Reasoning in Infinitely Valued G-$\mathcal{ALCQ}$

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Abstract
Fuzzy Description Logics (FDLs) are logic-based formalisms used to represent and reason with vague or imprecise knowledge. It has been recently shown that reasoning in most FDLs using truth values from the interval $[0,1]$ becomes undecidable in the presence of a negation constructor and general concept inclusion axioms. One exception to this negative result are FDLs whose semantics is based on the infinitely valued Gödel t-norm ($\mathbb{G}$). In this paper, we extend previous decidability results for G-$\mathcal{IALC}$ to deal also with qualified number restrictions. Our novel approach is based on a combination of the known crispification technique for finitely valued FDLs and the automata-based procedure originally developed for reasoning in G-$\mathcal{IALC}$. The proposed approach combines the advantages of these two methods, while removing their respective drawbacks.

1 Introduction
It is well-known that one of the main requirements for the development of an intelligent application is a formalism capable of representing and handling knowledge without ambiguity. Description Logics (DLs) are a well-studied family of knowledge representation formalisms [Baader et al., 2007]. They constitute the logical backbone of the standard Semantic Web ontology language OWL 2,

\[\text{http://www.w3.org/TR/owl2-overview/}\]

and its profiles, and have been successfully applied to represent the knowledge of many and diverse application domains, particularly in the bio-medical sciences.

DLs describe the domain knowledge using concepts (such as Patient) that represent sets of individuals, and roles (hasRelative) that represent connections between individuals. Ontologies are collections of axioms formulated over these concepts and roles, which restrict their possible interpretations. The typical axioms considered in DLs are assertions, like bob:Patient, providing knowledge about specific individuals; and general concept inclusions (GCIs), such as Patient $\subseteq$ Human, which express general relations between concepts. Different DLs are characterized by the constructors allowed to generate complex concepts and roles from atomic ones. $\mathcal{ALC}$ [Schmidt-Schauß and Smolka, 1991] is a prototypical DL of intermediate expressivity that contains the concept constructors conjunction ($C \land D$), negation ($\neg C$), and existential restriction ($\exists r.C$ for a role $r$). If additionally qualified number restrictions ($\geq n r.C$ for $n \in \mathbb{N}$) are allowed, the resulting logic is denoted by $\mathcal{ALCQ}$. Common reasoning problems in $\mathcal{ALCQ}$, such as consistency of ontologies or subsumption between concepts, are known to be EXPTime-complete [Schild, 1991; Tobies, 2001].

Fuzzy Description Logics (FDLs) have been introduced as extensions of classical DLs to represent and reason with vague knowledge. The main idea is to consider all the truth values from the interval $[0,1]$ instead of only true and false. In this way, it is possible give a more fine-grained semantics to inherently vague concepts like LowFrequency or HighConcentration, which can be found in biomedical ontologies like SNOMED CT,\footnote{http://www.ihtsdo.org/snomed-ct/} and Galen.\footnote{http://www.opengalen.org/} The different members of the family of FDLs are characterized not only by the constructors they allow, but also by the way these constructors are interpreted.

To interpret conjunction in complex concepts like

\[\exists \text{hasHeartRate}.\text{LowFrequency} \land \exists \text{hasBloodAlcohol}.\text{HighConcentration},\]

a popular approach is to use so-called t-norms [Klement et al., 2000]. The semantics of the other logical constructors can then be derived from these t-norms in a principled way, as suggested by Hájek [2001]. Following the principles of mathematical fuzzy logic, existential restrictions are interpreted as suprema of truth values. However, to avoid problems with infinitely many truth values, reasoning in fuzzy DLs is often restricted to so-called witnessed models [Hájek, 2005], in which these suprema are required to be maxima; i.e., the degree is witnessed by at least one domain element.

Unfortunately, reasoning in most FDLs becomes undecidable when the logic allows to use GCIs and negation under witnessed model semantics [Baader and Peñaloza, 2011; Cerami and Straccia, 2013; Borgwardt et al., 2015]. One of the few exceptions known are FDLs using the Gödel t-norm defined as $\min\{x, y\}$ to interpret conjunctions [Borgwardt et
al., 2014]. Despite not being as well-behaved as finitely valued FDLs, which use a finite total order of truth values instead of the infinite interval \([0, 1]\) [Borgwardt and Peñaloza, 2013], it has been shown using an automata-based approach that reasoning in Gödel extensions of \(\mathcal{ALC}\) exhibits the same complexity as in the classical case, i.e. it is \(\text{ExpTime}\)-complete. A major drawback of this approach is that it always has an exponential runtime, even when the input ontology has a simple form.

