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Extending the Description Logic $\tau\mathcal{EL}(deg)$ with Acyclic TBoxes

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Abstract. In a previous paper, we have introduced an extension of the lightweight Description Logic $\mathcal{EL}$ that allows us to define concepts in an approximate way. For this purpose, we have defined a graded membership function deg, which for each individual and concept yields a number in the interval $[0,1]$ expressing the degree to which the individual belongs to the concept. Threshold concepts $C_{\sim t}$ for $\sim\in\{<,\leq,>,\geq\}$ then collect all the individuals that belong to C with degree $\sim t$. We have then investigated the complexity of reasoning in the Description Logic $\tau\mathcal{EL}(deg)$, which is obtained from $\mathcal{EL}$ by adding such threshold concepts. In the present paper, we extend these results, which were obtained for reasoning without TBoxes, to the case of reasoning w.r.t. acyclic TBoxes. Surprisingly, this is not as easy as might have been expected. On the one hand, one must be quite careful to define acyclic TBoxes such that they still just introduce abbreviations for complex concepts, and thus can be unfolded. On the other hand, it turns out that, in contrast to the case of $\mathcal{EL}$, adding acyclic TBoxes to $\tau\mathcal{EL}(deg)$ increases the complexity of reasoning by at least on level of the polynomial hierarchy.

1 INTRODUCTION

Description logics (DLs) [3] allow their users to define the important notions of an application domain as concepts by stating necessary and sufficient conditions for an individual to belong to the concept. These conditions can be atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). The expressivity of a particular DL is determined on the one hand by what sort of properties can be required and how they can be combined. On the other hand, DLs provide their users with ways of stating terminological axioms in a so-called TBox. The simplest kind of TBoxes are called acyclic TBoxes, which consist of concept definitions without cyclic dependencies among the defined concepts. Basically, such a TBox introduces abbreviations for complex concept descriptions. But even this simple form of TBoxes may increase the complexity of reasoning, as is, for example the case for the DL $\mathcal{FL}_0$, for which the complexity of the subsumption problem increases from polynomial-time to coNP-complete if acyclic TBoxes are added [12].

The DL $\mathcal{EL}$, in which concepts can be built using concept names as well as the concept constructors conjunction ($\sqcap$), existential restriction ($\exists r.C$), and the top concept ($\top$),[2] has drawn considerable attention in the last decade since, on the one hand, important inference problems such as the subsumption problem are polynomial in $\mathcal{EL}$, not only w.r.t. acyclic TBoxes, but also w.r.t. more expressive terminological axioms called GCIs [8]. On the other hand, though quite inexpressive, $\mathcal{EL}$ underlies the OWL 2 EL profile[3] and can be used to define biomedical ontologies, such as the large medical ontology SNOMED CT,[4] which basically is an acyclic $\mathcal{EL}$ TBox. In $\mathcal{EL}$ we can, for example, define the concept of a *good movie* as a movie that is uplifting, has a simple, but original plot, a likable and an evil character, action and love scenes, and a happy ending.

$$\begin{aligned}\mathsf{Movie}\sqcap\mathsf{Uplifting}\sqcap\exists\mathsf{plot}.(\mathsf{Simple}\sqcap\mathsf{Original})\sqcap\\ \exists\mathsf{character}.\mathsf{Likeable}\sqcap\exists\mathsf{character}.\mathsf{Evil}\sqcap\\ \exists\mathsf{scene}.\mathsf{Action}\sqcap\exists\mathsf{scene}.\mathsf{Love}\sqcap\exists\mathsf{ending}.\mathsf{Happy}.\end{aligned}\tag{1}$$

For an individual to belong to this concept, all the stated properties need to be satisfied. However, maybe we would still want to call a movie good if most, though not all, of the properties hold.

In [2], we have introduced a DL extending $\mathcal{EL}$ that allows us to define concepts in such an approximate way. The main idea is to use a *graded membership function*, which instead of a Boolean membership value 0 or 1 yields a membership degree from the interval $[0,1]$. We can then require a good movie to belong to the $\mathcal{EL}$ concept (1) with degree at least .8. More generally, if C is an $\mathcal{EL}$ concept, then the *threshold concept* $C_{\geq t}$ for $t\in[0,1]$ collects all the individuals that belong to C with degree at least t. In addition to such upper threshold concepts, also lower threshold concepts $C_{\leq t}$ are considered, and strict inequalities may be used. For example, a bad movie could be required to belong to the $\mathcal{EL}$ concept (1) with a degree less than .2. In contrast to fuzzy DLs [7, 10, 16], which also yield membership degrees, we use classical crisp interpretations to define the semantics of the new logic. The membership degree of an individual d in a concept C is obtained by comparing the properties that the individual has with the properties that the concept requires. Moreover, the obtained threshold concepts are crisp rather than fuzzy.

There are, of course, different possibilities for how to define a graded membership function m based on the previous idea, and the semantics of the obtained new logic $\tau\mathcal{EL}(m)$ depends on m. In [2], we have not only introduced this general framework, but have also defined a specific graded membership function deg, and have investigated the complexity of reasoning in $\tau\mathcal{EL}(deg)$ in detail. More precisely, we have shown that the satisfiability and the ABox consistency problem in $\tau\mathcal{EL}(deg)$ are NP-complete, and the subsumption and the instance problem are coNP-complete (the latter w.r.t. data complexity). All these results are shown for the setting without TBoxes. From a technical point of view, we think that deg (but not our logic $\tau\mathcal{EL}(deg)$) can be expressed using the combination of *Aggregation Operators* (AOs) and fuzzy DLs [6]. Nevertheless, we

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[2] In $\mathcal{FL}_0$, we have value restrictions ($\forall r.C$) instead of existential restrictions.
[3] see http://www.w3.org/TR/owl2-profiles/
[4] see http://www.ihtsdo.org/snomed-ct/

believe that the existing results on reasoning in fuzzy DLs with AOs do not imply any of our results. In fact, the fuzzy DL introduced in [6] cannot express our threshold concepts, and we show exact complexity results for extensions of $\mathcal{EL}$.

The main contribution of the present paper is to investigate the complexity of reasoning in $\tau\mathcal{EL}(deg)$ w.r.t. acyclic TBoxes. Surprisingly, this is not as easy as might have been expected. The problem already starts with how to define acyclic TBoxes in $\tau\mathcal{EL}(deg)$. It turns out that simply replacing $\mathcal{EL}$ concepts by $\tau\mathcal{EL}(deg)$ concepts in the definition of an acyclic TBox does not yield the desired result. In fact, acyclic TBoxes are supposed to introduce concept names (defined concepts) as abbreviations for complex concepts, and these complex concepts can be obtained by *unfolding* defined concepts, i.e., by replacing defined concepts by their definitions until no more defined concepts occur. For the straightforward definition of acyclic $\tau\mathcal{EL}(deg)$ TBoxes mentioned above, this unfolding would actually yield concepts that are not syntactically correct $\tau\mathcal{EL}(deg)$ concepts since they may contain nested threshold operators, which is not allowed in $\tau\mathcal{EL}(deg)$.[5] Thus, we propose a more sophisticated notion of acyclic TBox for $\tau\mathcal{EL}(deg)$, which consists of an $\mathcal{EL}$ part and a $\tau\mathcal{EL}(deg)$ part satisfying certain properties. These properties ensures that unfolding yields a correct $\tau\mathcal{EL}(deg)$ concept. Of course, from a semantic point of view, we want defined concepts to have the same meaning as their unfolded counterparts. For this to hold, we need to require the graded membership function to "respect" the $\mathcal{EL}$ part of the TBox in an appropriate way. We show how deg can be modified such that it satisfies this requirement. This finally fixes syntax and semantics of acyclic $\tau\mathcal{EL}(deg)$ TBoxes. We then investigate reasoning w.r.t. such TBoxes. We show that, again quite surprisingly, the complexity increases by at least one level in the polynomial hierarchy when acyclic TBoxes are added: satisfiability and consistency are Π_2^P-hard and subsumption and the instance problem are Σ_2^P-hard. The best upper bound we can currently show for these problems is PSpace.

In the next section, we will sketch how the DL $\tau\mathcal{EL}(deg)$ was defined in [2] (more details and motivating discussions can be found in that paper). In Section 3, we introduce acyclic $\tau\mathcal{EL}(deg)$ TBoxes, and in Section 4 we sketch proofs of the mentioned complexity results (detailed proofs can be found in [4]).

2 THE DESCRIPTION LOGIC $\tau\mathcal{EL}(deg)$

We start by introducing the DL $\mathcal{EL}$ and some related notions that are needed in the rest of the paper. Afterwards, we present the abstract family of DLs $\tau\mathcal{EL}(m)$ that is obtained by extending $\mathcal{EL}$ with threshold concepts defined using a graded membership function m [2]. Finally, we recall the specific graded membership function deg, and briefly discuss the results obtained in [2] concerning the computational complexity of reasoning in $\tau\mathcal{EL}(deg)$.

