Łukasiewicz Fuzzy $\mathcal{EL}$ is Undecidable

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Abstract. Fuzzy Description Logics have been proposed as formalisms for representing and reasoning about imprecise knowledge by introducing intermediate truth degrees. Unfortunately, it has been shown that reasoning in these logics easily becomes undecidable, when infinitely many truth degrees are considered and conjunction is not idempotent. In this paper, we take those results to the extreme, and show that subsumption in fuzzy $\mathcal{EL}$ under Łukasiewicz semantics is undecidable. This provides the first instance of a Horn-style logic with polynomial-time reasoning whose fuzzy extension becomes undecidable.

1 Introduction

An important problem for practical AI applications is to represent and reason with imprecise knowledge in a formal way. Fuzzy Description Logics (FDLs) [3, 18, 22] extend classical DLs with the ideas and tools from Mathematical Fuzzy Logic to try to achieve this goal. The main premise of fuzzy logics is the use of more than two truth degrees to allow a more fine-grained analysis of dependencies between concepts [17]. For instance, a patient having a body temperature of 37.5 °C can have a degree of fever of 0.5, whereas a temperature of 39.2 °C may be interpreted as a fever with degree of 0.9. The so-called *standard semantics* of fuzzy logics uses the rational numbers in the interval $[0, 1]$ as infinitely many truth values. Consider the GCI

$$\exists \mathsf{hasDisease}.\mathsf{Flu} \sqsubseteq \exists \mathsf{hasSymptom}.\mathsf{Headache} \sqcap \exists \mathsf{hasSymptom}.\mathsf{Fever},$$

which may occur in a medical ontology like SNOMED CT.[4] The severity of the symptoms is certainly an indicator for the severity and progression of the disease, which means that truth degrees can be transferred between concepts. However, there are different choices of possible semantics for the logical constructors. The most general semantics are based on *triangular norms (t-norms)* that are used to interpret conjunctions [19]. Among these, the most prominent ones are the *Gödel*,

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[4] http://snomed.org/

Łukasiewicz, and *product* t-norms. All (continuous) t-norms can be expressed as combinations of these three basic ones.

Unfortunately, reasoning in many FDLs becomes undecidable [2,16], with the prominent exception of those based on the Gödel t-norm [6,12,20]; for a systematic overview on this topic, see [1,7]. In this paper, we study a fuzzy extension of the light-weight DL $\mathcal{EL}$ under the Łukasiewicz t-norm semantics. In previous work [5], we have already shown that subsumption checking in this logic is at least ExpTime-hard, but the precise complexity remained unclear. The results in [7] suggest the presence of a negation operator as the culprit for undecidability. However, the minimum expressivity necessary to trigger undecidability was still unknown. Using the framework developed in [7], we show that negation is not needed for undecidability; plain $\mathcal{EL}$ immediately becomes undecidable when endowed with the Łukasiewicz t-norm.

2 Preliminaries

In this section, we briefly recall the extension of $\mathcal{EL}$ under Łukasiewicz semantics, denoted by Ł-$\mathcal{EL}$. Let $\mathsf{N_I}$, $\mathsf{N_C}$, and $\mathsf{N_R}$ be countable sets of *individual*, *concept*, and *role names*, respectively. Ł-$\mathcal{EL}$ *concepts* are built from these sets through the grammar rule $C, D ::= \top \mid A \mid C \sqcap D \mid \exists r.C$, where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. That is, the syntax of Ł-$\mathcal{EL}$ concepts is the same as for classical $\mathcal{EL}$.

Semantically, Ł-$\mathcal{EL}$ differs from $\mathcal{EL}$ by considering all the values in the interval $[0,1]$ as truth degrees. These degrees are managed with the help of the *Łukasiewicz t-norm* ($*_{\text{Ł}}$) and its *residuum* ($\Rightarrow_{\text{Ł}}$), which are defined for every $x, y \in [0,1]$ by:

$$\begin{aligned} x *_{\text{Ł}} y &:= \max\{0, x+y-1\}, \\ x \Rightarrow_{\text{Ł}} y &:= \min\{1, 1-x+y\}. \end{aligned}$$

Some important properties of these operators are given the in following proposition (see [19] for details).

Proposition 1. *For all $x, y, x', y' \in [0,1]$, it holds that*

(a) $x *_{\text{Ł}} y = 1$ *iff both* $x = 1$ *and* $y = 1$;
(b) *if* $x \leq x'$ *and* $y \leq y'$, *then* $x * y \leq x' * y'$;
(c) *if* $x *_{\text{Ł}} x \leq x *_{\text{Ł}} x *_{\text{Ł}} x$, *then either* $x \leq \frac{1}{2}$ *or* $x = 1$;
(d) $x \Rightarrow_{\text{Ł}} y = 1$ *iff* $x \leq y$;
(e) $1 \Rightarrow_{\text{Ł}} y = y$;
(f) $x \Rightarrow_{\text{Ł}} y = \sup\{z \in [0,1] \mid x *_{\text{Ł}} z \leq y\}$.
(g) $x \Rightarrow_{\text{Ł}} 0 = 1 - x$.