In this paper, we extend the results of [Borgwardt et al., 2014] to deal with qualified number restrictions, showing again that the complexity of reasoning remains the same as for the classical case; i.e., it is \(\text{ExpTime}\)-complete. To this end, we focus on the problem of local consistency, which is a generalization of the classical concept satisfiability problem. We follow a more practical approach that combines the automata-based construction from [Borgwardt et al., 2014] with reduction techniques developed for finitely valued FDLs [Straccia, 2004; Bobillo et al., 2009; Bobillo and Straccia, 2013]. We exploit the forest model property of classical \(\mathcal{ALC}\) [Kazakov, 2004] to encode order relationships between concepts in a fuzzy interpretation in a manner similar to the Hintikka trees from [Borgwardt et al., 2014]. However, instead of using automata to determine the existence of such trees, we reduce the fuzzy ontology directly into a classical \(\mathcal{ALC}\) ontology whose local consistency is equivalent to that of the original ontology. This enables us to use optimized reasoners for classical DLs. In addition to the cut-concepts of the form \([C \geq q]\) for a fuzzy concept \(C\) and a value \(q\), which are used in the reductions for finitely valued DLs [Straccia, 2004; Bobillo et al., 2009; Bobillo and Straccia, 2013], we employ order concepts \([C \leq D]\) expressing relationships between fuzzy concepts. Contrary to the reductions for finitely valued Gödel FDLs presented by Bobillo et al. [2009; 2012], our reduction produces a classical ontology whose size is polynomial in the size of the input fuzzy ontology. Thus, we obtain tight complexity bounds for reasoning in this FDL [Tobies, 2001]. An extended version of this paper appears in [Borgwardt and Peñaloza, 2015].

2 Preliminaries

For the rest of this paper, we focus solely on vague statements that take truth degrees from the infinite interval \([0, 1]\), where the Gödel \(t\)-norm, defined by \(\min\{x, y\}\), is used to interpret logical conjunction. The semantics of implications is given by the residuum of this \(t\)-norm; that is,

\[
x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}
\]

We use both the residual negation \(\ominus x := x \Rightarrow 0\) and the involutive negation \(\sim x := 1 - x\) in the rest of this paper.

We first recall some basic definitions from [Borgwardt et al., 2014], which will be used extensively in the proofs throughout this work. An order structure \(S\) is a finite set containing at least the numbers 0, 0.5, and 1, together with an involutive unary operation \(\text{inv}: S \to S\) such that \(\text{inv}(x) = 1 - x\) for all \(x \in S \cap [0, 1]\). A total preorder over \(S\) is a transitive and total binary relation \(\preceq \subseteq S \times S\). For \(x, y \in S\), we write \(x \equiv y\) if \(x \preceq y\) and \(y \preceq x\). Notice that \(\equiv\) is an equivalence relation on \(S\). The total preorders considered in [Borgwardt et al., 2014] have to satisfy additional properties; for instance, that 0 and 1 are always the least and greatest elements, respectively. These properties can be found in our reduction in the axioms of \(\text{red}(\mathcal{U})\) (see Section 3 for more details).

The syntax of the FDL \(\mathcal{G}3\mathcal{ALCQ}\) is the same as that of classical \(\mathcal{ALC}\), with the addition of the implication constructor (denoted by the use of \(\mathcal{J}\) at the beginning of the name). This constructor is often added to FDLs, as the residuum cannot, in general, be expressed using only the \(t\)-norm and negation operators, in contrast to the classical semantics. In particular, this holds for the Gödel \(t\)-norm and its residuum, which is the focus of this work. Let now \(N_C, N_R, N_I\) be mutually disjoint sets of concept, role, and individual names, respectively. Concepts of \(\mathcal{G}3\mathcal{ALCQ}\) are built using the syntax rule

\[
C, D ::= \top | A | \neg C | C \cap D | C \rightarrow D | \forall r.C | \geq n r.C,
\]

where \(A \in N_C\), \(r \in N_R\), \(C, D\) are concepts, and \(n \in \mathbb{N}\). We use the abbreviations

\[
\bot := \neg \top,
\]

\[
C \cup D := \neg (\neg C \cap \neg D), \quad \exists r.C := \geq 1 r.C, \quad \leq n r.C := \neg (\geq (n + 1) r.C)
\]

Notice that Bobillo et al. consider a different definition of atmost restrictions, which uses the residual negation; that is, they define \(\leq n r.C := (\geq (n + 1) r.C) \cap \bot\). This has the strange side effect that the value of \(\leq n r.C\) is always either 0 or 1 (see the semantics below). However, this discrepancy in definitions is not an issue since our algorithm can handle both cases.

The semantics of this logic is based on interpretations. A \(G\)-interpretation is a pair \(I = (\Delta^\mathbb{X}, \mathbb{X})\), where \(\Delta^\mathbb{X}\) is a non-empty set called the domain, and \(\mathbb{X}\) is the interpretation function that assigns to each individual name \(a \in N_I\) an element \(a^\mathbb{X} \in \Delta^\mathbb{X}\), to each concept name \(A \in N_C\) a fuzzy set \(A^\mathbb{X}: \Delta^\mathbb{X} \to [0, 1]\), and to each role name \(r \in N_R\) a fuzzy binary relation \(r^\mathbb{X}: \Delta^\mathbb{X} \times \Delta^\mathbb{X} \to [0, 1]\). The interpretation of complex concepts is obtained from the semantics of first-order fuzzy logics via the well-known translation from DL concepts to first-order logic [Straccia, 2001; Bobillo et al., 2012], i.e. for all \(d \in \Delta^\mathbb{X}\),

\[
\begin{align*}
\top^\mathbb{X}(d) & := 1 \\
\neg C^\mathbb{X}(d) & := 1 - C^\mathbb{X}(d) \\
(C \cap D)^\mathbb{X}(d) & := \min\{C^\mathbb{X}(d), D^\mathbb{X}(d)\} \\
(C \rightarrow D)^\mathbb{X}(d) & := C^\mathbb{X}(d) \Rightarrow D^\mathbb{X}(d) \\
(\forall r.C)^\mathbb{X}(d) & := \inf_{e \in \Delta^\mathbb{X}} r^\mathbb{X}(d, e) \Rightarrow C^\mathbb{X}(e) \\
(\geq n r.C)^\mathbb{X}(d) & := \sup_{e_1, \ldots, e_n \in \Delta^\mathbb{X}} \min_{i=1}^n r^\mathbb{X}(d, e_i), C^\mathbb{X}(e_i)
\end{align*}
\]