2.1 The Description Logic $\mathcal{EL}$

Starting with finite sets of concept names $\mathsf{N_C}$ and role names $\mathsf{N_R}$, the set $\mathcal{C}_{\mathcal{EL}}$ of $\mathcal{EL}$ concept descriptions is obtained by combining the concept constructors *conjunction* ($C \sqcap D$), *existential restriction* ($\exists r.C$) and *top* ($\top$), in the following way:

$$C ::= \top \mid A \mid C \sqcap C \mid \exists r.C \tag{2}$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $C \in \mathcal{C}_{\mathcal{EL}}$.

The semantics of $\mathcal{EL}$ is given through standard *first-order* logic interpretations. An *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty domain $\Delta^\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$ that assigns subsets of $\Delta^\mathcal{I}$ to concept names in $\mathsf{N_C}$ and binary relations over $\Delta^\mathcal{I}$ to role names in $\mathsf{N_R}$. The function $\cdot^\mathcal{I}$ is inductively extended to arbitrary concept descriptions in the usual way, i.e.,

$$\top^\mathcal{I} := \Delta^\mathcal{I}, \quad (C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I},$$
$$(\exists r.C)^\mathcal{I} := \{x \in \Delta^\mathcal{I} \mid \exists y.((x,y) \in r^\mathcal{I} \wedge y \in C^\mathcal{I})\}.$$

Given two $\mathcal{EL}$ concept descriptions C and D, we say that C is *subsumed* by D (in symbols $C \sqsubseteq D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ for all interpretations $\mathcal{I}$. These two concepts are equivalent (in symbols $C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$. In addition, C is *satisfiable* iff $C^\mathcal{I} \neq \emptyset$ for some interpretation $\mathcal{I}$.

Information about specific individuals (represented by a set of individual names $\mathsf{N_I}$) can be expressed in an ABox, which is a finite set of *assertions* of the form $C(a)$ or $r(a,b)$, where $C \in \mathcal{C}_{\mathcal{EL}}$, $r \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. In addition to concept and role names, an interpretation $\mathcal{I}$ now assigns domain elements $a^\mathcal{I}$ to individual names a. We say that $\mathcal{I}$ satisfies an assertion $C(a)$ iff $a^\mathcal{I} \in C^\mathcal{I}$, and $r(a,b)$ iff $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$. Further, $\mathcal{I}$ is a model of $\mathcal{A}$ (denoted as $\mathcal{I} \models \mathcal{A}$) iff it satisfies all the assertions of $\mathcal{A}$. Then, an ABox $\mathcal{A}$ is *consistent* iff $\mathcal{I} \models \mathcal{A}$ for some interpretation $\mathcal{I}$. Finally, an individual a is an *instance* of C in $\mathcal{A}$ iff $a^\mathcal{I} \in C^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{A}$.

An $\mathcal{EL}$ TBox $\mathcal{T}$ is a finite set of concept definitions of the form $E \doteq C_E$, where $E \in \mathsf{N_C}$ and C_E is an $\mathcal{EL}$ concept description.[6] Additionally, we require that no concept name occurs more than once on the left hand side of a definition. Concept names occurring on the left hand side of a definition of $\mathcal{T}$ are called *defined concepts* while all other concept names are called *primitive concepts*. The sets of defined and primitive concepts of $\mathcal{T}$ are denoted as $\mathsf{N}_d^\mathcal{T}$ and $\mathsf{N}_{pr}^\mathcal{T}$, respectively. Note that $\mathsf{N}_{pr}^\mathcal{T} = \mathsf{N_C} \setminus \mathsf{N}_d^\mathcal{T}$, and thus also contains all concept names not occurring in $\mathcal{T}$. An interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ (in symbols $\mathcal{I} \models \mathcal{T}$) iff $E^\mathcal{I} = (C_E)^\mathcal{I}$ for all $E \doteq C_E \in \mathcal{T}$. The relations $\sqsubseteq$ and $\equiv$ are now defined *modulo* all models of $\mathcal{T}$ and denoted as $\sqsubseteq_\mathcal{T}$ and $\equiv_\mathcal{T}$, respectively. The satisfiability and the instance problem can be adapted accordingly to the presence of a TBox, and we then talk about satisfiability and instance w.r.t. $\mathcal{T}$.

TBoxes can be classified into being *acyclic* or *cyclic*, based on how their defined concepts depend on each other. A defined concept E_1 *directly* depends on a defined concept E_2 iff $E_1 \doteq C_{E_1} \in \mathcal{T}$ and E_2 occurs in C_{E_1}. Then, $\mathcal{T}$ is called cyclic iff it contains a defined concept E that depends *directly* or *indirectly* on itself. Otherwise, it is called *acyclic*. Given an acyclic TBox $\mathcal{T}$, the *unfolding* $u_\mathcal{T}(C)$ of an $\mathcal{EL}$ concept description C w.r.t. $\mathcal{T}$ is the concept description obtained by exhaustively replacing all occurrences of defined concepts E by their definitions C_E in $\mathcal{T}$. Based on this, the meaning of a concept C can always be determined from the meaning of its unfolded description: $C^\mathcal{I} = [u_\mathcal{T}(C)]^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{T}$, which means that $C \equiv_\mathcal{T} u_\mathcal{T}(C)$. From a model-theoretical point of view this is captured by the following proposition (see [13]).

Proposition 1 *Let $\mathcal{T}$ be an acyclic $\mathcal{EL}$ TBox. Any interpretation $\mathcal{I}$ of $\mathsf{N}_{pr}^\mathcal{T} \cup \mathsf{N_R}$ can be uniquely extended into a model of $\mathcal{T}$.*

The definition of the graded membership function deg that we will recall in Section 2.3 is based on the representation of concepts and interpretations as graphs, and homomorphisms between these graphs.

[5] In fact, the semantics of such nested concepts would not be well-defined since the graded membership function can only deal with $\mathcal{EL}$ concepts as input.

[6] In this paper, we do not consider so-called general concept inclusions (GCIs), which are of the form $C \sqsubseteq D$ for $C, D \in \mathcal{C}_{\mathcal{EL}}$.

An $\mathcal{EL}$ *description graph* is a tuple $G = (V_G, E_G, \ell_G)$, where ℓ_G labels each node v in V_G with a subset $\ell_G(v) \subseteq \mathsf{N_C}$ and each edge $vrw \in E_G$ is labeled with a role name r from $\mathsf{N_R}$. In [5] it is shown that every $\mathcal{EL}$ concept description C can be translated into an $\mathcal{EL}$ description tree T_C and vice versa. Moreover, in [1] interpretations $\mathcal{I}$ are translated into $\mathcal{EL}$ description graphs $G_\mathcal{I}$. For instance, the left-hand side of Figure 1 depicts the $\mathcal{EL}$ description tree corresponding to the concept description $A \sqcap \exists s.(B_1 \sqcap \exists r.B_3 \sqcap \exists r.B_2)$, whereas the right-hand side shows the description graph induced by an interpretation $\mathcal{I}$ whose domain consists of 6 elements, and where the extensions of concept and role names are given by the labels (the meaning of the lines going from T_C to $G_\mathcal{I}$ will be discussed later).

Homomorphisms between $\mathcal{EL}$ description trees were introduced in [5] to characterize subsumption in $\mathcal{EL}$: $C \sqsubseteq D$ iff there exists a homomorphism from T_D to T_C mapping the root of T_D to the root of T_C (Thm. 1, [5]). The following definition generalizes such homomorphisms to graphs.

Definition 2 Let $G = (V_G, E_G, \ell_G)$ and $H = (V_H, E_H, \ell_H)$ be two $\mathcal{EL}$ description graphs. A mapping $\varphi : V_G \to V_H$ is a *homomorphism* from G to H iff the following conditions are satisfied:

1. $\ell_G(v) \subseteq \ell_H(\varphi(v))$ for all $v \in V_G$, and
2. $vrw \in E_G$ implies $\varphi(v)r\varphi(w) \in E_H$.

Such homomorphisms can be used to characterize *membership* in $\mathcal{EL}$ concept descriptions.

Theorem 3 *Let $\mathcal{I}$ be an interpretation, $d \in \Delta^\mathcal{I}$, and C an $\mathcal{EL}$ concept description. Then, $d \in C^\mathcal{I}$ iff there exists a homomorphism φ from T_C to $G_\mathcal{I}$ such that $\varphi(v_0) = d$.*

One final technical notion: the *role depth* $\mathsf{rd}(C)$ of an $\mathcal{EL}$ concept description C is the maximal nesting of existential restrictions in C; equivalently, it is the height of the description tree T_C.

