SHI}^-$ cannot be determined by the number of zero divisors that the t-norm has, as was the case for the continuous t-norms over the interval $[0, 1]$.

We emphasize here that the restriction of disallowing the involutive negation operator in the logic is fundamental for the decidability results from Theorems 3.1 and 3.7. In fact, it is known that under the product t-norm, which has no zero divisors, involutive negation, conjunction, and existential restrictions suffice to make the logic undecidable [16]. Using this fact, and a reduction analogous to the proof of Theorem 3.3, it is possible to prove that L_∞-$\mathcal{SHI}$ is undecidable if the underlying t-norm $\otimes$ is the product t-norm.

Proposition 3.8. *If $\otimes$ is the product t-norm, then consistency of L_∞-$\mathcal{SHI}$ is undecidable.*

4. A Tableau Algorithm for Local Consistency

We now focus our attention to developing an algorithm for deciding local consistency for *finite* residuated De Morgan lattices. We will present a tableau algorithm [2] that can decide local consistency by constructing a model of a given L-$\mathcal{SHI}$ ontology containing a local ABox. Our algorithm is loosely based on the tableau algorithm developed for crisp $\mathcal{SHI}$ in [20].

We first recall two known results that will be useful for simplifying the algorithm. The first of these results is that, since L is finite, we can w.l.o.g. restrict our attention to reasoning w.r.t. n-witnessed models only, for some natural number n bounded by the size of the lattice.

Proposition 4.1 ([19]). *If the maximal cardinality of an antichain of L is n, then every interpretation in L-$\mathcal{SHI}$ is n-witnessed.*

For simplicity, we will present the algorithm only for the case $n = 1$, which corresponds to the case where L is a total order. For $n > 1$, the construction is similar, but n different witnesses have to be produced for satisfying each existential and value restriction. The necessary changes in the algorithm are described at the end of this section.

Another important assumption we can make for simplifying the description of the algorithm is that our RBoxes are acyclic. This assumption does not harm the generality of our method, as can be shown in an analogous manner to the corresponding result for crisp $\mathcal{SHI}$ [32].

Table 2: The tableaux conditions for L-$\mathcal{SHI}$.

	⟨trigger⟩	⟨values⟩	⟨assertions⟩
$\top$	$\langle x{:}\top = \ell\rangle$		$\langle x{:}\top = \mathbf{1}\rangle$
$\bot$	$\langle x{:}\bot = \ell\rangle$		$\langle x{:}\bot = \mathbf{0}\rangle$
$\sqcap$	$\langle x{:}C_1 \sqcap C_2 = \ell\rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \otimes \ell_2 = \ell$	$\langle x{:}C_1 = \ell_1\rangle$, $\langle x{:}C_2 = \ell_2\rangle$
$\sqcup$	$\langle x{:}C_1 \sqcup C_2 = \ell\rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \oplus \ell_2 = \ell$	$\langle x{:}C_1 = \ell_1\rangle$, $\langle x{:}C_2 = \ell_2\rangle$
$\to$	$\langle x{:}C_1 \to C_2 = \ell\rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 = \ell$	$\langle x{:}C_1 = \ell_1\rangle$, $\langle x{:}C_2 = \ell_2\rangle$
$\neg$	$\langle x{:}\neg C = \ell\rangle$		$\langle x{:}C = {\sim}\ell\rangle$
$\exists$	$\langle x{:}\exists r.C = \ell\rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \otimes \ell_2 = \ell$, individual y	$\langle (x,y){:}r = \ell_1\rangle$, $\langle y{:}C = \ell_2\rangle$
$\exists_\leq$	$\langle x{:}\exists r.C = \ell\rangle$, $\langle (x,y){:}r = \ell_1\rangle$	$\ell_2 \in L$ with $\ell_1 \otimes \ell_2 \leq \ell$	$\langle y{:}C = \ell_2\rangle$
$\exists_+$	$\langle x{:}\exists s.C = \ell\rangle$, $\langle (x,y){:}r = \ell_1\rangle$ with r transitive and $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \otimes \ell_2 \leq \ell$	$\langle y{:}\exists r.C = \ell_2\rangle$
$\forall$	$\langle x{:}\forall r.C = \ell\rangle$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 = \ell$, individual y	$\langle (x,y){:}r = \ell_1\rangle$, $\langle y{:}C = \ell_2\rangle$
$\forall_\geq$	$\langle x{:}\forall r.C = \ell\rangle$, $\langle (x,y){:}r = \ell_1\rangle$	$\ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle y{:}C = \ell_2\rangle$
$\forall_+$	$\langle x{:}\forall s.C = \ell\rangle$, $\langle (x,y){:}r = \ell_1\rangle$ with r transitive and $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle y{:}\forall r.C = \ell_2\rangle$
inv	$\langle (x,y){:}r = \ell_1\rangle$		$\langle (y,x){:}\overline{r} = \ell_1\rangle$
$\sqsubseteq_\mathcal{R}$	$\langle (x,y){:}r = \ell_1\rangle$, $r \sqsubseteq_\mathcal{R} s$	$\ell_2 \in L$ with $\ell_1 \leq \ell_2$	$\langle (x,y){:}s = \ell_2\rangle$
$\sqsubseteq_\mathcal{T}$	individual x, $\langle C_1 \sqsubseteq C_2 \geq \ell\rangle$ in $\mathcal{T}$	$\ell_1, \ell_2 \in L$ with $\ell_1 \Rightarrow \ell_2 \geq \ell$	$\langle x{:}C_1 = \ell_1\rangle$, $\langle x{:}C_2 = \ell_2\rangle$

Proposition 4.2 ([19]). *Deciding local consistency in L-$\mathcal{SHI}$ is polynomially equivalent to deciding local consistency in L-$\mathcal{SHI}$ w.r.t. acyclic RBoxes.*

In the following, $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ corresponds to an ontology where $\mathcal{A}$ is a local ABox that contains only the individual name a and $\mathcal{R}$ is an acyclic RBox. We first show that $\mathcal{O}$ has a model iff we can find a *tableau*, which intuitively corresponds to a (possibly infinite) "completed version" of $\mathcal{A}$. Later we describe an algorithm that constructs a finite representation of such a tableau, if it exists, and identifies when it does not exist.

Definition 4.3. A *tableau* for $\mathcal{O}$ is a set $\mathbf{T}$ of equality assertions over a set Ind of individuals such that $a \in \mathsf{Ind}$, $\mathcal{A} \subseteq \mathbf{T}$, and the following conditions are satisfied for all $C, C_1, C_2 \in \mathsf{sub}(\mathcal{O})$, $x, y \in \mathsf{Ind}$, $r, s \in \mathsf{N_R}$, and $\ell \in L$:

$\mathbf{T}$ is *clash-free:* If $\langle x{:}C = \ell\rangle \in \mathbf{T}$ or $\langle (x,y){:}r = \ell\rangle \in \mathbf{T}$, then there is no $\ell' \in L$ such that $\ell' \neq \ell$ and $\langle x{:}C = \ell'\rangle \in \mathbf{T}$ or $\langle (x,y){:}r = \ell'\rangle \in \mathbf{T}$, respectively.

$\mathbf{T}$ is *complete:* For every row of Table 2, the following condition holds: "If ⟨trigger⟩ is in $\mathbf{T}$, then there are ⟨values⟩ such that ⟨assertions⟩ are in $\mathbf{T}$."

In classical DLs, a clash is defined as the simultaneous presence of two assertions of the form $a{:}C$ and $a{:}\neg C$. Our definition generalizes this to fuzzy assertions: if $\langle a{:}C = \mathbf{1}\rangle$ and $\langle a{:}\neg C = \mathbf{1}\rangle$ are contained in $\mathbf{T}$, then by completeness $\mathbf{T}$ also contains $\langle a{:}C = \mathbf{0}\rangle$, and clearly $\mathbf{0} \neq \mathbf{1}$.

The conditions in Table 2 concerning the basic constructors, inverse roles, role inclusions, and GCIs are quite straightforward. For example, the condition $\top$ requires that individuals always belong to $\top$ to degree $\mathbf{1}$, while the condition $\sqsubseteq_\mathcal{T}$ ensures that a GCI $\langle C_1 \sqsubseteq C_2 \geq \ell\rangle$ is satisfied at an individual x by asserting appropriate values for C_1 and C_2 at x. The conditions for the existential and value restrictions deserve some more explanation. First, note that the semantics of $\forall$ is dual to that of $\exists$, and thus every rule for $\exists$ must have a dual counterpart for $\forall$ where the order is reversed and $\otimes$ is replaced by $\Rightarrow$.

In contrast to classical $\mathcal{SHI}$, where only the conditions $\exists$ and $\exists_+$ are needed to deal with existential restrictions [20], we need three rules in the fuzzy setting. The reason lies in the witnessed semantics of an assertion $\langle x\!:\!\exists r.C = \ell\rangle$. The condition $\exists$ ensures that a witness y with the correct value $r^{\mathbf{T}}(x,y) \otimes C^{\mathbf{T}}(y)$ exists (if we view $\mathbf{T}$ as an abstract description of an interpretation), while $\exists_\leq$ is needed to restrict all other individuals y' to not exceed this value. Finally, the conditions $\exists_+$ and $\forall_+$ specify how existential and value restrictions should be propagated along chains of successors through a transitive role, as shown in the following example.

Example 4.4. Consider the lattice L_4 from Figure 1, the individual names $\mathsf{my\_apartment}$ and $\mathsf{living\_room}$, and the transitive role $\mathsf{contains}$. Assume that the following assertions are in our tableau $\mathbf{T}$:

$$\langle \mathsf{my\_apartment}\!:\!\forall \mathsf{contains}.(\mathsf{Wall} \to \mathsf{White}) = \mathsf{i}\rangle \text{ and } \langle(\mathsf{my\_apartment}, \mathsf{living\_room})\!:\!\mathsf{contains} = \mathsf{t}\rangle.$$

The condition $\forall_\geq$ ensures that the value restriction is enforced at all $\mathsf{contains}$-successors of $\mathsf{my\_apartment}$, in particular at $\mathsf{living\_room}$. Thus, an assertion

$$\langle \mathsf{living\_room}\!:\!\mathsf{Wall} \to \mathsf{White} = \ell\rangle,$$

where ℓ is either t or i, must also be in $\mathbf{T}$.

Additionally, the condition $\forall_+$ transports the value restriction itself to $\mathsf{living\_room}$ in order to ensure that also all transitive sub-parts of $\mathsf{my\_apartment}$ satisfy the restriction, in particular all walls of the living room. Thus, also

$$\langle \mathsf{living\_room}\!:\!\forall \mathsf{contains}.(\mathsf{Wall} \to \mathsf{White}) = \ell'\rangle$$

must be in $\mathbf{T}$, where again ℓ' is either t or i.

The following lemma shows that the conditions of Definition 4.3 are sufficient to detect whether $\mathcal{O}$ has a model. The proof can be found in the appendix.

Lemma 4.5. *$\mathcal{O}$ is locally consistent iff it has a tableau.*

We now present a tableau algorithm for deciding local consistency. The algorithm starts with the local ABox $\mathcal{A}$, and nondeterministically expands it to a tree-like ABox $\widehat{\mathcal{A}}$ that represents a model of $\mathcal{O}$. It uses the tableau conditions from Table 2 and reformulates them into *expansion rules* of the form:

> "If there is $\langle\mathsf{trigger}\rangle$ in $\widehat{\mathcal{A}}$ and there are no $\langle\mathsf{values}\rangle$ such that $\langle\mathsf{assertions}\rangle$ are in $\mathcal{A}$,
> then introduce $\langle\mathsf{values}\rangle$ and add $\langle\mathsf{assertions}\rangle$ to $\widehat{\mathcal{A}}$."