A *(fuzzy) interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain*, and $\cdot^{\mathcal{I}}$ is a *fuzzy interpretation function* that assigns

- to each individual name $a \in \mathsf{N_I}$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,
- to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, and

– to each role name $r \in \mathsf{N_R}$ a fuzzy relation $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.

This function is extended to arbitrary concepts by setting, for all $x \in \Delta^{\mathcal{I}}$,

$$\begin{aligned}\top^{\mathcal{I}}(x) &:= 1,\\ (C \sqcap D)^{\mathcal{I}}(x) &:= C^{\mathcal{I}}(x) *_{\text{Ł}} D^{\mathcal{I}}(x),\\ (\exists r.C)^{\mathcal{I}}(x) &:= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) *_{\text{Ł}} C^{\mathcal{I}}(y).\end{aligned}$$

Notice that, according to this semantics, the concepts C and $C \sqcap C$ are not equivalent. We will exploit this fact in the following sections, and often use the abbreviation C^m, $m \geq 1$, for the m-ary conjunction; i.e. $C^1 := C$ and $C^{m+1} := C^m \sqcap C$. We then have that $(C^m)^{\mathcal{I}}(x) = \max\{0, m \cdot C^{\mathcal{I}}(x) - (m-1)\}$.

Fuzzy interpretations are often restricted to be *witnessed* [18], which means that for every existential restriction $\exists r.C$ and $x \in \Delta^{\mathcal{I}}$ there is an element $y \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y) *_{\text{Ł}} C^{\mathcal{I}}(y)$. This follows the intuition that an existential restriction actually forces the existence of a single individual that satisfies it, instead of infinitely many that only satisfy the restriction in the limit. In classical DLs, this property is always satisfied. We also adopt this restriction in the following, and whenever we speak of an interpretation, we implicitly assume that it is witnessed. Likewise, all reasoning problems we investigate are restricted to the class of witnessed interpretations.

In FDLs, axioms are usually assigned a minimum degree of truth to which they must be satisfied. Hence, *(fuzzy) general concept inclusions (GCIs)* are of the form $\langle C \sqsubseteq D \geq p\rangle$, where C and D are concepts and $p \in (0,1]$. A fuzzy interpretation $\mathcal{I}$ *satisfies* this axiom if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq p$ holds for all $x \in \Delta^{\mathcal{I}}$. A TBox is a finite set of GCIs, and a fuzzy interpretation $\mathcal{I}$ *satisfies* a TBox if it satisfies every axiom in it. A *crisp* GCI is of the form $\langle C \sqsubseteq D \geq 1\rangle$, and we usually abbreviate such an axiom by $\langle C \sqsubseteq D\rangle$, which has the semantics that $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ for all $x \in \Delta^{\mathcal{I}}$ (see Proposition 1(d)). We also use $\langle C \equiv D\rangle$ as a short-hand for the two axioms $\langle C \sqsubseteq D\rangle$ and $\langle D \sqsubseteq C\rangle$. Our goal is to analyze the computational complexity of deciding subsumption in fuzzy DLs. A concept C is *subsumed* by a concept D with respect to a TBox $\mathcal{T}$ if every fuzzy interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ also satisfies the GCI $\langle C \sqsubseteq D\rangle$.

We show that subsumption in Ł-$\mathcal{EL}$ is undecidable, using the framework from [7]. Since that framework considers a different reasoning problem, as a first step, we show that *ontology consistency* is undecidable in Ł-$\mathcal{EL}$ if we additionally use equality assertions. In this setting, an *ontology* is a finite set of GCIs and *equality assertions* of the form $\langle a{:}C = p\rangle$, where C is a concept, $p \in [0,1]$, and $a \in \mathsf{N_I}$. An ontology $\mathcal{O}$ is *consistent* if there is an interpretation that satisfies all its GCIs, and satisfies $C^{\mathcal{I}}(a^{\mathcal{I}}) = p$ for each $\langle a{:}C = p\rangle$ in $\mathcal{O}$. For disambiguation, we use the notation Ł-$\mathcal{EL}_=$ to refer to this modified setting. In contrast to classical $\mathcal{EL}$ and Ł-$\mathcal{EL}$, not every Ł-$\mathcal{EL}_=$ ontology is consistent, even though negation and bottom concept are absent in this logic. The reason is that equality in assertions can be viewed as a kind of negation; to see this, consider the simple ontology $\{\langle a{:}\top = 0.5\rangle\}$. Interestingly, we can show that consistency in Ł-$\mathcal{EL}_=$ is undecidable, even if all GCIs are restricted to be crisp.

In Section 5, we show how to modify this result to apply it to subsumption in Ł-$\mathcal{EL}$ (without equality assertions, but with non-crisp GCIs). This is the first instance of undecidability for a fuzzy description logic that does not allow for any negation constructor (and not even $\bot$). Indeed, the required expressivity is a consequence of the properties of the Łukasiewicz t-norm itself.