2.2 Adding threshold concepts to $\mathcal{EL}$

In [2], we have extended $\mathcal{EL}$ with a family of concept constructors of the form $C_{\sim t}$, such that C is an $\mathcal{EL}$ concept description, $\sim \in \{<, \leq, >, \geq\}$, and t is a *rational* number in $[0, 1]$. These new constructors can then be combined with the basic $\mathcal{EL}$ concept constructors (2) to form more complex concepts, e.g., $(\exists r.A)_{<1} \sqcap \exists r.(A \sqcap B)_{\geq .8} \sqcap B$. Concepts of the form $C_{\sim t}$ are called *threshold concepts*. The semantics of such concepts is based on a graded membership function m. The idea is that, given an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$, $m^\mathcal{I}(d, C)$ computes a value between 0 and 1 representing the extent to which d belongs to C in $\mathcal{I}$. For instance, the concept $C_{>.8}$ collects all the individuals that belong to C with degree greater than .8. To indicate which function m is used to obtain the semantics of threshold concepts, we call the extended logic $\tau\mathcal{EL}(m)$. We require such functions m to satisfy the following two properties.

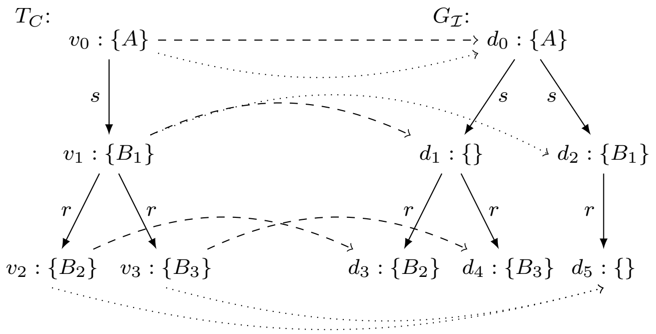
Definition 4 A *graded membership function* m is a family of functions that contains for every interpretation $\mathcal{I}$ a function $m^\mathcal{I} : \Delta^\mathcal{I} \times \mathcal{C}_\mathcal{EL} \to [0, 1]$ satisfying the following conditions (for $C, D \in \mathcal{C}_\mathcal{EL}$):

$$\textit{M1: } d \in C^\mathcal{I} \Leftrightarrow m^\mathcal{I}(d, C) = 1 \text{ for all } d \in \Delta^\mathcal{I},$$

$$\textit{M2: } C \equiv D \Leftrightarrow \forall \mathcal{I}\, \forall d \in \Delta^\mathcal{I} : m^\mathcal{I}(d, C) = m^\mathcal{I}(d, D).$$

The formal semantics of threshold concepts is then defined in terms of m as follows: $(C_{\sim t})^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid m^\mathcal{I}(d, C) \sim t\}$. Taking this into account, $\cdot^\mathcal{I}$ is extended in a natural way to interpret complex $\tau\mathcal{EL}(m)$ concept descriptions.

Figure 1. Partial mappings.



Coming back to Definition 4, on the one hand, property *M2* expresses the intuition that membership values should not depend on the syntactic form of a concept, but only on its semantics. On the other hand, requiring *M1* has the following consequences.

Proposition 5 *For every $\mathcal{EL}$ concept description C we have $C_{\geq 1} \equiv C$ and $C_{<1} \equiv \neg C$, where the semantics of negation is defined as usual, i.e., $(\neg C)^\mathcal{I} := \Delta^\mathcal{I} \setminus C^\mathcal{I}$.*

The equivalence $C_{<1} \equiv \neg C$ says that negation of $\mathcal{EL}$ concepts is expressible in $\tau\mathcal{EL}(m)$. This does not imply, however, that $\tau\mathcal{EL}(m)$ is closed under negation. Note that nesting of threshold constructors is not allowed. For example, strings like $((\exists r.A)_{<1})_{<1}$ or $(E_{\sim t})_{<1}$ do not constitute well-formed concepts in $\tau\mathcal{EL}(m)$. Thus, negation cannot be nested using these constructors.

Regarding notation, we will sometimes use $C_{=t}$ to abbreviate the concept description $C_{\leq t} \sqcap C_{\geq t}$. Symbols like $\widehat{C}$, $\widehat{D}$ will be used to refer to $\tau\mathcal{EL}(m)$ concept descriptions.

2.3 The graded membership function *deg*

In addition to defining the family of DLs $\tau\mathcal{EL}(m)$, in [2] we also define a concrete graded membership function deg and study its induced DL $\tau\mathcal{EL}(deg)$. Since the latter constitutes our main object of study, we shall briefly describe the principal components supporting the definition of deg.

Basically, we use the homomorphism characterization of membership in $\mathcal{EL}$ (Theorem 3) as a starting point. The computation of $deg^\mathcal{I}(d, C)$ relies on exploring the search space consisting of all partial mappings from T_C to $G_\mathcal{I}$ that map the root of T_C to d and respect the edge structure of T_C. Let us explain the reason for considering such partial mappings using the following example.

Example 6 Figure 1 shows a description tree T_C corresponding to the concept $C := A \sqcap \exists s.(B_1 \sqcap \exists r.B_2 \sqcap \exists r.B_3)$, and the description graph associated to an interpretation $\mathcal{I}$. Clearly, $d_0 \notin C^\mathcal{I}$, and thus there is no homomorphism that maps v_0 to d_0. Nevertheless, the mappings depicted in the figure (represented by the dashed lines and the dotted ones) provide two different views of how d_0 partially satisfies the properties required by C. The idea is then to calculate to which degree each partial mapping fulfills the homomorphism conditions (see Definition 2), and take the degree of the best one as the membership degree $deg^\mathcal{I}(d_0, C)$.

These partial mappings are formally defined in [2] (Def. 4) as *partial tree-to-graph homomorphisms* (*ptgh*). To measure to which degree a *ptgh* h satisfies the homomorphism conditions, a weighted function $h_w : \mathsf{dom}(h) \to [0,1]$ is defined and the value $h_w(v_0)$ considered as the corresponding degree. We use again Figure 1 to sketch how h_w calculates such degrees. The formal details, which are omitted due to space constraints, can be found in [2].

Example 7 Let h denote the mapping represented by the dashed lines and g the other one. To compute $h_w(v_0)$, we basically count the number of properties of v_0 (say ℓ), check how many of those d_0 actually has in $\mathcal{I}$ (say k) and give k/ℓ as the membership degree $h_w(v_0)$. In our example, v_0 has two properties, namely, A and the existence of an s-successor with a certain structure (the node v_1). In particular, the s-successor of d_0 selected by h to match v_1, does not satisfy all the conditions required by v_1. Now, instead of assuming that d_0 lacks the second property and setting $h_w(v_0) = 1/2$, $h_w(v_1)$ computes a value that expresses to which degree d_1 satisfies the conditions required by v_1. This is done by applying the same idea recursively. This procedure stops at nodes of T_C having no successors in $\mathsf{dom}(h)$.

Thus, the real computation is done in a bottom-up manner. First, we have $h_w(v_2)=1$ and $h_w(v_3)=1$. Based on these two values and the fact that $d_1 \notin (B_1)^{\mathcal{I}}$, we obtain $h_w(v_1) = 2/3$. Finally, since $d_0 \in A^{\mathcal{I}}$, we get $h_w(v_0) = (1 + h_w(v_1))/2 = 5/6$. Concerning the mapping g, the reader can verify that $g_w(v_0) < h_w(v_0)$, and thus *deg* sees h as a better approximation for membership in C.

Based on these ideas, we now define the graded membership function *deg*. However, in order to satisfy property *M2*, all concept descriptions C are transformed into an appropriate *reduced form* C^r before actually applying the computations sketched above. This reduced form, which was introduced in [11], removes redundancies from concepts, and has the property that $C \equiv D$ iff the description trees of C^r and D^r are isomorphic.

Definition 8 (Def. 6, [2]) Let $\mathcal{I}$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept with reduced form C^r. Moreover, let $\mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$ be the set of all *ptghs* h from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$. Then,

$$deg^{\mathcal{I}}(d, C) := \max\{q \mid h_w(v_0) = q \text{ and } h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)\}$$

We have shown in [2] that the maximum in the above expression always exists. This implies that the function *deg* is *well-defined*. In addition, we could show that the properties *M1* and *M2* are satisfied. Regarding the induced DL $\tau\mathcal{EL}(deg)$, we have investigated the computational complexity of the standard reasoning problems *satisfiability*, *subsumption*, ABox *consistency* and *instance checking*. In particular, the subsumption and the satisfiability problems are tackled by establishing the following polynomial model property for the satisfiability of concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$, for $\tau\mathcal{EL}(deg)$ concepts $\widehat{C}, \widehat{D}$. Note that this is equivalent to the non-subsumption problem and satisfiability is a special case.

