To appear in the *International Journal of General Systems*
Vol. 00, No. 00, Month 20XX, 1–33

Probabilistic Implicational Bases in FCA and Probabilistic Bases of GCIs in $\mathcal{EL}^\perp$

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(*Received 00 Month 20XX; accepted 00 Month 20XX*)

A probabilistic formal context is a triadic context the third dimension of which is a set of worlds equipped with a probability measure. After a formal definition of this notion, this document introduces probability of implications with respect to probabilistic formal contexts, and provides a construction for a base of implications the probabilities of which exceed a given lower threshold. A comparison between confidence and probability of implications is drawn, which yields the fact that both measures do not coincide. Furthermore, the results are extended towards the light-weight description logic $\mathcal{EL}^\perp$ with probabilistic interpretations, and a method for computing a base of general concept inclusions the probabilities of which are greater than a pre-defined lower bound is proposed. Additionally, we consider so-called probabilistic attributes over probabilistic formal contexts, and provide a method for the axiomatization of implications over probabilistic attributes.

Keywords: Formal Concept Analysis · Probabilistic Formal Context · Implication · Description Logics · Probabilistic Interpretation · General Concept Inclusion · Probabilistic Attribute

1. Introduction

Most data sets from real-world applications contain errors and noise. Hence, for mining them and axiomatize hidden terminological knowledge from them special techniques are necessary in order to circumvent the expression of the errors. This document focuses on rule mining, especially we attempt to extract rules that are approximately valid in data sets, or in families of data sets, respectively. There are at least two measures for the approximate soundness of rules, namely *confidence* and *probability*. While confidence expresses the number of counterexamples in a single data set, probability expresses the number of data sets in a data set family that do not contain any counterexample. More specifically, we consider implications in the Formal Concept Analysis setting (Ganter and Wille 1999), and general concept inclusions (GCIs) in the Description Logics setting (Baader et al. 2003), in particular in the light-weight description logic $\mathcal{EL}^\perp$.

Firstly, for axiomatizing rules from formal contexts possibly containing wrong incidences or having missing incidences the notion of a partial implication (also called association rule) as well as the notion of confidence were defined by Luxenburger (1993). Furthermore, Luxenburger introduced a method for the computation of a base of all partial implications the confidences of which in a given formal context is above a pre-defined threshold. Borchmann (2014) extended the results to the description logic $\mathcal{EL}^\perp$ by adjusting the notion of confidence to GCIs, and also proposed a method for the construction of a base of confident GCIs for an interpretation. Balcázar (2008) studies different notions of redundancy in sets of partial implications, and shows that there are essentially only two different variants of redundancy.

Furthermore, he presents corresponding constructions for minimal bases of partial implications. Atserias and Balcázar (2015) consider the problem of entailment between partial implications, and provide necessary and sufficient conditions for the entailment problem $\mathcal{L} \models_\gamma X \to Y$, i.e., for the question whether $X \to Y$ has a confidence of at least γ in all those formal contexts in which all implications contained in $\mathcal{L}$ possess a confidence of γ or greater.

Secondly, another perspective considers a family of data sets representing different views of the same domain, e.g., knowledge of different persons, or observations of an experiment that has been repeated several times, since some effects could not be observed in every run. In the field of Formal Concept Analysis, Demin, Ponomaryov, and Vityaev (2011) introduced probabilistic extensions of formal contexts as well as appropriate probabilistic variants of formal concepts and implications, and also provided some methods for their computation. Kriegel (2015a) showed some methods for the computation of a base of GCIs in probabilistic Description Logics where concept and role constructors are available to express probability directly in the concept descriptions. Here, we want to use another approach, and do not allow for probabilistic constructors, but define the notion of a probability of general concept inclusions in the light-weight description logic $\mathcal{EL}^\bot$. Furthermore, we provide a method for the computation of a base of GCIs satisfying a certain lower probability threshold. More specifically, we utilize the description logic $\mathcal{EL}^\bot$ with probabilistic interpretations that have been introduced by Lutz and Schröder (2010). Beforehand, we only consider conjunctions in the language of Formal Concept Analysis, define the notion of a probabilistic formal context in a more general form than by Demin, Ponomaryov, and Vityaev (2011), and provide a technique for the computation of a base of implications satisfying a given lower probability threshold.

At the end of this document, a new approach for the axiomatization of terminological knowledge from probabilistic formal contexts is proposed in Section 8. More specifically, we define the notion of a probabilistic attribute as an expression of the form $\mathsf{d} \geq p.\, B$ which is to be interpreted as having all attributes in B with a probability of at least p. This last section then provides a construction for a base of implications over probabilistic attributes.

The document is structured as follows. Section 2 repeats the basic notions of Formal Concept Analysis, and Section 3 briefly introduces the light-weight description logic $\mathcal{EL}^\bot$. Then, in Section 4 some notions for probabilistic extensions of Formal Concept Analysis are defined, and then in Section 5 a method for the computation of a base for all implications the probabilities of which exceed a given lower threshold in a probabilistic formal context is developed, and its correctness is proven. At the end of this section, a comparison of the notions of confidence and probability is drawn. The following sections then extend the results to the description logic $\mathcal{EL}^\bot$. In particular, Section 6 introduces the basic notions for handling probability in $\mathcal{EL}^\bot$. Section 7 shows a technique for the construction of a base of GCIs fulfilling a lower probability threshold in a probabilistic interpretation. Additionally, Section 8 proposes a technique for the axiomatization of implications over so-called probabilistic attributes.

2. Formal Concept Analysis

This section briefly introduces the standard notions of *Formal Concept Analysis* (abbr. *FCA*) (Ganter and Wille 1999). A *formal context* $\mathbb{K} \coloneqq (G, M, I)$ consists of a set G of *objects* (*Gegenstände* in German), a set M of *attributes* (*Merkmale* in German), and an *incidence relation* $I \subseteq G \times M$. For a pair $(g, m) \in I$, we say that g *has* m. The *derivation operators* of $\mathbb{K}$ are the mappings $\cdot^I\colon \wp(G) \to \wp(M)$ and $\cdot^I\colon \wp(M) \to \wp(G)$ such that for each object set $A \subseteq G$, the set A^I contains all attributes that are shared by all objects in A, and dually for each attribute set $B \subseteq M$, the set B^I contains all those objects that have all attributes from B. Formally, we define the derivation operators as follows.

$$A^I \coloneqq \{\, m \in M \mid \forall\, g \in A\colon (g, m) \in I \,\} \quad \text{for object sets } A \subseteq G,$$
$$\text{and} \quad B^I \coloneqq \{\, g \in G \mid \forall\, m \in B\colon (g, m) \in I \,\} \quad \text{for attribute sets } B \subseteq M.$$

$\mathbb{K}_1$	m_1	m_2	m_3	m_4	m_5	m_6	m_7
g_1	×	·	·	·	·	·	·
g_2	·	×	·	·	·	·	·
g_3	·	×	·	·	·	×	·
g_4	·	×	·	·	·	·	·
g_5	·	×	·	·	·	×	·
g_6	·	×	·	·	·	×	·
g_7	×	×	×	·	×	×	·
g_8	·	·	·	·	·	·	·
g_9	·	×	·	·	·	×	·
g_{10}	·	·	·	·	×	·	·
g_{11}	·	×	×	·	·	·	·
g_{12}	·	×	×	·	×	×	×
g_{13}	×	×	×	×	·	·	·

Figure 1. An exemplary formal context $\mathbb{K}_1$

For singleton sets, we may also use the abbreviations $g^I := \{g\}^I$ for all objects $g \in G$, as well as $m^I := \{m\}^I$ for all attributes $m \in M$.

It is well-known (Ganter and Wille 1999) that both derivation operators constitute a so-called *Galois connection* between the powersets $\wp(G)$ and $\wp(M)$, i.e., the following statements hold true for all subsets $A, A_1, A_2 \subseteq G$ and $B, B_1, B_2 \subseteq M$.

(1) $A \subseteq B^I$ if, and only if, $B \subseteq A^I$ if, and only if, $A \times B \subseteq I$
(2) $A \subseteq A^{II}$ (5) $B \subseteq B^{II}$
(3) $A^I = A^{III}$ (6) $B^I = B^{III}$
(4) $A_1 \subseteq A_2$ implies $A_2^I \subseteq A_1^I$ (7) $B_1 \subseteq B_2$ implies $B_2^I \subseteq B_1^I$

For obvious reasons, formal contexts can be represented as binary tables the rows of which are labeled with the objects, the columns of which are labeled with the attributes, and the occurrence of a cross $\times$ in the cell at row g and column m indicates that the object g has the attribute m. As an example, consider the formal context $\mathbb{K}_1$ in Figure 1.

An *intent* of $\mathbb{K}$ is an attribute set $B \subseteq M$ with $B = B^{II}$. The set of all intents of $\mathbb{K}$ is denoted by $\mathsf{Int}(\mathbb{K})$. An *implication* over M is an expression $X \to Y$ where $X, Y \subseteq M$. It is *valid* in $\mathbb{K}$, denoted as $\mathbb{K} \models X \to Y$, if $X^I \subseteq Y^I$, i.e., if each object of $\mathbb{K}$ that possesses all attributes in X also has all attributes in Y. An implication set $\mathcal{L}$ is *valid* in $\mathbb{K}$, denoted as $\mathbb{K} \models \mathcal{L}$, if all implications in $\mathcal{L}$ are valid in $\mathbb{K}$. Furthermore, the relation $\models$ is lifted to implication sets as follows: an implication set $\mathcal{L}$ *entails* an implication $X \to Y$, symbolized as $\mathcal{L} \models X \to Y$, if $X \to Y$ is valid in all formal contexts in which $\mathcal{L}$ is valid. More specifically, $\models$ is called the *semantic entailment relation*.

A *model* of $X \to Y$ is an attribute set $Z \subseteq M$ such that $X \subseteq Z$ implies $Y \subseteq Z$, and we shall then write $Z \models X \to Y$. Of course, then an implication $X \to Y$ is valid in $\mathbb{K}$ if, and only if, for each object $g \in G$, the *object intent* g^I is a model of $X \to Y$. It is furthermore straightforward to verify that the following statements are equivalent.

(1) $X \to Y$ is valid in $\mathbb{K}$.
(2) Each object intent of $\mathbb{K}$ is a model of $X \to Y$.
(3) Each intent of $\mathbb{K}$ is a model of $X \to Y$.
(4) $Y \subseteq X^{II}$.