The rules $\exists$ and $\forall$ always introduce new individuals y that do not appear in $\widehat{\mathcal{A}}$. Initially, the ABox $\mathcal{A}$ contains the single individual a. This ABox is expanded by the rules in a tree-like way: role connections are only created by adding new successors to existing individuals. If an individual y was created by a rule $\exists$ or $\forall$ that was applied to an assertion involving an individual x, then we say that y is a *successor* of x, and x is the *predecessor* of y; *ancestor* is the transitive closure of *predecessor*. Note that the presence of an assertion $\langle(x,y)\!:\!r = \ell\rangle$ in $\widehat{\mathcal{A}}$ does not imply that y is a successor of x—it could also be the case that this assertion was introduced by the inv-rule, which would mean that x is actually a successor of y.

We further denote by $\widehat{\mathcal{A}}_x$ the set of all concept assertions from $\widehat{\mathcal{A}}$ that involve the individual x, i.e. are of the form $\langle x\!:\!C = \ell\rangle$ for some concept C and $\ell \in L$. As is standard in DL, to ensure that the application of the rules terminates, we need to add a blocking condition. Here, we use *anywhere blocking* [33], which is based on the idea that it suffices to examine each set $\widehat{\mathcal{A}}_x$ only once in the whole ABox $\widehat{\mathcal{A}}$.

Let $\succ$ be a total order on the individuals of $\widehat{\mathcal{A}}$ such that whenever y is a successor of x, then $y \succ x$. An individual y is *directly blocked* if for some other individual x in $\widehat{\mathcal{A}}$ with $y \succ x$, $\widehat{\mathcal{A}}_x$ is equal to $\widehat{\mathcal{A}}_y$ modulo the individual names used; in this case, we write $\widehat{\mathcal{A}}_x \equiv \widehat{\mathcal{A}}_y$ and also say that *x blocks y*. It is *indirectly blocked* if its predecessor is either directly or indirectly blocked. An individual is *blocked* if it is either directly or indirectly blocked. The rules $\exists$ and $\forall$ are applied to $\widehat{\mathcal{A}}$ only if the individual x that triggers their execution is not blocked. All other rules are applied only if x is not indirectly blocked.

The total order $\succ$ is used to avoid cycles in the blocking relation in which two individuals are mutually blocking each other. One way to build this order is to simply use the order in which the individuals were created by the expansion rules. Note that the only individual a that occurs in $\mathcal{A}$, which is the root of the tree-like structure represented by $\widehat{\mathcal{A}}$, cannot be blocked since it is an ancestor of all other individuals in $\widehat{\mathcal{A}}$. With this blocking condition, we can show that the size of $\widehat{\mathcal{A}}$ is bounded exponentially in the size of $\mathcal{A}$, as in the crisp case [33].

Lemma 4.6. *Every sequence of applications of expansion rules to $\mathcal{A}$ terminates after at most exponentially many rule applications (measured in the size of $\mathcal{O}$).*

Proof. Let $\mathsf{sub}(\mathcal{O})$ denote the set of all subconcepts of concepts appearing in $\mathcal{O}$ and recall that every rule application expands $\widehat{\mathcal{A}}$ in a tree-like manner, where each node in this tree represents one individual. Note that there are at most $|L||\mathsf{sub}(\mathcal{O})|$ possible concept assertions for one individual x. Thus, every node in this tree has at most $|L||\mathsf{sub}(\mathcal{O})|$ successors: one for each possible assertion involving an existential or value restriction. Moreover, there can be at most $2^{|L||\mathsf{sub}(\mathcal{O})|}$ non-blocked nodes in $\widehat{\mathcal{A}}$ at any time, and thus, when a node becomes blocked, at most exponentially many nodes become indirectly blocked.

This bounds the total number of possible non-blocked, directly blocked, and indirectly blocked nodes by an exponential in the size of the input. Thus, we obtain a tree of at most exponential size before every rule application is disallowed by the blocking condition. The claim now follows from the fact that every rule application adds at least one assertion to $\widehat{\mathcal{A}}$ and cannot remove assertions from $\widehat{\mathcal{A}}$. $\square$

We say that $\widehat{\mathcal{A}}$ contains a *clash* if it contains two assertions that are equal except for their lattice value (cf. Definition 4.3). $\widehat{\mathcal{A}}$ is *complete* if it contains a clash or none of the expansion rules are applicable. The algorithm is correct in the sense that it produces a clash iff $\mathcal{O}$ is not locally consistent. The proof uses Lemma 4.5 to first abstract local consistency of $\mathcal{O}$ to the existence of a tableau for $\mathcal{O}$.

Lemma 4.7. *$\mathcal{O}$ is locally consistent iff some sequence of applications of the expansion rules to $\mathcal{A}$ yields a complete and clash-free ABox.*

Since the tableaux rules are nondeterministic, Lemmata 4.6 and 4.7 together imply that the tableau algorithm decides local consistency in NExpTime.

Theorem 4.8. *Local consistency in L-$\mathcal{SHI}$ w.r.t. witnessed models can be decided in* NExpTime.

As explained before, L-$\mathcal{SHI}$ has the n-witnessed model property for some $n \geq 1$ (see Lemma 4.1). We have so far restricted our description to the case where $n = 1$. If $n > 1$, it does not suffice to generate only one successor for every existential and universal restriction, but one must produce n different successors to ensure that the degrees guessed for these complex concepts are indeed witnessed by the model. The only required change to the algorithm is in the rules $\exists$ and $\forall$ (see Table 2), where we have to introduce n individuals $y_1, \ldots, y_n$, and $2n$ values $\ell_1^1, \ell_2^1, \ldots, \ell_1^n, \ell_2^n \in L$ that satisfy $\bigvee_{i=1}^n \ell_1^i \otimes \ell_2^i = \ell$ or $\bigwedge_{i=1}^n \ell_1^i \Rightarrow \ell_2^i = \ell$, respectively. The complexity of the algorithm as analyzed in Lemma 4.6 remains the same under this modification, as the number of successors of a node is still bounded polynomially, namely by $n|L||\mathsf{sub}(\mathcal{O})|$.

5. Local Completion and Other Black-Box Reductions

Now that we know how to decide local consistency w.r.t. witnessed models, we can try to reduce other reasoning problems to it. More precisely, we assume in the following that we have a black-box procedure that decides local consistency in a sublogic of L-$\mathcal{SHI}$. This procedure could be, for example, the tableaux-based algorithm from the previous section, the PSpace-decision procedure from [19] for deciding local consistency in L-$\mathcal{ALC}$ w.r.t. so-called *acyclic TBoxes*, or any other decision procedure for local consistency. We will show how to employ this procedure to solve other reasoning problems in this sublogic.

5.1. Consistency

We first show how to reduce ontology consistency to local consistency. Let $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be an ontology, where $\mathcal{A}$ is an arbitrary ABox; that is, $\mathcal{A}$ is not necessarily local. We first make sure that the information contained in $\mathcal{A}$ is consistent "in itself", i.e. that the knowledge that can be extracted from the individuals appearing in $\mathcal{A}$ without expanding the domain is not contradictory. It then suffices to check a local consistency condition for each of these individuals. This procedure is based on a similar idea developed for crisp description logics, called *pre-completion* [34].

Let $\mathsf{Ind}_\mathcal{A}$ denote the set of individual names occurring in $\mathcal{A}$ and $\mathsf{sub}(\mathcal{A}, \mathcal{T})$ the set of all subconcepts of concepts occurring in the ABox $\mathcal{A}$ or in the TBox $\mathcal{T}$. We first guess a set $\widehat{\mathcal{A}}$ of equality assertions of the forms $\langle a\!:\!C = \ell\rangle$ and $\langle (a,b)\!:\!r = \ell\rangle$ where $a, b \in \mathsf{Ind}_\mathcal{A}$, $C \in \mathsf{sub}(\mathcal{A}, \mathcal{T})$, $\ell \in L$, and r is a role name occurring in $\mathcal{O}$. Since there are at most polynomially many such assertions, this can be done in (nondeterministic) polynomial time in the size of $\mathcal{O}$ and L. We then check whether $\widehat{\mathcal{A}}$ is clash-free and satisfies the tableaux conditions listed in Table 2, except the witnessing conditions $\exists$ and $\forall$. Additionally, we impose the following condition to ensure that $\widehat{\mathcal{A}}$ satisfies $\mathcal{A}$:

"If there is an assertion $\langle \alpha \rhd \ell\rangle$ in $\mathcal{A}$, then there is $\ell' \in L$ such that $\ell' \rhd \ell$ and $\langle \alpha = \ell'\rangle$ is in $\widehat{\mathcal{A}}$."

We call $\widehat{\mathcal{A}}$ *locally complete* iff it is of the above form and satisfies all of the above conditions. Since the guessed set $\widehat{\mathcal{A}}$ contains at most polynomially many assertions, checking whether it is locally complete can be done in polynomial time in the size of $\mathcal{O}$ and L.

Lemma 5.1. *An ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ is consistent iff there exists a locally complete set $\widehat{\mathcal{A}}$ such that $\mathcal{O}_x = (\widehat{\mathcal{A}}_x, \mathcal{T}, \mathcal{R})$ is locally consistent for every $x \in \mathsf{Ind}_\mathcal{A}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $\widehat{\mathcal{A}}$ be the set of all assertions $\langle a\!:\!C = C^\mathcal{I}(a^\mathcal{I})\rangle$ and $\langle (a,b)\!:\!r = r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})\rangle$ for $a, b \in \mathsf{Ind}_\mathcal{A}$, $r \in \mathsf{N_R}$, and $C \in \mathsf{sub}(\mathcal{A}, \mathcal{T})$. Using the same arguments as in the proof of Lemma 4.5, we can show that $\widehat{\mathcal{A}}$ is locally complete. Furthermore, by construction $\mathcal{I}$ satisfies $\mathcal{O}_x$ for any $x \in \mathsf{Ind}_\mathcal{A}$.

Let now $\widehat{\mathcal{A}}$ be a locally complete set for $\mathcal{O}$ and $\mathcal{O}_x$ be locally consistent for every $x \in \mathsf{Ind}_\mathcal{A}$. By Lemma 4.5, for each $x \in \mathsf{Ind}_\mathcal{A}$ there is a tableau $\mathbf{T}_x$ for $\mathcal{O}_x$ over the set Ind_x of individuals. We can assume that the sets Ind_x are mutually disjoint. Note that $x \in \mathsf{Ind}_x$ for every $x \in \mathsf{Ind}_\mathcal{A}$.

We now define $C^\mathbf{T}(y) = \ell$ whenever $\langle y\!:\!C = \ell\rangle \in \mathbf{T}_x$ for some $x \in \mathsf{Ind}_\mathcal{A}$. Similarly, we set $r^\mathbf{T}(y,z) = \ell$ if $\langle (y,z)\!:\!r = \ell\rangle \in \mathbf{T}_x$ for some $x \in \mathsf{Ind}_\mathcal{A}$. Note that, since $\mathbf{T}$ is clash-free and the sets Ind_x are disjoint, these values are uniquely defined. To reconnect the individuals of $\mathsf{Ind}_\mathcal{A}$, we additionally define $r^\mathbf{T}(x,y) = \ell$ whenever $\langle (x,y)\!:\!r = \ell\rangle \in \widehat{\mathcal{A}}$.

As in the proof of Lemma 4.5, we can now define an interpretation $\mathcal{I}$ from these values by constructing the transitive closure of $r^\mathbf{T}$ if r is transitive. Then, $C^\mathcal{I}(x) = \ell$ whenever $\langle x\!:\!C = \ell\rangle \in \mathbf{T}$. Since the assertions in $\widehat{\mathcal{A}}$ satisfy $\mathcal{A}$, $\mathcal{I}$ also satisfies $\mathcal{A}$ and by the conditions $\sqsubseteq_\mathcal{T}$ and $\sqsubseteq_\mathcal{R}$, $\mathcal{I}$ satisfies $\mathcal{T}$ and $\mathcal{R}$. □

This shows that, if we have an algorithm that decides local consistency in a complexity class C, then we can decide consistency by additionally guessing polynomially many assertions.

Theorem 5.2. *If local consistency in L-$\mathcal{SHI}$ can be decided in a complexity class* C*, then consistency in L-$\mathcal{SHI}$ can be decided in any complexity class that contains both* NP *and* C.