Lemma 9 (Lem. 5, [2]) *Let $\widehat{C}$ and $\widehat{D}$ be $\tau\mathcal{EL}(deg)$ concepts of sizes $\mathsf{s}(\widehat{C})$ and $\mathsf{s}(\widehat{D})$. If $\widehat{C} \sqcap \neg\widehat{D}$ is satisfiable, then there exists an interpretation $\mathcal{I}$ such that $\widehat{C}^{\mathcal{I}} \setminus \widehat{D}^{\mathcal{I}} \neq \emptyset$ and $|\Delta^{\mathcal{I}}| \leq \mathsf{s}(\widehat{C}) \cdot \mathsf{s}(\widehat{D})$.*

A analogous property has been also proved for consistent ABoxes of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$, thus yielding a bounded model property for non-instance ($\mathcal{A} \not\models \widehat{C}(a)$). Unfortunately, in this case the bound on the model's size has the size of the concept $\widehat{C}$ in the exponent. Nevertheless, since *consistency* is a particular case where $\widehat{C}$ is not present, we have a polynomial model property for ABox consistency. In addition, checking whether a finite interpretation $\mathcal{I}$ satisfies a $\tau\mathcal{EL}(deg)$ concept/ABox can be done in polynomial time. Overall, we can thus employ a standard *guess-and-check* NP-algorithm to decide satisfiability, non-subsumption, and ABox consistency. For non-instance, this algorithm is only in NP if we consider *data complexity* as defined in [9].

Theorem 10 (Th. 5 and Th. 6, [2]) *In $\tau\mathcal{EL}(deg)$, satisfiability and consistency are NP-complete, whereas subsumption and instance checking (w.r.t. data complexity) are coNP-complete.*

3 ACYCLIC TBOXES FOR $\tau\mathcal{EL}(m)$

We now turn to introducing acyclic TBoxes for the whole family of DLs $\tau\mathcal{EL}(m)$, and hence also for $\tau\mathcal{EL}(deg)$. As with acyclic TBoxes in $\mathcal{EL}$, the purpose is to introduce abbreviations for composite $\tau\mathcal{EL}(m)$ concept descriptions. For instance, the $\mathcal{EL}$ concept definition $E \doteq \exists r.A \sqcap \exists r.B$ can be used to abbreviate the threshold concept $(\exists r.A \sqcap \exists r.B)_{\geq 1/2}$ as $E_{\geq 1/2}$. On top of this, we can then also introduce the abbreviation β for $E_{\geq 1/2}$ and use this abbreviation in other concept definitions, as done in the following TBox:

$$\left\{ \begin{array}{l} \alpha \doteq \exists s.A \sqcap \exists r.\beta \\ \beta \doteq E_{\geq 1/2} \\ E \doteq \exists r.A \sqcap \exists r.B \end{array} \right\} \tag{3}$$

Overall, the concept name α then abbreviates the $\tau\mathcal{EL}(m)$ concept description $\exists s.A \sqcap \exists r.(\exists r.A \sqcap \exists r.B)_{\geq 1/2}$, which can be obtained from α by unfolding.

However, we cannot use arbitrary acyclic sets of $\tau\mathcal{EL}(m)$ concept definitions. For example, suppose that α is now defined in the TBox (3) as $\alpha \doteq \exists s.A \sqcap \exists r.(\beta_{>.8})$ instead. Even though the right-hand side of this definition is a syntactically well-formed $\tau\mathcal{EL}(m)$ concept, unfolding α w.r.t. this new TBox yields

$$\exists s.A \sqcap \exists r.(((\exists r.A \sqcap \exists r.B)_{\geq 1/2})_{>.8}), \tag{4}$$

which is not a well-formed $\tau\mathcal{EL}(m)$ concept description since threshold operators are nested. The following definition is designed to avoid this problem.

Definition 11 An *acyclic $\tau\mathcal{EL}(m)$ TBox* $\widehat{\mathcal{T}}$ is a pair $(\mathcal{T}_\tau, \mathcal{T})$, where $\mathcal{T}$ is an acyclic $\mathcal{EL}$ TBox and $\mathcal{T}_\tau$ is a set of concept definitions of the form $\alpha \doteq \widehat{C}_\alpha$ satisfying the following conditions:

- $\widehat{C}_\alpha$ is a $\tau\mathcal{EL}(m)$ concept description,
- α does not *depend* on itself and it does not occur in $\mathcal{T}$,
- for all threshold concepts $C_{\sim t}$ occurring in $\widehat{C}_\alpha$, no defined concept of $\mathcal{T}_\tau$ occurs in C.

The TBox (3) can be seen as an acyclic $\tau\mathcal{EL}(m)$ TBox where the first two definitions belong to $\mathcal{T}_\tau$ and the last to $\mathcal{T}$. Notice that, although the syntax of the first and third axioms looks quite similar, they are actually different since the first one contains a concept name whose definition uses a threshold concept whereas the third one does not.

Given an acyclic $\tau\mathcal{EL}(m)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T})$, we define the set $\mathsf{N}_d^{\widehat{\mathcal{T}}}$ of defined concepts in $\widehat{\mathcal{T}}$ as the union $\mathsf{N}_d^{\mathcal{T}_\tau} \cup \mathsf{N}_d^{\mathcal{T}}$, where $\mathsf{N}_d^{\mathcal{T}_\tau}$ is the set of defined concepts in $\mathcal{T}_\tau$. We denote the set $\mathsf{N_C} \setminus \mathsf{N}_d^{\widehat{\mathcal{T}}}$ as $\mathsf{N}_{pr}^{\widehat{\mathcal{T}}}$. The notion of unfolding is extended to acyclic $\tau\mathcal{EL}(m)$ TBoxes in the obvious way. It is easy to see that the restrictions imposed in the previous definition guarantee that $\alpha \in \mathsf{N}_d^{\mathcal{T}_\tau}$ always unfolds into a well-formed $\tau\mathcal{EL}(m)$ concept description $u_{\widehat{\mathcal{T}}}(\alpha)$, whereas $E \in \mathsf{N}_d^{\mathcal{T}}$ unfolds into an $\mathcal{EL}$ concept. Regarding arbitrary $\tau\mathcal{EL}(m)$ concept descriptions $\widehat{C}$, we say that $\widehat{C}$ is *correctly defined* w.r.t $\widehat{\mathcal{T}}$ if the pair $(\mathcal{T}_\tau \cup \{\alpha \doteq \widehat{C}\}, \mathcal{T})$ is still an acyclic $\tau\mathcal{EL}(m)$ TBox, where α is a *fresh* concept name not occurring in $\widehat{\mathcal{T}}$ and $\widehat{C}$.

We are now ready to fix the semantics of acyclic $\tau\mathcal{EL}(m)$ TBoxes. To start with, we say that an interpretation $\mathcal{I}$ is a model of $\mathcal{T}_\tau$ (and write $\mathcal{I} \models \mathcal{T}_\tau$) iff $\alpha^{\mathcal{I}} = (\widehat{C}_\alpha)^{\mathcal{I}}$ in $\tau\mathcal{EL}(m)$ for all $\alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau$. Then, $\mathcal{I}$ satisfies $\widehat{\mathcal{T}}$ iff $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{T}_\tau$. The subsumption and equivalence relations $\sqsubseteq_{\widehat{\mathcal{T}}}$ and $\equiv_{\widehat{\mathcal{T}}}$ on correctly defined $\tau\mathcal{EL}(m)$ concepts are defined w.r.t. the set of models of $\widehat{\mathcal{T}}$. The next step is to ensure that defined concepts α and their unfolded counterparts $u_{\widehat{\mathcal{T}}}(\alpha)$ have the same meaning in all models of $\widehat{\mathcal{T}}$, i.e.,

$$\alpha \equiv_{\widehat{\mathcal{T}}} u_{\widehat{\mathcal{T}}}(\alpha). \tag{5}$$

Since this equivalence holds for $\mathcal{EL}$, the only constructor that might lead to a problem is the threshold constructor. More precisely, given a threshold concept $C_{\sim t}$ with $C \in \mathcal{C}_{\mathcal{EL}}$, for all models of $\widehat{\mathcal{T}}$ the following must hold:

$$(C_{\sim t})^{\mathcal{I}} = ((u_{\mathcal{T}}(C))_{\sim t})^{\mathcal{I}}. \tag{6}$$

Thus, we must turn our attention to the graded membership function m since m is providing the semantics for such concepts $C_{\sim t}$. In principle, the graded membership function m is defined on C since C is an $\mathcal{EL}$ concept description. However, this function (e.g., deg) is agnostic of the TBox and thus treats defined and primitive concepts alike: they are just concept names for m. In order to satisfy (6), the function needs to be aware of the TBox. Let us illustrate this using the graded membership function deg:

Example 12 Let $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T})$ be the $\tau\mathcal{EL}(m)$ acyclic TBox corresponding to the definitions in (3). In addition, let $\mathcal{I}$ be an interpretations such that $\Delta^{\mathcal{I}} = \{d_0, d_r, d_s\}$, $A^{\mathcal{I}} = \{d_s\}$, $B^{\mathcal{I}} = \{d_r\}$, and $r^{\mathcal{I}} = \{(d_0, d_0), (d_0, d_r)\}$, $s^{\mathcal{I}} = \{(d_0, d_s)\}$.