Recall that the usual duality between existential and value restrictions that appears in classical DLs does not hold in \(\mathcal{G}3\mathcal{ALCQ}\).
A classical interpretation is defined similarly, with the set of truth values restricted to 0 and 1. In this case, the semantics of a concept \( C \) is commonly viewed as a set \( C^\downarrow \subseteq \Delta^\downarrow \) instead of the characteristic function \( C^\downarrow : \Delta^\downarrow \rightarrow \{0,1\} \).

In the following, we restrict all reasoning problems to so-called witnessed \( \mathbb{G} \)-interpretations [Hajek, 2005], which intuitively require the suprema and infima in the semantics to be maxima and minima, respectively. More formally, the \( \mathbb{G} \)-interpretation \( I \) is witnessed if, for every \( d \in \Delta^\downarrow \), \( n \geq 0 \), \( r \in N_n \), and concept \( C \), there exist \( e, e_1, \ldots, e_n \in \Delta^\downarrow \) (where \( e_1, \ldots, e_n \) are pairwise different) such that

\[
(\forall r.C)^I(d) = r^\uparrow(d,e) \Rightarrow C^\uparrow(e) \quad \text{and} \quad (\geq n \cdot r.C)^I(d) = \min\{r^\uparrow(d,e_i), C^\uparrow(e_i)\}.
\]

The axioms of \( \mathbb{T}_{ALCQ} \) extend classical axioms by allowing to compare degrees of arbitrary assertions in so-called ordered ABoxes [Borgwardt et al., 2014], and to state inclusions relationships between fuzzy concepts that hold to a certain degree, instead of only 1. A classical assertion is an expression of the form \( a : C \) or \( (a,b) : r \) for \( a, b \in N_n \), \( r \in N_n \), and a concept \( C \). An order assertion is of the form \( (a \bowtie \beta) \) or \( (a \bowtie \beta) \) where \( \bowtie \in \{<,=,\geq,\geq\} \), \( \bowtie \) are classical assertions, and \( q \in [0,1] \). A fuzzy concept inclusion axiom (GCI) is of the form \( \langle C \subseteq D \mid q \rangle \) for concepts \( C, D \), and \( q \in [0,1] \). An ordered ABox is a finite set of order assertions, a TBox is a finite set of GICIs, and an ontology \( O = (A, T) \) consists of an ordered ABox \( A \) and a TBox \( T \). A \( \mathbb{G} \)-interpretation \( I \) satisfies (or is a model of) an order assertion \( (a \bowtie \beta) \) if \( a^\downarrow \bowtie \beta \) (where \( a : C \) := \( C^\downarrow(a^\downarrow) \)), \( (a,b) : r \) := \( r^\downarrow(a^\downarrow,b^\downarrow) \), and \( q^\downarrow := q \); it satisfies a GCI \( \langle C \subseteq D \mid q \rangle \) if \( C^\downarrow(d) \Rightarrow D^\downarrow(d) \geq q \) holds for all \( d \in \Delta^\downarrow \); and it satisfies an ordered ABox, TBox, or ontology if it satisfies all its axioms. An ontology is consistent if it has a (witnessed) model.

Given an ontology \( O \), we denote by \( \text{rol}(O) \) the set of all role names occurring in \( O \) and by \( \text{sub}(O) \) the closure under the union of the set of all subconcepts occurring in \( O \). We consider the concepts \( \neg C \) and \( C \) to be equal, and thus the latter set is of quadratic size in the size of \( O \). Moreover, we denote by \( \text{VOL} \) the closure under the involutive negation \( x \mapsto 1 - x \) of the set of all truth degrees appearing in \( O \), together with 0, 0.5, and 1. This set is of size linear on the size of \( O \). We sometimes denote the elements of \( \text{VOL} \subseteq [0,1] \) as \( 0 = q_0 < q_1 < \cdots < q_{k-1} < q_k = 1 \).

We stress that we do not consider the general consistency problem in this paper, but only a restricted version that uses only one individual name. An ordered ABox \( A \) is local if it contains no role assertions \( (a,b) : r \) and there is a single individual name \( a \in N_n \) such that all order assertions in \( A \) only use \( a \). The local consistency problem, i.e., deciding whether an ontology \( (A, T) \) with a local ordered ABox \( A \) is consistent, can be seen as a generalization of the classical concept satisfiability problem [Borgwardt and Peñaloza, 2013].

Other common reasoning problems for FDLs, such as concept satisfiability and subsumption can be reduced to local consistency [Borgwardt et al., 2014]: the subsumption between \( C \) and \( D \) to degree \( q \) w.r.t. a TBox \( T \) is equivalent to the (local) inconsistency of \( \{a : C \rightarrow D < q\}, T \), and the satisfiability of \( C \) to degree \( q \) w.r.t. \( T \) is equivalent to the (local) consistency of \( \{a : C \geq q\}, T \).