3 A General Framework for Undecidability

As we will use the framework originally presented in [7] for proving undecidability of FDLs, we recall the necessary definitions. To be precise, we present a slightly adapted version that suffices for the scope of this paper. For the full details of the general framework, we refer the reader to the original work. According to this framework, undecidability of an FDL can be shown by proving that it satisfies several properties, which together allow to construct an ontology that simulates an instance of the undecidable Post Correspondence Problem (PCP) [21]. For this purpose, we consider an arbitrary but fixed instance $\mathcal{P}$ of the PCP, which consists of pairs $(v_1, w_1), \dots, (v_n, w_n)$ of words over an alphabet $\Sigma = \{1, \dots, s\}$ for a natural number $s > 1$. The problem is to find a *solution* of $\mathcal{P}$, which is a finite sequence of the form $i_1 \dots i_k \in \{1, \dots, n\}^*$ such that $v_1 v_{i_1} \dots v_{i_k} = w_1 w_{i_1} \dots w_{i_k}$.[5] For any $\nu \in \{1, \dots, n\}^*$, we denote these words by v_ν and w_ν, respectively.

The first requirement of the framework is to provide an *encoding function* $\mathsf{enc}\colon \Sigma_0^* \to [0, 1]$ that allows us to represent words over the alphabet $\Sigma_0 \coloneqq \Sigma \cup \{0\}$ as truth degrees. For this encoding function to be *valid* (see [7, Definition 11]), there must exist two words $u_\varepsilon, u_+ \in \Sigma_0^*$ such that every candidate solution $\nu \in \{1, \dots, n\}^*$ satisfies the following properties:

- The word $u_\varepsilon \cdot u_+^{|\nu|}$ belongs to $\{\varepsilon\} \cup \Sigma\Sigma_0^*$, that is, it does not start with 0.
- Setting $p = \mathsf{enc}(v_\nu)$, $q = \mathsf{enc}(w_\nu)$, and $m = \mathsf{enc}(u_\varepsilon \cdot u_+^{|\nu|})$, we have

$$v_\nu \neq w_\nu \text{ iff } \min\{p \Rightarrow q, q \Rightarrow p\} \leq m.$$

  This means that one can use the encoding of $u_\varepsilon \cdot u_+^c$ to check the (in)equality of any two words v_ν, w_ν belonging to a candidate solution ν of length c.

Based on this encoding function, the following *canonical model* $\mathcal{I}_\mathcal{P}$ of $\mathcal{P}$ is used to encode the *search tree* for a solution of $\mathcal{P}$, as illustrated in Figure 1:

- the domain $\Delta^{\mathcal{I}_\mathcal{P}} \coloneqq \{1, \dots, n\}^*$ contains all candidate solutions for $\mathcal{P}$;
- we set $a^{\mathcal{I}_\mathcal{P}} \coloneqq \varepsilon$ for a distinguished individual name a that denotes the root node of the search tree;
- $V^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(v_\nu)$ and $W^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(w_\nu)$ represent the words v_ν and w_ν, respectively, of the candidate solution at a node $\nu \in \{1, \dots, n\}^*$;
- $V_i^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(v_i)$ and $W_i^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(w_i)$ for $i \in \{1, \dots, n\}$ encode the words v_i and w_i, respectively, at every node of the search tree;

[5] Without loss of generality, we can restrict to solutions that start with v_1 and w_1.

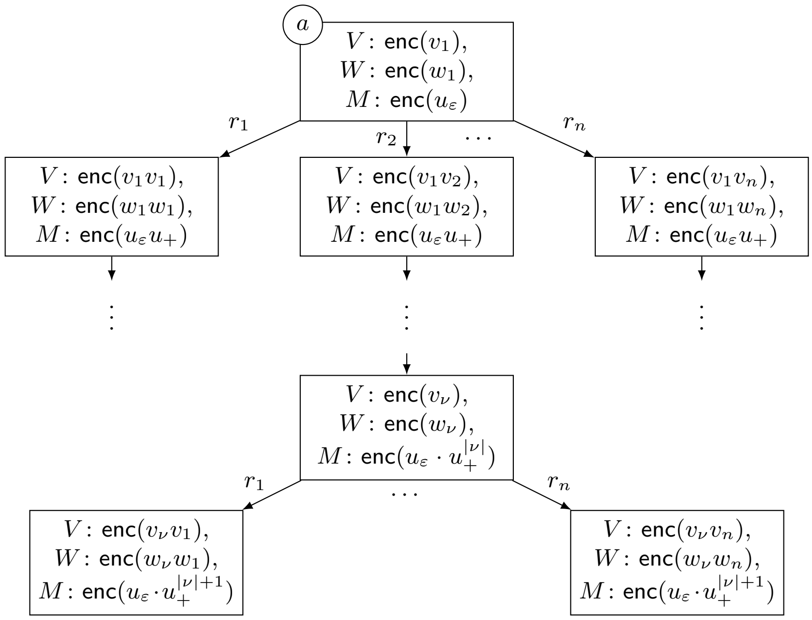

Fig. 1. The canonical model $\mathcal{I}_\mathcal{P}$ for an instance $\mathcal{P}$ of the PCP (taken from [7]).

- $M^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(u_\varepsilon \cdot u_+^{|\nu|})$ and $M_+^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(u_+)$ encode the words used to compare v_ν and w_ν;
- $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu i) \coloneqq 1$ and $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu') \coloneqq 0$ for $\nu' \neq \nu i$ are used to distinguish the successors in the search tree;
- $H^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq h$ holds an auxiliary constant value $h \in [0, 1]$ everywhere.

Strictly speaking, this construction is slightly different from the one described in [7], since the original construction does not contain the concept name H. It is easy to show, however, that this change does not affect the correctness of the approach.