When trying to extend $\mathcal{I}$ to a model of $\widehat{\mathcal{T}}$, we first note that we have $(\exists r.A \sqcap \exists r.B)^{\mathcal{I}} = \emptyset$, and hence $E^{\mathcal{I}}$ must be interpreted as the empty set. Then, since E is treated as a concept name by deg, this means that $deg^{\mathcal{I}}(d, E) = 0$ for all $d \in \Delta^{\mathcal{I}}$. Therefore, $(E_{\geq 1/2})^{\mathcal{I}} = \emptyset$, and consequently we must define $\beta^{\mathcal{I}} := \alpha^{\mathcal{I}} := \emptyset$. To see that (6) fails to hold, one can observe that in contrast to $deg^{\mathcal{I}}(d_0, E) = 0$, for d_0 we obtain $deg^{\mathcal{I}}(d_0, \exists r.A \sqcap \exists r.B) = 1/2$ (recall the ideas defining deg). This means that $(E_{\geq 1/2})^{\mathcal{I}} \neq ((u_{\mathcal{T}}(E))_{\geq 1/2})^{\mathcal{I}}$. Obviously, the problem propagates up to the more general requirement in (5). First, $d_0 \notin \beta^{\mathcal{I}}$ but $d_0 \in (u_{\widehat{\mathcal{T}}}(\beta))^{\mathcal{I}}$. Moreover, it is easy to check that $d_0 \in (u_{\widehat{\mathcal{T}}}(\alpha))^{\mathcal{I}}$, and thus $\alpha \not\equiv_{\widehat{\mathcal{T}}} u_{\widehat{\mathcal{T}}}(\alpha)$.

To avoid the problem demonstrated by this example, the graded membership function m needs to take into account the definitions in $\mathcal{T}$. This means that $\mathcal{T}$ must be a parameter of this function. Furthermore, to ensure that property (6) is satisfied, the membership degrees for an $\mathcal{EL}$ concept description C should be the same as for its unfolding $u_{\mathcal{T}}(C)$. Taking this into account, we extend a given graded membership function m such that it takes concept definitions in acyclic $\mathcal{EL}$ TBoxes into account.

Definition 13 For all graded membership functions m (in the sense of Definition 4), the extension of m computing membership degrees w.r.t. acyclic $\mathcal{EL}$ TBoxes is a family of functions containing for every interpretation $\mathcal{I}$ a function $\widehat{m}^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \mathcal{C}_{\mathcal{EL}} \times \mathfrak{T}(\mathcal{I}) \to [0, 1]$ satisfying

$$\widehat{m}^{\mathcal{I}}(d, C, \mathcal{T}) := m^{\mathcal{I}}(d, u_{\mathcal{T}}(C)),$$

where $\mathfrak{T}(\mathcal{I})$ is the set of all acyclic $\mathcal{EL}$ TBoxes $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$.

Clearly, well-definedness of m and acyclicity of $\mathcal{T}$ imply well-definedness of $\widehat{m}$. For the sake of simplicity, we will from now on use m both to denote the original graded membership function and its extension $\widehat{m}$.

The use of unfolding in the above definition ensures that, for all interpretations $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, we have $d \in (C_{\sim t})^{\mathcal{I}}$ iff $d \in ((u_{\mathcal{T}}(C))_{\sim t})^{\mathcal{I}}$. Consequently, (6) always holds, as does (5). Finally, it is easy to see that the analogon of Proposition 1 is also valid for acyclic $\tau\mathcal{EL}(m)$ TBoxes.

Proposition 14 *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(m)$ TBox. Any interpretation $\mathcal{I}$ of $\mathsf{N}_{pr}^{\widehat{\mathcal{T}}} \cup \mathsf{N}_{\mathsf{R}}$ can be uniquely extended into a model of $\widehat{\mathcal{T}}$.*

The following lemma is an easy consequence of Definition 13. It shows that graded membership functions constructed in such a way satisfy a generalization of the conditions stated in Definition 4.

Lemma 15 Let m be a graded membership function as in Definition 13. Then, for all acyclic $\mathcal{EL}$ TBoxes $\mathcal{T}$, we have:

$$M1^{\mathcal{T}}: d \in C^{\mathcal{I}} \Leftrightarrow m^{\mathcal{I}}(d, C, \mathcal{T}) = 1 \text{ for all } \mathcal{I} \models \mathcal{T} \text{ and } d \in \Delta^{\mathcal{I}}$$

$$M2^{\mathcal{T}}: C \equiv_{\mathcal{T}} D \Leftrightarrow \forall \mathcal{I} \models \mathcal{T}\, \forall d \in \Delta^{\mathcal{I}}: m^{\mathcal{I}}(d, C, \mathcal{T}) = m^{\mathcal{I}}(d, D, \mathcal{T})$$

where C and D are $\mathcal{EL}$ concept descriptions.

To sum up, we have introduced a notion of acyclic TBoxes for $\tau\mathcal{EL}(m)$ such that unfolding still works from a syntactic point of view, i.e., the unfolding of a defined concept is a syntactically correct $\tau\mathcal{EL}(m)$ concept description. To ensure that unfolding is also correct from the semantic point of view (i.e., (5) holds), we had to extend m such that it takes the $\mathcal{EL}$ part of the given acyclic TBox into account. In the following, we consider $m = deg$ and show that the presence of acyclic $\tau\mathcal{EL}(deg)$ TBoxes increases the complexity of reasoning.

4 REASONING WITH ACYCLIC $\tau\mathcal{EL}(deg)$ TBOXES

We will not only investigate the satisfiability and the subsumption problem, but also consistency and instance. In the presence of an acyclic $\tau\mathcal{EL}(deg)$ TBox, the concepts occurring in the ABox need to be correctly defined w.r.t. this TBox. An *acyclic $\tau\mathcal{EL}(deg)$ knowledge base* is a pair $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ where $\widehat{\mathcal{T}}$ is an acyclic $\tau\mathcal{EL}(deg)$ TBox, and $\mathcal{A}$ is a finite set of assertions $\widehat{C}(a)$ or $r(a, b)$, where $\widehat{C}$ is correctly defined w.r.t. $\widehat{\mathcal{T}}$. The satisfaction of such assertions and ABoxes by interpretations is defined in the obvious way.

Proposition 14 together with (5) tell us that reasoning w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes can be reduced to reasoning in the empty terminology, through unfolding. However, as shown by Nebel [12] for the DL $\mathcal{FL}_0$, unfolding may produce concept descriptions of exponential size. The following is an adaptation of Nebel's example to $\mathcal{EL}$.

Example 16 The TBox $\mathcal{T}_n$ is inductively defined as follows ($n \geq 0$):

$$\begin{aligned}
\mathcal{T}_0 &:= \{\alpha_0 \doteq \top\} \\
\mathcal{T}_1 &:= \mathcal{T}_0 \cup \{\alpha_1 \doteq \exists r.\alpha_0 \sqcap \exists s.\alpha_0\} \\
&\vdots \\
\mathcal{T}_n &:= \mathcal{T}_{n-1} \cup \{\alpha_n \doteq \exists r.\alpha_{n-1} \sqcap \exists s.\alpha_{n-1}\}
\end{aligned}$$

Obviously, the size of $\mathcal{T}_n$ is linear in n, but $\mathsf{s}(u_{\mathcal{T}_n}(\alpha_n)) \geq 2^n$.

Thus, by applying unfolding and then using the known NP decision procedures for satisfiability/non-subsumption in $\tau\mathcal{EL}(deg)$ [2], we obtain an NExpTime algorithm to decide the same problems w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes. The natural question to ask is thus: can we do better than NExpTime? We will show that this is indeed the case by providing a PSpace upper bound. At the moment, we do not have a matching lower bound. However, we can show that (unless NP$=\Pi_2^P$) the complexity of reasoning w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes is higher than of reasoning in $\tau\mathcal{EL}(deg)$ without a TBox. We start with showing the lower bounds.

4.1 Lower bounds

We reduce the problem $\forall\exists$3SAT to concept satisfiability with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. This problem is known to be complete for the class Π_2^P (see [15], Section 4).