In the following section we show how to decide local consistency of a \( \mathbb{T}_{\mathbb{G}ALCQ} \) ontology through a reduction to classical ontology consistency.

### 3 Deciding Local Consistency

Let \( O = (A, T) \) be a \( \mathbb{T}_{\mathbb{G}ALCQ} \) ontology where \( A \) is a local ordered ABox that uses only the individual name \( a \). The main ideas behind the reduction to classical \( \mathbb{ALCQ} \) are that it suffices to consider tree-shaped interpretations, where each domain element has a unique role predecessor, and that we only have to consider the order between values of concepts, instead of their precise values. This insight allows us to consider only finitely many different cases [Borgwardt et al., 2014].

To compare the values of the elements of \( \text{sub}(O) \) at different domain elements, we use the order structure

\[
\mathcal{U} := \text{VOL} \cup \text{sub}(O) \cup \text{sub}(O) \cup \{\lambda, -\lambda\},
\]

where \( \text{sub}(O) := \{C^\uparrow \mid C \in \text{sub}(O)\} \), \( \text{inv}(\lambda) := -\lambda \), \( \text{inv}(\lambda) := -\lambda \), and \( \text{inv}(C^\uparrow) := -C^\uparrow \), for all concepts \( C \subseteq \text{sub}(O) \). The idea is that total preorders over \( \mathcal{U} \) describe the relationships between the values of \( \text{sub}(O) \) and \( \text{VOL} \) at a single domain element. The elements of \( \text{sub}(O) \) allow us to additionally refer to the relevant values at the unique role predecessor of the current domain element (in a tree-shaped interpretation). The value \( \lambda \) represents the value of the role connection from this predecessor. For convenience, we define \( q^\downarrow := q \) for all \( q \in \text{VOL} \).

In order to describe such total preorders in a classical \( \mathbb{ALCQ} \) ontology, we employ special concept names of the form \( [\alpha \leq \beta] \) for \( \alpha, \beta \in \mathcal{U} \). This differs from previous reductions for finitely valued FDLs [Straccia, 2004; Bobillo and Straccia, 2011; Bobillo et al., 2012] in that we not only consider cut-concepts of the form \( [q \leq q] \) with \( q \in \text{VOL} \), but also relationships between different concepts.\footnote{For convenience, we introduce the abbreviations \( [\alpha \leq \beta] := [\alpha \leq \beta] \) and similarly for \( = \) and \( > \). Furthermore, we define the complex expressions

- \( [\alpha \geq \beta] := [\alpha \geq \beta] \)
- \( [\alpha \leq \beta] := [\alpha \leq \beta] \)
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- \( [\alpha \geq \beta] := [\alpha \geq \beta] \)
- \( [\alpha \leq \beta] := [\alpha \leq \beta] \)

and extend these notions to the expressions \([\alpha \bowtie \beta \Rightarrow \gamma]\) etc., for \( \bowtie \in \{<,=,\geq\} \), analogously. For each concept \( C \subseteq \text{sub}(O) \), we now define the classical \( \mathbb{ALCQ} \) TBox \( \text{red}(C) \), depending on the form of \( C \), as follows.

\[
\text{red}(\top) := \{T \subseteq \top \geq 1\}
\]
\[
\text{red}(\neg C) := \emptyset
\]
\[
\text{red}(C \land D) := \{T \subseteq (\top \land D) = \min(C, D)\}
\]
\[
\text{red}(C \rightarrow D) := \{T \subseteq (C \rightarrow D = C \Rightarrow D)\}
\]

\(4\)For the rest of this paper, expressions of the form \( [\alpha \leq \beta] \) denote (classical) concept names.
red(∀r.C) := \{ T \subseteq \exists r. [ [\forall r.C]_\epsilon \sqsubseteq \lambda \Rightarrow C] \cap \\
\forall r. [ [\forall r.C]_\epsilon \sqsubseteq \lambda \Rightarrow C] \}

red(\geq n r.C) := \{ T \subseteq \geq n r. [ [\geq n r.C]_\epsilon \sqsubseteq \min(\lambda, C)] \cap \\
\neg \geq n r. [ [\geq n r.C]_\epsilon \sqsubseteq \min(\lambda, C)] \}

Intuitively, \( \text{red}(C) \) describes the semantics of \( C \) in terms of its order relationships to other elements of \( \mathcal{U} \). Note that the semantics of the involutive negation \( \neg C = \text{inv}(C) \) is already handled by the operator \( \text{inv} \) (see also the last line of the definition of \( \text{red}(\mathcal{U}) \) below).