The equivalence between the first and the last statement indicates that X^{II} is the largest consequence of X in $\mathbb{K}$, i.e., $X \to X^{II}$ is valid in $\mathbb{K}$, and for each strict superset $Z \supsetneq X^{II}$, the implication $X \to Z$ is not valid in $\mathbb{K}$.

Consider an implication set $\mathcal{L} \cup \{X \to Y\} \subseteq \mathsf{Imp}(M)$. A *model* of $\mathcal{L}$ is an attribute set which is a simultaneous model of each implication in $\mathcal{L}$. In particular, each model Z of $\mathcal{L}$ satisfies that for each implication $X \to Y \in \mathcal{L}$, $X \subseteq Z$ implies $Y \subseteq Z$, i.e., Z is a fixed point of the operator

$$Z \mapsto Z^{\mathcal{L}(1)} := Z \cup \bigcup \{\, Y \mid \exists X \colon X \to Y \in \mathcal{L} \text{ and } X \subseteq Z \,\}.$$

The smallest model $Z^{\mathcal{L}}$ of $\mathcal{L}$ that contains Z is obtained by successive exhaustive application of the operator $\cdot^{\mathcal{L}(1)}$, i.e., $Z^{\mathcal{L}} = \bigcup \{\, Z^{\mathcal{L}(n)} \mid n \geq 1 \,\}$ where $Z^{\mathcal{L}(n+1)} := (Z^{\mathcal{L}(1)})^{\mathcal{L}(n)}$ for all $n \geq 1$. Additionally, the following statements are equivalent.

(1) $\mathcal{L}$ entails $X \to Y$.
(2) Each model of $\mathcal{L}$ is a model of $X \to Y$.
(3) $X \to Y$ is valid in all those formal contexts with attribute set M in which $\mathcal{L}$ is valid.
(4) $Y \subseteq X^{\mathcal{L}}$.

We then infer that $X^{\mathcal{L}}$ is the largest consequence of X with respect to the implication set $\mathcal{L}$, i.e., $\mathcal{L}$ entails $X \to X^{\mathcal{L}}$, and for all supersets $Y \supsetneq X^{\mathcal{L}}$, the implication $X \to Y$ does not follow from $\mathcal{L}$.

It was shown that entailment can also be decided *syntactically* by applying *deduction rules* to the implication set $\mathcal{L}$ without the requirement to consider all formal contexts in which $\mathcal{L}$ is valid, or all models of $\mathcal{L}$, respectively. Recall that an implication $X \to Y$ is *syntactically entailed* by an implication set $\mathcal{L}$, denoted as $\mathcal{L} \vdash X \to Y$, if $X \to Y$ can be constructed from $\mathcal{L}$ by the application of *inference axioms*, cf. (Maier 1983, Page 47), which are described as follows.

(F1) *Reflexivity:* $\emptyset \vdash X \to X$
(F2) *Augmentation:* $\{X \to Y\} \vdash X \cup Z \to Y$
(F3) *Additivity:* $\{X \to Y, X \to Z\} \vdash X \to Y \cup Z$
(F4) *Projectivity:* $\{X \to Y \cup Z\} \vdash X \to Y$
(F5) *Transitivity:* $\{X \to Y, Y \to Z\} \vdash X \to Z$
(F6) *Pseudotransitivity:* $\{X \to Y, Y \cup Z \to W\} \vdash X \cup Z \to W$

In the inference axioms above the symbols X, Y, Z, and W, denote arbitrary subsets of the considered set M of attributes. Formally, we define $\mathcal{L} \vdash X \to Y$ if there is a finite sequence of implications $X_0 \to Y_0, \ldots, X_n \to Y_n$ such that the following conditions hold.

(1) For each $i \in \{0, ..., n\}$, there is a subset $\mathcal{L}_i \subseteq \mathcal{L} \cup \{X_0 \to Y_0, \ldots, X_{i-1} \to Y_{i-1}\}$ such that $\mathcal{L}_i \vdash X_i \to Y_i$ matches one of the Axioms F1–F6.
(2) $X_n \to Y_n = X \to Y$.

Often, the Axioms F1, F2, and F6, are referred to as *Armstrong's axioms.* These three axioms constitute a *complete* and *independent* set of inference axioms for entailment, i.e., from it the other Axioms F3–F5 can be derived, and none of them is derivable from the others.

The semantic entailment and the syntactic entailment coincide, i.e., an implication $X \to Y$ is semantically entailed by an implication set $\mathcal{L}$ if, and only if, $\mathcal{L}$ syntactically entails $X \to Y$, cf. (Maier 1983, Theorem 4.1 on Page 50) as well as (Ganter and Wille 1999, Proposition 21 on Page 81). Consequently, we do not have distinguish between both entailment relations $\models$ and $\vdash$ when it is up to decide whether an implication follows from a set of implications.

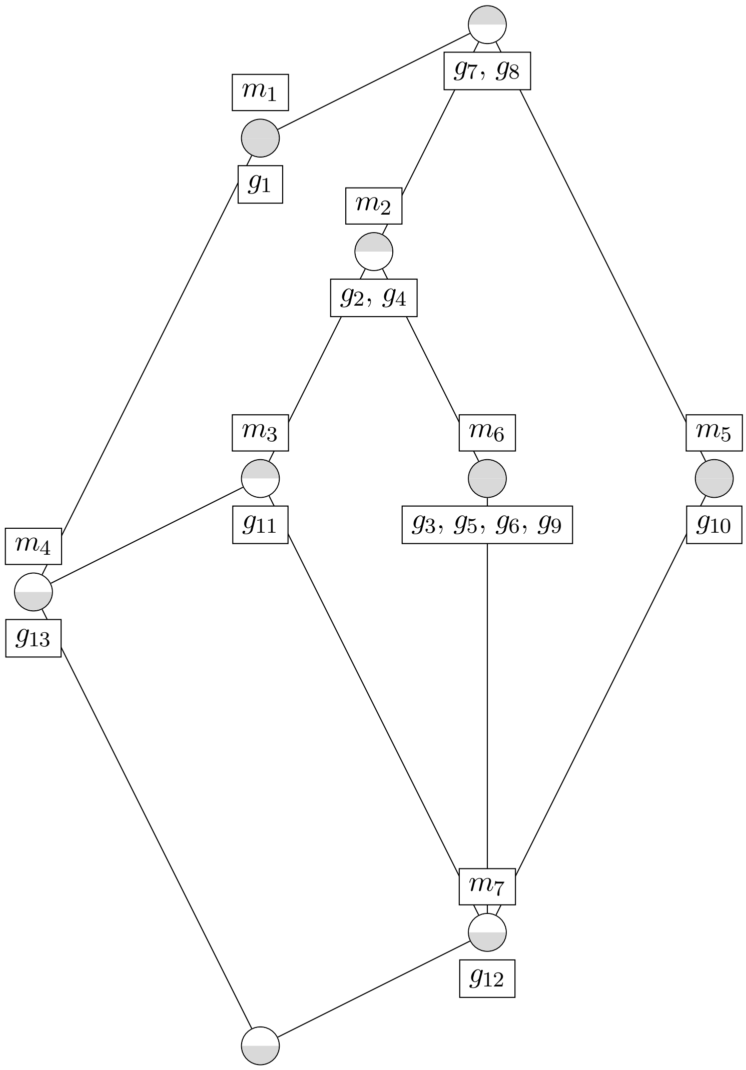


Figure 2. A line diagram of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_1)$ of the exemplary formal context in Figure 1

The data encoded in a formal context can be visualized as a *line diagram* of the corresponding *concept lattice*, which we shall shortly describe. A *formal concept* of a formal context $\mathbb{K} := (G, M, I)$ is a pair (A, B) consisting of a set $A \subseteq G$ of objects as well as a set $B \subseteq M$ of attributes such that $A^I = B$ and $B^I = A$. We then also refer to A as the *extent*, and to B as the *intent*, respectively, of (A, B). Another characterization of a formal concept is as follows: (A, B) is a formal concept of $\mathbb{K}$ if, and only if, $A \subseteq G$, $B \subseteq M$, and both A and B are maximal with respect to the property $A \times B \subseteq I$, i.e., for each strict superset $C \supsetneq A$, $C \times B \nsubseteq I$, and accordingly for each strict superset $D \supsetneq B$, $A \times D \nsubseteq I$. In the denotation of $\mathbb{K}$ as a cross table, those formal concepts are the maximal rectangles full of crosses (modulo reordering of rows and columns). Then, the set of all extents of $\mathbb{K}$ is symbolized as $\mathsf{Ext}(\mathbb{K})$, and the set of all formal concepts of $\mathbb{K}$ is denoted as $\mathfrak{B}(\mathbb{K})$, which is ordered by defining $(A, B) \leq (C, D)$ if, and only if, $A \subseteq C$. It was shown that this order always induces a complete lattice $\underline{\mathfrak{B}}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq, \wedge, \vee, \top, \bot)$, called the *concept lattice* of $\mathbb{K}$, cf. (Wille 1982; Ganter and Wille 1999), in which the infimum and the supremum operation satisfy the equations

$$\bigwedge\{\,(A_t, B_t) \mid t \in T\,\} = \big(\bigcap\{\,A_t \mid t \in T\,\}, (\bigcup\{\,B_t \mid t \in T\,\})^{II}\big),$$

$$\text{and} \quad \bigvee\{\,(A_t, B_t) \mid t \in T\,\} = \big((\bigcup\{\,A_t \mid t \in T\,\})^{II}, \bigcap\{\,B_t \mid t \in T\,\}\big),$$

and where $\top = (\emptyset^I, \emptyset^{II})$ is the greatest element, and where $\bot = (\emptyset^{II}, \emptyset^I)$ is the smallest element, respectively. The number of formal concepts can be exponential in the size of the formal context. Kuznetsov shows that determining this number is a #P-complete problem, cf. (Kuznetsov 2001). Furthermore, the problems of existence of a formal concept with restrictions on the size of the extent, intent, or both, respectively, are investigated in (Kuznetsov 2001) – Kuznetsov demonstrates that the existence of a formal concept (A, B) such that $|A| = k$, $|B| = k$, or $|A| + |B| = k$, respectively, are NP-complete problems; the similar problems with $\geq$ are all in P; and the problems with $\leq$ are also in P, except the problem where $|A| + |B| \leq k$ is NP-complete.