Definition 17 ($\forall\exists$3SAT) Let $u = \{u_1, \ldots, u_n\}$, $v = \{v_1, \ldots, v_\ell\}$ be two disjoint sets of propositional variables, and $\varphi(u, v)$ a 3CNF formula defined over $u \cup v$, i.e., a finite set of propositional clauses $\mathcal{C} = \{c_1, \ldots, c_q\}$ such that each c_k is a set of three literals $\{\gamma_1^k, \gamma_2^k, \gamma_3^k\}$ over $u \cup v$. A formula $\forall u \exists v.\varphi(u, v)$ is *valid* iff for all truth assignments θ of the variables in u there is an extension of θ for the variables in v satisfying $\varphi(u, v)$. $\forall\exists$3SAT is then the problem of deciding whether a formula $\forall u \exists v.\varphi(u, v)$ is valid or not.

The idea for the reduction goes as follows. Each formula $\forall u \exists v.\varphi(u, v)$ is translated into an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}_n^{\varphi}$ containing a defined concept α_n such that: $\forall u \exists v.\varphi(u, v)$ is valid iff α_n is satisfiable in $\widehat{\mathcal{T}}_n^{\varphi}$ (where n is the number of variables in u).

The first step consists of encoding $\varphi(u, v)$ into a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}_\varphi$. Literals defined over u and v are represented by concept names $A_i, \bar{A}_i$ $(1 \leq i \leq n)$ and $B_j, \bar{B}_j$ $(1 \leq j \leq \ell)$, respectively, according to the following mapping:

$$\eta(u_i) := A_i, \quad \eta(\neg u_i) := \bar{A}_i, \quad \eta(v_j) := B_j, \quad \eta(\neg v_j) := \bar{B}_j.$$

Using η, each clause c_k can be represented by the $\mathcal{EL}$ concept description $D_k := \eta(\gamma_1^k) \sqcap \eta(\gamma_2^k) \sqcap \eta(\gamma_3^k)$. The satisfiability of c_k can then be expressed by the threshold concept $(D_k)_{\geq 1/3}$. In fact, by the definition of deg, an individual d belongs to $(D_k)_{\geq 1/3}$ iff it belongs to at least one concept name in $\{\eta(\gamma_1^k), \eta(\gamma_2^k), \eta(\gamma_3^k)\}$. Therefore, the $\tau\mathcal{EL}(deg)$ concept $(D_1)_{\geq 1/3} \sqcap \ldots \sqcap (D_q)_{\geq 1/3}$ appears as a plausible choice to capture the satisfiability status of $\varphi(u, v)$. For this to work correctly, pairs of concepts $(A_i, \bar{A}_i)$ and $(B_j, \bar{B}_j)$ need to be complementary since they are meant to play the role of a literal u_i (v_j) and its negation. To this end, we define the TBox $\mathcal{T}_{n,\ell}^c$ as follows:

$$\mathcal{T}_{n,\ell}^c := \bigcup_{i=1}^{n} \{F_i \doteq A_i \sqcap \bar{A}_i\} \cup \bigcup_{j=1}^{\ell} \{G_j \doteq B_j \sqcap \bar{B}_j\}$$

Then, $(F_i)_{=1/2}$ collects the elements that are instances of exactly one concept in $\{A_i, \bar{A}_i\}$ (similarly for $(G_j)_{=1/2}$ and $\{B_j, \bar{B}_j\}$). Putting all these pieces together, $\widehat{C}_\varphi$ is defined as follows:

$$\widehat{C}_\varphi := \prod_{k=1}^{q} (D_k)_{\geq \frac{1}{3}} \sqcap \prod_{i=1}^{n} (F_i)_{=\frac{1}{2}} \sqcap \prod_{j=1}^{\ell} (G_j)_{=\frac{1}{2}}$$

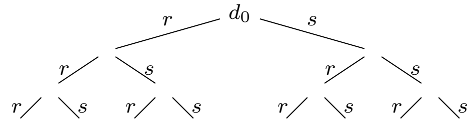
The following result is immediate given the construction of $\widehat{C}_\varphi$.

Lemma 18 *$\varphi(u, v)$ is satisfiable iff $\widehat{C}_\varphi$ is satisfiable w.r.t. $\mathcal{T}_{n,\ell}^c$.*

Obviously, this is not enough to achieve our main goal since $\forall\exists$3SAT asks for the satisfiability of $\varphi(u, v)$ in all truth assignments of u. To mimic this universal quantification, we extend the TBox $\mathcal{T}_n$ (from Example 16) into $\widehat{\mathcal{T}}_n^{\varphi}$ in such a way that for all models $\mathcal{I}$ of $\widehat{\mathcal{T}}_n^{\varphi}$, $(\alpha_n)^{\mathcal{I}} \neq \emptyset$ iff for all $X \subseteq u$ there exists $d_X \in \Delta^{\mathcal{I}}$ satisfying:

$$d_X \in (\widehat{C}_\varphi)^{\mathcal{I}} \text{ and for all } i, 1 \leq i \leq n\text{: } d_X \in (A_i)^{\mathcal{I}} \text{ iff } u_i \in X \quad (7)$$

For simplicity, we explain this step for $n = 3$. Let us start by looking at the interpretation $\mathcal{I}_3$ induced by the description tree representing the concept $u_{\mathcal{T}_3}(\alpha_3)$, which has the following shape:



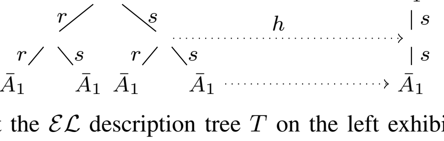
It is easy to see that the extension of $\mathcal{I}_3$ into a model of $\mathcal{T}_3$ is such that $d_0 \in (\alpha_3)^{\mathcal{I}_3}$. Moreover, a *one-to-one* correspondence can be established between the set of leaves of this tree and the words in $\{r, s\}^3$: for all words $x = x_1x_2x_3$ in $\{r, s\}^3$, the corresponding leaf d_x is the one reached from d_0 by the path $d_0x_1d_1x_2d_2x_3d_x$. Thus, the desired collection of elements satisfying (7) would exist if we could ensure the following: for each word $x \in \{r, s\}^3$ there is at least one path $d_0x_1 \ldots x_3d_x$ such that:

$$d_x \in (\widehat{C}_\varphi)^{\mathcal{I}_3} \text{ and } d_x \in (A_i)^{\mathcal{I}_3} \text{ iff } x_i = r \ (1 \leq i \leq 3) \quad (8)$$

The structure of $\mathcal{T}_3$ certainly guarantees that every model satisfying α_3 contains a path $d_0x_1 \ldots x_3d_x$ from a distinguished element d_0, for all $x \in \{r, s\}^3$. Moreover, the domain elements in such a path satisfy $d_0 \in (\alpha_3)^{\mathcal{I}_3}$, $d_1 \in (\alpha_2)^{\mathcal{I}_3}$, $d_2 \in (\alpha_1)^{\mathcal{I}_3}$ and $d_x \in (\alpha_0)^{\mathcal{I}_3}$. Hence, the first part of (8) can be satisfied by modifying the definition of α_0 to $\alpha_0 \doteq \widehat{C}_\varphi$. To fulfill the second part of (8), we must express within the logic the correct propagation of A_1, A_2, A_3 along each path. For example, for x_1 and A_1, a solution could be to redefine α_3 as $\alpha_3 \doteq \exists r.(\alpha_2 \sqcap \beta_2^r) \sqcap \exists s.(\alpha_2 \sqcap \beta_2^s)$, where:

$$\beta_2^r := \prod_{x_2, x_3 \in \{r,s\}} \forall x_2x_3.\neg\bar{A}_1 \qquad \beta_2^s := \prod_{x_2, x_3 \in \{r,s\}} \forall x_2x_3.\neg A_1$$

The definition of β_2^r implies that, if $d_0 \in (\alpha_3)^{\mathcal{I}_3}$, then all paths starting at d_0 following a word x of the form $x = rw$ with $w \in \{r, s\}^2$ must end at an element d_x such that $d_x \notin (\bar{A}_1)^{\mathcal{I}_3}$. If we also have $d_x \in (\alpha_0)^{\mathcal{I}_3}$, then this means that $d_x \in (A_1)^{\mathcal{I}_3}$. Now, β_2^r is obviously *not* a $\tau\mathcal{EL}(deg)$ concept, but its meaning can be equivalently expressed in the logic. We illustrate this with the help of the following diagram.