A direct consequence of this theorem and Theorem 4.8 is that consistency of L-$\mathcal{SHI}$-ontologies is decidable in NExpTime. As described before, the consistency algorithm simply uses a reasoner for local consistency as a black-box, which allows us to improve this upper bound to ExpTime if an ExpTime local consistency algorithm exists. Such an algorithm was presented in [19], where local consistency in L-$\mathcal{SHI}$ is reduced to the emptiness problem of an automaton whose size is exponential on the local ontology. Moreover, if the TBox satisfies some acyclicity conditions, this bound can be further improved to PSpace for the sublogics L-$\mathcal{ALCHI}$ and L-$\mathcal{SI}_\mathsf{c}$, where c denotes the restriction to crisp roles. With Theorem 5.2, this shows that consistency in these logics is of the same complexity.

5.2. Satisfiability, Instance Checking, and Subsumption

To decide whether a concept C is strongly ℓ-satisfiable w.r.t. $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$, we can simply check whether $(\mathcal{A} \cup \{a{:}C \geq \ell\}, \mathcal{T}, \mathcal{R})$ is consistent for a new individual name a not occurring in $\mathcal{A}$. Thus, strong ℓ-satisfiability is in the same complexity class as consistency. Moreover, we can easily compute the set of *all* values $\ell \in L$ such that the ontology $(\mathcal{A} \cup \{a{:}C \geq \ell\}, \mathcal{T}, \mathcal{R})$ is consistent by calling the decision procedure for consistency a constant number of times, i.e. once for each $\ell \in L$. We can use this set to compute the best bound for the satisfiability of C. Formally, the *best satisfiability degree* of a concept C is the join of all $\ell \in L$ such that C is ℓ-satisfiable w.r.t. $\mathcal{O}$. Since we can compute the set of all elements of L satisfying this property, obtaining the best satisfiability degree requires only a join computation. As the lattice L is fixed, this adds a constant factor to the complexity of checking consistency.

To check ℓ-instances, we can exploit the fact that a is *not* an ℓ-instance of C w.r.t. $\mathcal{O}$ iff there is a model $\mathcal{I}$ of $\mathcal{O}$ and a domain element $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a^{\mathcal{I}}) \not\geq \ell$. This is the case iff there is a value $\ell' \not\geq \ell$ such that the ontology $(\mathcal{A} \cup \{a{:}C = \ell'\}, \mathcal{T}, \mathcal{R})$ is consistent. Thus, ℓ-instances can be decided by calling the decision procedure for consistency a constant number of times, namely at most once for each $\ell' \in L$ with $\ell' \not\geq \ell$. We can also compute the *best instance degree* for a and C, which is the join of all $\ell \in L$ such that a is an ℓ-instance of C w.r.t. $\mathcal{O}$, as follows. Let $\mathfrak{L}$ denote the set of all ℓ' such that $(\mathcal{A} \cup \{a{:}C = \ell'\}, \mathcal{T}, \mathcal{R})$ is consistent. The best instance degree for a and C is the infimum of all $\ell' \in \mathfrak{L}$ since

$$\begin{aligned}\bigvee\{\ell \in L \mid a \text{ is an } \ell\text{-instance of } C\} &= \bigvee\{\ell \in L \mid \forall \ell' \not\geq \ell : \ell' \notin \mathfrak{L}\} \\ &= \bigvee\{\ell \in L \mid \forall \ell' \in \mathfrak{L} : \ell \leq \ell'\} = \bigwedge \mathfrak{L}.\end{aligned}$$

Finally, note that C is ℓ-subsumed by D iff a is an ℓ-instance of $C \to D$, where a is a new individual name. Thus, deciding ℓ-subsumption and computing the *best subsumption degree* can be done using the same approach as above. By Theorem 5.2, we now get the following complexity results.

Lemma 5.3. *If local consistency in L-$\mathcal{SHI}$ can be decided in a complexity class* C*, then strong satisfiability, instance checking, and subsumption in L-$\mathcal{SHI}$ can be decided in any complexity class that contains both* NP *and* C*.*

This shows that strong satisfiability, instance checking, and subsumption in L-$\mathcal{SHI}$ are in ExpTime [19]. As explained before, this bound can be improved to PSpace if we consider L-$\mathcal{ALCHI}$ or L-$\mathcal{SI}_{\mathsf{c}}$ w.r.t. acyclic TBoxes. Note that all previous complexity results also hold if, rather than assuming the lattice L to be fixed, L is of polynomial size in the size of the input and all lattice operations are computable in polynomial time (see also [19]).

6. Related Work

Deduction systems for logics based on multi-valued semantics have a long history [35], and in particular for fuzzy DLs their study goes back at least twenty years [6]. We do not attempt to survey all the work related to ours, but rather provide a brief description of the most closely related papers from the area of fuzzy DLs.

6.1. Reasoning over the Standard Chain

Most of the literature on fuzzy description logics focuses on the total order defined over the interval $[0, 1]$ of real numbers (see [36] for a comparison of fuzzy, probabilistic, and possibilistic extensions of DLs). The first tableau algorithms were developed in [7, 37] to decide ontology consistency and other reasoning problems in fuzzy extensions of $\mathcal{ALC}$ under the Gödel t-norm and involutive negation, but only for acyclic TBoxes. The restriction to witnessed models was later introduced in [15] to correct these algorithms. This last paper also introduced the first t-norm-based fuzzy DLs and proved decidability of (witnessed) strong 1-satisfiability and 1-subsumption for fuzzy $\mathcal{ALC}$ under any t-norm, but without a background ontology. In particular, this also holds for the Łukasiewicz t-norm without the restriction to witnessed models since in this case the two semantics coincide [15]. In [38], it is proved that 1-subsumption is also decidable under the product t-norm, but still without a background ontology. Decidability of strong satisfiability under the restriction to so-called *quasi-witnessed* models is also shown there. In [39], the approach of [15] is extended to deal with ontologies, and axiomatizations of t-norm-based fuzzy DLs are investigated.

Following the approach of [15], several tableau algorithms were proposed to deal with GCIs in t-norm-based fuzzy DLs [40–42]. However, these approaches were recently shown to be incorrect in the presence of GCIs, and are correct only for acyclic or unfoldable TBoxes [43, 44]. Motivated by these results, it was later shown that the ontology consistency problem is undecidable for many fuzzy DLs over the interval $[0, 1]$ in the presence of GCIs [16, 24, 44]. Decidability is known only if the underlying t-norm has no zero divisors and no equality assertions nor involutive negations are used in the construction of concepts [23] (compare to Theorem 3.1), or if the t-norm is the Gödel t-norm [21, 22, 45]. In both cases, reasoning becomes effectively finite-valued and can be decided using a crisp reasoner as a black-box procedure.

6.2. Reasoning over Finite Lattices

There is little research dealing with fuzzy semantics based on general lattices that are not just total orders. In [12], certainty lattices are introduced and it is shown that consistency of L-$\mathcal{ALC}$ ontologies with acyclic TBoxes is in PSpace. In [13], a tableau algorithm is presented for the more expressive DL L-$\mathcal{SHIN}$. However, both of these papers deal only with the simple Gödel t-norm and do not allow the implication constructor and therefore have no residual negation. Furthermore, the presented tableau algorithms cannot deal with GCIs. In [46], a tableau algorithm is developed for L-$\mathcal{ALC}$ under finite-valued Łukasiewicz semantics, and it is shown that strong satisfiability and 1-subsumption w.r.t. empty ontologies is PSpace-complete. This is a special case of a result obtained in [19], where PSpace-completeness is shown for strong satisfiability and subsumption in L-$\mathcal{ALC}$ w.r.t. arbitrary finite residuated De Morgan lattices. As mentioned before, this holds even in the more expressive logics L-$\mathcal{ALCHI}$ and L-$\mathcal{SI}_\mathsf{c}$, while in the presence of GCIs these problems become ExpTime-complete in all logics from L-$\mathcal{ALC}$ to L-$\mathcal{SHI}$.

A popular method for deciding ontology consistency over finitely many membership degrees is the reduction of fuzzy ontologies into crisp ones. This method has so far only been described for finite total orders [12, 47, 48], but can be extended to lattices as described below. Reasoning can then be performed through calls to an existing highly-optimized reasoner for crisp DLs. The main idea of the method is to translate every concept name A into finitely many crisp concept names $A_{\geq \ell}$, one for each degree $\ell \in L$, where $A_{\geq \ell}$ collects all those individuals that belong to A with a membership degree $\geq \ell$. The lattice structure is expressed through GCIs of the form $A_{\geq \ell_2} \sqsubseteq A_{\geq \ell_1}$, where ℓ_2 is a minimal element above ℓ_1, and analogously for the role names. All axioms are then recursively translated into crisp axioms that use only the introduced crisp concept and role names. The resulting crisp ontology is consistent iff the original fuzzy ontology is consistent.

In general, such a translation is exponential in the size of the concepts that occur in the fuzzy ontology. The reason is a combination of two factors. First, complex concepts can be built from several nested constructors. Second, depending on the t-norm used, there might be $|L|^2$ different combinations of elements ℓ_1, ℓ_2 for C, D, respectively, that lead to $C \sqcap D$ having the value $\ell = \ell_1 \otimes \ell_2$, and similarly for the other constructors. All these possibilities have to be expressed in the translation, which then produces a crisp ontology whose size is exponential in the size of the original ontology. Since ontology consistency in crisp $\mathcal{SHI}$ is ExpTime-hard, this can at best yield a 2-ExpTime reasoning procedure.[13] Moreover, DL reasoners usually implement tableau algorithms with a worst-case complexity above NExpTime; in that case, one gets a 2-NExpTime reasoning procedure. In contrast, our tableau algorithm has a worst-case complexity of NExpTime, matching that of the tableau algorithms for crisp $\mathcal{SHI}$ that are used in practice (e.g. in RacerPro).

To the best of our knowledge, at the moment there exists only one (correct) tableau algorithm that can deal with a finite total order of truth values and GCIs [49], without relying on a crispification of the input.

[13]For special lattices, in particular total orders with the so-called *Zadeh* semantics that use the Gödel t-norm and involutive negation, this blowup can be avoided [12].

However, this algorithm is restricted to the Gödel t-norm. The main difference between the algorithm from [49] and ours is that we nondeterministically guess the degree of membership of each individual to every relevant concept, while the approach from [49] sets only lower and upper bounds for these degrees; this greatly reduces the amount of nondeterminism encountered, but introduces several complications when a t-norm different from the Gödel t-norm is used.

6.3. Crisp Tableau Algorithms

When restricted to the lattice **2** that has only the elements $\{\mathbf{0},\mathbf{1}\}$, our tableau algorithm basically behaves like the algorithm for crisp $\mathcal{SHI}$ from [20]. One notorious difference is that our algorithm also creates assertions that hold with degree **0**. We could easily disallow such assertions in the description of our algorithm, but we decided against that to simplify the description of the general algorithm and the proofs of correctness. Another difference is that the algorithm from [20] does not need the rule $\exists_{\leq}$: once a crisp existential restriction is satisfied by introducing an appropriate r-successor using the rule $\exists$, it does not further restrict the other r-successors. Likewise, the rule $\forall$ is unnecessary: a restriction $\langle x : \forall r.C = \mathbf{1}\rangle$ is trivially witnessed by any individual y that is *not* connected to x by r; such a y can always be introduced without affecting the rest of the model.

The above modifications would eliminate the nondeterminism of our algorithm except in the rules $\sqcup$, $\rightarrow$, and $\sqsubseteq_{\mathcal{T}}$. This corresponds to the fact that the nondeterminism of the algorithm in [20] is only due to the crisp disjunctions and the implications in the TBox (i.e. GCIs). A final difference is that we opted not to use the pair-wise blocking condition from [20], but rather the anywhere blocking condition from [33].