The reduced classical \( \mathcal{ALCQ} \) ontology \( \text{red}(O) \) is defined as follows:

\[
\begin{align*}
\text{red}(O) &:= (\text{red}(A), \text{red}(\mathcal{U}) \cup \text{red}(\forall) \cup \text{red}(\exists)) , \\
\text{red}(A) &:= \{ a : [\mathcal{C} \sqcap q] \in A \} \cup \\
&\{ a : [\mathcal{C} \sqcap D] \mid [a : \mathcal{C} \sqcap a : \mathcal{D}] \in A \} , \\
\text{red}(\mathcal{U}) &:= \{ \alpha \sqsubseteq \beta \mid \beta \sqsubseteq \gamma \sqsubseteq \alpha \} \cup \\
&\{ T \subseteq \alpha \sqsubseteq \beta \} \cup \{ \alpha, \beta \in \mathcal{U} \} , \\
\text{red}(\forall) &:= \{ \forall r. [ (\alpha \sqcap [\mathcal{C} \sqcap \beta])] \mid \\
\text{red}(\exists) &:= \{ T \subseteq \mathcal{C} \cap D \} \cup \{ C \subseteq D \cap q \} \in T \cup \\
\text{red}(C) \}. \\
\end{align*}
\]

We briefly explain this construction. The reductions of the order assertions and fuzzy GCIs in \( O \) are straightforward; the former expresses that the individual \( a \) must belong to the corresponding order concept \( C \sqcap q \) or \( C \sqcap D \), while the latter expresses that every element of the domain must satisfy the restriction provided by the fuzzy GCI. The axioms of \( \text{red}(\mathcal{U}) \) intuitively ensure that the relation \( \sqsubseteq \) forms a total preorder that is compatible with all the values in \( \mathcal{V} \), and that \( \text{inv} \) is an antitone operator. Finally, the TBox \( \text{red}(\forall) \) expresses a connection between the orders of a domain element and those of its role successors.

The following lemmata show that this reduction is correct; i.e., that it preserves local consistency.

**Lemma 1.** If \( \text{red}(O) \) has a classical model, then \( O \) has a \( \mathcal{G} \)-model.

**Proof.** By [Kazakov, 2004], \( \text{red}(O) \) must have a tree model \( \mathcal{I} \), i.e. we can assume that \( \Delta^2 \) is a prefix-closed set of \( \mathbb{N}^* \), \( a^2 = \epsilon \), for all \( n_1, \ldots, n_k \in \mathbb{N}, k \geq 1 \), with \( u := n_1 \ldots n_k \in \Delta^2 \), the element \( u := n_1 \ldots n_k \in \Delta^2 \) is an \( r \)-predecessor of \( u \) for some \( r \in \text{rol}(O) \), and there are no other role connections. For any \( u \in \Delta^2 \), we define by \( \epsilon \) the corresponding total preorder on \( \mathcal{U} \), that is, we define \( \alpha \sqsubseteq \beta \) iff \( u \in [\alpha \sqsubseteq \beta]^2 \), and by \( \equiv \) the induced equivalence relation.

As a first step in the construction of a \( \mathcal{G} \)-model of \( O \), we define the auxiliary function \( v : \mathcal{U} \times \Delta^2 \rightarrow [0, 1] \) that satisfies the following conditions for all \( u \in \Delta^2 \):

- (P1) for all \( q \in \mathcal{V} \), we have \( v(q, u) = q \).
- (P2) for all \( \alpha, \beta \in \mathcal{U} \), we have \( v(\alpha, u) \leq v(\beta, u) \) iff \( \alpha \equiv u \beta \).
- (P3) for all \( \alpha \in \mathcal{U} \), we have \( v(\text{inv}(\alpha), u) = 1 - v(\alpha, u) \).
- (P4) if \( u \neq \epsilon \), then for all \( C \in \text{sub}(O) \) it holds that \( v(\mathcal{C}_\epsilon, u) = v(C, u) ) \).

We define \( v \) by induction on the structure of \( \Delta^2 \) starting with \( \epsilon \). Let \( \mathcal{U} \equiv \epsilon \) be the set of all equivalence classes of \( \equiv \). Then \( \equiv \) yields a total order \( \leq \epsilon \) on \( \mathcal{U} \equiv \epsilon \). Since \( \approx \) satisfies \( \text{red}(\mathcal{U}) \), we have

\[
[0]_\epsilon \leq [1]_\epsilon \leq [q_1]_\epsilon \leq \cdots \leq [q_{k-1}]_\epsilon \leq [1]_\epsilon
\]

w.r.t. this order. For every \([\alpha]_\epsilon \in \mathcal{U} \equiv \epsilon \), we now set \( \text{inv}(\alpha)_\epsilon := [\text{inv}(\alpha)]_\epsilon \). This function is well-defined by the axioms in \( \text{red}(\mathcal{U}) \). On all \( \alpha \in [q_i]_\epsilon \) for \( q \in \mathcal{V} \), we now define \( v(\alpha, \epsilon) := q \), which ensures that (P1) holds. For the equivalence classes that do not contain a value from \( \mathcal{V} \), note that by \( \text{red}(\mathcal{U}) \), every such class must be strictly between \([q_i]_\epsilon \) and \([q_{i+1}]_\epsilon \) for \( q_i, q_{i+1} \in \mathcal{V} \). We denote the \( n \) equivalence classes between \([q_i]_\epsilon \) and \([q_{i+1}]_\epsilon \) as follows:

\[
[q_i]_\epsilon \leq [q_i]_\epsilon \leq \cdots \leq [q_{i+1}]_\epsilon
\]

For every \( \alpha \in [q_i]_\epsilon \), we set \( v(\alpha, \epsilon) := q_i + \frac{j}{n} (q_{i+1} - q_i) \), which ensures that (P2) is also satisfied. Furthermore, observe that \( 1 - q_{i+1} + 1 - q_i \) are also adjacent in \( \mathcal{V} \) and we have

\[
[1 - q_{i+1}]_\epsilon \leq \text{inv}(E_i) \leq \cdots \leq \text{inv}(E_i) \leq [1 - q_i]_\epsilon
\]

by the axioms in \( \text{red}(\mathcal{U}) \). Hence, it follows from the definition of \( v(\alpha, \epsilon) \) that (P3) holds.