Furthermore, the concept lattice of $\mathbb{K}$ can be nicely represented as a *line diagram* as follows: each formal concept is depicted as a vertex. Furthermore, there is an upward directed edge from each formal concept to its upper neighbors, i.e., to all those formal concepts which are greater with respect to $\leq$, but for which there is no other formal concept in between. The nodes are labeled as follows: an attribute $m \in M$ is an upper label of the *attribute concept* (m^I, m^{II}), and an object $g \in G$ is a lower label of the *object concept* (g^{II}, g^I). Then, the extent of the formal concept represented by a vertex consists of all objects which label vertices reachable by a downward directed path, and dually the intent is obtained by gathering all attribute labels of vertices reachable by an upward directed path. The concept lattice of the exemplary formal context from Figure 1 is depicted in Figure 2.

Let $\mathbb{K} \models \mathcal{L}$. A *pseudo-intent* of a formal context $\mathbb{K}$ relative to an implication set $\mathcal{L}$ is an attribute set $P \subseteq M$ which is no intent of $\mathbb{K}$, but is a model of $\mathcal{L}$, and satisfies $Q^{II} \subseteq P$ for all pseudo-intents $Q \subsetneq P$. The set of all those pseudo-intents is symbolized by $\mathsf{PsInt}(\mathbb{K}, \mathcal{L})$. Then the implication set

$$\mathsf{Can}(\mathbb{K}, \mathcal{L}) := \{\, P \to P^{II} \mid P \in \mathsf{PsInt}(\mathbb{K}, \mathcal{L}) \,\}$$

constitutes an *implication base* of $\mathbb{K}$ relative to $\mathcal{L}$, i.e., for each implication $X \to Y$ over M, the following equivalence is satisfied.

$$\mathbb{K} \models X \to Y \text{ if, and only if, } \mathsf{Can}(\mathbb{K}, \mathcal{L}) \cup \mathcal{L} \models X \to Y$$

$\mathsf{Can}(\mathbb{K}, \mathcal{L})$ is called the *canonical base* of $\mathbb{K}$ relative to $\mathcal{L}$. It can be shown that it is a *minimal* implication base of $\mathbb{K}$ relative to $\mathcal{L}$, i.e., there is no implication base of $\mathbb{K}$ relative to $\mathcal{L}$ with smaller cardinality. Further information is given in (Guigues and Duquenne 1986; Ganter 1984, 2010; Stumme 1996). For the given example of a formal context in Figure 1, the canonical base with respect to the empty background knowledge $\emptyset$ is shown in Figure 3. The most prominent algorithm for computing the canonical base is certainly *NextClosure* developed by Ganter (Ganter 1984, 2010). Bazhanov and Obiedkov propose an optimized version of *NextClosure* in (Bazhanov and Obiedkov 2014) which speeds up the computation of the lectically next closure, and furthermore they then perform some benchmarks to compare both versions. Additionally, they also utilize three different algorithms for computing closures with respect to implication sets, i.e., firstly the already presented and straight-forward algorithm which computes the (least) fixed point of the operator $X \mapsto X^{\mathcal{L}(1)}$, see also (Maier 1983), secondly the *LinClosure* algorithm (Beeri and Bernstein 1979), which computes $X^{\mathcal{L}}$ in linear time, and thirdly *Wild's Closure* algorithm (Wild 1995), which is essentially an improved version of *LinClosure*. Please note that *LinClosure* is not always faster than computing the least fixed point of $X \mapsto X^{\mathcal{L}(1)}$, due to its intialization overhead. Furthermore, Obiedkov and Duquenne constitute an attribute-incremental algorithm for constructing the canonical base, cf. (Obiedkov and Duquenne 2007). A parallel algorithm called *NextClosures* is also available (Kriegel 2015b; Kriegel and Borchmann 2015), and an implementation is provided in *Concept Explorer FX* (Kriegel 2010–2017); its advantage is that its processing time scales almost inverse linear with respect to the number of available CPU cores.

There are some important complexity problems related to the pseudo-intents and canonical bases. Kuznetsov, and later together with Obiedkov, has proven in (Kuznetsov 2004;

$$\mathsf{Can}(\mathbb{K}_1, \emptyset) = \left\{ \begin{array}{r} \{m_1, m_2\} \to \{m_3\}, \\ \{m_1, m_2, m_3, m_4, m_5, m_6\} \to \{m_7\}, \\ \{m_1, m_2, m_3, m_5, m_6, m_7\} \to \{m_4\}, \\ \{m_1, m_5\} \to \{m_2, m_3, m_6\}, \\ \{m_2, m_3, m_6\} \to \{m_5\}, \\ \{m_2, m_5\} \to \{m_3, m_6\}, \\ \{m_3\} \to \{m_2\}, \\ \{m_4\} \to \{m_1, m_2, m_3\}, \\ \{m_6\} \to \{m_2\}, \\ \{m_7\} \to \{m_2, m_3, m_5, m_6\} \end{array} \right\}$$

Figure 3. The canonical base of the exemplary formal context from Figure 1

Kuznetsov and Obiedkov 2006, 2008) that the number of pseudo-intents can be exponential in $|M|$ as well as in $|G| \cdot |M|$ or in $|I|$, and determining this number is $\#\mathsf{P}$-hard, furthermore that recognizing a pseudo-intent is in coNP, and that determining the number of non-pseudo-intents is $\#\mathsf{P}$-complete. Sertkaya and Distel demonstrated in (Sertkaya 2009a,b; Distel 2010; Distel and Sertkaya 2011) that the number of intents can be exponential in the number of pseudo-intents, i.e., the set of pseudo-intents cannot be enumerated in output-polynomial time by utilizing one of the existing algorithms, which all enumerate the closure system of both intents and pseudo-intents, and that the lectically first pseudo-intent can be computed in polynomial time, but recognizing the first n pseudo-intents is coNP-complete. Consequently, the pseudo-intents of a given formal context cannot be enumerated in the lectic order with polynomial delay, unless $\mathsf{P} = \mathsf{NP}$. Enumeration of pseudo-intents (in an arbitrary order) was also investigated, but concrete complexity results are outstanding. Babin and Kuznetsov showed in (Babin and Kuznetsov 2010, 2013) that recognizing a pseudo-intent is coNP-complete, and furthermore that recognizing the lectically largest pseudo-intent is coNP-hard. Hence, computing pseudo-intents in the dual lectic order is also intractable, i.e., not possible with polynomial delay, unless $\mathsf{P} = \mathsf{NP}$. As a corollary Babin and Kuznetsov conclude that the maximal pseudo-intents cannot be enumerated with polynomial delay, unless $\mathsf{P} = \mathsf{NP}$. Further consequences which they found are, for example, that premises of minimal implication bases cannot be tractably recognized, since this problem is coNP-complete, and that there cannot be an algorithm that outputs a random pseudo-intent in polynomial time, unless $\mathsf{NP} = \mathsf{coNP}$.

Eventually, in case a given formal context is not complete in the sense that it does not contain enough objects to refute invalid implications, i.e., only contains some observed objects in the domain of interest, but one aims at exploring all valid implications over the given attribute set, a technique called *Attribute Exploration* can be utilized, which guides the user through the process of axiomatizing an implication base for the underlying domain in a way the number of questions posed to the user is minimal. For a sophisticated introduction as well as for theoretical and technical details, the interested reader is rather referred to (Ganter 1984; Stumme 1996; Ganter 1999, 2010; Kriegel 2016b). A parallel variant of the *Attribute Exploration* also exists, cf. (Kriegel 2015b, 2016b), which is implemented in *Concept Explorer FX* (Kriegel 2010–2017).

3. The Description Logic $\mathcal{EL}^\perp$

This section gives a brief overview on the light-weight description logic $\mathcal{EL}^\perp$ (Baader et al. 2003). First, assume that (N_C, N_R) is a signature, i.e., N_C is a set of *concept names*, and

N_R is a set of *role names*, respectively. Then $\mathcal{EL}^\bot$*-concept descriptions* C over (N_C, N_R) may be constructed according to the following inductive rule (where $A \in N_C$ and $r \in N_R$):

$$C ::= \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C.$$

We shall denote the set of all $\mathcal{EL}^\bot$-concept descriptions over (N_C, N_R) by $\mathcal{EL}^\bot(N_C, N_R)$. Second, the semantics of $\mathcal{EL}^\bot$-concept descriptions is defined by means of interpretations: An *interpretation* is a tuple $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ that consists of a set $\Delta^\mathcal{I}$, called *domain*, and an *extension function* $\cdot^\mathcal{I}\colon N_C \cup N_R \to \wp(\Delta^\mathcal{I}) \cup \wp(\Delta^\mathcal{I} \times \Delta^\mathcal{I})$ that maps concept names $A \in N_C$ to subsets $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and role names $r \in N_R$ to binary relations $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The extension function is extended to all $\mathcal{EL}^\bot$-concept descriptions as follows:

$$\begin{aligned}
\bot^\mathcal{I} &:= \emptyset,\\
\top^\mathcal{I} &:= \Delta^\mathcal{I},\\
(C \sqcap D)^\mathcal{I} &:= C^\mathcal{I} \cap D^\mathcal{I},\\
(\exists\, r.\, C)^\mathcal{I} &:= \{\, d \in \Delta^\mathcal{I} \mid \exists e \in \Delta^\mathcal{I}\colon (d, e) \in r^\mathcal{I} \text{ and } e \in C^\mathcal{I} \,\}.
\end{aligned}$$

A *general concept inclusion (GCI)* in $\mathcal{EL}^\bot$ is of the form $C \sqsubseteq D$ where C and D are $\mathcal{EL}^\bot$-concept descriptions. It is *valid* in an interpretation $\mathcal{I}$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ is satisfied, and we then also write $\mathcal{I} \models C \sqsubseteq D$, and say that $\mathcal{I}$ is a *model* of $C \sqsubseteq D$. Furthermore, C is *subsumed* by D if $C \sqsubseteq D$ is valid in all interpretations, and we shall denote this by $C \sqsubseteq D$, too. A *TBox* is a set of GCIs, and a *model* of a TBox is a model of all its GCIs. A TBox $\mathcal{T}$ *entails* a GCI $C \sqsubseteq D$, denoted by $\mathcal{T} \models C \sqsubseteq D$, if every model of $\mathcal{T}$ is a model of $C \sqsubseteq D$.

