Reasoning in Infinitely Valued G-$\mathcal{IALCQ}$

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Abstract

Fuzzy Description Logics (FDLs) are logic-based formalisms used to represent and reason with vague or imprecise knowledge. It has been recently shown that reasoning in most FDLs using truth values from the interval $[0,1]$ becomes undecidable in the presence of a negation constructor and general concept inclusion axioms. One exception to this negative result are FDLs whose semantics is based on the infinitely valued Gödel t-norm (G). In this paper, we extend previous decidability results for G-$\mathcal{IALC}$ to deal also with qualified number restrictions. Our novel approach is based on a combination of the known crispification technique for finitely valued FDLs and the automata-based procedure originally developed for reasoning in G-$\mathcal{IALC}$. The proposed approach combines the advantages of these two methods, while removing their respective drawbacks.

1 Introduction

It is well-known that one of the main requirements for the development of an intelligent application is a formalism capable of representing and handling knowledge without ambiguity. Description Logics (DLs) are a well-studied family of knowledge representation formalisms [Baader *et al.*, 2007]. They constitute the logical backbone of the standard Semantic Web ontology language OWL 2,[1] and its profiles, and have been successfully applied to represent the knowledge of many and diverse application domains, particularly in the bio-medical sciences.

DLs describe the domain knowledge using *concepts* (such as $\mathsf{Patient}$) that represent sets of individuals, and *roles* ($\mathsf{hasRelative}$) that represent connections between individuals. *Ontologies* are collections of axioms formulated over these concepts and roles, which restrict their possible interpretations. The typical axioms considered in DLs are *assertions*, like $\mathsf{bob}{:}\mathsf{Patient}$, providing knowledge about specific individuals; and *general concept inclusions (GCIs)*, such as $\mathsf{Patient} \sqsubseteq \mathsf{Human}$, which express general relations between concepts. Different DLs are characterized by the constructors allowed to generate complex concepts and roles from atomic ones. $\mathcal{ALC}$ [Schmidt-Schauß and Smolka, 1991] is a prototypical DL of intermediate expressivity that contains the concept constructors conjunction ($C \sqcap D$), negation ($\neg C$), and existential restriction ($\exists r.C$ for a role r). If additionally qualified number restrictions (${\geqslant} n\, r.C$ for $n \in \mathbb{N}$) are allowed, the resulting logic is denoted by $\mathcal{ALCQ}$. Common reasoning problems in $\mathcal{ALCQ}$, such as consistency of ontologies or subsumption between concepts, are known to be ExpTime-complete [Schild, 1991; Tobies, 2001].

Fuzzy Description Logics (FDLs) have been introduced as extensions of classical DLs to represent and reason with vague knowledge. The main idea is to consider all the truth values from the interval $[0,1]$ instead of only *true* and *false*. In this way, it is possible give a more fine-grained semantics to inherently vague concepts like $\mathsf{LowFrequency}$ or $\mathsf{HighConcentration}$, which can be found in biomedical ontologies like Snomed CT,[2] and Galen.[3] The different members of the family of FDLs are characterized not only by the constructors they allow, but also by the way these constructors are interpreted.

To interpret conjunction in complex concepts like

$$\exists \mathsf{hasHeartRate}.\mathsf{LowFrequency} \sqcap \\ \exists \mathsf{hasBloodAlcohol}.\mathsf{HighConcentration},$$

a popular approach is to use so-called *t-norms* [Klement *et al.*, 2000]. The semantics of the other logical constructors can then be derived from these t-norms in a principled way, as suggested by Hájek [2001]. Following the principles of mathematical fuzzy logic, existential restrictions are interpreted as suprema of truth values. However, to avoid problems with infinitely many truth values, reasoning in fuzzy DLs is often restricted to so-called *witnessed models* [Hájek, 2005], in which these suprema are required to be maxima; i.e., the degree is witnessed by at least one domain element.

Unfortunately, reasoning in most FDLs becomes undecidable when the logic allows to use GCIs and negation under witnessed model semantics [Baader and Peñaloza, 2011; Cerami and Straccia, 2013; Borgwardt *et al.*, 2015]. One of the few exceptions known are FDLs using the *Gödel* t-norm defined as $\min\{x,y\}$ to interpret conjunctions [Borgwardt *et*

[1] http://www.w3.org/TR/owl2-overview/

[2] http://www.ihtsdo.org/snomed-ct/

[3] http://www.opengalen.org/

al., 2014]. Despite not being as well-behaved as finitely valued FDLs, which use a finite total order of truth values instead of the infinite interval $[0, 1]$ [Borgwardt and Peñaloza, 2013], it has been shown using an automata-based approach that reasoning in Gödel extensions of $\mathcal{ALC}$ exhibits the same complexity as in the classical case, i.e. it is ExpTime-complete. A major drawback of this approach is that it always has an exponential runtime, even when the input ontology has a simple form.

In this paper, we extend the results of [Borgwardt *et al.*, 2014] to deal with qualified number restrictions, showing again that the complexity of reasoning remains the same as for the classical case; i.e., it is ExpTime-complete. To this end, we focus only on the problem of *local consistency*, which is a generalization of the classical concept satisfiability problem. We follow a more practical approach that combines the automata-based construction from [Borgwardt *et al.*, 2014] with reduction techniques developed for finitely valued FDLs [Straccia, 2004; Bobillo *et al.*, 2009; Bobillo and Straccia, 2013]. We exploit the forest model property of classical $\mathcal{ALCQ}$ [Kazakov, 2004] to encode order relationships between concepts in a fuzzy interpretation in a manner similar to the Hintikka trees from [Borgwardt *et al.*, 2014]. However, instead of using automata to determine the existence of such trees, we reduce the fuzzy ontology directly into a classical $\mathcal{ALCQ}$ ontology whose local consistency is equivalent to that of the original ontology. This enables us to use optimized reasoners for classical DLs. In addition to the *cut-concepts* of the form $\boxed{C \geqslant q}$ for a fuzzy concept C and a value q, which are used in the reductions for finitely valued DLs [Straccia, 2004; Bobillo *et al.*, 2009; Bobillo and Straccia, 2013], we employ *order concepts* $\boxed{C \leqslant D}$ expressing relationships between fuzzy concepts. Contrary to the reductions for finitely valued Gödel FDLs presented by Bobillo *et al.* [2009; 2012], our reduction produces a classical ontology whose size is polynomial in the size of the input fuzzy ontology. Thus, we obtain tight complexity bounds for reasoning in this FDL [Tobies, 2001]. An extended version of this paper appears in [Borgwardt and Peñaloza, 2015].

2 Preliminaries

For the rest of this paper, we focus solely on vague statements that take truth degrees from the infinite interval $[0, 1]$, where the *Gödel t-norm*, defined by $\min\{x, y\}$, is used to interpret logical conjunction. The semantics of implications is given by the *residuum* of this t-norm; that is,

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

We use both the *residual negation* $\ominus x := x \Rightarrow 0$ and the *involutive negation* $\sim x := 1 - x$ in the rest of this paper.