Let now \( u \in \Delta^2 \) be such that the function \( v \), satisfying the properties (P1)-(P4), has already been defined for \( u \). Since \( \mathcal{I} \) is a tree model, there must be an \( r \in \mathbb{N}^* \) such that \( (u_r, u) \in r^2 \). We again consider the set of equivalence classes \( \mathcal{U} \equiv \epsilon \) and set \( v(\alpha, u) := q \) for all \( q \in \mathcal{V} \) and \( \alpha \in [q_i]_\epsilon \), and \( v(\alpha, u) := v(C, u) \) for all \( C \in \text{sub}(O) \) and \( \alpha \in [(C)]_\epsilon \). To see that this is well-defined, consider the case that \([C]_\epsilon \equiv [D]_\epsilon \), i.e. \( u \in [(C)]_\epsilon \equiv [D]_\epsilon \). From the axioms in \( \text{red}(\forall) \) and the fact that \( (u_r, u) \in r^2 \), it follows that \( u_r \in [(C)]_\epsilon \equiv [D]_\epsilon \), and thus \([C]_\epsilon \equiv [D]_\epsilon \). Since \( (P2) \) is satisfied for \( u_r \), we get \( v(C, u_r) = v(D, u_r) \). The same argument shows that \([q_i]_\epsilon \equiv [(q_i)]_\epsilon \equiv [(C)]_\epsilon \) implies \( v(q_i, u_r) = v(C, u_r) \). For the remaining equivalence classes, we can use a construction analogous to the case for \( \epsilon \) by considering the two unique neighboring equivalence classes that contain an element of \( \mathcal{V} \cup \text{sub}(O) \) (for which \( v \) has already been defined). This construction ensures that (P1)-(P4) hold for \( u \).

Based on the function \( v \), we define the \( \mathcal{G} \)-interpretation \( \mathcal{I} \) over the domain \( \Delta^2 := \Delta^2 \), where \( a^2 := a^2 = \epsilon \):

\[
A^2(u) := \begin{cases} v(A, u) & \text{if } A \in \text{sub}(O), \\ 0 & \text{otherwise} \end{cases} \\
r^2(u, w) := \begin{cases} v(\lambda, u) & \text{if } (u, w) \in r^2, \\ 0 & \text{otherwise} \end{cases}
\]
We show by induction on the structure of $C$ that

$$C^I(u) = v(C, u) \quad \text{for all } C \in \text{sub}(O) \text{ and } u \in \Delta^I. \quad (1)$$

For concept names, this holds by the definition of $I$. For $T$, we know that $T^I(u) = 1 = v(T, u)$ by the definition of $\text{red}(T)$ and (P2). For $\neg C$, we have

$$(-C)^I(u) = 1 - C^I(u) = 1 - v(C, u) = v(-C, u)$$

by the induction hypothesis and (P3). For conjunctions $C \sqcap D$, we know that

$$(C \cap D)^I(u) = \min\{C^I(u), D^I(u)\} = \min\{v(C, u), v(D, u)\} = v(C \cap D, u)$$

by the definition of $\text{red}(C \cap D)$ and (P2). Implications can be treated similarly.

Consider a value restriction $\forall r.C \in \text{sub}(O)$. For every $w \in \Delta^I$ with $(u, w) \in r^I$, we have $w \in \{\forall r.C\}_\Rightarrow \lambda \Rightarrow C^I$ since $I$ satisfies $\text{red}(\forall r.C)$. By the induction hypothesis, the fact that $w_\tau = u$, (P2), and (P4), this implies that $v(\forall r.C, u) \leq v(\lambda, w) \Rightarrow v(C, w) = r^I(u, w) = C^I(w)$, and thus

$$(\forall r.C)^I(u) = \inf_{w \in \Delta^I, (u, w) \in r^I} r^I(u, w) = C^I(w)$$

$$\forall v(\forall r.C, u),$$

Furthermore, by the existential restriction introduced in $\text{red}(\forall r.C)$, we know that there exists a $w_0 \in \Delta^I$ such that $(u, w_0) \in r^I$ and $w_0 \in \{\forall r.C\}_\Rightarrow \lambda \Rightarrow C^I$. By the same arguments as above, we get

$$v(\forall r.C, u) \geq r^I(u, w_0) = C^I(w_0)$$

which concludes the proof of (1) for $\forall r.C$. As a by-product, we have found in the element $w_0$ the witness required for satisfying the concept $\forall r.C$ at $u$.