The following property expresses that the logic is capable of constructing the canonical model.

The Canonical Model Property:

There is an ontology $\mathcal{O}_\mathcal{P}$ such that every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ admits a mapping $g \colon \Delta^{\mathcal{I}_\mathcal{P}} \to \Delta^\mathcal{I}$ that satisfies

$$A^{\mathcal{I}_\mathcal{P}}(\nu) = A^\mathcal{I}(g(\nu)) \quad \text{and} \quad H^\mathcal{I}(g(\nu)) = h$$

for every $A \in \{V, W, M, M_+\} \cup \bigcup_{i=1}^n \{V_i, W_i\}$ and $\nu \in \{1, \dots, n\}^*$.

In other words, the ontology $\mathcal{O}_\mathcal{P}$ required by this property enforces that the canonical model can be *embedded* into every interpretation satisfying it. As shown in [7, Theorem 12], the canonical model property is implied by the following four simpler properties, which are used, in that order, to initialize the values of the concept names at the root node, to enforce the existence of the r_i-successors, to construct the encodings of the next candidate solutions $(v_{\nu i}, w_{\nu i})$ by concatenation, and to transfer these encodings along the r_i-connections to the successors.

The Initialization Property:

> Let C be a concept, $a \in \mathsf{N_I}$, and $u \in \Sigma_0^*$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ it holds that $C^{\mathcal{I}}(a^{\mathcal{I}}) = \mathsf{enc}(u)$.

The Successor Property:

> Let $r \in \mathsf{N_R}$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x \in \Delta^{\mathcal{I}}$ with $H^{\mathcal{I}}(x) = h$ there is a $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$ and $H^{\mathcal{I}}(y) = h$.

The Concatenation Property:

> Let $u \in \Sigma_0^*$, and C and C_u be concepts. There is an ontology $\mathcal{O}$ and a concept name D such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x \in \Delta^{\mathcal{I}}$, if $C_u^{\mathcal{I}}(x) = \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, then $D^{\mathcal{I}}(x) = \mathsf{enc}(u'u)$.

The Transfer Property:

> Let C, D be concepts and $r \in \mathsf{N_R}$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x, y \in \Delta^{\mathcal{I}}$, if $H^{\mathcal{I}}(x) = h$, $r^{\mathcal{I}}(x, y) = 1$, $H^{\mathcal{I}}(y) = h$, and $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$ for some $u \in \Sigma_0^*$, then $D^{\mathcal{I}}(y) = \mathsf{enc}(u)$.

In a logic satisfying these four properties, it is possible to create an ontology $\mathcal{O}_{\mathcal{P}}$ whose models all embed the search tree for a solution of a PCP instance $\mathcal{P}$. To obtain undecidability, one further needs to guarantee that the existence of such a solution can be decided. We achieve this through the following property, which intuitively states that no node of the search tree is a solution; thus, the ontology is inconsistent if and only if $\mathcal{P}$ has a solution [7, Theorem 13].

The Solution Property:

> $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, and there is an ontology $\mathcal{O}$ such that:
>
> 1. For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$ and every $\nu \in \{1, \dots, n\}^*$,
>
> $$\min\left\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)),\ W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\right\} \leq M^{\mathcal{I}}(g(\nu)).$$
>
> 2. If for every $\nu \in \{1, \dots, n\}^*$ we have
>
> $$\min\left\{V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu),\ W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)\right\} \leq M^{\mathcal{I}_{\mathcal{P}}}(\nu),$$
>
> then $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$.

4 Consistency is Undecidable

We now consider Ł-$\mathcal{EL}_{=}$, and verify that it satisfies all the properties introduced in the previous section. Thus, as explained before, ontology consistency in this logic is undecidable.

Encoding function. To encode the words for the PCP, we use the function

$$\mathsf{enc}(u) := 1 - \tfrac{1}{2}\big(0.\overleftarrow{u}\big),$$

where $\overleftarrow{u}$ is the word u written in reverse and interpreted as a sequence of digits in base $s+1$, and $0.\overleftarrow{u}$ is the number represented by this sequence of digits when written after the decimal point. For example, if $s = 9$, then we have $\mathsf{enc}(1) = 0.95$ and $\mathsf{enc}(81) = 0.91$. It is an important property of this function that the encoding of every word is always strictly greater than $\frac{1}{2}$; that is, $\mathsf{enc}(u) \in (0.5, 1]$ for all words u.

To see that this encoding function is valid, consider an instance $\mathcal{P}$ of the PCP, and let k be the maximal length of any word v_i, w_i appearing in $\mathcal{P}$. Choose $u_\varepsilon := 1 \cdot 0^k$—that is, the word consisting of the digit 1 followed by k zeros—and $u_+ := 0^k$. It can be verified as in [7, Lemma 14] that the two required conditions hold. In particular, if $v_\nu \neq w_\nu$, then these words must differ in one of the first $K := (|\nu| + 1)k$ symbols. Thus, either $\mathsf{enc}(v_\nu) > \mathsf{enc}(w_\nu)$, and hence

$$\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) = 1 + \tfrac{1}{2}0.\overleftarrow{v_\nu} - \tfrac{1}{2}0.\overleftarrow{w_\nu} \leq 1 - \tfrac{1}{2}0.\overleftarrow{1 \cdot 0^K} = \mathsf{enc}\big(u_\varepsilon \cdot u_+^{|\nu|}\big),$$

or else $\mathsf{enc}(v_\nu) < \mathsf{enc}(w_\nu)$ and $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}\big(u_\varepsilon \cdot u_+^{|\nu|}\big)$.