We first recall some basic definitions from [Borgwardt *et al.*, 2014], which will be used extensively in the proofs throughout this work. An *order structure* S is a finite set containing at least the numbers 0, 0.5, and 1, together with an involutive unary operation $\mathsf{inv}\colon S \to S$ such that $\mathsf{inv}(x) = 1-x$ for all $x \in S \cap [0, 1]$. A *total preorder* over S is a transitive and total binary relation $\preccurlyeq \subseteq S \times S$. For $x, y \in S$, we write $x \equiv y$ if $x \preccurlyeq y$ and $y \preccurlyeq x$. Notice that $\equiv$ is an equivalence relation on S. The total preorders considered in [Borgwardt *et al.*, 2014] have to satisfy additional properties; for instance, that 0 and 1 are always the least and greatest elements, respectively. These properties can be found in our reduction in the axioms of $\mathsf{red}(\mathcal{U})$ (see Section 3 for more details).

The syntax of the FDL G-$\mathfrak{I}\mathcal{ALCQ}$ is the same as that of classical $\mathcal{ALCQ}$, with the addition of the implication constructor (denoted by the use of $\mathfrak{I}$ at the beginning of the name). This constructor is often added to FDLs, as the residuum cannot, in general, be expressed using only the t-norm and negation operators, in contrast to the classical semantics. In particular, this holds for the Gödel t-norm and its residuum, which is the focus of this work. Let now $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be mutually disjoint sets of *concept*, *role*, and *individual names*, respectively. *Concepts* of G-$\mathfrak{I}\mathcal{ALCQ}$ are built using the syntax rule

$$C, D ::= \top \mid A \mid \neg C \mid C \sqcap D \mid C \to D \mid \forall r.C \mid {\geqslant} n\, r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, C, D are concepts, and $n \in \mathbb{N}$. We use the abbreviations

$$\begin{aligned} \bot &:= \neg\top, \\ C \sqcup D &:= \neg(\neg C \sqcap \neg D), \\ \exists r.C &:= {\geqslant} 1\, r.C, \qquad \text{and} \\ {\leqslant} n\, r.C &:= \neg({\geqslant}(n+1)\, r.C) \end{aligned}$$

Notice that Bobillo *et al.* consider a different definition of *at-most restrictions*, which uses the residual negation; that is, they define ${\leqslant} n\, r.C := ({\geqslant}(n+1)\, r.C) \to \bot$ [2012]. This has the strange side effect that the value of ${\leqslant} n\, r.C$ is always either 0 or 1 (see the semantics below). However, this discrepancy in definitions is not an issue since our algorithm can handle both cases.

The semantics of this logic is based on interpretations. A G-*interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is the *interpretation function* that assigns to each individual name $a \in \mathsf{N_I}$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0, 1]$, and to each role name $r \in \mathsf{N_R}$ a fuzzy binary relation $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$. The interpretation of complex concepts is obtained from the semantics of first-order fuzzy logics via the well-known translation from DL concepts to first-order logic [Straccia, 2001; Bobillo *et al.*, 2012], i.e. for all $d \in \Delta^{\mathcal{I}}$,

$$\begin{aligned} \top^{\mathcal{I}}(d) &:= 1 \\ (\neg C)^{\mathcal{I}}(d) &:= 1 - C^{\mathcal{I}}(d) \\ (C \sqcap D)^{\mathcal{I}}(d) &:= \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\ (C \to D)^{\mathcal{I}}(d) &:= C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d) \\ (\forall r.C)^{\mathcal{I}}(d) &:= \inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d, e) \Rightarrow C^{\mathcal{I}}(e) \\ ({\geqslant} n\, r.C)^{\mathcal{I}}(d) &:= \sup_{\substack{e_1, \dots, e_n \in \Delta^{\mathcal{I}} \\ \text{pairwise different}}} \min_{i=1}^{n} \min\{r^{\mathcal{I}}(d, e_i), C^{\mathcal{I}}(e_i)\} \end{aligned}$$

Recall that the usual duality between existential and value restrictions that appears in classical DLs does not hold in G-$\mathfrak{I}\mathcal{ALCQ}$.

A *classical interpretation* is defined similarly, with the set of truth values restricted to 0 and 1. In this case, the semantics of a concept C is commonly viewed as a set $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$ instead of the characteristic function $C^\mathcal{I}\colon \Delta^\mathcal{I} \to \{0,1\}$.

In the following, we restrict all reasoning problems to so-called *witnessed* G-interpretations [Hájek, 2005], which intuitively require the suprema and infima in the semantics to be maxima and minima, respectively. More formally, the G-interpretation $\mathcal{I}$ is *witnessed* if, for every $d \in \Delta^\mathcal{I}$, $n \geqslant 0$, $r \in \mathsf{N_R}$, and concept C, there exist $e, e_1, \dots, e_n \in \Delta^\mathcal{I}$ (where $e_1, \dots, e_n$ are pairwise different) such that

$$(\forall r.C)^\mathcal{I}(d) = r^\mathcal{I}(d,e) \Rightarrow C^\mathcal{I}(e) \text{ and}$$
$$({\geqslant} n\, r.C)^\mathcal{I}(d) = \min_{i=1}^{n} \min\{r^\mathcal{I}(d,e_i), C^\mathcal{I}(e_i)\}.$$

The axioms of G-$\mathfrak{I}\mathcal{ALCQ}$ extend classical axioms by allowing to compare degrees of arbitrary assertions in so-called *ordered ABoxes* [Borgwardt *et al.*, 2014], and to state inclusions relationships between fuzzy concepts that hold to a certain degree, instead of only 1. A *classical assertion* is an expression of the form $a{:}C$ or $(a,b){:}r$ for $a,b \in \mathsf{N_I}$, $r \in \mathsf{N_R}$, and a concept C. An *order assertion* is of the form $\langle \alpha \bowtie q\rangle$ or $\langle \alpha \bowtie \beta\rangle$ where $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$, α, β are classical assertions, and $q \in [0,1]$. A *(fuzzy) general concept inclusion axiom (GCI)* is of the form $\langle C \sqsubseteq D \geqslant q\rangle$ for concepts C, D and $q \in [0,1]$. An *ordered ABox* is a finite set of order assertions, a *TBox* is a finite set of GCIs, and an *ontology* $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ consists of an ordered ABox $\mathcal{A}$ and a TBox $\mathcal{T}$. A G-interpretation $\mathcal{I}$ *satisfies* (or is a *model* of) an order assertion $\langle \alpha \bowtie \beta\rangle$ if $\alpha^\mathcal{I} \bowtie \beta^\mathcal{I}$ (where $(a{:}C)^\mathcal{I} := C^\mathcal{I}(a^\mathcal{I})$, $((a,b){:}r)^\mathcal{I} := r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})$, and $q^\mathcal{I} := q$); it *satisfies* a GCI $\langle C \sqsubseteq D \geqslant q\rangle$ if $C^\mathcal{I}(d) \Rightarrow D^\mathcal{I}(d) \geqslant q$ holds for all $d \in \Delta^\mathcal{I}$; and it *satisfies* an ordered ABox, TBox, or ontology if it satisfies all its axioms. An ontology is *consistent* if it has a (witnessed) model.

Given an ontology $\mathcal{O}$, we denote by $\mathsf{rol}(\mathcal{O})$ the set of all role names occurring in $\mathcal{O}$ and by $\mathsf{sub}(\mathcal{O})$ the closure under negation of the set of all subconcepts occurring in $\mathcal{O}$. We consider the concepts $\neg\neg C$ and C to be equal, and thus the latter set is of quadratic size in the size of $\mathcal{O}$. Moreover, we denote by $\mathcal{V}_\mathcal{O}$ the closure under the involutive negation $x \mapsto 1-x$ of the set of all truth degrees appearing in $\mathcal{O}$, together with 0, 0.5, and 1. This set is of size linear on the size of $\mathcal{O}$. We sometimes denote the elements of $\mathcal{V}_\mathcal{O} \subseteq [0,1]$ as $0 = q_0 < q_1 < \dots < q_{k-1} < q_k = 1$.