Similar to the implicational bases in Formal Concept Analysis, so-called *bases of GCIs* were defined in Description Logic, more specifically for our considered description logic $\mathcal{EL}^\bot$. The first works in this direction are Baader and Distel (2008); Distel (2011). Later, the same problem is considered with respect to a bound on the role depths in Borchmann, Distel, and Kriegel (2016),

A *base of GCIs*, for an interpretation $\mathcal{I}$ *relative to* a TBox $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$ is a TBox $\mathcal{B}$ that satisfies the following equivalence for all general concept inclusions $C \sqsubseteq D$:

$$\mathcal{I} \models C \sqsubseteq D \text{ if, and only if, } \mathcal{B} \cup \mathcal{T} \models C \sqsubseteq D.$$

The *canonical base* for $\mathcal{I}$ relative to $\mathcal{T}$ is defined as follows:

$$\mathsf{Can}(\mathcal{I}, \mathcal{T}) := \{\, \textstyle\bigsqcap P \sqsubseteq \bigsqcap P^{II} \mid P \in \mathsf{PsInt}(\mathbb{K}(\mathcal{I}, \mathcal{T}), \mathcal{L}) \,\}$$

where the formal context is defined by $\mathbb{K}(\mathcal{I}, \mathcal{T}) := (\Delta^\mathcal{I}, \mathcal{M}(\mathcal{I}, \mathcal{T}), I)$ where

$$\mathcal{M}(\mathcal{I}, \mathcal{T}) := \{\bot\} \cup N_C \cup \{\, \exists\, r.\, X^{\mathcal{I}(\mathcal{T})} \mid r \in N_R \text{ and } \emptyset \neq X \subseteq \Delta^\mathcal{I} \,\}$$

and $(d, C) \in I$ if, and only if, $d \in C^\mathcal{I}$. For a subset $X \subseteq \Delta^\mathcal{I}$, a concept description C is called *model-based most specific concept description (mmsc)* of X in $\mathcal{I}$ *relative* to $\mathcal{T}$ if it satisfies the following conditions:

(1) $X \subseteq C^\mathcal{I}$, and
(2) for each concept description D, $X \subseteq D^\mathcal{I}$ implies $\mathcal{T} \models C \sqsubseteq D$.

It is readily verified that – provided the existence – the mmsc of X in $\mathcal{I}$ relative to $\mathcal{T}$ is unique modulo equivalence with respect to $\mathcal{T}$, and thus we shall denote it by $X^{\mathcal{I}(\mathcal{T})}$. Please note that these relative mmsc-s need not exist for all interpretations $\mathcal{I}$ and TBoxes $\mathcal{T}$, in particular if the interpretation $\mathcal{I}$ contains a cycle which is not axiomatized within the

$\mathbb{K}_2(w_1)$	m_1	m_2	m_3
g_1	×	·	×
g_2	·	×	×
g_3	·	·	×

$\mathbb{P}_2(w_1) \coloneqq \frac{1}{2}$

$\mathbb{K}_2(w_2)$	m_1	m_2	m_3
g_1	×	·	×
g_2	·	×	·
g_3	×	·	×

$\mathbb{P}_2(w_2) \coloneqq \frac{1}{3}$

$\mathbb{K}_2(w_3)$	m_1	m_2	m_3
g_1	×	·	×
g_2	·	×	×
g_3	·	×	·

$\mathbb{P}_2(w_3) \coloneqq \frac{1}{6}$

Figure 4. An exemplary probabilistic formal context $\mathbb{K}_2$

TBox $\mathcal{T}$. There are two solutions for guaranteeing the existence of the mmsc-s: On the one hand, we could equip $\mathcal{EL}^\perp$ with *greatest fixpoint semantics (gfp-semantics)* allowing for the expression of cycles within the concept description; on the other hand, we could restrict the *role depth* of the candidate concept descriptions for the mmsc-s. For the simpler case without a TBox, the first method was presented in Baader and Distel (2008); Distel (2011), and the second method was presented in Borchmann, Distel, and Kriegel (2016).

4. Probabilistic Formal Concept Analysis

A *probability measure* $\mathbb{P}$ on a countable set W is a mapping $\mathbb{P}\colon \wp(W) \to [0,1]$ such that $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(W) = 1$, and $\mathbb{P}$ is σ*-additive*, i.e., for all countable families $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets $U_n \subseteq W$ it holds that $\mathbb{P}(\bigcup_{n\in\mathbb{N}} U_n) = \sum_{n\in\mathbb{N}} \mathbb{P}(U_n)$. A world $w \in W$ is *possible* if $\mathbb{P}\{w\} > 0$, and *impossible* otherwise. The set of all possible worlds is denoted by W_ε, and the set of all impossible worlds is denoted by W_0. Obviously, $W_\varepsilon \uplus W_0$ is a partition of W. Of course, such a probability measure can be completely characterized by the definition of the probabilities of the singleton subsets of W, since it holds true that $\mathbb{P}(U) = \mathbb{P}(\bigcup_{w\in U}\{w\}) = \sum_{w\in U} \mathbb{P}(\{w\})$.

Definition 1. *A* probabilistic formal context $\mathbb{K}$ *is a tuple* $(G, M, W, I, \mathbb{P})$ *that consists of a set* G *of* objects*, a set* M *of* attributes*, a countable set* W *of* worlds*, an* incidence relation $I \subseteq G \times M \times W$*, and a probability measure* $\mathbb{P}$ *on* W*. For a triple* $(g, m, w) \in I$ *we say that* object g has attribute m in world w*. Furthermore, we define the* derivations in world w *as operators* $\cdot^{I(w)}\colon \wp(G) \to \wp(M)$ *and* $\cdot^{I(w)}\colon \wp(M) \to \wp(G)$ *where*

$$A^{I(w)} \coloneqq \{\, m \in M \mid \forall g \in A\colon (g, m, w) \in I \,\} \quad \textit{for object sets } A \subseteq G,$$

$$\textit{and} \quad B^{I(w)} \coloneqq \{\, g \in G \mid \forall m \in B\colon (g, m, w) \in I \,\} \quad \textit{for attribute sets } B \subseteq M,$$

i.e., $A^{I(w)}$ *is the set of all common attributes of all objects in* A *in the world* w*, and* $B^{I(w)}$ *is the set of all objects that have all attributes in* B *in* w*. The formal context induced by a world* $w \in W$ *is defined as* $\mathbb{K}(w) \coloneqq (G, M, I(w))$.

As a running example for the up-coming part on probabilistic formal concept analysis, we consider the probabilistic formal context presented in Figure 4. It consists of three objects g_1, g_2, g_3, three attributes m_1, m_2, m_3, and three worlds w_1, w_2, w_3. In $\mathbb{K}_2$ it holds true that the object g_1 has the attribute m_1 in all three worlds, and the object g_3 has the attribute m_3 in all worlds except in w_3.

Definition 2. *Let* $\mathbb{K} = (G, M, W, I, \mathbb{P})$ *be a probabilistic formal context. The* probability *of an implication* $X \to Y$ *over* M *is defined as the measure of the set of worlds it is* *valid in, i.e.,*

$$\mathbb{P}(X \to Y) := \mathbb{P}\{\, w \in W \mid X^{I(w)} \subseteq Y^{I(w)} \,\}.$$

Furthermore, we define the following properties for an implication $X \to Y$*:*

1. $X \to Y$ *is* valid *in world* w *of* $\mathbb{K}$ *if* $X^{I(w)} \subseteq Y^{I(w)}$ *is satisfied.*
2. $X \to Y$ *is* certain *in* $\mathbb{K}$ *if it is valid in all worlds of* $\mathbb{K}$*.*
3. $X \to Y$ *is* almost certain *in* $\mathbb{K}$ *if it is valid in all possible worlds of* $\mathbb{K}$*.*
4. $X \to Y$ *is* possible *in* $\mathbb{K}$ *if it is valid in a possible world of* $\mathbb{K}$*.*
5. $X \to Y$ *is* impossible *in* $\mathbb{K}$ *if it is not valid in any possible world of* $\mathbb{K}$*.*
6. $X \to Y$ *is* refuted *by* $\mathbb{K}$ *if it is not valid in any world of* $\mathbb{K}$*.*

It is readily verified that

$$\mathbb{P}(X \to Y) = \mathbb{P}\{\, w \in W_\varepsilon \mid X^{I(w)} \subseteq Y^{I(w)} \,\} = \sum \{\, \mathbb{P}\{w\} \mid w \in W_\varepsilon \text{ and } X^{I(w)} \subseteq Y^{I(w)} \,\}.$$

An implication $X \to Y$ is almost certain if $\mathbb{P}(X \to Y) = 1$, is possible if $\mathbb{P}(X \to Y) > 0$, and is impossible if $\mathbb{P}(X \to Y) = 0$. If $X \to Y$ is certain, then it is almost certain, and if $X \to Y$ is refuted, then it is impossible.

Returning back to our running example from Figure 4, we can easily verify that the implication $\{m_1\} \to \{m_3\}$ has a probability of 1 in $\mathbb{K}_2$, i.e., is certain. Furthermore, the implication $\{m_2\} \to \{m_3\}$ is possible in $\mathbb{K}_2$, and in particular has a probability of $\mathbb{P}(w_1) + \mathbb{P}(w_3) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, since it is valid in worlds w_1 and w_3. The implication $\{m_1\} \to \{m_2\}$ is refuted by $\mathbb{K}_2$.

5. Implicational Bases for Probabilistic Formal Contexts

At first we introduce the notion of an implicational base with respect to a probability threshold. Then we are going to develop and prove a construction of such bases for probabilistic formal contexts. If the underlying context is finite, then the base is computable.

Definition 3. *Let* $\mathbb{K} = (G, M, W, I, \mathbb{P})$ *be a probabilistic formal context, and* $p \in [0, 1]$ *a threshold. An* implicational base *for* $\mathbb{K}$ *and* p *is an implication set* $\mathcal{B}$ *over* M *that satisfies the following properties:*

1. $\mathcal{B}$ *is* sound *for* $\mathbb{K}$ *and* p*, i.e.,* $\mathbb{P}(X \to Y) \geq p$ *for all implications* $X \to Y \in \mathcal{B}$*, and*
2. $\mathcal{B}$ *is* complete *for* $\mathbb{K}$ *and* p*, i.e., if* $\mathbb{P}(X \to Y) \geq p$*, then* $X \to Y$ *follows from* $\mathcal{B}$*.*

An implicational base is irredundant *if none of its implications follows from the others, and is* minimal *if it has minimal cardinality among all bases for* $\mathbb{K}$ *and* p*.*

It is readily verified that the above definition is a straight-forward generalization of implicational bases (as defined by Ganter and Wille 1999, Definition 37), in particular formal contexts coincide with probabilistic formal contexts having only one possible world, and implications valid in the formal context coincide with implications having probability 1.

We now define a transformation from probabilistic formal contexts to formal contexts. It allows to decide whether an implication is (almost) certain, and furthermore it can be utilized to construct an implicational base for the (almost) certain implications.