Initialization property. This property follows trivially from the availability of equality assertions in Ł-$\mathcal{EL}_=$: we can simply set $\mathcal{O} := \{\langle a{:}C = \mathsf{enc}(u)\rangle\}$.

Successor property. For this, we choose $h := \frac{1}{2}$ as the constant for the concept name H, and consider the ontology $\{\langle H \equiv G^2\rangle, \langle G \sqsubseteq \exists r.G\rangle, \langle \exists r.H \sqsubseteq H\rangle\}$. Since we assume that $H^{\mathcal{I}}(x) = \frac{1}{2}$, the first axiom yields that $G^{\mathcal{I}}(x) = \frac{3}{4}$. Then, by the second axiom and the assumption that our interpretations are witnessed, we find an element $y \in \Delta^{\mathcal{I}}$ such that $\frac{3}{4} \leq r^{\mathcal{I}}(x, y) *_{Ł} G^{\mathcal{I}}(y)$. By the third axiom, $r^{\mathcal{I}}(x, y) *_{Ł} (G^2)^{\mathcal{I}}(y) \leq (\exists r.H)^{\mathcal{I}}(x) \leq H^{\mathcal{I}}(x) = \frac{1}{2}$. Since $*_{Ł}$ is monotone (Proposition 1(b)), we get

$$r^{\mathcal{I}}(x, y) *_{Ł} (G^2)^{\mathcal{I}}(y) \leq \tfrac{1}{2} = \tfrac{3}{4} *_{Ł} \tfrac{3}{4} \leq r^{\mathcal{I}}(x, y) *_{Ł} r^{\mathcal{I}}(x, y) *_{Ł} (G^2)^{\mathcal{I}}(y) \qquad (1)$$

This implies that $r^{\mathcal{I}}(x, y) = 1$, since otherwise we would have

$$r^{\mathcal{I}}(x, y) *_{Ł} \big(r^{\mathcal{I}}(x, y) *_{Ł} (G^2)^{\mathcal{I}}(y)\big) \leq r^{\mathcal{I}}(x, y) *_{Ł} \tfrac{1}{2} = \max\{0, r^{\mathcal{I}}(x, y) - \tfrac{1}{2}\} < \tfrac{1}{2},$$

in contradiction to (1). From this, we obtain $H^{\mathcal{I}}(y) = (G^2)^{\mathcal{I}}(y) = \frac{1}{2}$, as required.

Concatenation property. Consider $\mathcal{O} := \{\langle C'^{(s+1)^{|u|}} \equiv C\rangle, \langle D \equiv C' \sqcap C_u\rangle\}$, where C' is a fresh auxiliary concept name. Let $\mathcal{I}$ be a model of $\mathcal{O}$ and $x \in \Delta^{\mathcal{I}}$ such that $C_u^{\mathcal{I}}(x) = \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$. Suppose first that $u' \neq \varepsilon$. Then from the first axiom it follows that both $C^{\mathcal{I}}(x)$ and $C'^{\mathcal{I}}(x)$ belong to the interval $(0, 1)$, and hence $C'^{\mathcal{I}}(x) = 1 - \frac{(s+1)^{-|u|}}{2}0.\overleftarrow{u'}$. If $u \notin \{0\}^*$, then

$$D^{\mathcal{I}}(x) = 1 - \tfrac{1}{2}0.\overleftarrow{u} - \tfrac{(s+1)^{-|u|}}{2}0.\overleftarrow{u'} = \mathsf{enc}(u'u).$$

Otherwise, $C_u^{\mathcal{I}}(x) = 1$ and thus $D^{\mathcal{I}}(x) = C'^{\mathcal{I}}(x) = \mathsf{enc}(u'u)$. If $u' = \varepsilon$, then $C^{\mathcal{I}}(x) = 1$ which implies that $C'^{\mathcal{I}}(x) = 1$ by Proposition 1(a), and hence $D^{\mathcal{I}}(x) = \mathsf{enc}(u) = \mathsf{enc}(\varepsilon u)$.

Transfer property. Consider concepts C, D, a role name r, and let $\overline{C}$ be a fresh concept name. For every model $\mathcal{I}$ of $\langle H \equiv C \sqcap \overline{C}\rangle$ and every $x \in \Delta^{\mathcal{I}}$ with $H^{\mathcal{I}}(x) = \frac{1}{2}$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u) \in (\frac{1}{2}, 1]$, we get

$$C^{\mathcal{I}}(x) + \overline{C}^{\mathcal{I}}(x) - 1 = H^{\mathcal{I}}(x) = \tfrac{1}{2},$$

and hence $\overline{C}^{\mathcal{I}}(x) = \frac{3}{2} - C^{\mathcal{I}}(x) \in$ and $C^{\mathcal{I}}(x) = \frac{3}{2} - \overline{C}^{\mathcal{I}}(x)$. That is, $\overline{C}$ simulates an involutive negation of C. Since $H^{\mathcal{I}}(y) = \frac{1}{2}$, we can similarly use the axiom $\langle H \equiv D \sqcap \overline{D}\rangle$ to simulate the involutive negation of D at y, i.e. $\overline{D}^{\mathcal{I}}(y) = \frac{3}{2} - D^{\mathcal{I}}(y)$.