We stress that we do not consider the general consistency problem in this paper, but only a restricted version that uses only one individual name. An ordered ABox $\mathcal{A}$ is *local* if it contains no role assertions $(a,b){:}r$ and there is a single individual name $a \in \mathsf{N_I}$ such that all order assertions in $\mathcal{A}$ only use a. The *local consistency* problem, i.e. deciding whether an ontology $(\mathcal{A}, \mathcal{T})$ with a local ordered ABox $\mathcal{A}$ is consistent, can be seen as a generalization of the classical concept satisfiability problem [Borgwardt and Peñaloza, 2013].

Other common reasoning problems for FDLs, such as concept satisfiability and subsumption can be reduced to local consistency [Borgwardt *et al.*, 2014]: the subsumption between C and D to degree q w.r.t. a TBox $\mathcal{T}$ is equivalent to the (local) inconsistency of $(\{\langle a{:}C \to D < q\rangle\}, \mathcal{T})$, and the satisfiability of C to degree q w.r.t. $\mathcal{T}$ is equivalent to the (local) consistency of $(\{\langle a{:}C \geqslant q\rangle\}, \mathcal{T})$.

In the following section we show how to decide local consistency of a G-$\mathfrak{I}\mathcal{ALCQ}$ ontology through a reduction to classical ontology consistency.

3 Deciding Local Consistency

Let $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ be a G-$\mathfrak{I}\mathcal{ALCQ}$ ontology where $\mathcal{A}$ is a local ordered ABox that uses only the individual name a. The main ideas behind the reduction to classical $\mathcal{ALCQ}$ are that it suffices to consider tree-shaped interpretations, where each domain element has a unique role predecessor, and that we only have to consider the *order* between values of concepts, instead of their precise values. This insight allows us to consider only finitely many different cases [Borgwardt *et al.*, 2014].

To compare the values of the elements of $\mathsf{sub}(\mathcal{O})$ at different domain elements, we use the order structure

$$\mathcal{U} := \mathcal{V}_\mathcal{O} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_\uparrow(\mathcal{O}) \cup \{\lambda, \neg\lambda\},$$

where $\mathsf{sub}_\uparrow(\mathcal{O}) := \{\langle C\rangle_\uparrow \mid C \in \mathsf{sub}(\mathcal{O})\}$, $\mathsf{inv}(\lambda) := \neg\lambda$, $\mathsf{inv}(C) := \neg C$, and $\mathsf{inv}(\langle C\rangle_\uparrow) := \langle \neg C\rangle_\uparrow$, for all concepts $C \in \mathsf{sub}(\mathcal{O})$. The idea is that total preorders over $\mathcal{U}$ describe the relationships between the values of $\mathsf{sub}(\mathcal{O})$ and $\mathcal{V}_\mathcal{O}$ at a single domain element. The elements of $\mathsf{sub}_\uparrow(\mathcal{O})$ allow us to additionally refer to the relevant values at the unique role predecessor of the current domain element (in a tree-shaped interpretation). The value λ represents the value of the role connection from this predecessor. For convenience, we define $\langle q\rangle_\uparrow := q$ for all $q \in \mathcal{V}_\mathcal{O}$.

In order to describe such total preorders in a classical $\mathcal{ALCQ}$ ontology, we employ special concept names of the form $\boxed{\alpha \leqslant \beta}$ for $\alpha, \beta \in \mathcal{U}$. This differs from previous reductions for finitely valued FDLs [Straccia, 2004; Bobillo and Straccia, 2011; Bobillo *et al.*, 2012] in that we not only consider *cut-concepts* of the form $\boxed{q \leqslant \alpha}$ with $q \in \mathcal{V}_\mathcal{O}$, but also relationships between different concepts.[4] For convenience, we introduce the abbreviations $\boxed{\alpha \geqslant \beta} := \boxed{\beta \leqslant \alpha}$, $\boxed{\alpha < \beta} := \neg\boxed{\alpha \geqslant \beta}$, and similarly for $=$ and $>$. Furthermore, we define the complex expressions

- $\boxed{\alpha \geqslant \min\{\beta,\gamma\}} := \boxed{\alpha \geqslant \beta} \sqcup \boxed{\alpha \geqslant \gamma}$,
- $\boxed{\alpha \leqslant \min\{\beta,\gamma\}} := \boxed{\alpha \leqslant \beta} \sqcap \boxed{\alpha \leqslant \gamma}$,
- $\boxed{\alpha \geqslant \beta \Rightarrow \gamma} := (\boxed{\beta \leqslant \gamma} \to \boxed{\alpha \geqslant 1}) \sqcap (\boxed{\beta > \gamma} \to \boxed{\alpha \geqslant \gamma})$,
- $\boxed{\alpha \leqslant \beta \Rightarrow \gamma} := \boxed{\beta \leqslant \gamma} \sqcup \boxed{\alpha \leqslant \gamma}$,

and extend these notions to the expressions $\boxed{\alpha \bowtie \beta \Rightarrow \gamma}$ etc., for $\bowtie \in \{<, =, >\}$, analogously.

For each concept $C \in \mathsf{sub}(\mathcal{O})$, we now define the classical $\mathcal{ALCQ}$ TBox $\mathsf{red}(C)$, depending on the form of C, as follows.

$$\begin{aligned}
\mathsf{red}(\top) &:= \{\top \sqsubseteq \boxed{\top \geqslant 1}\} \\
\mathsf{red}(\neg C) &:= \emptyset \\
\mathsf{red}(C \sqcap D) &:= \{\top \sqsubseteq \boxed{C \sqcap D = \min\{C, D\}}\} \\
\mathsf{red}(C \to D) &:= \{\top \sqsubseteq \boxed{C \to D = C \Rightarrow D}\}
\end{aligned}$$

[4]For the rest of this paper, expressions of the form $\boxed{\alpha \leqslant \beta}$ denote (classical) concept names.

$$\mathsf{red}(\forall r.C) := \{\top \sqsubseteq \exists r.\boxed{\langle \forall r.C\rangle_{\uparrow} \geqslant \lambda \Rightarrow C} \sqcap \forall r.\boxed{\langle \forall r.C\rangle_{\uparrow} \leqslant \lambda \Rightarrow C}\}$$

$$\mathsf{red}({\geqslant} n\, r.C) := \{\top \sqsubseteq {\geqslant} n\, r.\boxed{\langle {\geqslant} n\, r.C\rangle_{\uparrow} \leqslant \min\{\lambda, C\}} \sqcap \neg{\geqslant} n\, r.\boxed{\langle {\geqslant} n\, r.C\rangle_{\uparrow} < \min\{\lambda, C\}}\}$$

Intuitively, $\mathsf{red}(C)$ describes the semantics of C in terms of its order relationships to other elements of $\mathcal{U}$. Note that the semantics of the involutive negation $\neg C = \mathsf{inv}(C)$ is already handled by the operator inv (see also the last line of the definition of $\mathsf{red}(\mathcal{U})$ below).