7. Conclusions

In this paper we have studied fuzzy description logics with semantics based on complete residuated De Morgan lattices. Lattice-based semantics generalize those based on chains, and hence this study has direct consequences on chain-based fuzzy DLs. Moreover, the lattice-based semantics allow for reasoning with incomparable membership degrees; a task that is impossible for chains. Incomparable membership degrees arise naturally when knowledge is acquired from different sources, and it is impossible to ascertain that a value given by one source is necessarily larger (or smaller) than a value from another source (see Examples 2.5 and 2.12). A similar phenomenon occurs if one considers different scales for measuring different aspects of an observation.

We showed that for every lattice L without zero divisors, the problem of ontology consistency in L-$\mathcal{SHI}^-$ is decidable in ExpTime. If the lattice has zero divisors, even finitely many, we have provided examples of infinite families of lattices where the consistency problem becomes undecidable. On the other hand, there exist also infinite families of residuated lattices for which reasoning is reducible to reasoning in a finite lattice. This was later used as a basis for showing that the complexity of ontology consistency matches the complexity of crisp reasoning.

To obtain an effective reasoning mechanism, we have also proposed a tableaux-based procedure for deciding consistency of L-$\mathcal{SHI}$ ontologies, when L is a finite lattice. As a first step, we restricted to the case of *local* consistency, where the ABox has only equality concept assertions that refer to one individual name. As is common for tableaux-based algorithms, the method proposed in this paper does not match the known upper bound for local consistency, shown previously through an automata-based approach [19] to be ExpTime. However, the tableaux-based algorithm is more goal-directed, and hence may be suitable for further optimizations. Afterwards we have shown that the tableaux rules can be also applied to reduce consistency of arbitrary ontologies to a linear number of instances of local consistency. This reduction yields a decision procedure for ontology consistency, which was not possible using only the automata-based approach.

While the proposed reduction is based on the tableaux rules, it is independent of the algorithm used for deciding local consistency. This means that the complexity-optimal automata-based procedures can be called for each of the local consistency instances generated. In particular, we obtain tight complexity bounds for deciding consistency in the logics L-$\mathcal{SHI}$ w.r.t. general TBoxes and in L-$\mathcal{ALCHI}$ and L-$\mathcal{SI}_{\mathsf{c}}$

w.r.t. acyclic TBoxes—ExpTime and PSpace, respectively. The precise complexity of full $L\text{-}\mathcal{SI}$ w.r.t. acyclic TBoxes is still open, but using our reduction it will suffice to provide a local consistency reasoner to get a corresponding upper bound for consistency. We also demonstrated how other standard decision and computation problems, such as subsumption and instance checking, can be solved using a local consistency reasoner as a black box.

The presented tableau algorithm is highly nondeterministic; in fact, almost every rule needs to make a nondeterministic choice, and in many cases, several different outcomes need to be considered. For that reason, the algorithm as described in the paper is not suitable for an efficient implementation, and adequate optimizations need to be studied. Most of the optimizations developed for tableau algorithms for crisp DLs, like the use of an optimized rule-application ordering to reduce the number of generated individuals, can be transferred to our setting. However, the most important task is to reduce the search space created by the choice of lattice values in most of the rules. The number of values that are applicable highly depends on the structure of the lattice and the t-norm, and thus tableaux rules that are tailored towards a specific family of lattices might improve the performance.

As future work, we plan to study different optimizations that will reduce the number of choices in the application of rules, both in general and for specific kinds of lattices. We also plan to implement a tableaux-based fuzzy reasoner and compare its performance to other existing reasoners. Additionally, we will extend our analysis of infinite lattices, trying to fully characterize the cases for which reasoning is decidable, and to find practical cases where infinite membership structures are useful.

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References

[1] F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi, P. F. Patel-Schneider (Eds.), The Description Logic Handbook: Theory, Implementation, and Applications, 2nd Edition, Cambridge University Press, 2007.

[2] F. Baader, U. Sattler, An overview of tableau algorithms for description logics, Studia Logica 69 (1) (2001) 5–40. doi:10.1023/A:1013882326814.

[3] F. Baader, S. Brandt, C. Lutz, Pushing the $\mathcal{EL}$ envelope., in: L. P. Kaelbling, A. Saffiotti (Eds.), Proc. of the 19th Int. Joint Conf. on Artificial Intelligence (IJCAI'05), Professional Book Center, 2005, pp. 364–369.

[4] D. Calvanese, G. De Giacomo, M. Lenzerini, Reasoning in expressive description logics with fixpoints based on automata on infinite trees, in: T. Dean (Ed.), Proc. of the 16th Int. Joint Conf. on Artificial Intelligence (IJCAI'99), Morgan Kaufmann, 1999, pp. 84–89.

[5] L. A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353. doi:10.1016/S0019-9958(65)90241-X.

[6] J. Yen, Generalizing term subsumption languages to fuzzy logic, in: J. Mylopoulos, R. Reiter (Eds.), Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI'91), Morgan Kaufmann, 1991, pp. 472–477.

[7] C. B. Tresp, R. Molitor, A description logic for vague knowledge, in: H. Prade (Ed.), Proc. of the 13th Eur. Conf. on Artificial Intelligence (ECAI'98), John Wiley and Sons, 1998, pp. 361–365.

[8] R. Molitor, C. B. Tresp, Extending description logics to vague knowledge in medicine, in: P. S. Szczepaniak, P. J. G. Lisboa, J. Kacprzyk (Eds.), Fuzzy Systems in Medicine, Vol. 41 of Studies in Fuzziness and Soft Computing, Springer-Verlag, 2000, pp. 617–635.

[9] M. Gao, C. Liu, Extending owl by fuzzy description logic, in: Proc. of the 17th IEEE Int. Conf. on Tools with Artificial Intelligence (ICTAI'05), IEEE Computer Society Press, 2005, pp. 562–567. doi:10.1109/ICTAI.2005.65.

[10] G. Stoilos, G. B. Stamou, J. Z. Pan, Fuzzy extensions of OWL: Logical properties and reduction to fuzzy description logics, International Journal of Approximate Reasoning 51 (6) (2010) 656–679. doi:10.1016/j.ijar.2010.01.005.

[11] F. Bobillo, U. Straccia, Fuzzy ontology representation using OWL 2, International Journal of Approximate Reasoning 52 (7) (2011) 1073–1094. doi:10.1016/j.ijar.2011.05.003.

[12] U. Straccia, Description logics over lattices, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 14 (1) (2006) 1–16. doi:10.1142/S0218488506003807.

[13] Y. Jiang, Y. Tang, J. Wang, P. Deng, S. Tang, Expressive fuzzy description logics over lattices, Knowledge-Based Systems 23 (2) (2010) 150–161. doi:10.1016/j.knosys.2009.11.002.

[14] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1) (1967) 145–174. doi:10.1016/0022-247X(67)90189-8.

[15] P. Hájek, Making fuzzy description logic more general, Fuzzy Sets and Systems 154 (1) (2005) 1–15. doi:10.1016/j.fss.2005.03.005.

[16] S. Borgwardt, R. Peñaloza, Undecidability of fuzzy description logics, in: G. Brewka, T. Eiter, S. A. McIlraith (Eds.), Proc. of the 13th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'12), AAAI Press, 2012, pp. 232–242.

[17] S. Borgwardt, R. Peñaloza, Fuzzy ontologies over lattices with t-norms, in: R. Rosati, S. Rudolph, M. Zakharyaschev (Eds.), Proc. of the 2011 Int. Workshop on Description Logics (DL'11), Vol. 745 of CEUR Workshop Proceedings, 2011, pp. 70–80.

[18] S. Borgwardt, R. Peñaloza, A tableau algorithm for fuzzy description logics over residuated De Morgan lattices, in: M. Krötzsch, U. Straccia (Eds.), Proc. of the 6th Int. Conf. on Web Reasoning and Rule Systems (RR'12), Vol. 7497 of Lecture Notes in Computer Science, Springer-Verlag, 2012, pp. 9–24. doi:10.1007/978-3-642-33203-6_3.

[19] S. Borgwardt, R. Peñaloza, The complexity of lattice-based fuzzy description logics, Journal on Data Semantics 2 (1) (2013) 1–19. doi:10.1007/s13740-012-0013-x.

[20] I. Horrocks, U. Sattler, A description logic with transitive and inverse roles and role hierarchies, Journal of Logic and Computation 9 (3) (1999) 385–410. doi:10.1093/logcom/9.3.385.

[21] F. Bobillo, M. Delgado, J. Gómez-Romero, U. Straccia, Fuzzy description logics under Gödel semantics, International Journal of Approximate Reasoning 50 (3) (2009) 494–514. doi:10.1016/j.ijar.2008.10.003.

[22] F. Bobillo, M. Delgado, J. Gómez-Romero, U. Straccia, Joining Gödel and Zadeh fuzzy logics in fuzzy description logics, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 20 (4) (2012) 475–508. doi:10.1142/S0218488512500249.

[23] S. Borgwardt, F. Distel, R. Peñaloza, How fuzzy is my fuzzy description logic?, in: B. Gramlich, D. Miller, U. Sattler (Eds.), Proc. of the 6th Int. Joint Conf. on Automated Reasoning (IJCAR'12), Vol. 7364 of Lecture Notes in Artificial Intelligence, Springer-Verlag, 2012, pp. 82–96. doi:10.1007/978-3-642-31365-3_9.

[24] M. Cerami, U. Straccia, On the (un)decidability of fuzzy description logics under Łukasiewicz t-norm, Information Sciences 227 (2013) 1–21.

[25] G. De Cooman, E. E. Kerre, Order norms on bounded partially ordered sets, Journal of Fuzzy Mathematics 2 (1993) 281–310.

[26] P. Hájek, Metamathematics of Fuzzy Logic (Trends in Logic), Springer, 2001.

[27] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Vol. 151 of Studies in Logic and the Foundations of Mathematics, Elsevier, 2007.

[28] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Trends in Logic, Studia Logica Library, Springer-Verlag, 2000.

[29] P. S. Mostert, A. L. Shields, On the structure of semigroups on a compact manifold with boundary, Annals of Mathematics 65 (1) (1957) 117–143.

[30] N. D. Belnap, A useful four-valued logic, in: Modern Uses of Multiple-Valued Logic, Reidel Publishing Company, 1977, pp. 7–37.

[31] H. Viana, J. Alcântara, A. T. C. Martins, Paraconsistent rough description logic, in: R. Rosati, S. Rudolph, M. Zakharyaschev (Eds.), Proc. of the 2011 Int. Workshop on Description Logics (DL'11), Vol. 745 of CEUR Workshop Proceedings, 2011, pp. 587–597.

[32] S. Tobies, Complexity results and practical algorithms for logics in knowledge representation, Ph.D. thesis, RWTH Aachen (2001).

[33] B. Motik, R. Shearer, I. Horrocks, Optimized reasoning in description logics using hypertableaux, in: F. Pfenning (Ed.), Proc. of the 21st Int. Conf. on Automated Deduction (CADE'07), Vol. 4603 of Lecture Notes in Computer Science, Springer-Verlag, 2007, pp. 67–83. doi:10.1007/978-3-540-73595-3_6.

[34] B. Hollunder, Consistency checking reduced to satisfiability of concepts in terminological systems, Annals of Mathematics and Artificial Intelligence 18 (2-4) (1996) 133–157. doi:10.1007/BF02127745.

[35] R. Hähnle, G. Escalada-Imaz, Deduction in many-valued logics: a survey, Mathware and Soft Computing IV (2) (1997) 69–97.

[36] T. Lukasiewicz, U. Straccia, Managing uncertainty and vagueness in description logics for the semantic web, Journal of Web Semantics 6 (4) (2008) 291–308. doi:10.1016/j.websem.2008.04.001.

[37] U. Straccia, Reasoning within fuzzy description logics, Journal of Artificial Intelligence Research 14 (2001) 137–166. doi:10.1613/jair.813.

[38] M. Cerami, F. Esteva, F. Bou, Decidability of a description logic over infinite-valued product logic, in: F. Lin, U. Sattler, M. Truszczynski (Eds.), Proc. of the 12th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'10), AAAI Press, 2010, pp. 203–213.