Consider now the axioms $\langle \exists r.D \sqsubseteq C\rangle$ and $\langle \exists r.\overline{D} \sqsubseteq \overline{C}\rangle$. The first axiom implies that $D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x, y) *_{\text{Ł}} D^{\mathcal{I}}(y) \leq (\exists r.D)^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$. From the second axiom, we similarly get that $\frac{3}{2} - D^{\mathcal{I}}(y) = \overline{D}^{\mathcal{I}}(y) \leq \overline{C}^{\mathcal{I}}(x) = \frac{3}{2} - C^{\mathcal{I}}(x)$, and thus $D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x) = \mathsf{enc}(u)$.

We obtain the required ontology $\mathcal{O}_{\mathcal{P}}$ by collecting all the axioms introduced for the four properties described above.

Solution property. The first part of this property, namely that $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$, can easily be verified as in [7]. The remaining two conditions again require a more intricate proof. Consider the ontology

$$\begin{aligned} \mathcal{O} := \{&\langle X^2 \sqsubseteq X^3\rangle,\ \langle H \equiv X \sqcap \overline{X}\rangle, && (2)\\ &\langle X^2 \sqcap V \sqsubseteq W \sqcap M\rangle, && (3)\\ &\langle \overline{X}^2 \sqcap W \sqsubseteq V \sqcap M\rangle\} && (4) \end{aligned}$$

Since $H^{\mathcal{I}}(g(\nu)) = \frac{1}{2}$, the canonical model property implies that for every model $\mathcal{I}$ of (2) it holds that $X^{\mathcal{I}}(g(\nu)) \in \{\frac{1}{2}, 1\}$ and $\overline{X}^{\mathcal{I}}(g(\nu)) \in \{\frac{1}{2}, 1\}$ (see Proposition 1(c)). Furthermore, X and $\overline{X}$ complement each other, i.e. $X^{\mathcal{I}}(g(\nu)) = \frac{1}{2}$ iff $\overline{X}^{\mathcal{I}}(g(\nu)) = 1$.

Let $\mathcal{I}$ be a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$ and $\nu \in \{1, \ldots, n\}^*$. If $X^{\mathcal{I}}(g(\nu)) = 1$, then the axiom (3) implies that $V^{\mathcal{I}}(g(\nu)) \leq W^{\mathcal{I}}(g(\nu)) *_{\text{Ł}} M^{\mathcal{I}}(g(\nu))$, while (4) is trivially satisfied since $(\overline{X}^2)^{\mathcal{I}}(g(\nu)) = 0$. Consider again $K = (|\nu|+1)k$ from above. Since $|w_\nu| \leq K$, we have

$$W^{\mathcal{I}}(g(\nu)) *_{\text{Ł}} M^{\mathcal{I}}(g(\nu)) = 1 - \tfrac{1}{2}0.\overleftarrow{w_\nu} - \tfrac{1}{2}0.\overleftarrow{1 \cdot 0^K} \in (\tfrac{1}{2}, 1).$$

Since $*_{\text{Ł}}$ is strictly increasing in that interval, for any $z > M^{\mathcal{I}}(g(\nu))$ we know that $W^{\mathcal{I}}(g(\nu)) * z > V^{\mathcal{I}}(g(\nu))$. By Proposition 1(f), we obtain

$$\begin{aligned} W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) &= \sup\{z \in [0,1] \mid W^{\mathcal{I}}(g(\nu)) *_{\text{Ł}} z \leq V^{\mathcal{I}}(g(\nu))\}\\ &\leq \inf\{z \in [0,1] \mid z > M^{\mathcal{I}}(g(\nu))\} = M^{\mathcal{I}}(g(\nu)). \end{aligned}$$

Dually, if $X^{\mathcal{I}}(g(\nu)) = \frac{1}{2}$, then $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$. Thus,

$$\min\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)),\ W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\} \leq M^{\mathcal{I}}(g(\nu)).$$

Assume now that $\min\{V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu), W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\} \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$, and let $\mathcal{I}$ be an extension of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$. We show that $\mathcal{I}$ can be extended to satisfy $\mathcal{O}$. We only need to provide the adequate interpretation of the concept names X and $\overline{X}$ on the elements $\nu \in \{1, \ldots, n\}^*$. If $V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu) = 1$, we set $X^{\mathcal{I}}(\nu) := 1$, which requires $\overline{X}^{\mathcal{I}}(\nu) := \frac{1}{2}$ and trivially satisfies (4). We must then have $W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$, which shows that (3) is also satisfied. In the remaining case, setting $X^{\mathcal{I}}(\nu) := \frac{1}{2}$ yields the desired result.

Undecidability now follows from [7, Theorem 13]; note that in the above constructions we used only crisp GCIs.

Theorem 2. *Ontology consistency in Ł-$\mathcal{EL}_=$ is undecidable. This undecidability result holds even if all GCIs are crisp.*

We emphasise here that Theorem 2 improves on previous undecidability results for fuzzy DLs, where the presence of some kind of negation concept constructor was always required.