The reduced classical $\mathcal{ALCQ}$ ontology $\mathsf{red}(\mathcal{O})$ is defined as follows:

$$\begin{aligned}
\mathsf{red}(\mathcal{O}) &:= (\mathsf{red}(\mathcal{A}), \mathsf{red}(\mathcal{U}) \cup \mathsf{red}(\uparrow) \cup \mathsf{red}(\mathcal{T})),\\
\mathsf{red}(\mathcal{A}) &:= \{a{:}\boxed{C \bowtie q} \mid \langle a{:}C \bowtie q\rangle \in \mathcal{A}\} \cup \\
&\quad\ \{a{:}\boxed{C \bowtie D} \mid \langle a{:}C \bowtie a{:}D\rangle \in \mathcal{A}\},\\
\mathsf{red}(\mathcal{U}) &:= \{\boxed{\alpha \leqslant \beta} \sqcap \boxed{\beta \leqslant \gamma} \sqsubseteq \boxed{\alpha \leqslant \gamma} \mid \alpha, \beta, \gamma \in \mathcal{U}\} \cup \\
&\quad\ \{\top \sqsubseteq \boxed{\alpha \leqslant \beta} \sqcup \boxed{\beta \leqslant \alpha} \mid \alpha, \beta \in \mathcal{U}\} \cup \\
&\quad\ \{\top \sqsubseteq \boxed{0 \leqslant \alpha} \sqcap \boxed{\alpha \leqslant 1} \mid \alpha \in \mathcal{U}\} \cup \\
&\quad\ \{\top \sqsubseteq \boxed{\alpha \bowtie \beta} \mid \alpha, \beta \in \mathcal{V}_{\mathcal{O}},\ \alpha \bowtie \beta\} \cup \\
&\quad\ \{\boxed{\alpha \leqslant \beta} \sqsubseteq \boxed{\mathsf{inv}(\beta) \leqslant \mathsf{inv}(\alpha)} \mid \alpha, \beta \in \mathcal{U}\},\\
\mathsf{red}(\uparrow) &:= \{\boxed{\alpha \bowtie \beta} \sqsubseteq \forall r.\boxed{\langle\alpha\rangle_{\uparrow} \bowtie \langle\beta\rangle_{\uparrow}} \mid \\
&\qquad\qquad \alpha, \beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\},\\
\mathsf{red}(\mathcal{T}) &:= \{\top \sqsubseteq \boxed{q \leqslant C \Rightarrow D} \mid \langle C \sqsubseteq D \geqslant q\rangle \in \mathcal{T}\} \cup \\
&\qquad \bigcup_{C \in \mathsf{sub}(\mathcal{O})} \mathsf{red}(C).
\end{aligned}$$

We briefly explain this construction. The reductions of the order assertions and fuzzy GCIs in $\mathcal{O}$ are straightforward; the former expresses that the individual a must belong to the corresponding order concept $C \bowtie q$ or $C \bowtie D$, while the latter expresses that every element of the domain must satisfy the restriction provided by the fuzzy GCI. The axioms of $\mathsf{red}(\mathcal{U})$ intuitively ensure that the relation "$\leqslant$" forms a total preorder that is compatible with all the values in $\mathcal{V}_{\mathcal{O}}$, and that inv is an antitone operator. Finally, the TBox $\mathsf{red}(\uparrow)$ expresses a connection between the orders of a domain element and those of its role successors.

The following lemmata show that this reduction is correct; i.e., that it preserves local consistency.

Lemma 1. *If $\mathsf{red}(\mathcal{O})$ has a classical model, then $\mathcal{O}$ has a* G*-model.*

Proof. By [Kazakov, 2004], $\mathsf{red}(\mathcal{O})$ must have a *tree model* $\mathcal{I}$, i.e. we can assume that $\Delta^{\mathcal{I}}$ is a prefix-closed subset of $\mathbb{N}^*$, $a^{\mathcal{I}} = \varepsilon$, for all $n_1, \dots, n_k \in \mathbb{N}$, $k \geqslant 1$, with $u := n_1 \dots n_k \in \Delta^{\mathcal{I}}$, the element $u_{\uparrow} := n_1 \dots n_{k-1} \in \Delta^{\mathcal{I}}$ is an r-predecessor of u for some $r \in \mathsf{rol}(\mathcal{O})$, and there are no other role connections. For any $u \in \Delta^{\mathcal{I}}$, we denote by $\preccurlyeq_u$ the corresponding total preorder on $\mathcal{U}$, that is, we define $\alpha \preccurlyeq_u \beta$ iff $u \in \boxed{\alpha \leqslant \beta}^{\mathcal{I}}$, and by $\equiv_u$ the induced equivalence relation.

As a first step in the construction of a G-model of $\mathcal{O}$, we define the auxiliary function $v\colon \mathcal{U} \times \Delta^{\mathcal{I}} \to [0,1]$ that satisfies the following conditions for all $u \in \Delta^{\mathcal{I}}$:

(P1) for all $q \in \mathcal{V}_{\mathcal{O}}$, we have $v(q, u) = q$,

(P2) for all $\alpha, \beta \in \mathcal{U}$, we have $v(\alpha, u) \leqslant v(\beta, u)$ iff $\alpha \preccurlyeq_u \beta$,

(P3) for all $\alpha \in \mathcal{U}$, we have $v(\mathsf{inv}(\alpha), u) = 1 - v(\alpha, u)$,

(P4) if $u \neq \varepsilon$, then for all $C \in \mathsf{sub}(\mathcal{O})$ it holds that $v(C, u_{\uparrow}) = v(\langle C\rangle_{\uparrow}, u)$.

We define v by induction on the structure of $\Delta^{\mathcal{I}}$ starting with ε. Let $\mathcal{U}/{\equiv_{\varepsilon}}$ be the set of all equivalence classes of $\equiv_{\varepsilon}$. Then $\preccurlyeq_{\varepsilon}$ yields a total order $\leqslant_{\varepsilon}$ on $\mathcal{U}/{\equiv_{\varepsilon}}$. Since $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{U})$, we have

$$[0]_{\varepsilon} <_{\varepsilon} [q_1]_{\varepsilon} <_{\varepsilon} \cdots <_{\varepsilon} [q_{k-1}]_{\varepsilon} <_{\varepsilon} [1]_{\varepsilon}$$

w.r.t. this order. For every $[\alpha]_{\varepsilon} \in \mathcal{U}/{\equiv_{\varepsilon}}$, we now set $\mathsf{inv}([\alpha]_{\varepsilon}) := [\mathsf{inv}(\alpha)]_{\varepsilon}$. This function is well-defined by the axioms in $\mathsf{red}(\mathcal{U})$. On all $\alpha \in [q]_{\varepsilon}$ for $q \in \mathcal{V}_{\mathcal{O}}$, we now define $v(\alpha, \varepsilon) := q$, which ensures that (P1) holds. For the equivalence classes that do not contain a value from $\mathcal{V}_{\mathcal{O}}$, note that by $\mathsf{red}(\mathcal{U})$, every such class must be strictly between $[q_i]_{\varepsilon}$ and $[q_{i+1}]_{\varepsilon}$ for $q_i, q_{i+1} \in \mathcal{V}_{\mathcal{O}}$. We denote the n_i equivalence classes between $[q_i]_{\varepsilon}$ and $[q_{i+1}]_{\varepsilon}$ as follows:

$$[q_i]_{\varepsilon} <_{\varepsilon} E^i_1 <_{\varepsilon} \cdots <_{\varepsilon} E^i_{n_i} <_{\varepsilon} [q_{i+1}]_{\varepsilon}.$$

For every $\alpha \in E^i_j$, we set $v(\alpha, \varepsilon) := q_i + \frac{j}{n_i+1}(q_{i+1} - q_i)$, which ensures that (P2) is also satisfied. Furthermore, observe that $1 - q_{i+1}$ and $1 - q_i$ are also adjacent in $\mathcal{V}_{\mathcal{O}}$ and we have

$$[1 - q_{i+1}]_{\varepsilon} <_{\varepsilon} \mathsf{inv}(E^i_{n_i}) <_{\varepsilon} \cdots <_{\varepsilon} \mathsf{inv}(E^i_1) <_{\varepsilon} [1 - q_i]_{\varepsilon}$$

by the axioms in $\mathsf{red}(\mathcal{U})$. Hence, it follows from the definition of $v(\alpha, \varepsilon)$ that (P3) holds.