Notice that the $\mathcal{EL}$ description tree T on the left exhibits all (*and only*) paths falsifying β_2^r. Moreover, if d_0 has an r-successor leading to one such path (d_1 on the *right-hand* side), then there is always a *ptgh* h such that $h(v) = d_1$ and $h_w(v) > 0$. Conversely, it is not hard to show that, if no such path exists, then any possible *ptgh* h satisfies $h_w(v) = 0$. Hence, β_2^r is equivalent to the threshold concept $(E_2^3)_{\leq 0}$, where E_2^3 is the concept corresponding to T. Similarly, we can also deal with the requirements for $\bar{A}_2$ and $\bar{A}_3$ by using analogous concepts E_1^2 and E_0^1. Finally, to succinctly represent these concepts (which would be exponentially large for general n), we employ the $\mathcal{EL}$ TBox consisting of the following definitions:

$$\begin{array}{lll} E_2^3 \doteq \exists r.E_1^3 \sqcap \exists s.E_1^3 & E_1^2 \doteq \exists r.E_0^2 \sqcap \exists s.E_0^2 & E_0^1 \doteq \bar{A}_3 \\ E_1^3 \doteq \exists r.E_0^3 \sqcap \exists s.E_0^3 & E_0^2 \doteq \bar{A}_2 & \\ E_0^3 \doteq \bar{A}_1 & & \end{array}$$

To express β_2^s, similar definitions using A_i instead of $\bar{A}_i$ can be given. Let us call $\mathcal{T}_{3,p}$ the collection of all these definitions. Then, the final acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}_3^\varphi$ is the pair $(\mathcal{T}_{3,\tau}, \mathcal{T}_{3,\ell}^c \cup \mathcal{T}_{3,p})$, where $\mathcal{T}_{3,\tau}$ consists of the following definitions ($1 \leq i \leq 3$):

$$\alpha_i \doteq \exists r.(\alpha_{i-1} \sqcap (E_{i-1}^i)_{\leq 0}) \sqcap \exists s.(\alpha_{i-1} \sqcap (E_{i-1}^{\bar{i}})_{\leq 0})$$
$$\alpha_0 \doteq \widehat{C}_\varphi$$

The construction can easily be generalized to arbitrary n without losing the properties exhibited for $n = 3$ (see[4] for details).

Lemma 19 *$\forall u \exists v.\varphi(u,v)$ is valid iff α_n is satisfiable in $\widehat{\mathcal{T}}_n^\varphi$.*

In addition, it is easy to see that $\widehat{\mathcal{T}}_n^\varphi$ is an acyclic $\tau\mathcal{EL}(deg)$ TBox of size polynomial in the size of $\forall u \exists v.\varphi(u,v)$, Therefore, $\forall\exists$3SAT is polynomial-time reducible to concept satisfiability w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes. In addition, non-satisfiability can be reduced to the subsumption and the instance problem, and satisfiability to the consistency problem, by the same arguments used in [2] for the setting without TBoxes.

Theorem 20 *In $\tau\mathcal{EL}(deg)$, satisfiability and consistency are Π_2^P-hard and the subsumption and the instance problem are Σ_2^P-hard, with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.*

4.2 A PSpace upper bound

We now present a PSpace procedure that decides satisfiability of concepts of the form $\alpha_1 \sqcap \neg\alpha_2$ w.r.t. an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$, where $\alpha_1, \alpha_2 \in \mathsf{N}_d^{\widehat{\mathcal{T}}}$. The restriction to defined concepts is without loss of generality since every $\tau\mathcal{EL}(m)$ concept $\widehat{C}$ correctly defined w.r.t. $\widehat{\mathcal{T}}$ can be equivalently replaced with a fresh concept name $\alpha_{\widehat{C}}$, by adding $\alpha_{\widehat{C}} \doteq \widehat{C}$ to $\mathcal{T}_\tau$.

As mentioned earlier, by using unfolding, we can reduce this problem to satisfiability of the concept $u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)$. Therefore, the application of Lemma 9 yields that $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable in $\widehat{\mathcal{T}}$ iff there exists an interpretation $\mathcal{I}$ over $\mathsf{N}_{pr}^{\widehat{\mathcal{T}}} \cup \mathsf{N}_\mathsf{R}$ such that:

$$[u_{\widehat{\mathcal{T}}}(\alpha_1)]^\mathcal{I} \setminus [u_{\widehat{\mathcal{T}}}(\alpha_2)]^\mathcal{I} \neq \emptyset \text{ and } |\Delta^\mathcal{I}| \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \cdot \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$$

Since the sizes of $u_{\widehat{\mathcal{T}}}(\alpha_1)$ and $u_{\widehat{\mathcal{T}}}(\alpha_2)$ may be exponential in the size $\mathsf{s}(\widehat{\mathcal{T}})$ of $\widehat{\mathcal{T}}$, this provides us with an exponential bounded model property, and hence an NExpTime upper bound for satisfiability. However, the construction used to prove Lemma 9 in [2] provides additional information about $\mathcal{I}$, which allows us to improve on this upper bound:

- $\mathcal{I}$ is *tree-shaped*,
- the depth of its description tree $T_\mathcal{I}$ is bounded by:

$$\mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2)), \tag{9}$$

- the element $d_0 \in \Delta^\mathcal{I}$ corresponding to the root of $T_\mathcal{I}$ satisfies

$$d_0 \in [u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)]^\mathcal{I}.$$

Fortunately, unfolding increases the role depth only polynomially, and thus the depth (9) of $T_\mathcal{I}$ is polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. Thus, despite its size, one can non-deterministically generate and explore $\mathcal{I}$ in a top-down manner, while keeping the used space polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. Let $\mathfrak{d} \geq 0$ and $\mathfrak{b} > 0$ be natural numbers. Then, each run ρ of the procedure *Gen* described below generates a *tree-shaped* interpretation $\mathcal{I}_\rho$ over $\mathsf{N}_{pr}^{\widehat{\mathcal{T}}} \cup \mathsf{N}_\mathsf{R}$, such that $|\Delta^{\mathcal{I}_\rho}| \leq \mathfrak{b}$ and the depth of $T_{\mathcal{I}_\rho}$ is not greater than $\mathfrak{d}$.

```
procedure Gen(𝔡 : ℕ, 𝔟 : binary)
    𝔟 := 𝔟 − 1
    non-deterministically choose a subset 𝒫 of N_pr^T̂
    if (𝔡 ≠ 0) and (𝔟 ≠ 0) then
        for all r ∈ N_R do
            non-deterministically choose 0 ≤ 𝔟_r ≤ 𝔟
            𝔟 := 𝔟 − 𝔟_r
            for all 1 ≤ i ≤ 𝔟_r do
                non-deterministically choose 0 ≤ 𝔟_r^i ≤ 𝔟
                𝔟 := 𝔟 − 𝔟_r^i
                Gen(𝔡 − 1, 𝔟_r^i + 1)
            end for
        end for
    end if
end procedure
```

Note that each recursive call decreases the value of $\mathfrak{d}$, and therefore it is a *terminating* procedure executing at most $\mathfrak{d}$ nested recursive calls. Moreover, as evidenced by the parameter declaration $\mathfrak{b}$: *binary*, *Gen* uses the binary representation of the value $\mathfrak{b}$ (similarly for the variables $\mathfrak{b}_r$ and $\mathfrak{b}_r^i$). Finally, the set of variables $\mathfrak{b}_r$ and $\mathfrak{b}_r^i$ can be reduced to two variables since they are only used within the scope of the **for** loops. Therefore, each run of *Gen* uses space polynomial in $\mathfrak{d}$ and the number of bits needed to represent $\mathfrak{b}$.

The general idea of the procedure is as follows: each recursive call represents an individual of $\Delta^{\mathcal{I}_\rho}$ and the recursion tree lays out the tree-shaped form of $\mathcal{I}_\rho$. The set $\mathcal{P}$ contains the primitive concept names that a domain element is an instance of, the number $\mathfrak{b}_r$ stands for the number of r-successors, and $\mathfrak{b}_r^i$ means that the interpretation rooted at the i-th r-successor has at most $\mathfrak{b}_r^i + 1$ elements. Figure 2 shows a run ρ of *Gen* and its induced interpretation $\mathcal{I}_\rho$. In fact, up to isomorphism $\simeq$, every interpretation satisfying the size and depth constraints imposed by $\mathfrak{b}$ and $\mathfrak{d}$ is generated by such a run.

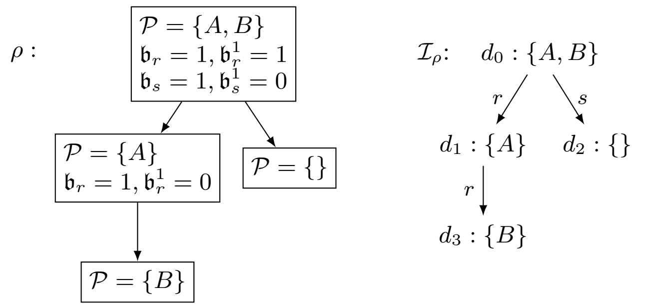


Figure 2. A run ρ of *Gen* on $(2, 4)$ and its induced interpretation $\mathcal{I}_\rho$.