Definition 4. *Let* $\mathbb{K}$ *be a probabilistic formal context. The* certain scaling *of* $\mathbb{K}$ *is the formal context* $\mathbb{K}^\times := (G \times W, M, I^\times)$ *where* $((g, w), m) \in I^\times$ *if* $(g, m, w) \in I$*, and the* almost certain scaling *of* $\mathbb{K}$ *is the subcontext* $\mathbb{K}^\times_\varepsilon := (G \times W_\varepsilon, M, I^\times_\varepsilon)$ *of* $\mathbb{K}^\times$*.*

Lemma 5. *Let* $\mathbb{K} = (G, M, W, I, \mathbb{P})$ *be a probabilistic formal context, and let* $X \to Y$ *be an implication. Then the following statements are satisfied:*

1. $X \to Y$ *is certain in* $\mathbb{K}$ *if, and only if,* $X \to Y$ *is valid in* $\mathbb{K}^\times$*.*

$(\mathbb{K}_2)^\times$	m_1	m_2	m_3
(g_1, w_1)	×	·	×
(g_2, w_1)	·	×	×
(g_3, w_1)	·	·	×
(g_1, w_2)	×	·	×
(g_2, w_2)	·	×	·
(g_3, w_2)	×	·	×
(g_1, w_3)	×	·	×
(g_2, w_3)	·	×	×
(g_3, w_3)	·	×	·

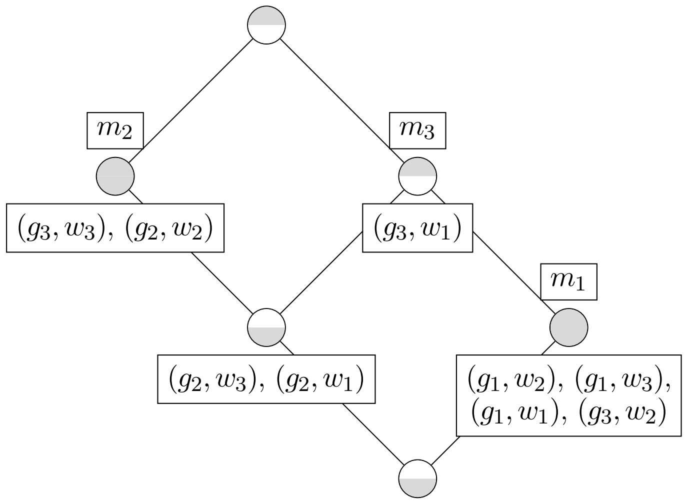


Figure 5. The certain scaling of $\mathbb{K}_2$ from Figure 4 and its concept lattice

(2) $X \to Y$ is almost certain in $\mathbb{K}$ if, and only if, $X \to Y$ is valid in $\mathbb{K}_\varepsilon^\times$.

Proof. The following equivalences can be readily verified:

$$\begin{aligned}\mathbb{P}(X \to Y) = 1 &\Leftrightarrow \forall w \in W\colon X^{I(w)} \subseteq Y^{I(w)} \\ &\Leftrightarrow X^{I^\times} = \biguplus_{w \in W} X^{I(w)} \times \{w\} \subseteq \biguplus_{w \in W} Y^{I(w)} \times \{w\} = Y^{I^\times} \\ &\Leftrightarrow \mathbb{K}^\times \models X \to Y.\end{aligned}$$

The second statement can be proven analogously. □

For our running example $\mathbb{K}_2$, the certain scaling is displayed in Figure 5. As it can be read off it essentially consists of the subposition of the three formal contexts $\mathbb{K}_2(w_1)$, $\mathbb{K}_2(w_2)$, and $\mathbb{K}_2(w_3)$. Due to the non-existence of impossible worlds in $\mathbb{K}_2$, i.e., worlds with a probability of 0, the certain scaling $(\mathbb{K}_2)^\times$ and the almost certain scaling $(\mathbb{K}_2)_\varepsilon^\times$ coincide. We have noted before that the implication $\{m_1\} \to \{m_3\}$ is certain in $\mathbb{K}_2$ – hence according to Lemma 5 it must be valid in $(\mathbb{K}_2)^\times$, which is indeed true.

The next corollary is an immediate consequence of Lemma 5.

Corollary 6. *Let $\mathbb{K}$ be a probabilistic formal context. Then the following statements hold:*

(1) An implicational base for $\mathbb{K}^\times$ is an implicational base for the certain implications of $\mathbb{K}$, in particular this is true for the following implication set:

$$\mathcal{B}(\mathbb{K}) := \{\, P \to P^{I^\times I^\times} \mid P \in \mathsf{PsInt}(\mathbb{K}^\times) \,\}.$$

(2) An implicational base for $\mathbb{K}_\varepsilon^\times$ relative to the background knowledge $\mathcal{B}(\mathbb{K})$ is an implicational base for the almost certain implications of $\mathbb{K}$, in particular this is true for the following implication set:

$$\mathcal{B}(\mathbb{K}, 1) := \mathcal{B}(\mathbb{K}) \cup \{\, P \to P^{I_\varepsilon^\times I_\varepsilon^\times} \mid P \in \mathsf{PsInt}(\mathbb{K}_\varepsilon^\times, \mathcal{B}(\mathbb{K})) \,\}.$$

For the certain scaling $(\mathbb{K}_2)^\times$ in Figure 5, there is only one pseudo-intent, namely $\{m_1\}$. It holds true that $\{m_1\}^{I^\times I^\times} = \{(g_1, w_1), (g_1, w_2), (g_3, w_2), (g_1, w_3)\}^{I^\times} = \{m_1, m_3\}$.

Consequently, we derive the following base for the (almost) certain implications of $\mathbb{K}_2$:

$$\mathcal{B}(\mathbb{K}_2) = \mathcal{B}(\mathbb{K}_2, 1) = \{\{m_1\} \to \{m_3\}\}.$$

Lemma 7. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. Then the following statements are satisfied:*

(1) $Y \subseteq X$ implies that $X \to Y$ is certain in $\mathbb{K}$.
(2) $X_1 \subseteq X_2$ and $Y_1 \supseteq Y_2$ imply $\mathbb{P}(X_1 \to Y_1) \leq \mathbb{P}(X_2 \to Y_2)$.
(3) $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n$ implies $\mathbb{P}(X_0 \to X_n) \leq \bigwedge_{i=1}^{n} \mathbb{P}(X_{i-1} \to X_i)$.

Proof. (1) If $Y \subseteq X$, then it follows that $X^{I(w)} \subseteq Y^{I(w)}$ for all worlds $w \in W$.

(2) Assume $X_1 \subseteq X_2$ and $Y_2 \subseteq Y_1$. Then $X_1^{I(w)} \supseteq X_2^{I(w)}$ and $Y_2^{I(w)} \supseteq Y_1^{I(w)}$ for all worlds $w \in W$. Consider a world $w \in W$ where $X_1^{I(w)} \subseteq Y_1^{I(w)}$. Of course, we may conclude that $X_2^{I(w)} \subseteq Y_2^{I(w)}$. As a consequence we get $\mathbb{P}(X_1 \to Y_1) \leq \mathbb{P}(X_2 \to Y_2)$.

(3) We prove the third claim by induction on n. For $n = 0$ there is nothing to show, and the case $n = 1$ is trivial. Hence, consider $n = 2$ for the induction base, and let $X_0 \subseteq X_1 \subseteq X_2$. Then we have that $X_0^{I(w)} \supseteq X_1^{I(w)} \supseteq X_2^{I(w)}$ is satisfied in all worlds $w \in W$. Now consider a world $w \in W$ where $X_0^{I(w)} \subseteq X_2^{I(w)}$ is true. Of course, it then follows that $X_0^{I(w)} \subseteq X_1^{I(w)} \subseteq X_2^{I(w)}$. Consequently, we conclude $\mathbb{P}(X_0 \to X_2) \leq \mathbb{P}(X_0 \to X_1)$ and $\mathbb{P}(X_0 \to X_2) \leq \mathbb{P}(X_1 \to X_2)$.

For the induction step let $n > 2$. The induction hypothesis yields that

$$\mathbb{P}(X_0 \to X_{n-1}) \leq \bigwedge_{i=1}^{n-1} \mathbb{P}(X_{i-1} \to X_i).$$

Of course, it also holds that $X_0 \subseteq X_{n-1} \subseteq X_n$, and it follows by induction hypothesis and the previous inequality that

$$\mathbb{P}(X_0 \to X_n) \leq \mathbb{P}(X_0 \to X_{n-1}) \wedge \mathbb{P}(X_{n-1} \to X_n) \leq \bigwedge_{i=1}^{n} \mathbb{P}(X_{i-1} \to X_i). \qquad \square$$

Lemma 8. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. Then for all implications $X \to Y$, the following equalities are valid:*

$$\mathbb{P}(X \to Y) = \mathbb{P}(X^{I^\times I^\times} \to Y^{I^\times I^\times}) = \mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}).$$

Proof. Let $X \to Y$ be an implication. Then for all worlds $w \in W$ it holds that

$$g \in X^{I(w)} \Leftrightarrow \forall m \in X\colon (g, m, w) \in I \Leftrightarrow \forall m \in X\colon ((g, w), m) \in I^\times \Leftrightarrow (g, w) \in X^{I^\times},$$

and we conclude that $X^{I(w)} = \pi_1(X^{I^\times} \cap (G \times \{w\}))$. Furthermore, we then infer $X^{I(w)} = X^{I^\times I^\times I(w)}$, and thus the following equations hold:

$$\begin{aligned}\mathbb{P}(X \to Y) &= \mathbb{P}\{\, w \in W \mid X^{I(w)} \subseteq Y^{I(w)} \,\} \\ &= \mathbb{P}\{\, w \in W \mid X^{I^\times I^\times I(w)} \subseteq Y^{I^\times I^\times I(w)} \,\} = \mathbb{P}(X^{I^\times I^\times} \to Y^{I^\times I^\times}).\end{aligned}$$

In particular, for all possible worlds $w \in W_\varepsilon$ it holds that $g \in X^{I(w)} \Leftrightarrow (g, w) \in X^{I_\varepsilon^\times}$, and thus $X^{I(w)} = \pi_1(X^{I_\varepsilon^\times} \cap (G \times \{w\}))$ and $X^{I(w)} = X^{I_\varepsilon^\times I_\varepsilon^\times I(w)}$ are satisfied. It may be concluded that $\mathbb{P}(X \to Y) = \mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times})$. $\square$

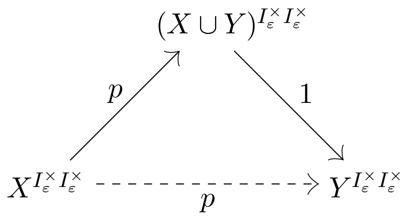
Lemma 9. *Let $\mathbb{K}$ be a probabilistic formal context. Then the following statements hold:*

(1) *If $\mathcal{B}$ is an implicational base for the certain implications of $\mathbb{K}$, then the implication $X \to Y$ follows from $\mathcal{B} \cup \{X^{I^\times I^\times} \to Y^{I^\times I^\times}\}$.*
(2) *If $\mathcal{B}$ is an implicational base for the almost certain implications of $\mathbb{K}$, then the implication $X \to Y$ follows from $\mathcal{B} \cup \{X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}\}$.*

Proof. Of course, the implication $X \to X^{I^\times I^\times}$ is valid in $\mathbb{K}^\times$, i.e., it is certain in $\mathbb{K}$ by Lemma 5, and hence follows from $\mathcal{B}$. Thus, the implication $X \to Y^{I^\times I^\times}$ is entailed by $\mathcal{B} \cup \{X^{I^\times I^\times} \to Y^{I^\times I^\times}\}$, and because of $Y \subseteq Y^{I^\times I^\times}$ the claim follows.
The second statement follows analogously. □

Lemma 10. *For each probabilistic formal context $\mathbb{K}$, the following statements are true:*

(1) $\mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}) = \mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times})$,
(2) $(X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ *is certain in $\mathbb{K}$, and*
(3) $X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ *is entailed by* $\{X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}, (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}\}$.