Let now $u \in \Delta^{\mathcal{I}}$ be such that the function v, satisfying the properties (P1)–(P4), has already been defined for $u_{\uparrow}$. Since $\mathcal{I}$ is a tree model, there must be an $r \in \mathsf{N_R}$ such that $(u_{\uparrow}, u) \in r^{\mathcal{I}}$. We again consider the set of equivalence classes $\mathcal{U}/{\equiv_u}$ and set $v(\alpha, u) := q$ for all $q \in \mathcal{V}_{\mathcal{O}}$ and $\alpha \in [q]_u$, and $v(\alpha, u) := v(C, u_{\uparrow})$ for all $C \in \mathsf{sub}(\mathcal{O})$ and $\alpha \in [\langle C\rangle_{\uparrow}]_u$. To see that this is well-defined, consider the case that $[\langle C\rangle_{\uparrow}]_u = [\langle D\rangle_{\uparrow}]_u$, i.e. $u \in \boxed{\langle C\rangle_{\uparrow} = \langle D\rangle_{\uparrow}}^{\mathcal{I}}$. From the axioms in $\mathsf{red}(\uparrow)$ and the fact that $(u_{\uparrow}, u) \in r^{\mathcal{I}}$, it follows that $u_{\uparrow} \in \boxed{C = D}^{\mathcal{I}}$, and thus $[C]_{u_{\uparrow}} = [D]_{u_{\uparrow}}$. Since (P2) is satisfied for $u_{\uparrow}$, we get $v(C, u_{\uparrow}) = v(D, u_{\uparrow})$. The same argument shows that $[q]_u = [\langle q\rangle_{\uparrow}]_u = [\langle C\rangle_{\uparrow}]_u$ implies $v(q, u_{\uparrow}) = v(C, u_{\uparrow})$. For the remaining equivalence classes, we can use a construction analogous to the case for ε by considering the two unique neighboring equivalence classes that contain an element of $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ (for which v has already been defined). This construction ensures that (P1)–(P4) hold for u.

Based on the function v, we define the G-interpretation $\mathcal{I}_{\mathsf{f}}$ over the domain $\Delta^{\mathcal{I}_{\mathsf{f}}} := \Delta^{\mathcal{I}}$, where $a^{\mathcal{I}_{\mathsf{f}}} := a^{\mathcal{I}} = \varepsilon$;

$$A^{\mathcal{I}_{\mathsf{f}}}(u) := \begin{cases} v(A, u) & \text{if } A \in \mathsf{sub}(\mathcal{O}),\\ 0 & \text{otherwise; and} \end{cases}$$

$$r^{\mathcal{I}_{\mathsf{f}}}(u, w) := \begin{cases} v(\lambda, w) & \text{if } (u, w) \in r^{\mathcal{I}},\\ 0 & \text{otherwise.} \end{cases}$$

We show by induction on the structure of C that

$$C^{\mathcal{I}_\mathsf{f}}(u) = v(C,u) \text{ for all } C \in \mathsf{sub}(\mathcal{O}) \text{ and } u \in \Delta^{\mathcal{I}}. \qquad (1)$$

For concept names, this holds by the definition of $\mathcal{I}_\mathsf{f}$. For $\top$, we know that $\top^{\mathcal{I}_\mathsf{f}}(u) = 1 = v(\top,u)$ by the definition of $\mathsf{red}(\top)$ and (P2). For $\neg C$, we have

$$(\neg C)^{\mathcal{I}_\mathsf{f}}(u) = 1 - C^{\mathcal{I}_\mathsf{f}}(u) = 1 - v(C,u) = v(\neg C,u)$$

by the induction hypothesis and (P3). For conjunctions $C \sqcap D$, we know that

$$\begin{aligned}(C \sqcap D)^{\mathcal{I}_\mathsf{f}}(u) &= \min\{C^{\mathcal{I}_\mathsf{f}}(u), D^{\mathcal{I}_\mathsf{f}}(u)\}\\ &= \min\{v(C,u), v(D,u)\}\\ &= v(C \sqcap D,u)\end{aligned}$$

by the definition of $\mathsf{red}(C \sqcap D)$ and (P2). Implications can be treated similarly.

Consider a value restriction $\forall r.C \in \mathsf{sub}(\mathcal{O})$. For every $w \in \Delta^{\mathcal{I}}$ with $(u,w) \in r^{\mathcal{I}}$, we have $w \in \boxed{\langle\forall r.C\rangle_\uparrow \leqslant \lambda \Rightarrow C}^{\mathcal{I}}$ since $\mathcal{I}$ satisfies $\mathsf{red}(\forall r.C)$. By the induction hypothesis, the fact that $w_\uparrow = u$, (P2), and (P4), this implies that $v(\forall r.C,u) \leqslant v(\lambda,w) \Rightarrow v(C,w) = r^{\mathcal{I}_\mathsf{f}}(u,w) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(w)$, and thus

$$\begin{aligned}(\forall r.C)^{\mathcal{I}_\mathsf{f}}(u) &= \inf_{w \in \Delta^{\mathcal{I}},\ (u,w) \in r^{\mathcal{I}}} r^{\mathcal{I}_\mathsf{f}}(u,w) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(w)\\ &\geqslant v(\forall r.C,u).\end{aligned}$$

Furthermore, by the existential restriction introduced in $\mathsf{red}(\forall r.C)$, we know that there exists a $w_0 \in \Delta^{\mathcal{I}}$ such that $(u,w_0) \in r^{\mathcal{I}}$ and $w_0 \in \boxed{\langle\forall r.C\rangle_\uparrow \geqslant \lambda \Rightarrow C}^{\mathcal{I}}$. By the same arguments as above, we get

$$\begin{aligned}v(\forall r.C,u) &\geqslant r^{\mathcal{I}_\mathsf{f}}(u,w_0) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(w_0)\\ &\geqslant (\forall r.C)^{\mathcal{I}_\mathsf{f}}(u),\end{aligned}$$

which concludes the proof of (1) for $\forall r.C$. As a by-product, we have found in the element w_0 the witness required for satisfying the concept $\forall r.C$ at u.