Lemma 21 *Let $\mathfrak{d} \geq 0$ and $\mathfrak{b} > 0$ be two natural numbers. For all tree-shaped interpretations $\mathcal{I}$ over $\mathsf{N}_{pr}^{\widehat{\mathcal{T}}} \cup \mathsf{N}_\mathsf{R}$ with $|\Delta^\mathcal{I}| \leq \mathfrak{b}$ and depth $\leq \mathfrak{d}$, there is a run ρ of Gen on $(\mathfrak{d}, \mathfrak{b})$ such that $\mathcal{I} \simeq \mathcal{I}_\rho$.*

Lemma 21 ensures that, by choosing $\mathfrak{d}$ as in (9) and $\mathfrak{b}$ as $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1))\cdot\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$, the set of runs of *Gen* on $(\mathfrak{d}, \mathfrak{b})$ covers a set of interpretations that suffices to find out whether $u_{\widehat{\mathcal{T}}}(\alpha_1)\sqcap\neg u_{\widehat{\mathcal{T}}}(\alpha_2)$ is satisfiable. Hence, it remains to see how to check for a run ρ of *Gen*, whether $d_0 \in [u_{\widehat{\mathcal{T}}}(\alpha_1)\sqcap\neg u_{\widehat{\mathcal{T}}}(\alpha_2)]^{\mathcal{I}_\rho}$. Fortunately, the unique extension of $\mathcal{I}_\rho$ satisfying $\widehat{\mathcal{T}}$ (recall Proposition 14) can actually be computed along the run ρ. In fact, since $\mathcal{I}_\rho$ is tree-shaped, this extension can be computed in a *bottom-up* manner. Therefore, by doing that we can then simply check whether $d_0 \in (\alpha_1)^{\mathcal{I}_\rho}$ and $d_0 \notin (\alpha_2)^{\mathcal{I}_\rho}$.

To simplify this task, we extend the normal form for $\mathcal{EL}$ TBoxes presented in [1] to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. A TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T})$ is said to be *normalized* iff $\mathcal{T}$ is in normal form, and $\alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau$ implies that $\widehat{C}_\alpha$ has the following structure:

$$\widehat{P}_1 \sqcap \ldots \sqcap \widehat{P}_k \sqcap \exists r_1.\beta_1 \sqcap \ldots \sqcap \exists r_n.\beta_n \tag{10}$$

where $k, n \geq 0$, $\widehat{P}_j$ is either of the form $A \in \mathsf{N}_{pr}^{\widehat{\mathcal{T}}}$ or $E_{\sim t}$ with $E \in \mathsf{N}_d^{\mathcal{T}}$, and $\beta_1, \ldots, \beta_n \in \mathsf{N}_d^{\widehat{\mathcal{T}}}$. As shown in [4], there is a polynomial translation of acyclic $\tau\mathcal{EL}(deg)$ TBoxes into normalized ones that preserves inferences such as satisfiability and subsumption. Thus, we can restrict our attention to TBoxes in normal form. Assuming that $\widehat{\mathcal{T}}$ is normalized, we can transform *Gen* into a function *Gen+* that returns, for each run ρ, a pair (Ex, D) such that

$$\mathsf{Ex} = \{\alpha \mid \alpha \in \mathsf{N}_d^{\widehat{\mathcal{T}}} \text{ and } d_0 \in \alpha^{\mathcal{I}_\rho}\} \tag{11}$$

and $D : \mathsf{N}_d^{\mathcal{T}} \to [0, 1]$ satisfies

$$D(E) = deg^{\mathcal{I}_\rho}(d_0, u_{\mathcal{T}}(E)). \tag{12}$$

Checking whether $d_0 \in A^{\mathcal{I}_\rho}$ for $A \in \mathsf{N}_{pr}^{\widehat{\mathcal{T}}}$ is easy by using the set $\mathcal{P}$. Hence, according to (10), the computation of Ex relies on computing D and verifying if d_0 has an r_i-successor d_i in $\mathcal{I}_\rho$ such that $d_i \in (\beta_i)^{\mathcal{I}_\rho}$. The latter can be done based on the following observations:

- Each successor e of d_0 is the root of an interpretation $\mathcal{I}_{\rho_e}$ induced by a run ρ_e corresponding to a recursive call triggered by ρ.
- The description tree $T_{\mathcal{I}_{\rho_e}}$ is the one rooted at e in $T_{\mathcal{I}_\rho}$.

This means that the set Ex_e computed by ρ_e is also correct for e in the context of $\mathcal{I}_\rho$. Therefore, all the needed information to compute Ex comes in the sets Ex_e returned by ρ_e. To avoid storing (possibly) exponentially many sets Ex_e, *Gen* constructs a binary relation $z : \mathsf{N_R} \times \mathsf{N}_d^{\widehat{\mathcal{T}}}$ such that: $(r, \alpha) \in z$ iff there exists $e \in \Delta^{\mathcal{I}_\rho}$ satisfying $(d_0, e) \in r^{\mathcal{I}_\rho}$ and $e \in \alpha^{\mathcal{I}_\rho}$. The same idea can be used to compute D for d_0, based on the sets D_e returned by the recursive calls. In [4] we show in detail how to extend *Gen* to *Gen+* and show that the tuples (Ex, D) computed by *Gen+* are correct, i.e., they satisfy (11) and (12). Thus, the final non-deterministic algorithm testing satisfiability of $\alpha_1 \sqcap \neg\alpha_2$ w.r.t. $\widehat{\mathcal{T}}$ looks as follows.

Based on the correctness of the computed tuple (Ex, D), it is easy to show that Algorithm 1 is sound and complete. Moreover, since $\mathfrak{b}$ is stored in binary, each run of it uses space polynomial in the size of $\widehat{\mathcal{T}}$. Hence, satisfiability and non-subsumption are in NPSpace. By Savitch's theorem [14] and since PSpace is closed under complement, we thus obtain the following results.

Theorem 22 *In $\tau\mathcal{EL}(deg)$, satisfiability and subsumption w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes are in PSpace.*

The PSpace upper bound carries over to *reasoning w.r.t. acyclic $\tau\mathcal{EL}(deg)$ knowledge bases*. In this setting, the interesting reasoning tasks are *consistency* and *instance checking*. These two reasoning tasks can be reduced to consistency of a knowledge base of the form $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ where $a \in \mathsf{N_I}$ and $\alpha \in \mathsf{N}_d^{\widehat{\mathcal{T}}}$. A PSpace procedure to decide this task can be obtained by extending the ideas used to design *Gen+* and Algorithm 1. The specific details showing how this can be achieved are contained in [4].

Algorithm 1 Satisf. of $\alpha_1 \sqcap \neg\alpha_2$ w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Input: An acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$ and $\alpha_1, \alpha_2 \in \mathsf{def}(\widehat{\mathcal{T}})$.
Output: "yes", if $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable in $\widehat{\mathcal{T}}$, "no" otherwise.

$\mathfrak{b} := \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \cdot \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$ // $\mathfrak{b}$ is stored in binary
$\mathfrak{d} := \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$
$(\mathsf{Ex}, D) :=$ *Gen+*$(\mathfrak{d}, \mathfrak{b})$
if $\alpha_1 \in \mathsf{Ex}$ **and** $\alpha_2 \notin \mathsf{Ex}$ **then**
  return "yes"
end if
return "no"

Theorem 23 *In $\tau\mathcal{EL}(deg)$, consistency and instance checking w.r.t. acyclic $\tau\mathcal{EL}(deg)$ knowledge bases are in PSpace.*

5 CONCLUSION

We have introduced a notion of acyclic TBoxes for $\tau\mathcal{EL}(m)$ such that unfolding still works both from the syntactic and the semantic point of view. For the special case of $\tau\mathcal{EL}(deg)$, we have investigated the complexity of reasoning w.r.t. such acyclic TBoxes. In contrast to the case of $\mathcal{EL}$, in $\tau\mathcal{EL}(deg)$ the presence of acyclic TBoxes increases the complexity.

Regarding future research, we will try to close the gap between Π_2^P/Σ_2^P and PSpace. Unfortunately, it is not clear to us how the construction employed in the hardness proof could be extended to higher levels of the polynomial hierarchy, let alone to PSpace. Conversely, it is also not clear how to generate and test an exponentially large model on some fixed level of the polynomial hierarchy. Another interesting and non-trivial problem is to extend our approach to more general forms of TBoxes (e.g., GCIs). As demonstrated by the semantic problems for unrestricted sets of concept definitions shown in this paper, naive extensions will probably lead to unintuitive results. For example, we have seen that, embedded in a threshold concept, a concept name and its definition need not lead to the same result. We have overcome this problem by modifying the graded membership function using unfolding. For TBoxes that are not acyclic, or do not even consist of concept definitions, this simple solution is not possible. Other interesting open problems are, for instance, to provide an intuitive semantics for nested threshold operators, and to apply our approach of approximately defining concepts to other DLs.

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