[39] À. García-Cerdaña, E. Armengol, F. Esteva, Fuzzy description logics and t-norm based fuzzy logics, International Journal of Approximate Reasoning 51 (6) (2010) 632–655. doi:10.1016/j.ijar.2010.01.001.

[40] F. Bobillo, U. Straccia, A fuzzy description logic with product t-norm, in: Proc. of the 2007 IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'07), IEEE Computer Society Press, 2007, pp. 1–6. doi:10.1109/FUZZY.2007.4295443.

[41] U. Straccia, F. Bobillo, Mixed integer programming, general concept inclusions and fuzzy description logics, in: M. Stepnicka, V. Novák, U. Bodenhofer (Eds.), Proc. of the 5th EUSFLAT Conf. (EUSFLAT'07), Universitas Ostraviensis, 2007, pp. 213–220.

[42] V. Haarslev, H.-I. Pai, N. Shiri, A formal framework for description logics with uncertainty, International Journal of Approximate Reasoning 50 (9) (2009) 1399–1415. doi:10.1016/j.ijar.2009.04.009.

[43] F. Bobillo, F. Bou, U. Straccia, On the failure of the finite model property in some fuzzy description logics, Fuzzy Sets and Systems 172 (1) (2011) 1–12. doi:10.1016/j.fss.2011.02.012.

[44] F. Baader, R. Peñaloza, Are fuzzy description logics with general concept inclusion axioms decidable?, in: Proc. of the 2011 IEEE Int. Conf. on Fuzzy Systems (FUZZ-IEEE'11), 2011, pp. 1735–1742. doi:10.1109/FUZZY.2011.6007520.
[45] G. Stoilos, G. B. Stamou, Reasoning with fuzzy extensions of OWL and OWL 2, Knowledge and Information Systems. In press. doi:10.1007/s10115-013-0641-y.
[46] F. Bou, M. Cerami, F. Esteva, Finite-valued Lukasiewicz modal logic is PSPACE-complete, in: T. Walsh (Ed.), Proc. of the 22nd Int. Joint Conf. on Artificial Intelligence (IJCAI'11), AAAI Press, 2011, pp. 774–779. doi:10.5591/978-1-57735-516-8/IJCAI11-136.
[47] F. Bobillo, U. Straccia, Finite fuzzy description logics: A crisp representation for finite fuzzy $\mathcal{ALCH}$, in: F. Bobillo, R. Carvalho, P. C. G. da Costa, C. D'Amato, N. Fanizzi, K. B. Laskey, K. J. Laskey, T. Lukasiewicz, T. Martin, M. Nickles, M. Pool (Eds.), Proc. of the 6th Int. Workshop on Uncertainty Reasoning for the Semantic Web (URSW'10), Vol. 654 of CEUR Workshop Proceedings, 2010, pp. 61–72.
[48] F. Bobillo, U. Straccia, Reasoning with the finitely many-valued Łukasiewicz fuzzy description logic $\mathcal{SROIQ}$, Information Sciences 181 (2011) 758–778. doi:j.ins.2010.10.020.
[49] G. Stoilos, U. Straccia, G. Stamou, J. Z. Pan, General concept inclusions in fuzzy description logics, in: G. Brewka, S. Coradeschi, A. Perini, P. Traverso (Eds.), Proc. of the 17th Eur. Conf. on Artificial Intelligence (ECAI'06), IOS Press, 2006, pp. 457–461.

Appendix A. Proofs

Proof of Theorem 3.1

Let $\mathcal{O}$ and $\mathcal{O}'$ be as constructed in Section 3. We need to show that $\mathcal{O}$ is consistent in L-$\mathcal{SHI}^-$ iff $\mathcal{O}'$ is consistent in $\mathbf{2}$-$\mathcal{SHI}^-$. Let first $\mathcal{J}$ be a model of $\mathcal{O}'$ in $\mathbf{2}$-$\mathcal{SHI}^-$, i.e. every membership degree is either $\mathbf{0}$ or $\mathbf{1}$. Since $\{\mathbf{0}, \mathbf{1}\}$ is closed under the lattice operations of L, the values $C^{\mathcal{J}}$ interpreted in L are the same as in $\mathbf{2}$. Any assertion $\langle \alpha \geq \ell \rangle \in \mathcal{A}$ with $\ell > \mathbf{0}$ is thus satisfied by $\mathcal{J}$, since it even satisfies the stronger assertion $\langle \alpha \geq \mathbf{1} \rangle$. Note that if $\ell = \mathbf{0}$, then the assertion is trivially satisfied by any interpretation. A similar argument holds for $\mathcal{T}$, and thus $\mathcal{J}$ is also a model of $\mathcal{O}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}$ and define the crisp model $\mathcal{J}$ as follows:

- $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$;
- $A^{\mathcal{J}}(x) := \mathbb{1}(A^{\mathcal{I}}(x))$ for all $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$;
- $r^{\mathcal{J}}(x, y) := \mathbb{1}(r^{\mathcal{I}}(x, y))$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$;
- $a^{\mathcal{J}} := a^{\mathcal{I}}$ for all $a \in \mathsf{N_I}$,

where the function $\mathbb{1}\colon L \to \{\mathbf{0}, \mathbf{1}\}$ is defined as follows (see Proposition 2.3):

$$\mathbb{1}(x) := (x \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0} = \begin{cases} \mathbf{1} & \text{if } x > \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

It is easy to see that $s^{\mathcal{J}}(x, y) = \mathbb{1}(s^{\mathcal{I}}(x, y))$ also holds for any *complex* role s and that $s^{\mathcal{J}}$ is transitive if $s^{\mathcal{I}}$ is. Furthermore, we now show by induction on the structure of the concept C that $C^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x))$ holds for all concepts C and $x \in \Delta^{\mathcal{I}}$.

- For the concept names, the claim holds by definition of $\mathcal{J}$.
- We have $\top^{\mathcal{J}}(x) = \mathbf{1} = \mathbb{1}(\mathbf{1}) = \mathbb{1}(\top^{\mathcal{I}}(x))$ and $\bot^{\mathcal{J}}(x) = \mathbf{0} = \mathbb{1}(\mathbf{0}) = \mathbb{1}(\bot^{\mathcal{I}}(x)$.
- For concepts of the form $C \sqcap D$, we have

  $$(C \sqcap D)^{\mathcal{J}}(x) = C^{\mathcal{J}}(x) \otimes D^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x)) \otimes \mathbb{1}(D^{\mathcal{I}}(x)).$$

  Since L has no zero divisors, we know that $C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x) > \mathbf{0}$ iff both $C^{\mathcal{I}}(x) > \mathbf{0}$ and $D^{\mathcal{I}}(x) > \mathbf{0}$. We conclude that

  $$(C \sqcap D)^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)) = \mathbb{1}((C \sqcap D)^{\mathcal{I}}(x).$$

- For concepts of the form $C \sqcup D$, we have

$$(C \sqcup D)^{\mathcal{J}}(x) = C^{\mathcal{J}}(x) \oplus D^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x)) \oplus \mathbb{1}(D^{\mathcal{I}}(x)).$$

  Since the t-conorm is always bounded from below by the join, we know that $C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x) = \mathbf{0}$ iff $C^{\mathcal{I}}(x) = \mathbf{0}$ or $D^{\mathcal{I}}(x) = \mathbf{0}$. As in the previous case, we conclude that

$$(C \sqcup D)^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)) = \mathbb{1}((C \sqcup D)^{\mathcal{I}}(x)).$$

- For concepts of the form $C \to D$, we have

$$(C \to D)^{\mathcal{J}}(x) = C^{\mathcal{J}}(x) \Rightarrow D^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x)) \Rightarrow \mathbb{1}(D^{\mathcal{I}}(x)).$$

  This value is $\mathbf{1}$ iff $\mathbb{1}(C^{\mathcal{I}}(x)) = \mathbf{0}$ or $\mathbb{1}(D^{\mathcal{I}}(x)) = \mathbf{1}$, i.e. either $C^{\mathcal{I}}(x) = \mathbf{0}$ or $D^{\mathcal{I}}(x) > \mathbf{0}$. By Proposition 2.3, this is the case iff $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) > \mathbf{0}$, i.e. $\mathbb{1}(C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)) = \mathbf{1}$. We conclude

$$(C \to D)^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)) = \mathbb{1}((C \to D)^{\mathcal{I}}(x)).$$

- For concepts of the form $\exists s.C$, we have

$$(\exists s.C)^{\mathcal{J}}(x) = \bigvee_{y \in \Delta^{\mathcal{J}}} s^{\mathcal{J}}(x,y) \otimes C^{\mathcal{J}}(y) = \bigvee_{y \in \Delta^{\mathcal{I}}} \mathbb{1}(s^{\mathcal{I}}(x,y)) \otimes \mathbb{1}(C^{\mathcal{I}}(y)) = \bigvee_{y \in \Delta^{\mathcal{I}}} \mathbb{1}(s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y))$$

  by the same arguments as for the case $C \sqcap D$ above. This value is $\mathbf{0}$ iff we have $s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) = \mathbf{0}$ for all $y \in \Delta^{\mathcal{I}}$ iff the join of all these values is $\mathbf{0}$. Thus,

$$(\exists s.C)^{\mathcal{J}}(x) = \mathbb{1}\Big(\bigvee_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\Big) = \mathbb{1}((\exists s.C)^{\mathcal{I}}(x)).$$

- For concepts of the form $\forall s.C$, we have

$$(\forall s.C)^{\mathcal{J}}(x) = \bigwedge_{y \in \Delta^{\mathcal{J}}} s^{\mathcal{J}}(x,y) \Rightarrow C^{\mathcal{J}}(y) = \bigwedge_{y \in \Delta^{\mathcal{I}}} \mathbb{1}(s^{\mathcal{I}}(x,y)) \Rightarrow \mathbb{1}(C^{\mathcal{I}}(y)) = \bigwedge_{y \in \Delta^{\mathcal{I}}} \mathbb{1}(s^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y))$$

  by the same arguments as for the case $C \to D$ above. Since $\mathcal{I}$ is witnessed, there is $y' \in \Delta^{\mathcal{I}}$ such that $(\forall s.C)^{\mathcal{I}}(x) = s^{\mathcal{I}}(x,y') \Rightarrow C^{\mathcal{I}}(y')$. Thus, we know that $(\forall s.C)^{\mathcal{J}}(x) = \mathbf{0}$ iff $s^{\mathcal{I}}(x,y') \Rightarrow C^{\mathcal{I}}(y') = \mathbf{0}$, i.e. $\mathbb{1}(s^{\mathcal{I}}(x,y') \Rightarrow C^{\mathcal{I}}(y')) = \mathbf{0}$. Thus,

$$(\forall s.C)^{\mathcal{J}}(x) = \mathbb{1}(s^{\mathcal{I}}(x,y') \Rightarrow C^{\mathcal{I}}(y')) = \mathbb{1}((\forall s.C)^{\mathcal{I}}(x)).$$

Consider now an assertion $\langle a : C \geq \ell \rangle \in \mathcal{A}$ with $\ell > \mathbf{0}$. Since $\mathcal{I}$ satisfies $\mathcal{A}$, we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \ell$, and thus $C^{\mathcal{J}}(a^{\mathcal{J}}) = \mathbf{1}$, i.e. $\mathcal{J}$ satisfies the crisp axiom $\langle a : C \geq \mathbf{1} \rangle$. A similar argument can be made for role assertions. If $\langle C \sqsubseteq D \geq \ell \rangle \in \mathcal{T}$ with $\ell > \mathbf{0}$, then we know that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \ell$ for all $x \in \Delta^{\mathcal{I}}$. As in the case for $C \to D$ above, we can show that

$$C^{\mathcal{J}}(x) \Rightarrow D^{\mathcal{J}}(x) = \mathbb{1}(C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)) = \mathbf{1}$$

holds for all $x \in \Delta^{\mathcal{J}}$, and thus $\mathcal{J}$ satisfies the crisp GCI $\langle C \sqsubseteq D \geq \mathbf{1} \rangle$. Finally, the role inclusions in $\mathcal{R}$ are also satisfied by $\mathcal{J}$ since $\mathbb{1}$ is monotone. □

Proof of Lemma 3.4

We first show by induction that for every $\mathfrak{NEL}$ concept C such that Bot and Nil do not appear in C, and every $x \in \Delta^{\mathcal{I}}$ it holds that $C^{\mathcal{I}}(x) = \min\{(\rho(C))^{\mathcal{J}}(x), 1\}$. In particular, this means that $(\rho(C))^{\mathcal{J}}(x) \geq 0$. If C is a concept name or $\top$, then the claim holds by definition of ρ. Assume that this holds for the concepts D and E.