Consider now $\geqslant n\, r.C \in \mathsf{sub}(\mathcal{O})$. For any n-tuple $(w_1,\dots,w_n)$ of different domain elements with $(u,w_1),\dots,(u,w_n) \in r^{\mathcal{I}}$, by $\mathsf{red}(\geqslant n\, r.C)$ there must be an index i, $1 \leqslant i \leqslant n$, such that $w_i \notin \boxed{\langle\geqslant n\, r.C\rangle_\uparrow < \min\{\lambda,C\}}^{\mathcal{I}}$. Using arguments similar to those introduced above, we obtain that

$$\begin{aligned}v(\geqslant n\, r.C,u) &\geqslant \min\{r^{\mathcal{I}_\mathsf{f}}(u,w_i), C^{\mathcal{I}_\mathsf{f}}(w_i)\\ &\geqslant \min_{j=1}^{n} \min\{r^{\mathcal{I}_\mathsf{f}}(u,w_j), C^{\mathcal{I}_\mathsf{f}}(w_j)\}.\end{aligned}$$

On the other hand, we know that there are n different elements $w_1^0,\dots,w_n^0 \in \Delta^{\mathcal{I}}$ such that $(u,w_j^0) \in r^{\mathcal{I}}$ and $w_j \in \boxed{\langle\geqslant n\, r.C\rangle_\uparrow \leqslant \min\{\lambda,C\}}^{\mathcal{I}}$ for all j, $1 \leqslant j \leqslant n$. As in the case of $\forall r.C$ above, we conclude that

$$\begin{aligned}v(\geqslant n\, r.C,u) &\leqslant \min_{j=1}^{n} \min\{r^{\mathcal{I}_\mathsf{f}}(u,w_j^0), C^{\mathcal{I}_\mathsf{f}}(w_j^0)\}\\ &\leqslant (\geqslant n\, r.C)^{\mathcal{I}_\mathsf{f}}(u) \leqslant v(\geqslant n\, r.C,u),\end{aligned}$$

as required. Furthermore, $w_1^0,\dots,w_n^0$ are the required witnesses for $\geqslant n\, r.C$ at u. This concludes the proof of (1).

It remains to be shown that $\mathcal{I}_\mathsf{f}$ is a model of $\mathcal{O}$. For every $\langle a{:}C \bowtie q\rangle \in \mathcal{A}$, we have $a^{\mathcal{I}} = \varepsilon \in [C \bowtie q]^{\mathcal{I}}$, and thus $C^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}}) = v(C,\varepsilon) \bowtie v(q,\varepsilon) = q$ by (1), (P1), and (P2). A similar argument works for handling order assertions of the form $\langle a{:}C \bowtie a{:}D\rangle$. To conclude, consider an arbitrary GCI $\langle C \sqsubseteq D \geqslant q\rangle \in \mathcal{T}$ and $u \in \Delta^{\mathcal{I}}$. By the definition of $\mathsf{red}(\mathcal{T})$ and (P1), we have $v(q,u) \leqslant v(C,u) \Rightarrow v(D,u)$. Thus, (1) and (P2) yield $C^{\mathcal{I}_\mathsf{f}}(u) \Rightarrow D^{\mathcal{I}_\mathsf{f}}(u) \geqslant q$. Thus, $\mathcal{I}_\mathsf{f}$ satisfies all the axioms in $\mathcal{O}$, which concludes the proof. □

For the converse direction, we now show that it is possible to unravel every G-model of $\mathcal{O}$ into a classical tree model of $\mathsf{red}(\mathcal{O})$.

Lemma 2. *If $\mathcal{O}$ has a* G*-model, then* $\mathsf{red}(\mathcal{O})$ *has a classical model.*

Proof. Given a G-model $\mathcal{I}$ of $\mathcal{O}$, we define a classical interpretation $\mathcal{I}_\mathsf{c}$ over the domain $\Delta^{\mathcal{I}_\mathsf{c}}$ of all paths of the form $\varrho = r_1 d_1 \dots r_m d_m$ with $r_i \in \mathsf{N_R}$, $d_i \in \Delta^{\mathcal{I}}$, $m \geqslant 0$. We set $a^{\mathcal{I}_\mathsf{c}} := \varepsilon$ and

$$r^{\mathcal{I}_\mathsf{c}} := \{(\varrho, \varrho r d) \mid \varrho \in \Delta^{\mathcal{I}_\mathsf{c}},\ d \in \Delta^{\mathcal{I}}\}$$

for all $r \in \mathsf{N_R}$. We denote by $\mathsf{tail}(r_1 d_1 \dots r_m d_m)$ the element d_m if $m > 0$, and $a^{\mathcal{I}}$ if $m = 0$. Similarly, we set $\mathsf{prev}(r_1 d_1 \dots r_m d_m)$ to d_{m-1} if $m > 1$, and to $a^{\mathcal{I}}$ if $m = 1$. Finally, $\mathsf{role}(r_1 d_1 \dots r_m d_m)$ denotes r_m whenever $m > 0$. For any $\alpha \in \mathcal{U}$ and $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, we define $\alpha^{\mathcal{I}}(\varrho)$ as

$$\begin{aligned}C^{\mathcal{I}}(\mathsf{tail}(\varrho)) &\text{ if } \alpha = C \in \mathsf{sub}(\mathcal{O});\\ C^{\mathcal{I}}(\mathsf{prev}(\varrho)) &\text{ if } \alpha = \langle C\rangle_\uparrow, C \in \mathsf{sub}(\mathcal{O});\\ q &\text{ if } \alpha = q \in \mathcal{V}_{\mathcal{O}};\\ \mathsf{role}(\varrho)^{\mathcal{I}}(\mathsf{prev}(\varrho),\mathsf{tail}(\varrho)) &\text{ if } \alpha = \lambda;\\ 1 - \mathsf{role}(\varrho)^{\mathcal{I}}(\mathsf{prev}(\varrho),\mathsf{tail}(\varrho)) &\text{ if } \alpha = \neg\lambda.\end{aligned}$$

Note that for $\varrho = \varepsilon$ this expression is only defined for $\alpha \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$. We fix the value of $\alpha^{\mathcal{I}}(\varepsilon)$ for all other α arbitrarily, in such a way that for all $\alpha,\beta \in \mathcal{U}$ we have $\alpha^{\mathcal{I}}(\varepsilon) \leqslant \beta^{\mathcal{I}}(\varepsilon)$ iff $\mathsf{inv}(\beta)^{\mathcal{I}}(\varepsilon) \leqslant \mathsf{inv}(\alpha)^{\mathcal{I}}(\varepsilon)$. We can now define the interpretation of all concept names $\boxed{\alpha \leqslant \beta}$ with $\alpha,\beta \in \mathcal{U}$ as

$$\boxed{\alpha \leqslant \beta}^{\mathcal{I}_\mathsf{c}} := \{\varrho \mid \alpha^{\mathcal{I}}(\varrho) \leqslant \beta^{\mathcal{I}}(\varrho)\}.$$

It is easy to see that we have $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$ iff $\alpha^{\mathcal{I}}(\varrho) \bowtie \beta^{\mathcal{I}}(\varrho)$ also for all other order expressions $\bowtie$, and that $\mathcal{I}_\mathsf{c}$ satisfies $\mathsf{red}(\mathcal{U})$. We now show that $\mathcal{I}_\mathsf{c}$ satisfies the remaining parts of $\mathsf{red}(\mathcal{O})$.