Proof. First note that $X^{I(w)} \subseteq Y^{I(w)}$ if, and only if, $X^{I(w)} \subseteq X^{I(w)} \cap Y^{I(w)} = (X \cup Y)^{I(w)}$. Hence, the implication $X \to Y$ has the same probability as $X \to X \cup Y$. Consequently, we may conclude by means of Lemma 7 that

$$\mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}) = \mathbb{P}(X \to Y) = \mathbb{P}(X \to X \cup Y) = \mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}).$$

Furthermore, we have that $(X^{I_\varepsilon^\times I_\varepsilon^\times} \cup Y^{I_\varepsilon^\times I_\varepsilon^\times})^{I_\varepsilon^\times I_\varepsilon^\times} = (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}$. As $Y^{I_\varepsilon^\times I_\varepsilon^\times}$ is a subset of $(X^{I_\varepsilon^\times I_\varepsilon^\times} \cup Y^{I_\varepsilon^\times I_\varepsilon^\times})^{I_\varepsilon^\times I_\varepsilon^\times}$, the implication $(X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ is certain in $\mathbb{K}$, cf. Statement 1 in Lemma 7. Obviously, $\{X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}, (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}\}$ entails $X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$. □

Lemma 11. *Let $\mathbb{K}$ be a probabilistic formal context, and X, Y be intents of $\mathbb{K}_\varepsilon^\times$ such that $X \subseteq Y$ and $\mathbb{P}(X \to Y) \geq p$. Then the following statements hold true:*

(1) *There is a chain $X = X_0 \prec X_1 \prec X_2 \prec \ldots \prec X_n = Y$ of neighboring intents in $\mathbb{K}_\varepsilon^\times$,*[1]
(2) $\mathbb{P}(X_{i-1} \to X_i) \geq p$ *for all $i \in \{1, \ldots, n\}$, and*
(3) $X \to Y$ *is entailed by* $\{\, X_{i-1} \to X_i \mid i \in \{1, \ldots, n\} \,\}$.

Proof. The existence of a chain $X = X_0 \prec X_1 \prec X_2 \prec \ldots \prec X_{n-1} \prec X_n = Y$ of neighboring intents between X and Y in $\mathbb{K}_\varepsilon^\times$ follows from $X \subseteq Y$.
From Statement 3 in Lemma 7 it follows that all implications $X_{i-1} \to X_i$ have a probability of at least p in $\mathbb{K}$. It is trivial that they entail $X \to Y$. □

Theorem 12. *Let $\mathbb{K}$ be a probabilistic formal context, and $p \in [0, 1)$ a probability threshold. Then the following implication set is an implicational base for $\mathbb{K}$ and p:*

$$\mathcal{B}(\mathbb{K}, p) := \mathcal{B}(\mathbb{K}, 1) \cup \{\, X \to Y \mid X, Y \in \mathsf{Int}(\mathbb{K}_\varepsilon^\times) \text{ and } X \prec Y \text{ and } \mathbb{P}(X \to Y) \geq p \,\}.$$

[1] For two intents X and Y, we say that X is a *lower intent-neighbor* of Y if $X \subsetneq Y$ and there is no intent between X and Y. Furthermore, we denote this as $X \prec Y$.

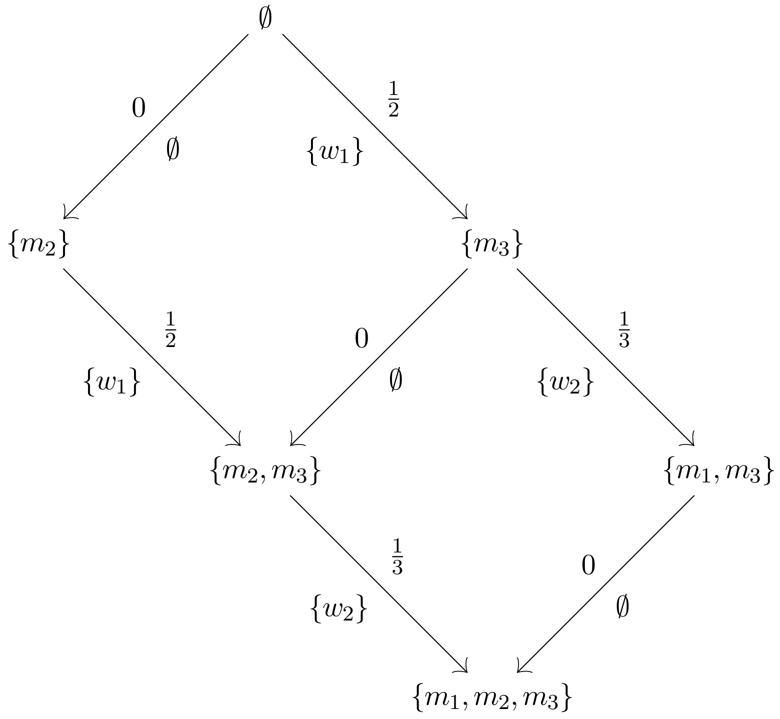


Figure 6. Computation of $\mathcal{B}(\mathbb{K}_2, p) \setminus \mathcal{B}(\mathbb{K}_2, 1)$

Proof. All implications in $\mathcal{B}(\mathbb{K}, 1)$ are almost certain in $\mathbb{K}$, and thus have probability 1. By construction, all other implications $X \to Y$ in the second subset have a probability $\geq p$. Hence, Statement 1 in Definition 3 is satisfied.

Now consider an implication $X \to Y$ over M such that $\mathbb{P}(X \to Y) \geq p$. We have to prove Statement 2 of Definition 3, i.e., that $X \to Y$ is entailed by $\mathcal{B}(\mathbb{K}, p)$.

Lemma 8 yields that both implications $X \to Y$ and $X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ have the same probability. Lemma 9 states that $X \to Y$ follows from $\mathcal{B}(\mathbb{K}, 1) \cup \{X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}\}$. According to Lemma 10, the implication $X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ follows from $\{X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}, (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}\}$. Furthermore, it holds that

$$\mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}) = \mathbb{P}(X^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}) = \mathbb{P}(X \to Y) \geq p,$$

and the second implication $(X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times} \to Y^{I_\varepsilon^\times I_\varepsilon^\times}$ is certain, i.e., follows from $\mathcal{B}(\mathbb{K}, 1)$. Finally, Lemma 11 states that there is a chain of neighboring intents of $\mathbb{K}_\varepsilon^\times$ starting at $X^{I_\varepsilon^\times I_\varepsilon^\times}$ and ending at $(X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}$, i.e.,

$$X^{I_\varepsilon^\times I_\varepsilon^\times} = X_0^{I_\varepsilon^\times I_\varepsilon^\times} \prec X_1^{I_\varepsilon^\times I_\varepsilon^\times} \prec X_2^{I_\varepsilon^\times I_\varepsilon^\times} \prec \ldots \prec X_n^{I_\varepsilon^\times I_\varepsilon^\times} = (X \cup Y)^{I_\varepsilon^\times I_\varepsilon^\times}$$

such that all implications $X_{i-1}^{I_\varepsilon^\times I_\varepsilon^\times} \to X_i^{I_\varepsilon^\times I_\varepsilon^\times}$ have a probability $\geq p$, and are thus contained in $\mathcal{B}(\mathbb{K}, p)$. Hence, $\mathcal{B}(\mathbb{K}, p)$ entails the implication $X \to Y$. □

For the running example $\mathbb{K}_2$ from Figure 4, we are now going to apply Theorem 12 and construct a base for the possible but not certain implications. The concept lattice of the appropriate certain scaling is presented in Figure 5, and the induced implications as well as their probabilities are shown in Figure 6. The figure visualizes the neighborhood relation between the intents of the certain scaling, and the edges now represent implications where the labels below the edges show the worlds in which the corresponding implication is valid, and where the upper labels indicate the probability in $\mathbb{K}_2$. For example, the implication

$\mathbb{K}_3(w_1)$	m_1	m_2
g_1	×	×
g_2	·	×

$\mathbb{P}_3(w_1) := \frac{1}{2}$

$\mathbb{K}_3(w_2)$	m_1	m_2
g_1	×	×
g_2	×	×

$\mathbb{P}_3(w_2) := \frac{1}{2}$

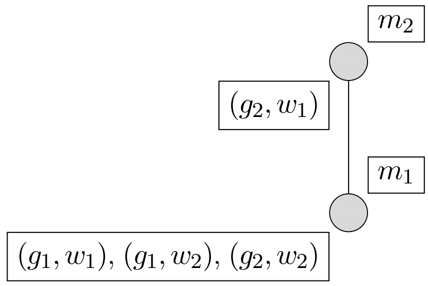


Figure 7. A first counterexample

$\{m_3\} \to \{m_1, m_3\}$ is valid only in world w_2, and hence has a probability of $\frac{1}{3}$. For the threshold of $p := \frac{1}{2}$ we read off the following implicational base for $\mathbb{K}_2$ and p:

$$\begin{aligned}\mathcal{B}(\mathbb{K}_2, \tfrac{1}{2}) &= \mathcal{B}(\mathbb{K}_2, 1) \cup \{\emptyset \to \{m_3\}, \{m_2\} \to \{m_2, m_3\}\} \\ &= \{\{m_1\} \to \{m_3\}, \emptyset \to \{m_3\}, \{m_2\} \to \{m_2, m_3\}\}.\end{aligned}$$

Corollary 13. *Let $\mathbb{K}$ be a probabilistic formal context. Then the following set is an implicational base for the possible implications of $\mathbb{K}$:*

$$\mathcal{B}(\mathbb{K}, \varepsilon) := \mathcal{B}(\mathbb{K}, 1) \cup \{\, X \to Y \mid X, Y \in \mathsf{Int}(\mathbb{K}_\varepsilon^\times) \text{ and } X \prec Y \text{ and } \mathbb{P}(X \to Y) > 0 \,\}.$$

According to the previous Corollary 13, a base for the possible implications of our example $\mathbb{K}_2$ consists of the implications in $\mathcal{B}(\mathbb{K}_2, \frac{1}{2})$, and additionally contains the implications $\{m_3\} \to \{m_1, m_3\}$ and $\{m_2, m_3\} \to \{m_1, m_2, m_3\}$, as those are the remaining implications in Figure 6 that possess a probability exceeding 0.