- Let C be of the form $D \sqcap E$. Then $(\rho(C))^{\mathcal{J}}(x) = (\rho(D))^{\mathcal{J}}(x) \otimes_\infty (\rho(E))^{\mathcal{J}}(x)$. If $(\rho(C))^{\mathcal{J}}(x) > 1$, then $(\rho(D))^{\mathcal{J}}(x) > 1$ and $(\rho(E))^{\mathcal{J}}(x) > 1$. By induction, $D^{\mathcal{I}}(x) = E^{\mathcal{I}}(x) = 1$ and hence $C^{\mathcal{I}}(x) = 1$. If $(\rho(C))^{\mathcal{J}}(x) \leq 1$, then (i) $(\rho(D))^{\mathcal{J}}(x) \leq 1$ or (ii) $(\rho(E))^{\mathcal{J}}(x) \leq 1$. Thus,
$$(\rho(C))^{\mathcal{J}}(x) \quad = \quad \min\{(\rho(D))^{\mathcal{J}}(x), 1\} \otimes \min\{(\rho(E))^{\mathcal{J}}(x), 1\} = D^{\mathcal{I}}(x) \otimes E^{\mathcal{I}}(x) = C^{\mathcal{I}}(x).$$
- If C is of the form $\exists r.D$, then
$$\begin{aligned}\min\{(\rho(C))^{\mathcal{J}}(x), 1\} &= \min\left\{\bigvee_{y\in\Delta^{\mathcal{J}}} r^{\mathcal{J}}(x,y) \otimes_\infty (\rho(D))^{\mathcal{J}}(y), 1\right\}\\ &= \bigvee_{y\in\Delta^{\mathcal{J}}} r^{\mathcal{J}}(x,y) \otimes_\infty \min\{(\rho(D))^{\mathcal{J}}(y), 1\}\\ &= \bigvee_{y\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes_\infty D^{\mathcal{I}}(y) = \bigvee_{y\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x).\end{aligned}$$
- If C is of the form $D \to \bot$, then
$$\begin{aligned}\min\{(\rho(C))^{\mathcal{J}}(x), 1\} &= \min\{(\rho(D))^{\mathcal{J}}(x) \Rightarrow_\infty 0, 1\}\\ &= \begin{cases}1 & \text{if } (\rho(D))^{\mathcal{J}}(x) = 0\\ \min\{(\rho(D))^{\mathcal{J}}(x), 1\} \Rightarrow 0 & \text{otherwise}\end{cases}\\ &= D^{\mathcal{I}}(x) \Rightarrow 0 = C^{\mathcal{I}}(x).\end{aligned}$$

Suppose now that $\mathcal{J}$ is not a model of $\mathcal{T}_\infty$. This implies that there must be an axiom $\langle C \sqsubseteq D \geq 1\rangle \in \mathcal{T}$ such that $\mathcal{J} \not\models \langle \rho(C) \sqsubseteq \rho(D) \geq 1\rangle$. In particular, this means that $(\rho(C))^{\mathcal{J}}(x) \Rightarrow_\infty (\rho(D))^{\mathcal{J}}(x) < 1$ for some $x \in \Delta^{\mathcal{I}}$ and hence $(\rho(D))^{\mathcal{J}}(x) < 1$, which implies that $D^{\mathcal{I}}(x) = (\rho(D))^{\mathcal{J}}(x) \in [0, 1]$. If $(\rho(C))^{\mathcal{J}}(x) < 1$, then $C^{\mathcal{I}}(x) = (\rho(C))^{\mathcal{J}}(x) \in [0, 1]$, and $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) = (\rho(C))^{\mathcal{J}}(x) \Rightarrow_\infty (\rho(D))^{\mathcal{J}}(x) < 1$; if $(\rho(C))^{\mathcal{J}}(x) \geq 1$, then $C^{\mathcal{I}}(x) = 1$ and $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) = D^{\mathcal{I}}(x) < 1$. Both cases violate the assumption that $\mathcal{I}$ is a model of the axiom $\langle C \sqsubseteq D \geq 1\rangle$. A similar argument shows that $\mathcal{J}$ satisfies $\mathcal{A}_\infty$. □

Proof of Lemma 3.5

We first show that the transformation ρ is compatible with the approximation $\mathcal{J}$ of $\mathcal{I}$. Formally, we show that for every $\mathfrak{NEL}$ concept C and every $x \in \Delta^{\mathcal{I}}$, it holds that
$$C^{\mathcal{J}}(x) = \begin{cases}0 & \text{if } (\rho(C))^{\mathcal{I}}(x) \leq 0\\ 1 & \text{if } (\rho(C))^{\mathcal{I}}(x) \geq 1\\ (\rho(C))^{\mathcal{I}}(x) & \text{otherwise.}\end{cases}$$

The proof is by induction on the structure of the concept C. For all concept names A and for $\top$, the claim holds by construction. Suppose now it holds for concepts D and E.

- Let C be of the form $D \sqcap E$. If $(\rho(C))^{\mathcal{I}}(x) = (\rho(D))^{\mathcal{I}}(x) \otimes_\infty (\rho(E))^{\mathcal{I}}(x) < 0$, then $(\rho(D))^{\mathcal{I}}(x) < 0$ or $(\rho(E))^{\mathcal{I}}(x) < 0$. By induction, $D^{\mathcal{J}}(x) = 0$ or $E^{\mathcal{J}}(x) = 0$ and hence $C^{\mathcal{J}}(x) = D^{\mathcal{J}}(x) \otimes E^{\mathcal{J}}(x) = 0$.

  If $(\rho(C))^{\mathcal{I}}(x) \geq 1$, then $(\rho(D))^{\mathcal{I}}(x) \geq 1$ and $(\rho(E))^{\mathcal{I}}(x) \geq 1$. By induction, $D^{\mathcal{J}}(x) = 1 = E^{\mathcal{J}}(x)$ and hence $C^{\mathcal{J}}(x) = D^{\mathcal{J}}(x) \otimes E^{\mathcal{J}}(x) = 1$.

  Finally, if $(\rho(C))^{\mathcal{I}}(x) \in [0, 1)$, then we also have $(\rho(D))^{\mathcal{I}}(x), (\rho(E))^{\mathcal{I}}(x) \in [0, 1)$. It follows that $(\rho(C))^{\mathcal{I}}(x) = (\rho(D))^{\mathcal{I}}(x) \otimes_\infty (\rho(E))^{\mathcal{I}}(x) = D^{\mathcal{J}}(x) \otimes E^{\mathcal{J}}(x) = C^{\mathcal{J}}(x)$.

- Consider now the concept $C = \exists r.D$. If $(\rho(C))^{\mathcal{I}}(x) < 0$, then for every $y \in \Delta^{\mathcal{I}}$, $r^{\mathcal{I}}(x,y) \otimes_\infty D^{\mathcal{I}}(y) < 0$. By induction, this implies that $r^{\mathcal{J}}(x,y) \otimes D^{\mathcal{J}}(y) = 0$ for every $y \in \Delta^{\mathcal{J}}$, and hence $C^{\mathcal{J}}(x) = 0$.

  If $(\rho(C))^{\mathcal{I}}(x) > 1$, then there exists a $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x,y) \otimes_\infty D^{\mathcal{I}}(y) > 1$. This implies that $r^{\mathcal{J}}(x,y) = 1 = D^{\mathcal{J}}(y)$ and hence $1 \geq C^{\mathcal{J}}(x) \geq r^{\mathcal{J}}(x,y) \otimes D^{\mathcal{J}}(y) = 1$.

  Otherwise, $(\rho(C))^{\mathcal{I}}(x) = \bigvee_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes_\infty D^{\mathcal{I}}(y) = \bigvee_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{J}}(x,y) \otimes D^{\mathcal{J}}(y) = C^{\mathcal{J}}(x)$.

- For $C = D \to \bot$, by definition, $(\rho(C))^{\mathcal{I}}(x) = (\rho(D))^{\mathcal{I}}(x) \Rightarrow_\infty 0 \geq 0$.

  If $(\rho(C))^{\mathcal{I}}(x) > 1$, then $(\rho(D))^{\mathcal{I}}(x) \leq 0$ and hence $D^{\mathcal{J}}(x) = 0$, which yields $C^{\mathcal{J}}(x) = 1$.

  Otherwise, $(\rho(D))^{\mathcal{I}}(x) > 0$. If $(\rho(D))^{\mathcal{I}}(x) \geq 1$, then $D^{\mathcal{J}}(x) = 1$ and $C^{\mathcal{J}}(x) = 0 = (\rho(C))^{\mathcal{I}}(x)$; otherwise, we have $D^{\mathcal{J}}(x) = (\rho(D))^{\mathcal{I}}(x)$, which yields the result.

Suppose now that there is an axiom $\langle C \sqsubseteq D \geq 1\rangle \in \mathcal{T}$ and an $x \in \Delta^{\mathcal{J}}$ such that $C^{\mathcal{J}}(x) \Rightarrow D^{\mathcal{J}}(x) < 1$. This means that $0 \leq D^{\mathcal{J}}(x) < 1$ and $C^{\mathcal{J}}(x) > D^{\mathcal{J}}(x)$. But then, $(\rho(D))^{\mathcal{I}}(x) < 1$ and $(\rho(C))^{\mathcal{I}}(x) > (\rho(D))^{\mathcal{I}}(x)$. This implies then that $(\rho(C))^{\mathcal{I}}(x) \Rightarrow_\infty (\rho(D))^{\mathcal{I}}(x) < 1$, which means that $\mathcal{I}$ does not satisfy the axiom $\langle \rho(C) \sqsubseteq \rho(D) \geq 1\rangle$ violating the assumption that $\mathcal{I}$ is a model of $\mathcal{T}_\infty$. Again, a similar argument shows that $\mathcal{J}$ must satisfy $\mathcal{A}$. □

Proof of Lemma 3.6

Let $\mathcal{O}$ and $\mathcal{O}'$ be as defined in Section 3. We show that $\mathcal{O}$ is consistent in L_∞-$\mathcal{SHI}^-$ iff $\mathcal{O}'$ is consistent in $\mathbf{4}$-$\mathcal{SHI}^-$. Let $\mathcal{J}$ be a model of $\mathcal{O}'$ in $\mathbf{4}$-$\mathcal{SHI}^-$. As $\mathbf{4}$ is closed under the lattice operations of L_∞, all the $\mathcal{SHI}$-concepts get the exact same interpretation in L_∞ as in $\mathbf{4}$. As all the axioms in $\mathcal{O}'$ are stronger than those in $\mathcal{O}$, $\mathcal{J}$ is also a model of $\mathcal{O}$.

To prove the converse, for an arbitrary model $\mathcal{I}$ of $\mathcal{O}$, we define the $\mathbf{4}$-interpretation $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ with $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}}$ such that for every individual name a we have $a^{\mathcal{J}} := a^{\mathcal{I}}$, and for every role name r, concept name A and $x, y \in \Delta^{\mathcal{J}}$

$$A^{\mathcal{J}}(x) := \mathbb{4}(A^{\mathcal{I}}(x)) \qquad\qquad r^{\mathcal{J}}(x,y) := \mathbb{4}(r^{\mathcal{I}}(x,y)).$$

We show by induction on the structure that for every concept C and $x \in \Delta^{\mathcal{I}}$, it holds that $C^{\mathcal{J}}(x) = \mathbb{4}(C^{\mathcal{I}}(x))$. For concept names, $\top$, and $\bot$, this holds by definition of $\mathcal{J}$ and $\mathbb{4}$. Suppose now that it holds for C and D.