Consider now $\forall n.r.C \in \text{sub}(O)$. For any $n$-tuple $(w_1, \ldots, w_n)$ of different domain elements with $(u, w_1), \ldots, (u, w_n) \in r^I$, by red($\forall n.r.C$), there must be an index $i$, $1 \leq i \leq n$, such that $w_i \notin \{\forall n.r.C\}_\Rightarrow \lambda \Rightarrow C^I$. Using arguments similar to those introduced above, we obtain that

$$v(\forall n.r.C, u) \geq \min\{r^I(u, w_i), C^I(w_i)\}$$

$$\geq \min_j \min\{r^I(u, w_j), C^I(w_j)\}.$$

On the other hand, we know that there are $n$ different elements $w_0^1, \ldots, w_0^n \in \Delta^I$ such that $(u, w_0^j) \in r^I$ and $w_j \in \{\forall n.r.C\}_\Rightarrow \lambda \Rightarrow C^I$ for all $j$, $1 \leq j \leq n$. As in the case of $\forall r.C$ above, we conclude that

$$v(\forall n.r.C, u) \leq \min_j \min\{r^I(u, w_0^j), C^I(w_0^j)\}$$

$$\leq (\forall n.r.C)^I(u) \leq v(\forall n.r.C, u),$$

as required. Furthermore, $w_0^1, \ldots, w_0^n$ are the required witnesses for $\forall n.r.C$ at $u$. This concludes the proof of (1).

It remains to be shown that $\forall r.C$ is a model of $O$. For every $(a:C \bowtie q) \in A$, we have $a^\Rightarrow = \epsilon = [C \bowtie q]^I$, and thus $C^I(a^\Rightarrow) = v(C, \epsilon) = v(q, \epsilon) = q$ by (1), (P1), and (P2). A similar argument works for handling order assertions of the form $(a:C \bowtie a:D)$. To conclude, consider an arbitrary GCI $(C \subseteq D) \in T$ and $u \in \Delta^I$. By the definition of $\text{red}(T)$ and (P1), we have $v(q, u) \leq v(C, u) = v(D, u)$. Thus, (1) and (P2) yield $C^I(u) = D^I(u) = q$. Thus, $I$ satisfies all the axioms in $O$, which concludes the proof.

For the converse direction, we now show that it is possible to unravel every G-model of $O$ into a classical tree model of $\text{red}(O)$.

**Lemma 2.** If $O$ has a G-model, then $\text{red}(O)$ has a classical model.

**Proof.** Given a G-model $I$ of $O$, we define a classical interpretation $I_c$ over the domain $\Delta^I_c$ of all paths of the form $v = r_1d_1 \ldots r_md_m$ with $r_i \in \text{Nr}, d_i \in \Delta^I, m \geq 0$. We set $a^I_c := \epsilon$ and

$$r^I_c := \{(q, r, c) \mid q \in \Delta^I_c, d \in \Delta^I\}$$

for all $r \in \text{Nr}$. We denote by tail$(r_1d_1 \ldots r_md_m)$ the element $d_m$ if $m > 0$, and $\epsilon$ if $m = 0$. Similarly, we set prev$(r_1d_1 \ldots r_md_m)$ to $d_{m-1}$ if $m > 1$, and to $\epsilon$ if $m = 1$. Finally, role$(r_1d_1 \ldots r_md_m)$ denotes $d_m$ whenever $m > 0$.

For any $\alpha \in U$ and $q \in \Delta^I_c$, we define $\alpha^I(q)$ as

$$C^I(q)$$

for all $\alpha \in U$ and $q \in \Delta^I_c$. We fix $\alpha^I(q)$ and $\alpha^I(q)$ for all $q \in \Delta^I_c$. Thus, it holds that $\alpha^I(q) = \epsilon$ for all $q \in \Delta^I_c$. We can now define the interpretation of all concept names $\alpha \bowtie \beta$ with $\alpha, \beta \in U$ as

$$\alpha \bowtie \beta := \{q \mid \alpha^I(q) \bowtie \beta^I(q)\}.$$
\[
\langle \alpha \rangle_T^X (\varnothing d) = \alpha_T^X (\varnothing) \sqsupset \beta_T^X (\varnothing) = (\beta_T^X (\varnothing d), \text{we know that all } r\text{-successors of } \varnothing \text{ satisfy } \langle \alpha \rangle_T^{r} \sqsupset \langle \beta \rangle_T^{r}. \]

It remains to be shown that \( I_x \) satisfies \( \text{red}(C) \) for all concepts \( C \in \text{sub}(O) \). For \( C = \top \), the claim follows from the fact that \( T^X (\varnothing) = T^X (\text{tail}(\varnothing)) = 1 \). For \( \neq C \), the result is trivial, and for conjunctions and implications, it follows from the semantics of \( \land \) and \( \rightarrow \) and the properties of \( \sqcap \) and \( \Rightarrow \), respectively.

Consider the case of \( \forall r.C \) and an arbitrary domain element \( \varnothing \in \Delta^X \), and set \( d := \text{tail}(\varnothing) \). Since \( I_x \) is witnessed, there must be an \( e \in \Delta^X \) such that
\[
(\forall r.C) T^X (\varnothing, \varnothing) = (\forall r.C) T^X (d, e) = r^X (d, e) \Rightarrow C^X (e) = \lambda T^X (\varnothing) \Rightarrow C^X (\varnothing). \]

Since \( (\varnothing, \varnothing) \in r^X \), this shows that \( \exists r. [\forall r.C] \geq \lambda \Rightarrow \top \) is satisfied by \( \varnothing \) in \( I_x \). Additionally, for any \( r\)-successor \( \varnothing, \varnothing \) of \( \varnothing \) we have
\[
(\forall r.C) T^X (\varnothing, \varnothing) = (\forall r.C) T^X (d, e) \leq r^X (d, e) \Rightarrow C^X (e) = \lambda T^X (\varnothing) \Rightarrow C^X (\varnothing), \]
and thus \( \forall r. [\forall r.C] \geq \lambda \Rightarrow \top \) is also satisfied.