5 Subsumption is Undecidable

We now turn our attention to the problem of deciding subsumption between two concepts. Notice first that we can use the results from the previous section to show directly that subsumption in Ł-$\mathcal{EL}_=$ is undecidable. Indeed, consider an arbitrary Ł-$\mathcal{EL}_=$ ontology $\mathcal{O}$ and let A be a concept name not appearing in $\mathcal{O}$. Then, $\top$ is subsumed by A w.r.t. $\mathcal{O}$ iff $\mathcal{O}$ is inconsistent. We are interested, however, in the problem of deciding subsumption w.r.t. a TBox, without any assertions.

We now use Theorem 2 to show that subsumption in Ł-$\mathcal{EL}$ is also undecidable. Notice first that in the construction from Section 4, the equality assertions are only used for the initialization property. In the overall proof of undecidability from [7], this property is used to ensure that a can serve as the root of the search tree for $\mathcal{P}$, which requires initializing the interpretation of several concept names at a (see Figure 1). Using this insight, we show that undecidability arises already if *only one* equality assertion is allowed in the ontology $\mathcal{O}_\mathcal{P}$. It suffices to show that the initialization property can be obtained using one fixed equality assertion. However, in the following we also use a single non-crisp GCI.

Lemma 3. *Given a concept C and $u \in \Sigma_0^*$, there exists a TBox $\mathcal{T}$ such that for every model $\mathcal{I}$ of $\mathcal{T} \cup \{\langle a{:}Y = \frac{1}{2}\rangle\}$ it holds that $C^{\mathcal{I}}(a^{\mathcal{I}}) = \mathsf{enc}(u)$.*

Proof. For any model $\mathcal{I}$ of $\mathcal{T}_0 := \{\langle Y^2 \sqsubseteq Y^3\rangle, \langle \top \sqsubseteq H \geq \frac{1}{2}\rangle, \langle H \equiv Y \sqcap \overline{Y}\rangle\}$ and $\langle a{:}Y = \frac{1}{2}\rangle$, it holds that $\frac{1}{2} \leq H^{\mathcal{I}}(a^{\mathcal{I}}) \leq (Y \sqcap \overline{Y})^{\mathcal{I}}(a^{\mathcal{I}}) \leq Y^{\mathcal{I}}(a^{\mathcal{I}}) = \frac{1}{2}$. In particular, this initializes the concept name H as desired. In addition, we have that $\overline{Y}^{\mathcal{I}}(a^{\mathcal{I}}) = 1$. To ensure $C^{\mathcal{I}}(a^{\mathcal{I}}) \in \mathsf{enc}(u)$, let $\mathcal{T} := \mathcal{T}_0 \cup \{\langle H \equiv A^{(s+1)^{|u|}}\rangle, \langle \overline{Y}^2 \sqcap C \equiv \overline{Y}^2 \sqcap A^{\overleftarrow{u}}\rangle\}$, where A is an auxiliary concept name. The first axiom implies that $(s+1)^{|u|}(A^{\mathcal{I}}(a^{\mathcal{I}})-1)+1 = \frac{1}{2}$, and thus $A^{\mathcal{I}}(a^{\mathcal{I}}) = 1-\frac{1}{2(s+1)^{|u|}}$. Since $(\overline{Y}^2)^{\mathcal{I}}(a^{\mathcal{I}}) = 1$, the second axiom entails that either $C^{\mathcal{I}}(a^{\mathcal{I}})$ and $(A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}})$ are both equal to 1, or $C^{\mathcal{I}}(a^{\mathcal{I}}) = (A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}}) < 1$. If $u \in \{0\}^*$, then $A^{\overleftarrow{u}}$ is equivalent to $\top$, and hence we get $C^{\mathcal{I}}(a^{\mathcal{I}}) = 1 = \mathsf{enc}(u)$. Otherwise, we obtain $C^{\mathcal{I}}(a^{\mathcal{I}}) = (A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}}) = 1 - \frac{1}{2}0.\overleftarrow{u} = \mathsf{enc}(u)$. □

We have hence re-proven the canonical model property and obtained undecidability of consistency in Ł-$\mathcal{EL}_=$ with only one equality assertion ($\langle a{:}Y = \frac{1}{2}\rangle$).

We now use this to prove undecidability of subsumption in Ł-$\mathcal{EL}$.[6] Consider the ontology $\mathcal{O}_{\mathcal{P}}$ used in the new proof of undecidability of Ł-$\mathcal{EL}_=$, and define $\mathcal{T}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P}} \setminus \{\langle a{:}Y = \frac{1}{2}\rangle\}$. Due to the axioms $\langle Y^2 \sqsubseteq Y^3\rangle$, $\langle H \equiv Y \sqcap \overline{Y}\rangle$, and $\langle \top \sqsubseteq H \geq \frac{1}{2}\rangle$, the interpretation of Y in a model of $\mathcal{T}_{\mathcal{P}}$ is always in $\{\frac{1}{2}, 1\}$. Hence, $\mathcal{O}_{\mathcal{P}}$ is consistent iff $\top$ is not subsumed by Y w.r.t. $\mathcal{T}_{\mathcal{P}}$ (since in the latter case there must be a model $\mathcal{I}$ of $\mathcal{T}_{\mathcal{P}}$ such that $Y^{\mathcal{I}}(x) = \frac{1}{2}$ for some domain element $x \in \Delta^{\mathcal{I}}$, i.e. x can serve the function of $a^{\mathcal{I}}$).