5.1. *Some Remarks*

However, it is not possible to show irredundancy or minimality for the base of probabilistic implications given above in Theorem 12. Consider the probabilistic formal context $\mathbb{K}_3 := (\{g_1, g_2\}, \{m_1, m_2\}, \{w_1, w_2\}, I, \{\{w_1\} \mapsto \frac{1}{2}, \{w_2\} \mapsto \frac{1}{2}\})$ the incidence relation I of which is defined in Figure 7. The only pseudo-intent of $(\mathbb{K}_3)^\times$ is $\emptyset$, and the concept lattice of $(\mathbb{K}_3)^\times$ is shown above. Hence, we have the following implicational base for $p := \frac{1}{2}$:

$$\mathcal{B}(\mathbb{K}_3, \tfrac{1}{2}) = \{\emptyset \to \{m_2\}, \{m_2\} \to \{m_1, m_2\}\}.$$

However, the set $\mathcal{B} := \{\emptyset \to \{m_1, m_2\}\}$ is also a probabilistic implicational base for $\mathbb{K}_3$ and $\frac{1}{2}$, but has less elements.

In order to compute a minimal base for the implications holding in a probabilistic formal context with a probability $\geq p$, one can for example determine the above given base, and minimize it by means of constructing the Duquenne-Guigues base of it. This either requires the transformation of the implication set into a formal context that has this implication set as an implicational base, or directly compute all pseudo-closures of the closure operator induced by the (probabilistic) implicational base.

Recall that the confidence of an implication $X \to Y$ in a formal context (G, M, I) is defined as

$$\mathsf{conf}(X \to Y) := \frac{\left|(X \cup Y)^I\right|}{|X^I|},$$

$\mathbb{K}_4(w_1)$	m_1	m_2
g_1	×	×
g_2	·	·

$\mathbb{P}_4(w_1) \coloneqq \frac{1}{2}$

$\mathbb{K}_4(w_2)$	m_1	m_2
g_1	·	×
g_2	·	×

$\mathbb{P}_4(w_2) \coloneqq \frac{1}{2}$

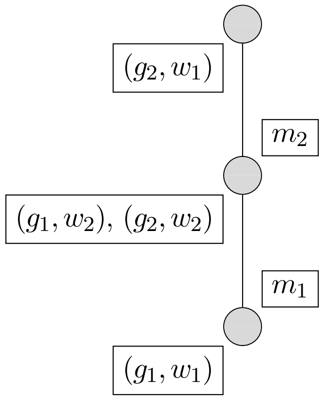


Figure 8. A second counterexample

cf. Luxenburger (1993). In general, there is no correspondence between the probability of an implication in $\mathbb{K}$ and its confidence in $\mathbb{K}^{\times}$ or $\mathbb{K}^{\times}_{\varepsilon}$. To prove this we will provide two counterexamples. As first counterexample we consider the context $\mathbb{K}_3$ above. It is readily verified that $\mathbb{P}_3(\{m_2\} \to \{m_1\}) = \frac{1}{2}$ and $\mathsf{conf}(\{m_2\} \to \{m_1\}) = \frac{3}{4}$, i.e., the confidence is greater than the probability.

Furthermore, consider the modification $\mathbb{K}_4$ in Figure 8 as second counterexample. Then we have that $\mathbb{P}_4(\{m_2\} \to \{m_1\}) = \frac{1}{2}$ and $\mathsf{conf}(\{m_2\} \to \{m_1\}) = \frac{1}{3}$, i.e., the confidence is smaller than the probability.

6. Probabilistic Interpretations for the Description Logic $\mathcal{EL}^{\bot}$

To introduce probability into the description logic $\mathcal{EL}^{\bot}$, we now present the notion of a probabilistic interpretation, as introduced by Lutz and Schröder (2010). It is simply a family of interpretations over the same domain and the same signature, indexed by a set of worlds that is equipped with a probability measure.

Definition 14. *Let (N_C, N_R) be a signature. A* probabilistic interpretation *$\mathcal{I}$ is a tuple $(\Delta^{\mathcal{I}}, (\cdot^{\mathcal{I}(w)})_{w \in W}, W, \mathbb{P})$ consisting of a set $\Delta^{\mathcal{I}}$, called* domain, *a countable set W of* worlds, *a probability measure $\mathbb{P}$ on W, and an* extension function *$\cdot^{\mathcal{I}(w)}$ for each world $w \in W$, i.e., $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}(w)})$ is an interpretation for each $w \in W$.*

For a general concept inclusion $C \sqsubseteq D$ its probability *in $\mathcal{I}$ is defined as follows:*

$$\mathbb{P}(C \sqsubseteq D) \coloneqq \mathbb{P}\{\, w \in W \mid C^{\mathcal{I}(w)} \subseteq D^{\mathcal{I}(w)} \,\}.$$

Furthermore, for a GCI $C \sqsubseteq D$ we define the following properties (as for probabilistic formal contexts):

(1) $C \sqsubseteq D$ is valid *in world w if $C^{\mathcal{I}(w)} \subseteq D^{\mathcal{I}(w)}$.*
(2) $C \sqsubseteq D$ is certain *in $\mathcal{I}$ if it is valid in all worlds.*
(3) $C \sqsubseteq D$ is almost certain *in $\mathcal{I}$ if it is valid in all possible worlds.*
(4) $C \sqsubseteq D$ is possible *in $\mathcal{I}$ if it is valid in a possible world.*
(5) $C \sqsubseteq D$ is impossible *in $\mathcal{I}$ if it is not valid in any possible world.*
(6) $C \sqsubseteq D$ is refuted *by $\mathcal{I}$ if it is not valid in any world.*

It is readily verified that for all GCIs $C \sqsubseteq D$, the following equations are satisfied:

$$\mathbb{P}(C \sqsubseteq D) = \mathbb{P}\{\, w \in W_{\varepsilon} \mid C^{\mathcal{I}(w)} \subseteq D^{\mathcal{I}(w)} \,\} = \sum \{\, \mathbb{P}\{w\} \mid w \in W_{\varepsilon} \text{ and } C^{\mathcal{I}(w)} \subseteq D^{\mathcal{I}(w)} \,\}.$$

Figure 9 provides an exemplary probabilistic interpretation over the signature

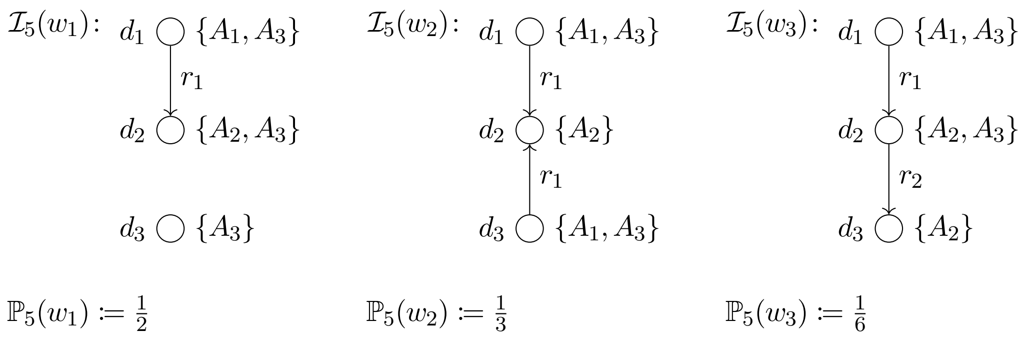


Figure 9. An exemplary probabilistic interpretation $\mathcal{I}_5$

$(\{A_1, A_2, A_3\}, \{r_1, r_2\})$. It essentially consists of the data described by the probabilistic formal context $\mathbb{K}_2$ in Figure 4, but where some of the objects are connected via a role name. For example, the general concept inclusion $A_1 \sqsubseteq A_3 \sqcap \exists\, r_1.\, A_2$ is certain in $\mathcal{I}$, as it is valid in all worlds. The GCI $A_2 \sqcap A_3 \sqsubseteq \exists\, r_2.\, A_2$ is neither certain, nor almost certain, nor impossible, nor refuted in $\mathcal{I}$, but is possible with a probability of $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$, since it is valid in all worlds except in w_1.

7. Bases of GCIs for Probabilistic Interpretations

In what follows we construct from a probabilistic interpretation $\mathcal{I}$ a base of GCIs that entails all GCIs the probability in $\mathcal{I}$ of which exceeds a pre-defined threshold $p \in [0, 1]$.