For any order assertion $\langle a{:}C \bowtie a{:}D\rangle \in \mathcal{A}$ we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie D^{\mathcal{I}}(a^{\mathcal{I}})$. This implies that $C^{\mathcal{I}}(\varepsilon) \bowtie D^{\mathcal{I}}(\varepsilon)$, and thus $a^{\mathcal{I}} = \varepsilon \in \boxed{C \bowtie D}^{\mathcal{I}_\mathsf{c}}$, as required. A similar argument works for assertions of the form $\langle a{:}C \bowtie q\rangle$. Consider now a GCI $\langle C \sqsubseteq D \geqslant q\rangle \in \mathcal{T}$ and any $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$. We know that $C^{\mathcal{I}}(\mathsf{tail}(\varrho)) \Rightarrow D^{\mathcal{I}}(\mathsf{tail}(\varrho)) \geqslant q$, and thus $\varrho \in \boxed{q \leqslant C \Rightarrow D}^{\mathcal{I}_\mathsf{c}}$.

For $\mathsf{red}(\uparrow)$, consider any $\alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$, $r \in \mathsf{rol}(\mathcal{O})$, and $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$. Thus, it holds that $\alpha^{\mathcal{I}}(\varrho) \bowtie \beta^{\mathcal{I}}(\varrho)$. Every r-successor of ϱ in $\mathcal{I}_\mathsf{c}$ must be of the form $\varrho r d$. Since

$\langle\alpha\rangle_\uparrow^\mathcal{I}(\varrho r d) = \alpha^\mathcal{I}(\varrho) \bowtie \beta^\mathcal{I}(\varrho) = \langle\beta\rangle_\uparrow^\mathcal{I}(\varrho r d)$, we know that all r-successors of ϱ satisfy $\boxed{\langle\alpha\rangle_\uparrow \bowtie \langle\beta\rangle_\uparrow}$.

It remains to be shown that $\mathcal{I}_\mathsf{c}$ satisfies $\mathsf{red}(C)$ for all concepts $C \in \mathsf{sub}(\mathcal{O})$. For $C = \top$, the claim follows from the fact that $\top^\mathcal{I}(\varrho) = \top^\mathcal{I}(\mathsf{tail}(\varrho)) = 1$. For $\neg C$, the result is trivial, and for conjunctions and implications, it follows from the semantics of $\sqcap$ and $\to$ and the properties of $\min$ and $\Rightarrow$, respectively.

Consider the case of $\forall r.C$ and an arbitrary domain element $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, and set $d := \mathsf{tail}(\varrho)$. Since $\mathcal{I}$ is witnessed, there must be an $e \in \Delta^\mathcal{I}$ such that

$$\begin{aligned}\langle\forall r.C\rangle_\uparrow^\mathcal{I}(\varrho r e) &= (\forall r.C)^\mathcal{I}(d)\\ &= r^\mathcal{I}(d,e) \Rightarrow C^\mathcal{I}(e)\\ &= \lambda^\mathcal{I}(\varrho r e) \Rightarrow C^\mathcal{I}(\varrho r e).\end{aligned}$$

Since $(\varrho, \varrho r e) \in r^{\mathcal{I}_\mathsf{c}}$, this shows that $\exists r.\boxed{\langle\forall r.C\rangle_\uparrow \geqslant \lambda \Rightarrow C}$ is satisfied by ϱ in $\mathcal{I}_\mathsf{c}$. Additionally, for any r-successor $\varrho r e$ of ϱ we have

$$\begin{aligned}\langle\forall r.C\rangle_\uparrow^\mathcal{I}(\varrho r e) &= (\forall r.C)^\mathcal{I}(d)\\ &\leqslant r^\mathcal{I}(d,e) \Rightarrow C^\mathcal{I}(e)\\ &= \lambda^\mathcal{I}(\varrho r e) \Rightarrow C^\mathcal{I}(\varrho r e),\end{aligned}$$

and thus $\forall r.\boxed{\langle\forall r.C\rangle_\uparrow \leqslant \lambda \Rightarrow C}$ is also satisfied.

For at-least restrictions ${\geqslant} n\, r.C$, we similarly know that there are n different elements $e_1, \dots, e_n$ such that, for all i, $1 \leqslant i \leqslant n$,

$$\begin{aligned}\langle{\geqslant} n\, r.C\rangle_\uparrow^\mathcal{I}(\varrho r e_i) &= ({\geqslant} n\, r.C)^\mathcal{I}(d)\\ &= \min_{j=1}^{n} \min\{r^\mathcal{I}(d,e_j), C^\mathcal{I}(e_j)\}\\ &\leqslant \min\{r^\mathcal{I}(d,e_i), C^\mathcal{I}(e_i)\}\\ &= \min\{\lambda^\mathcal{I}(\varrho r e_i), C^\mathcal{I}(\varrho r e_i)\}.\end{aligned}$$

Since also the elements $\varrho r e_1, \dots, \varrho r e_n$ are different, this shows that the at-least restriction ${\geqslant} n\, r.\boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow \leqslant \min\{\lambda, C\}}$ is satisfied by $\mathcal{I}_\mathsf{c}$ at ϱ. On the other hand, for *all* n-tuples $(\varrho r e_1, \dots, \varrho r e_n)$ of different r-successors of ϱ and all i, $1 \leqslant i \leqslant n$, we must have

$$\begin{aligned}\langle{\geqslant} n\, r.C\rangle_\uparrow^\mathcal{I}(\varrho r e_i) &= ({\geqslant} n\, r.C)^\mathcal{I}(d)\\ &\geqslant \min_{j=1}^{n} \min\{r^\mathcal{I}(d,e_j), C^\mathcal{I}(e_j)\}\\ &= \min_{j=1}^{n} \min\{\lambda^\mathcal{I}(\varrho r e_j), C^\mathcal{I}(\varrho r e_j)\},\end{aligned}$$

and thus there must be at least one j, $1 \leqslant j \leqslant n$, such that

$$\varrho r e_j \in \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow \geqslant \min\{\lambda, C\}}^{\mathcal{I}_\mathsf{c}}.$$

In other words, there can be no n different elements of the form $\varrho r e$ that satisfy $\varrho r e \in \boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow < \min\{\lambda, C\}}^{\mathcal{I}_\mathsf{c}}$, i.e. $\varrho \notin {\geqslant} n\, r.\boxed{\langle{\geqslant} n\, r.C\rangle_\uparrow < \min\{\lambda, C\}}^{\mathcal{I}_\mathsf{c}}$. □

In contrast to the reductions for finitely valued Gödel FDLs [Bobillo *et al.*, 2009; 2012], the size of $\mathsf{red}(\mathcal{O})$ is always polynomial in the size of $\mathcal{O}$. The reason is that we do not translate the concepts occurring in the ontology recursively, but rather introduce a polynomial-sized subontology $\mathsf{red}(C)$ for each relevant subconcept C. Moreover, we do not need to introduce role hierarchies for our reduction, since the value of role connections is expressed using the special element λ. ExpTime-completeness of concept satisfiability in classical $\mathcal{ALCQ}$ [Schild, 1991; Tobies, 2001] now yields the following result.