- For concepts of the form $C \sqcap D$, we have $(C \sqcap D)^{\mathcal{J}}(x) = C^{\mathcal{J}}(x) \otimes_\infty D^{\mathcal{J}}(x) = \mathbb{4}(C^{\mathcal{I}}(x)) \otimes_\infty \mathbb{4}(D^{\mathcal{I}}(x))$. Since $\otimes$ has no zero divisors, if $C^{\mathcal{I}}(x) > 0$ and $D^{\mathcal{I}}(x) > 0$, then $(C \sqcap D)^{\mathcal{I}}(x) > 0$. This implies that $(C \sqcap D)^{\mathcal{J}}(x) = \infty = \mathbb{4}((C \sqcap D)^{\mathcal{I}}(x))$.

  Otherwise, we have $(C \sqcap D)^{\mathcal{I}}(x) \leq 0$, and thus $\mathbb{4}((C \sqcap D)^{\mathcal{I}}(x)) = C^{\mathcal{I}}(x) \otimes_\infty D^{\mathcal{I}}(x)$. Since either $C^{\mathcal{I}}(x) \leq 0$ or $D^{\mathcal{I}}(x) \leq 0$, this is in turn equal to $\mathbb{4}(C^{\mathcal{I}}(x)) \otimes_\infty \mathbb{4}(D^{\mathcal{I}}(x)) = (C \sqcap D)^{\mathcal{J}}(x)$.

- For the case of $C \to D$, we have $(C \to D)^{\mathcal{J}}(x) = C^{\mathcal{J}}(x) \Rightarrow_\infty D^{\mathcal{J}}(x) = \mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x))$. If $C^{\mathcal{I}}(x) \geq 0$ and $D^{\mathcal{I}}(x) \geq 0$, then $\mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x))$ and $\mathbb{4}((C \to D)^{\mathcal{I}}(x))$ are either 0 or ∞. Furthermore, $\mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x)) = \infty$ iff $\mathbb{4}(C^{\mathcal{I}}(x)) = 0$ or $\mathbb{4}(D^{\mathcal{I}}(x)) = \infty$ iff $C^{\mathcal{I}}(x) = 0$ or $D^{\mathcal{I}}(x) > 0$ iff $(C \to D)^{\mathcal{I}}(x) > 0$ iff $\mathbb{4}((C \to D)^{\mathcal{I}}(x)) = \infty$.

  If $C^{\mathcal{I}}(x) < 0$ and $D^{\mathcal{I}}(x) \geq 0$, then $C^{\mathcal{I}}(x) \Rightarrow_\infty D^{\mathcal{I}}(x) = \mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x)) = \infty$. If $C^{\mathcal{I}}(x) \geq 0$ and $D^{\mathcal{I}}(x) < 0$, then $C^{\mathcal{I}}(x) \Rightarrow_\infty D^{\mathcal{I}}(x) = \mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x)) = \mathbb{4}(D^{\mathcal{I}}(x))$.

  Finally, if both $C^{\mathcal{I}}(x) < 0$ and $D^{\mathcal{I}}(x) < 0$, then $\mathbb{4}(C^{\mathcal{I}}(x)) = C^{\mathcal{I}}(x)$ and $\mathbb{4}(D^{\mathcal{I}}(x)) = D^{\mathcal{I}}(x)$, which implies that $C^{\mathcal{I}}(x) \Rightarrow_\infty D^{\mathcal{I}}(x) = \mathbb{4}(C^{\mathcal{I}}(x)) \Rightarrow_\infty \mathbb{4}(D^{\mathcal{I}}(x))$.

  Since $\mathbf{4}$ is closed under $\Rightarrow_\infty$, in all cases we have that $\mathbb{4}((C \to D)^{\mathcal{I}}(x)) = (C \to D)^{\mathcal{J}}(x)$.

- For concepts of the form $\forall s.C$, as in the previous case we know that

  $$(\forall s.C)^{\mathcal{J}}(x) = \bigwedge_{y \in \Delta^{\mathcal{J}}} s^{\mathcal{J}}(x,y) \Rightarrow_\infty C^{\mathcal{J}}(y) = \bigwedge_{y \in \Delta^{\mathcal{I}}} \mathbb{4}(s^{\mathcal{I}}(x,y) \Rightarrow_\infty C^{\mathcal{I}}(y)).$$

  Since $\mathcal{I}$ is witnessed, there is a $y' \in \Delta^{\mathcal{I}}$ such that $(\forall s.C)^{\mathcal{I}}(x) = s^{\mathcal{I}}(x,y') \Rightarrow_\infty C^{\mathcal{I}}(y')$. From monotonicity of $\mathbb{4}$ it follows that $(\forall s.C)^{\mathcal{J}}(x) = \mathbb{4}(s^{\mathcal{I}}(x,y') \Rightarrow_\infty C^{\mathcal{I}}(y')) = \mathbb{4}((\forall s.C)^{\mathcal{I}}(x))$.

All the other cases can be treated similarly as above (see the proof of Theorem 3.1 for details). We now show that $\mathcal{J}$ is a model of $\mathcal{O}'$. Given an assertion $\langle a\!:\!C \geq \ell\rangle \in \mathcal{A}$, we know that $C^\mathcal{I}(a^\mathcal{I}) \geq \ell$, since $\mathcal{I}$ is a model of $\mathcal{O}$. Thus, $C^\mathcal{J}(a^\mathcal{J}) \geq \mathbb{4}(\ell)$, and hence $\mathcal{J}$ satisfies the assertion $\langle a\!:\!C \geq \mathbb{4}(\ell)\rangle \in \mathcal{A}'$. The case of role assertions follows from a similar argument. For GCIs $\langle C \sqsubseteq D \geq \ell\rangle \in \mathcal{T}$ we know that $C^\mathcal{I}(x) \Rightarrow_\infty D^\mathcal{I}(x) \geq \ell$ for all $x \in \Delta^\mathcal{I}$. We can then show that $C^\mathcal{J}(x) \Rightarrow_\infty D^\mathcal{J}(x) = \mathbb{4}(C^\mathcal{I}(x) \Rightarrow_\infty D^\mathcal{I}(x)) \geq \mathbb{4}(\ell)$ holds for every $x \in \Delta^\mathcal{J} = \Delta^\mathcal{I}$ (see the case for the implication constructor above). Thus, $\mathcal{J}$ satisfies the axiom $\langle C \sqsubseteq D \geq \mathbb{4}(\ell)\rangle \in \mathcal{T}'$. The fact that all role inclusions in $\mathcal{R}$ are satisfied follows trivially from the monotonicity of the function $\mathbb{4}$. □

Proof of Lemma 4.5

Let $\mathbf{T}$ be a tableau for $\mathcal{O}$ over the set Ind of individuals. For each role name r, we define a fuzzy binary relation $r^\mathbf{T}$ over Ind as follows: $r^\mathbf{T}(x,y) := \ell$ if $\langle (x,y)\!:\!r = \ell\rangle \in \mathbf{T}$; and $r^\mathbf{T}(x,y) := \mathbf{0}$ otherwise. Note that these values are either unique or undefined since $\mathbf{T}$ is clash-free. If they are undefined in $\mathbf{T}$, then we set them to $\mathbf{0}$ for now. In this way, $\mathbf{T}$ immediately defines a rudimentary interpretation of the role names. However, transitive roles are not yet interpreted by transitive fuzzy relations. In the following, we denote by $r^\mathbf{T}(z_1,\dots,z_n)$ the value $r^\mathbf{T}(z_1,z_2) \otimes \dots \otimes r^\mathbf{T}(z_{n-1},z_n)$ for any sequence $z_1,\dots,z_n \in \mathsf{Ind}$. This value is $\mathbf{1}$ if $n = 1$ since $\mathbf{1}$ is the unit element for $\otimes$.

We now define a proper model $\mathcal{I}$ of $\mathcal{O}$ as follows:

- $\Delta^\mathcal{I} := \mathsf{Ind}$;
- for all concept names A and $x \in \mathsf{Ind}$, $A^\mathcal{I}(x) := \ell$ if $\langle x\!:\!A = \ell\rangle \in \mathbf{T}$; and $A^\mathcal{I}(x) := \mathbf{0}$ otherwise; and
- for all role names r and $x,y \in \mathsf{Ind}$,

$$r^\mathcal{I}(x,y) := \bigvee_{n\geq 0}\ \bigvee_{z_1,\dots,z_n\in\mathsf{Ind}} r^\mathbf{T}(x,z_1,\dots,z_n,y) \text{ if the role } r \text{ is transitive, and}$$
$$r^\mathcal{I}(x,y) := r^\mathbf{T}(x,y) \vee \bigvee_{s\sqsubseteq_\mathcal{R} r,\ s\neq r} s^\mathcal{I}(x,y) \text{ otherwise.}$$

This complex expression is necessary to account for the transitive sub-roles of r. If r itself is transitive, then $r^\mathcal{I}$ is the transitive closure of $r^\mathbf{T}$. By the condition inv, it is easy to show that the same equations hold for an inverse role $s = r^-$ if we define $s^\mathbf{T}(x,y) := r^\mathbf{T}(y,x)$ for all $x,y \in \mathsf{Ind}$.

This construction of $\mathcal{I}$ was inspired by a similar one used for crisp $\mathcal{SHI}$ in [20]. It is well-defined since $\mathcal{R}$ is acyclic (see Lemma 4.2). We can show that for every concept C we have $C^\mathcal{I}(x) = \ell$ whenever $\langle x\!:\!C = \ell\rangle \in \mathbf{T}$ for any $x \in \mathsf{Ind}$ and $\ell \in L$. Together with the conditions $\sqsubseteq_\mathcal{R}$ and $\sqsubseteq_\mathcal{T}$ and the fact that $\mathcal{A} \subseteq \mathbf{T}$, this shows that $\mathcal{I}$ satisfies all axioms of $\mathcal{O}$. We show the claim by induction on the structure of C.

- The claim for $\top$, $\bot$ and concept names follows from the conditions $\top$, $\bot$, clash-freeness of $\mathbf{T}$, and definition of $\mathcal{I}$.
- If $\langle x\!:\!\neg C = \ell\rangle \in \mathbf{T}$, then by condition $\neg$ and induction we have $(\neg C)^\mathcal{I}(x) = {\sim} C^\mathcal{I}(x) = {\sim}{\sim}\ell = \ell$.
- The claims for $C \sqcap D$, $C \sqcup D$, and $C \to D$ follow similarly.
- If $\langle x\!:\!\exists r.C = \ell\rangle \in \mathbf{T}$, then by condition $\exists$ there must be $y \in \mathsf{Ind}$ and $\ell_1,\ell_2 \in L$ such that $\ell_1 \otimes \ell_2 = \ell$ and $\langle (x,y)\!:\!r = \ell_1\rangle, \langle y\!:\!C = \ell_2\rangle \in \mathbf{T}$. By induction, we have $\ell = r^\mathbf{T}(x,y) \otimes C^\mathcal{I}(y) \leq r^\mathcal{I}(x,y) \otimes C^\mathcal{I}(y)$. We now show that for every $z \in \mathsf{Ind}$ we have $r^\mathcal{I}(x,z) \otimes C^\mathcal{I}(z) \leq \ell$, which in particular implies that y is a witness for $(\exists r.C)^\mathcal{I}(x)$.

  By definition of $r^\mathcal{I}$ and monotonicity of $\otimes$, it suffices to show that (a) $r^\mathbf{T}(x,z) \otimes C^\mathcal{I}(z) \leq \ell$ and (b) $s^\mathbf{T}(x,y_1,\dots,y_n,z) \otimes C^\mathcal{I}(z) \leq \ell$ for all transitive roles $s \sqsubseteq_\mathcal{R} r$ and all $y_1,\dots,y_n \in \mathsf{Ind}$, $n \geq 1$.