Theorem 4. *Subsumption in Ł-$\mathcal{EL}$ is undecidable.*

Interestingly, this provides the first known instance of a Horn-like logic allowing for polynomial-time reasoning, whose fuzzy extension becomes undecidable.

6 Discussion and Related Work

While the results presented in Theorems 2 and 4 are significant by themselves, it is possible to generalize their proofs to infinitely many other continuous t-norms. Specifically, subsumption in $*$-$\mathcal{EL}$ is undecidable for every t-norm $*$ that *contains the Łukasiewicz t-norm*, i.e. is isomorphic to $*_{Ł}$ on some interval $[a, b] \subseteq [0, 1]$, (see [19] for details). For the special case of t-norms *starting with the Łukasiewicz t-norm* (where a is equal to 0) the proofs can be easily adapted by scaling the encoding function to $[0, b]$ and replacing $\frac{1}{2}$ by $\frac{b}{2}$. We can then use the following result from [9, Theorem 13] to extend this result to arbitrary intervals $[a, b]$: If $*$ is isomorphic to $*_1$ on $[0, c]$ and to $*_2$ on $[c, 1]$, then subsumption in $*$-$\mathcal{EL}$ is at least as hard as subsumption in $*_2$-$\mathcal{EL}$. Since any t-norm $*$ that contains $*_{Ł}$ can be decomposed into $*_1$ and $*_2$ as above such that $*_2$ starts with $*_{Ł}$ [19], this shows that subsumption in fuzzy $\mathcal{EL}$ is undecidable for all such t-norms $*$.

These arguments similarly apply to ontology consistency in $*$-$\mathcal{EL}_=$ with crisp GCIs; that is, consistency in all these logics is undecidable. Hence, we have significantly strengthened an undecidability result from [7], which states that $*$-$\mathfrak{N}\mathcal{EL}$ is undecidable if $*$ starts with $*_{Ł}$, where $\mathfrak{N}$ denotes the presence of the negation constructor $\boxminus$ with the semantics $(\boxminus C)^{\mathcal{I}}(x) := C^{\mathcal{I}}(x) \Rightarrow 0$ (see Proposition 1(g)). However, the proof used in that previous work requires only *crisp* assertions of the form $\langle a{:}C = 1\rangle$.

[6] Notice that we are removing the equality assertions at this step.

We now briefly discuss some related work dealing with other semantics for fuzzy DLs.

Product t-norm. In contrast, for the *product t-norm* defined by $x *_{\Pi} y := x \cdot y$, ontology consistency in Π-$\mathfrak{NEL}$ with crisp assertions is decidable. It is still unknown, however, whether equality assertions make consistency in Π-$\mathcal{EL}_=$ or Π-$\mathfrak{NEL}_=$ undecidable. The same question remains open for t-norms that contain $*_{\Pi}$ (but not $*_{\text{Ł}}$). As a partial result, we can adapt our undecidability proof to show that Π-$\mathcal{ELU}_=$ becomes undecidable, where the semantics of the additional disjunction constructor is defined as $(C \sqcup D)^{\mathcal{I}}(x) := 1 - (1 - C^{\mathcal{I}}(x)) \cdot (1 - D^{\mathcal{I}}(x))$. This also slightly strengthens previous results from [7], where this was shown for the negation constructor $(\neg C)^{\mathcal{I}}(x) := 1 - C^{\mathcal{I}}(x)$ (but again with crisp assertions).[7]

Gödel t-norm. According to a known classification of continuous t-norms [19], the only continuous t-norm that we have not yet discussed is the *Gödel t-norm* (or *minimum t-norm*), defined as $x *_{\mathsf{G}} y := \min\{x, y\}$. In this case, it is known that reasoning $\mathcal{EL}$ and even very expressive FDLs is still decidable, and has the same complexity as reasoning in the underlying classical DLs [6, 12, 20].

Finitely valued t-norms. A lot of research has been undertaken on fuzzy extensions of DLs with *finitely valued* t-norms, where the domain of truth degrees is restricted to a finite subset of $[0, 1]$, usually of the form $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ for some $n \geq 2$. In such logics, decidability can be established by either simulating the fuzzy semantics by classical ontologies, or by employing multi-valued extensions of classical reasoning algorithms such as automata- or tableau-based approaches [4, 8, 10, 11, 13–15].

Open problems. The results in this paper show that infinitely-valued fuzzy semantics can make reasoning even in fairly inexpressive logics undecidable. Although the boundaries of decidability and undecidability in fuzzy DLs have been extensively mapped, there are a few remaining open problems. A problem that remains open from this work is whether undecidability in Ł-$\mathcal{EL}$ requires fuzzy GCIs. In addition, other inexpressive DLs like $\mathcal{FL}_0$ or members of the DL-Lite family may provide examples where reasoning remains decidable, despite the use of intermediate truth degrees.

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[7] The details of these proofs are in a manuscript currently under review.

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