For at-least restrictions \( \geq r.C \), we similarly know that there are \( n \) different elements \( e_1, \ldots, e_n \) such that, for all \( i, 1 \leq i \leq n, \)
\[
\langle \geq n r.C \rangle_T^X (\varnothing, \varnothing) = (\geq n r.C) T^X (d) = \min \{ r^X (d, e_i), C^X (e_i) \} \leq \min \{ r^X (d, e_i), C^X (e_i) \} = \min \{ \lambda T^X (\varnothing, \varnothing), C^X (\varnothing, \varnothing) \}. \]
Since also the elements \( \varnothing, \varnothing \) are different, this shows that the at-least restriction \( \geq n r. [\geq n r.C] \leq \lambda \Rightarrow \top \) is satisfied by \( I_x \) at \( \varnothing \).

On the other hand, for all \( n \)-tuples \( (\varnothing, \varnothing) \) of different \( r\)-successors of \( \varnothing \) and all \( i, 1 \leq i \leq n \), we must have
\[
\langle \geq n r.C \rangle_T^X (\varnothing, \varnothing) = (\geq n r.C) T^X (d) = \min \{ r^X (d, e_i), C^X (e_i) \} = \min \{ \lambda T^X (\varnothing, \varnothing), C^X (\varnothing, \varnothing) \}, \]
and thus there must be at least one \( j, 1 \leq j \leq n \), such that
\[
\varnothing, \varnothing \in [\geq n r.C] \geq \lambda \Rightarrow \top \text{, } \varnothing \notin [\geq n r.C] \Rightarrow \lambda \Rightarrow \top. \]

In other words, there can be no \( n \) different elements of the form \( \varnothing, \varnothing \) that satisfy \( \varnothing, \varnothing \in [\geq n r.C] \leq \lambda \Rightarrow \top \text{, } \varnothing \notin [\geq n r.C] \Rightarrow \lambda \Rightarrow \top. \)

In contrast to the reductions for finitely valued Gödel FDLs [Bobillo et al., 2009; 2012], the size of \( \text{red}(O) \) is always polynomial in the size of \( O \). The reason is that we do not translate the concepts occurring in the ontology recursively, but rather introduce a polynomial-sized subontology \( \text{red}(C) \) for each relevant subconcept \( C \). Moreover, we do not need to introduce role hierarchies for our reduction, since the value of role connections is expressed using the special element \( \lambda \). \( \text{ExpTime}\text{-completeness} \text{ of concept satisfiability in classical } \text{ALCQ} \text{ [Schild, 1991; Tobies, 2001] now yields the following result.}

**Theorem 3.** Local consistency in \( \text{G-}\text{ALCQ} \text{ is } \text{ExpTime}\text{-complete.}

**4 Conclusions**

Using a combination of techniques developed for infinitely valued Gödel extensions of \( \text{ALC} \) [Borgwardt et al., 2014] and for finitely valued Gödel extensions of \( \text{SROIQ} \) [Bobillo et al., 2009; 2012], we have shown that local consistency in infinitely valued \( \text{G-}\text{ALCQ} \text{ is } \text{ExpTime}\text{-complete. Our reduction is more practical than the automata-based approach proposed by Borgwardt et al. [2014] and does not exhibit the exponential blowup of the reductions developed by Bobillo et al. [2009; 2012]. Beyond the complexity results, an important benefit of our approach is that it does not need the development of a specialized fuzzy DL reasoner, but can use any state-of-the-art reasoner for classical \( \text{ALCQ} \text{ without modifications. For that reason, this new reduction aids to shorten the gap between efficient classical and fuzzy DL reasoners.}

In future work, we want to extend this result to full consistency, possibly using the notion of a \textit{pre-completion} as introduced in [Borgwardt et al., 2014]. Our ultimate goal is to provide methods for reasoning efficiently in infinitely valued Gödel extensions of the very expressive DL \( \text{SROIQ} \), underlying OWL 2 DL. We believe that it is possible to treat transitive roles, inverse roles, role hierarchies, and nominals using the extensions of the automata-based approach developed originally for finitely valued FDLs in [Borgwardt and Peñaloza, 2013; 2014; Borgwardt, 2014].

As done previously in [Bobillo et al., 2012], we can also combine our reduction with the one for infinitely-valued Zadeh semantics. Although Zadeh semantics is not based on \( \tau\)-norms, it nevertheless is important to handle it correctly, as it is one of the most widely used semantics for fuzzy applications. It also has some properties that make it closer to the classical semantics, and hence become a natural choice for simple applications.

A different direction for future research would be to integrate our reduction directly into a classical tableaux reasoner. Observe that the definition of \( \text{red}(C) \) is already very close to the rules employed in (classical and fuzzy) tableaux algorithms (see, e.g. [Baader and Sattler, 2001; Bobillo and Straccia, 2009]). However, the tableaux procedure would need to deal with total preorders in each node, possibly using an external solver.

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References