  (a) If $r^\mathbf{T}(x,z) = \mathbf{0}$, the claim is trivial; otherwise, there must be an assertion $\langle (x,z)\!:\!r = \ell'\rangle \in \mathbf{T}$ with $\ell' = r^\mathbf{T}(x,z)$. By condition $\exists_\leq$, we now have $\langle z\!:\!C = \ell''\rangle \in \mathbf{T}$ with $\ell' \otimes \ell'' \leq \ell$. By induction, $r^\mathbf{T}(x,z) \otimes C^\mathcal{I}(z) = \ell' \otimes \ell'' \leq \ell$.

(b) Again, the claim for $s^{\mathbf{T}}(x, y_1, \ldots, y_n, z) = \mathbf{0}$ is trivial. If this is not the case, then $s^{\mathbf{T}}(x, y_1)$, $s^{\mathbf{T}}(y_1, y_2)$, …, $s^{\mathbf{T}}(y_{n-1}, y_n)$, $s^{\mathbf{T}}(y_n, z)$ are all greater than $\mathbf{0}$.
Since $s^{\mathbf{T}}(x, y_1) > \mathbf{0}$, there must be an assertion $\langle (x, y_1) : s = \ell' \rangle \in \mathbf{T}$. By condition $\exists_+$, we have $\langle y_1 : \exists s.C = \ell^{(1)} \rangle \in \mathbf{T}$ with $s^{\mathbf{T}}(x, y_1) \otimes \ell^{(1)} = \ell' \otimes \ell^{(1)} \leq \ell$.
Analogously, one can show that for every $i \in \{1, \ldots, n-1\}$ we have $\langle y_{i+1} : \exists s.C = \ell^{(i+1)} \rangle \in \mathbf{T}$ with $s^{\mathbf{T}}(y_i, y_{i+1}) \otimes \ell^{(i+1)} \leq \ell^{(i)}$. Additionally, as in case (a) it holds that $s^{\mathbf{T}}(y_n, z) \otimes C^{\mathcal{I}}(z) \leq \ell(n)$.
This implies that $s^{\mathbf{T}}(x, y_1, \ldots, y_n, z) \otimes D^{\mathcal{I}}(z) \leq \ell$ by monotonicity of $\otimes$.

- The case of $\forall r.C$ can be handled similarly to the previous case since $\mathbf{T}$ satisfies the dual conditions for value restrictions.

For the other direction, let $\mathcal{I}$ be a model of $\mathcal{O}$. We can easily construct a tableau $\mathbf{T}$ over the set $\Delta^{\mathcal{I}}$ of individuals as follows. For every concept C and $x \in \Delta^{\mathcal{I}}$, we add $\langle x : C = \ell \rangle$ to $\mathbf{T}$ if $C^{\mathcal{I}}(x) = \ell$. Similarly, for every role r and $x, y \in \Delta^{\mathcal{I}}$, we add the assertion $\langle (x, y) : r = r^{\mathcal{I}}(x, y) \rangle$ to $\mathbf{T}$. We have $\mathcal{A} \subseteq \mathbf{T}$ since $\mathcal{I}$ satisfies $\mathcal{A}$. Moreover, $\mathbf{T}$ is clash-free since the values are uniquely defined by $\mathcal{I}$.

Furthermore, the semantics of L-$\mathcal{SHI}$ concepts and axioms yield completeness: consider for instance the condition $\exists_+$ and assume that $(\exists s.C)^{\mathcal{I}}(x) = \ell$, $r^{\mathcal{I}}(x, y) = \ell_1$ with r transitive, and $r \sqsubseteq_{\mathcal{R}} s$. Since the value $\ell_2 = (\exists r.C)^{\mathcal{I}}(y)$ is defined, by monotonicity of $\otimes$ this value satisfies

$$\begin{aligned}
\ell_1 \otimes \ell_2 = r^{\mathcal{I}}(x, y) \otimes (\exists r.C)^{\mathcal{I}}(y) &= \bigvee_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes r^{\mathcal{I}}(y, z) \otimes C^{\mathcal{I}}(z) \\
&\leq \bigvee_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) \otimes C^{\mathcal{I}}(z) \leq \bigvee_{z \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, z) \otimes C^{\mathcal{I}}(z) = (\exists s.C)^{\mathcal{I}}(x) = \ell.
\end{aligned}$$

Similar arguments show that $\mathbf{T}$ satisfies the other completeness conditions. □

Proof of Lemma 4.7

By Lemma 4.5, $\mathcal{O}$ is locally consistent iff it has a tableau. Assume first that $\mathbf{T}$ is a tableau for $\mathcal{O}$ over the set Ind of individuals. We show how to guide the application of the expansion rules in such a way that no clash is produced.

Observe that the initial ABox $\mathcal{A}$ is included in $\mathbf{T}$ by definition. We will ensure that the expansion rules add only assertions to $\widehat{\mathcal{A}}$ that are also in $\mathbf{T}$. Assume that, for some row of Table 2, an expansion rule is applicable, i.e. $\langle$trigger$\rangle$ is in $\widehat{\mathcal{A}}$ and there are no $\langle$values$\rangle$ such that $\langle$assertions$\rangle$ are in $\widehat{\mathcal{A}}$ and the blocking condition does not apply. Since $\langle$trigger$\rangle$ is also in the tableau $\mathbf{T}$, there must be $\langle$values$\rangle$ such that $\langle$assertions$\rangle$ are in $\mathbf{T}$, and thus we can add $\langle$assertions$\rangle$ to $\widehat{\mathcal{A}}$.

Since $\mathbf{T}$ is clash-free, this process cannot create any clashes in $\widehat{\mathcal{A}}$. Lemma 4.6 shows that at some point $\widehat{\mathcal{A}}$ must also be complete.

For the other direction, assume now that the expansion rules have produced a complete and clash-free ABox $\widehat{\mathcal{A}}$. We define a tableau $\mathbf{T}$ for $\mathcal{O}$ over the set

$$\mathsf{Ind} := \{x \in \mathsf{N_I} \mid x \text{ occurs in } \widehat{\mathcal{A}} \text{ and is not blocked}\}$$

of individuals as follows:

$$\begin{aligned}
\mathbf{T} := &\{\langle x : C = \ell \rangle \in \widehat{\mathcal{A}} \mid x \in \mathsf{Ind}\} \\
&\cup \{\langle (x, y) : r = \ell \rangle \in \widehat{\mathcal{A}} \mid x, y \in \mathsf{Ind}\} \\
&\cup \{\langle (x, y) : r = \ell \rangle \mid x, y \in \mathsf{Ind}, \langle (x, z) : r = \ell \rangle \in \widehat{\mathcal{A}}, \text{ and } y \text{ blocks } z\} \\
&\cup \{\langle (x, y) : r = \ell \rangle \mid x, y \in \mathsf{Ind}, \langle (z, y) : r = \ell \rangle \in \widehat{\mathcal{A}}, \text{ and } x \text{ blocks } z\}.
\end{aligned}$$

Thus, whenever y blocks z and z is not indirectly blocked, then all incoming role connections of z are "re-routed" back to y. Since the root a of the tree-like structure $\widehat{\mathcal{A}}$ has no predecessors, it cannot be blocked, and thus the initial ABox $\mathcal{A}$ is still contained in $\mathbf{T}$. Furthermore, since $\widehat{\mathcal{A}}$ is clash-free, $\mathbf{T}$ is also clash-free.

It remains to show completeness of $\mathbf{T}$. For any row of Table 2, we distinguish three cases based on the form of $\langle$trigger$\rangle$.

a) If $\langle \mathsf{trigger} \rangle$ involves only assertions from $\widehat{\mathcal{A}}$, then the corresponding expansion rule was applied at some point and introduced $\langle \mathsf{values} \rangle$ and $\langle \mathsf{assertions} \rangle$. If no new individual was introduced, all $\langle \mathsf{assertions} \rangle$ must also be in $\mathbf{T}$. We consider now the case of the rule $\exists$; the rule $\forall$ can be handled similarly.

   Assume that $\langle x : \exists r.C = \ell \rangle \in \widehat{\mathcal{A}}$ and x is not blocked. Then a new individual y was introduced, together with the assertions $\langle (x, y) : r = \ell_1 \rangle$ and $\langle y : C = \ell_2 \rangle$, where $\ell_1 \otimes \ell_2 = \ell$. If y is not blocked, these assertions are also in $\mathbf{T}$. If y is blocked by an individual z, then the assertion $\langle (x, z) : r = \ell_2 \rangle$ is in $\mathbf{T}$. Additionally, we have $\widehat{\mathcal{A}}_y \equiv \widehat{\mathcal{A}}_z$, and thus also $\langle z : C = \ell_2 \rangle$ is in $\mathbf{T}$.

b) If $\langle \mathsf{trigger} \rangle$ involves a role assertion $\langle (x, y) : r = \ell_1 \rangle$ where $\langle (x, z) : r = \ell_1 \rangle \in \widehat{\mathcal{A}}$ and y blocks z, then x is not blocked and the corresponding expansion rule was applied to $\widehat{\mathcal{A}}$ with z instead of y.

   Consider the case of the rule $\exists_{\leq}$. Then the assertions $\langle x : \exists r.C = \ell \rangle$ and $\langle z : C = \ell_2 \rangle$ must be in $\widehat{\mathcal{A}}$ with $\ell_1 \otimes \ell_2 \leq \ell$. Since $\widehat{\mathcal{A}}_z \equiv \widehat{\mathcal{A}}_y$, we have $\langle y : C = \ell_2 \rangle$ in $\widehat{\mathcal{A}}$ and also in $\mathbf{T}$. The rules $\exists_{+}$, $\forall_{\geq}$, and $\forall_{+}$ behave similarly.

   If the rule inv was applied, then we have $\langle (z, x) : \overline{r} = \ell_1 \rangle \in \widehat{\mathcal{A}}$, and thus $\langle (y, x) : \overline{r} = \ell_1 \rangle$ is in $\mathbf{T}$.

   If the rule $\sqsubseteq_{\mathcal{R}}$ was applied with $r \sqsubseteq_{\mathcal{R}} s$, then $\langle (x, z) : s = \ell_2 \rangle \in \widehat{\mathcal{A}}$ with some $\ell_2 \in L$ such that $\ell_1 \leq \ell_2$. Thus, we have $\langle (x, y) : s = \ell_2 \rangle$ in $\mathbf{T}$.

c) If $\langle \mathsf{trigger} \rangle$ involves a role assertion $\langle (x, y) : r = \ell_1 \rangle$ where $\langle (z, y) : r = \ell_1 \rangle \in \widehat{\mathcal{A}}$ and x blocks z, then consider the concrete condition concerned.

   If it is the condition $\exists_{\leq}$, then we have $\langle x : \exists r.C = \ell \rangle$ in $\mathbf{T}$ and also in $\widehat{\mathcal{A}}$. Since $\widehat{\mathcal{A}}_x \equiv \widehat{\mathcal{A}}_z$, this implies that $\langle z : \exists r.C = \ell \rangle$ is in $\widehat{\mathcal{A}}$. Since z must be a successor of y, z is not indirectly blocked, and thus by the rule $\exists_{\leq}$ there is $\langle y : C = \ell_2 \rangle$ in $\widehat{\mathcal{A}}$ with $\ell_1 \otimes \ell_2 \leq \ell$. The same assertion must also be present in $\mathbf{T}$ since y is not blocked. Again, the conditions $\exists_{+}$, $\forall_{\geq}$, and $\forall_{+}$ can be handled similarly.

   If it is the condition inv, then since z is not indirectly blocked, we have $\langle (y, z) : \overline{r} = \ell_1 \rangle \in \widehat{\mathcal{A}}$, and thus $\langle (y, x) : \overline{r} = \ell_1 \rangle$ in $\mathbf{T}$.

   If it is the condition $\sqsubseteq_{\mathcal{R}}$ with $r \sqsubseteq_{\mathcal{R}} s$, then since z is not indirectly blocked, there must be a value $\ell_2 \in L$ with $\ell_1 \leq \ell_2$ such that $\langle (z, y) : s = \ell_2 \rangle$ is in $\widehat{\mathcal{A}}$, and thus $\langle (x, y) : s = \ell_2 \rangle$ is in $\mathbf{T}$. $\square$