Theorem 3. *Local consistency in* G-$\mathfrak{I}\mathcal{ALCQ}$ *is* ExpTime*-complete.*

4 Conclusions

Using a combination of techniques developed for infinitely valued Gödel extensions of $\mathcal{ALC}$ [Borgwardt *et al.*, 2014] and for finitely valued Gödel extensions of $\mathcal{SROIQ}$ [Bobillo *et al.*, 2009; 2012], we have shown that local consistency in infinitely valued G-$\mathfrak{I}\mathcal{ALCQ}$ is ExpTime-complete. Our reduction is more practical than the automata-based approach proposed by Borgwardt *et al.* [2014] and does not exhibit the exponential blowup of the reductions developed by Bobillo *et al.* [2009; 2012]. Beyond the complexity results, an important benefit of our approach is that it does not need the development of a specialized fuzzy DL reasoner, but can use any state-of-the-art reasoner for classical $\mathcal{ALCQ}$ without modifications. For that reason, this new reduction aids to shorten the gap between efficient classical and fuzzy DL reasoners.

In future work, we want to extend this result to full consistency, possibly using the notion of a *pre-completion* as introduced in [Borgwardt *et al.*, 2014]. Our ultimate goal is to provide methods for reasoning efficiently in infinitely valued Gödel extensions of the very expressive DL $\mathcal{SROIQ}$, underlying OWL 2 DL. We believe that it is possible to treat transitive roles, inverse roles, role hierarchies, and nominals using the extensions of the automata-based approach developed originally for finitely valued FDLs in [Borgwardt and Peñaloza, 2013; 2014; Borgwardt, 2014].

As done previously in [Bobillo *et al.*, 2012], we can also combine our reduction with the one for infinitely-valued Zadeh semantics. Although Zadeh semantics is not based on t-norms, it nevertheless is important to handle it correctly, as it is one of the most widely used semantics for fuzzy applications. It also has some properties that make it closer to the classical semantics, and hence become a natural choice for simple applications.

A different direction for future research would be to integrate our reduction directly into a classical tableaux reasoner. Observe that the definition of $\mathsf{red}(C)$ is already very close to the rules employed in (classical and fuzzy) tableaux algorithms (see, e.g. [Baader and Sattler, 2001; Bobillo and Straccia, 2009]). However, the tableaux procedure would need to deal with total preorders in each node, possibly using an external solver.

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References

[Baader and Peñaloza, 2011] Franz Baader and Rafael Peñaloza. On the undecidability of fuzzy description logics with GCIs and product t-norm. In *Proc. of the 8th Int. Symp. on Frontiers of Combining Systems (FroCoS'11)*, volume 6989 of *Lecture Notes in Computer Science*, pages 55–70. Springer-Verlag, 2011.

[Baader and Sattler, 2001] Franz Baader and Ulrike Sattler. An overview of tableau algorithms for description logics. *Studia Logica*, 69(1):5–40, 2001.

[Baader *et al.*, 2007] Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, 2nd edition, 2007.

[Bobillo and Straccia, 2009] Fernando Bobillo and Umberto Straccia. Fuzzy description logics with general t-norms and datatypes. *Fuzzy Sets and Systems*, 160(23):3382–3402, 2009.

[Bobillo and Straccia, 2011] Fernando Bobillo and Umberto Straccia. Reasoning with the finitely many-valued Łukasiewicz fuzzy description logic $\mathcal{SROIQ}$. *Information Sciences*, 181:758–778, 2011.

[Bobillo and Straccia, 2013] Fernando Bobillo and Umberto Straccia. Finite fuzzy description logics and crisp representations. In *Uncertainty Reasoning for the Semantic Web II*, volume 7123 of *Lecture Notes in Computer Science*, pages 102–121. Springer-Verlag, 2013.

[Bobillo *et al.*, 2009] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia. Fuzzy description logics under Gödel semantics. *International Journal of Approximate Reasoning*, 50(3):494–514, 2009.

[Bobillo *et al.*, 2012] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia. Joining Gödel and Zadeh fuzzy logics in fuzzy description logics. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 20(4):475–508, 2012.

[Borgwardt and Peñaloza, 2013] Stefan Borgwardt and Rafael Peñaloza. The complexity of lattice-based fuzzy description logics. *Journal on Data Semantics*, 2(1):1–19, 2013.

[Borgwardt and Peñaloza, 2014] Stefan Borgwardt and Rafael Peñaloza. Finite lattices do not make reasoning in $\mathcal{ALCOI}$ harder. In *Uncertainty Reasoning for the Semantic Web III*, volume 8816 of *Lecture Notes in Artificial Intelligence*, pages 122–141. Springer-Verlag, 2014.

[Borgwardt and Peñaloza, 2015] Stefan Borgwardt and Rafael Peñaloza. Reasoning in expressive description logics under infinitely valued gödel semantics. In *Proc. of FroCoS 2015*, Lecture Notes in Computer Science. Springer-Verlag, 2015. To appear.

[Borgwardt *et al.*, 2014] Stefan Borgwardt, Felix Distel, and Rafael Peñaloza. Decidable Gödel description logics without the finitely-valued model property. In *Proc. of the 14th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR'14)*, pages 228–237. AAAI Press, 2014.

[Borgwardt *et al.*, 2015] Stefan Borgwardt, Felix Distel, and Rafael Peñaloza. The limits of decidability in fuzzy description logics with general concept inclusions. *Artificial Intelligence*, 218:23–55, 2015.

[Borgwardt, 2014] Stefan Borgwardt. Fuzzy DLs over finite lattices with nominals. In *Proc. of the 27th Int. Workshop on Description Logics (DL'14)*, volume 1193 of *CEUR Workshop Proceedings*, pages 58–70, 2014.

[Cerami and Straccia, 2013] Marco Cerami and Umberto Straccia. On the (un)decidability of fuzzy description logics under Łukasiewicz t-norm. *Information Sciences*, 227:1–21, 2013.

[Hájek, 2001] Petr Hájek. *Metamathematics of Fuzzy Logic (Trends in Logic)*. Springer-Verlag, 2001.

[Hájek, 2005] Petr Hájek. Making fuzzy description logic more general. *Fuzzy Sets and Systems*, 154(1):1–15, 2005.

[Kazakov, 2004] Yevgeny Kazakov. A polynomial translation from the two-variable guarded fragment with number restrictions to the guarded fragment. In *Proc. of the 9th Eur. Conf. on Logics in Artificial Intelligence (JELIA'04)*, volume 3229 of *Lecture Notes in Computer Science*, pages 372–384. Springer-Verlag, 2004.

[Klement *et al.*, 2000] Erich Peter Klement, Radko Mesiar, and Endre Pap. *Triangular Norms*. Trends in Logic, Studia Logica Library. Springer-Verlag, 2000.

[Schild, 1991] Klaus Schild. A correspondence theory for terminological logics: Preliminary report. In *Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI'91)*, pages 466–471. Morgan Kaufmann, 1991.

[Schmidt-Schauß and Smolka, 1991] Manfred Schmidt-Schauß and Gert Smolka. Attributive concept descriptions with complements. *Artificial Intelligence*, 48(1):1–26, 1991.

[Straccia, 2001] Umberto Straccia. Reasoning within fuzzy description logics. *Journal of Artificial Intelligence Research*, 14:137–166, 2001.

[Straccia, 2004] Umberto Straccia. Transforming fuzzy description logics into classical description logics. In *Proc. of the 9th Eur. Conf. on Logics in Artificial Intelligence (JELIA'04)*, volume 3229 of *Lecture Notes in Computer Science*, pages 385–399. Springer-Verlag, 2004.

[Tobies, 2001] Stephan Tobies. *Complexity Results and Practical Algorithms for Logics in Knowledge Representation*. PhD thesis, RWTH Aachen, Germany, 2001.