Definition 15. *Let $\mathcal{I}$ be a probabilistic interpretation, and $p \in [0, 1]$ a threshold. A* base of GCIs *for $\mathcal{I}$ and p is a TBox $\mathcal{B}$ that satisfies the following conditions:*

(1) $\mathcal{B}$ is sound *for $\mathcal{I}$ and p, i.e., $\mathbb{P}(C \sqsubseteq D) \geq p$ for all GCIs $C \sqsubseteq D \in \mathcal{B}$, and*
(2) $\mathcal{B}$ is complete *for $\mathcal{I}$ and p, i.e., if $\mathbb{P}(C \sqsubseteq D) \geq p$, then $\mathcal{B} \models C \sqsubseteq D$.*

A base $\mathcal{B}$ is irredundant *if none of its GCIs follows from the others, and is* minimal *if it has minimal cardinality among all bases of GCIs for $\mathcal{I}$ and p.*

For a probabilistic interpretation $\mathcal{I}$ we define its *certain scaling* as the disjoint union of all interpretations $\mathcal{I}(w)$ with $w \in W$, i.e., as the interpretation $\mathcal{I}^\times \coloneqq (\Delta^\mathcal{I} \times W, \cdot^{\mathcal{I}^\times})$ the extension mapping of which is given as follows:

$$A^{\mathcal{I}^\times} \coloneqq \{\, (d, w) \mid d \in A^{\mathcal{I}(w)} \,\} \qquad \text{for concept names } A \in N_C,$$
$$\text{and} \quad r^{\mathcal{I}^\times} \coloneqq \{\, ((d, w), (e, w)) \mid (d, e) \in r^{\mathcal{I}(w)} \,\} \quad \text{for role names } r \in N_R.$$

Furthermore, the *almost certain scaling* $\mathcal{I}^\times_\varepsilon$ of $\mathcal{I}$ is the disjoint union of all interpretations $\mathcal{I}(w)$ where $w \in W_\varepsilon$ is a possible world. Analogously to Lemma 5, a GCI $C \sqsubseteq D$ is certain in $\mathcal{I}$ if, and only if, it is valid in $\mathcal{I}^\times$, and is almost certain in $\mathcal{I}$ if, and only if, it is valid in $\mathcal{I}^\times_\varepsilon$.

The certain scaling of our illustrative probabilistic interpretation from Figure 9 is presented in Figure 10. Please note that due to the non-existence of impossible worlds, i.e., worlds the probability of which is 0, the certain and the almost certain scaling are equal.

The so-called *model-based most-specific concept descriptions (mmscs)* w.r.t. greatest fixpoint semantics have been defined by Baader and Distel (2008); Distel (2011) as follows: Let $\mathcal{J}$ be an interpretation, and $X \subseteq \Delta^\mathcal{J}$. Then a concept description C is an *mmsc* of X in $\mathcal{J}$, if $X \subseteq C^\mathcal{J}$ is satisfied, and $\emptyset \models C \sqsubseteq D$ for all concept descriptions D with $X \subseteq D^\mathcal{J}$. It is easy to see that all mmscs of X are unique up to equivalence, and hence we denote *the* mmsc of X in $\mathcal{J}$ by $X^\mathcal{J}$. Please note that there is also a role-depth bounded

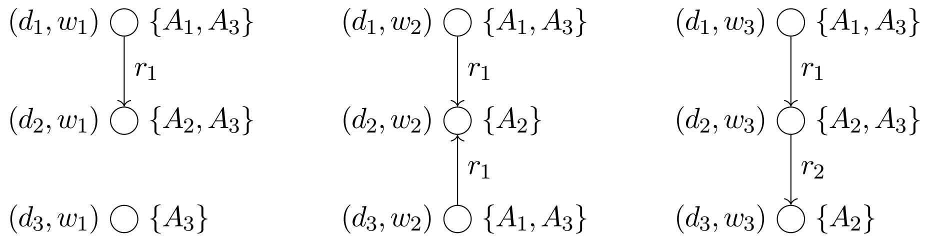


Figure 10. The (almost) certain scaling of the exemplary probabilistic interpretation from Figure 9

variant w.r.t. descriptive semantics as introduced by Borchmann, Distel, and Kriegel (2016).

Lemma 16. *Let $\mathcal{I}$ be a probabilistic interpretation. Then the following statements hold:*

(1) $C^{\mathcal{I}(w)} \times \{w\} = C^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\})$ *for all concept descriptions C and worlds $w \in W$.*
(2) $C^{\mathcal{I}(w)} \times \{w\} = C^{\mathcal{I}^\times_\varepsilon} \cap (\Delta^{\mathcal{I}} \times \{w\})$ *for all concept descriptions C and possible worlds $w \in W_\varepsilon$.*
(3) $\mathbb{P}(C \sqsubseteq D) = \mathbb{P}(C^{\mathcal{I}^\times \mathcal{I}^\times} \sqsubseteq D^{\mathcal{I}^\times \mathcal{I}^\times}) = \mathbb{P}(C^{\mathcal{I}^\times_\varepsilon \mathcal{I}^\times_\varepsilon} \sqsubseteq D^{\mathcal{I}^\times_\varepsilon \mathcal{I}^\times_\varepsilon})$ *for all GCIs $C \sqsubseteq D$.*

Proof. (1) We prove the claim by structural induction on C. By definition, the statement holds for $\bot$, $\top$, and all concept names $A \in N_C$. Consider a conjunction $C \sqcap D$, then

$$\begin{aligned}(C \sqcap D)^{\mathcal{I}(w)} \times \{w\} &= (C^{\mathcal{I}(w)} \cap D^{\mathcal{I}(w)}) \times \{w\} \\ &= C^{\mathcal{I}(w)} \times \{w\} \cap D^{\mathcal{I}(w)} \times \{w\} \\ &\overset{\text{I.H.}}{=} C^{\mathcal{I}^\times} \cap D^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}) \\ &= (C \sqcap D)^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}).\end{aligned}$$

For an existential restriction $\exists\, r.\, C$ the following equalities are satisfied:

$$\begin{aligned}&(\exists\, r.\, C)^{\mathcal{I}(w)} \times \{w\} \\ &= \{\, d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}} \colon (d,e) \in r^{\mathcal{I}(w)} \text{ and } e \in C^{\mathcal{I}(w)} \,\} \times \{w\} \\ &= \{\, (d,w) \mid \exists (e,w) \colon ((d,w),(e,w)) \in r^{\mathcal{I}^\times} \text{ and } (e,w) \in C^{\mathcal{I}(w)} \times \{w\} \,\} \\ &\overset{\text{I.H.}}{=} \{\, (d,w) \mid \exists (e,w) \colon ((d,w),(e,w)) \in r^{\mathcal{I}^\times} \text{ and } (e,w) \in C^{\mathcal{I}^\times} \,\} \\ &= (\exists\, r.\, C)^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}).\end{aligned}$$

(2) analogously.
(3) Using the first statement we may conclude that the following equalities hold:

$$\begin{aligned}&\mathbb{P}(C \sqsubseteq D) \\ &= \mathbb{P}\{\, w \in W \mid C^{\mathcal{I}(w)} \times \{w\} \subseteq D^{\mathcal{I}(w)} \times \{w\} \,\} \\ &= \mathbb{P}\{\, w \in W \mid C^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}) \subseteq D^{\mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}) \,\} \\ &= \mathbb{P}\{\, w \in W \mid C^{\mathcal{I}^\times \mathcal{I}^\times \mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}) \subseteq D^{\mathcal{I}^\times \mathcal{I}^\times \mathcal{I}^\times} \cap (\Delta^{\mathcal{I}} \times \{w\}) \,\} \\ &= \mathbb{P}\{\, w \in W \mid C^{\mathcal{I}^\times \mathcal{I}^\times \mathcal{I}(w)} \times \{w\} \subseteq D^{\mathcal{I}^\times \mathcal{I}^\times \mathcal{I}(w)} \times \{w\} \,\} \\ &= \mathbb{P}(C^{\mathcal{I}^\times \mathcal{I}^\times} \sqsubseteq D^{\mathcal{I}^\times \mathcal{I}^\times}).\end{aligned}$$

The second equality follows analogously. □

For a probabilistic interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, W, \mathbb{P})$ and a set M of $\mathcal{EL}^{\bot}$-concept descriptions we define their *induced context* as the probabilistic formal context $\mathbb{K}(\mathcal{I}, M) \coloneqq (\Delta^{\mathcal{I}}, M, W, I, \mathbb{P})$ where $(d, C, w) \in I$ if, and only if, $d \in C^{\mathcal{I}(w)}$.

Lemma 17. *Let $\mathcal{I}$ be a probabilistic interpretation, M a set of $\mathcal{EL}^{\bot}$-concept descriptions, and $X, Y \subseteq M$. Then the probability of the implication $X \to Y$ in the induced context $\mathbb{K}(\mathcal{I}, M)$ equals the probability of the GCI $\bigsqcap X \sqsubseteq \bigsqcap Y$ in $\mathcal{I}$, i.e., it holds that*

$$\mathbb{P}(X \to Y) = \mathbb{P}(\bigsqcap X \sqsubseteq \bigsqcap Y).$$

Proof. The following equivalences are satisfied for all $Z \subseteq M$ and worlds $w \in W$:

$$d \in Z^{I(w)} \Leftrightarrow \forall C \in Z \colon (d, C, w) \in I \Leftrightarrow \forall C \in Z \colon d \in C^{\mathcal{I}(w)} \Leftrightarrow d \in (\bigsqcap Z)^{\mathcal{I}(w)}.$$

Now consider two subsets $X, Y \subseteq M$, then it holds that

$$\begin{aligned} \mathbb{P}(X \to Y) &= \mathbb{P}\{\, w \in W \mid X^{I(w)} \subseteq Y^{I(w)} \,\} \\ &= \mathbb{P}\{\, w \in W \mid (\bigsqcap X)^{\mathcal{I}(w)} \subseteq (\bigsqcap Y)^{\mathcal{I}(w)} \,\} = \mathbb{P}(\bigsqcap X \sqsubseteq \bigsqcap Y). \end{aligned}$$

□

Generalizing the notion of Baader and Distel (2008); Distel (2011), the *induced probabilistic formal concept* $\mathbb{K}(\mathcal{I})$ of $\mathcal{I}$ is defined as $\mathbb{K}(\mathcal{I}, M(\mathcal{I}))$ with the attribute set

$$M(\mathcal{I}) \coloneqq \{\bot\} \cup N_C \cup \{\, \exists\, r.\, X^{\mathcal{I}_\varepsilon^\times} \mid r \in N_R \text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \times W_\varepsilon \,\}.$$

Please note that the model-based most specific concept descriptions $X^{\mathcal{I}_\varepsilon^\times}$ do not exist with respect to *descriptive semantics* in case of a cyclic interpretation $\mathcal{I}$. In order to circumvent this problem, either the role depth of the mmscs must be restricted, or *greatest fixpoint semantics (gfp-semantics)* have to be used instead. More sophisticated explanations are given in Baader and Distel (2008); Distel (2011) for the case of gfp-semantics, and in Borchmann, Distel, and Kriegel (2016) for the case of bounds on the role depth in descriptive semantics.

Of course, when axiomatizing probable GCIs in $\mathcal{I}$, it is not necessary to compute trivial axioms which are valid in arbitrary interpretations. Hence, we use the following set of *background implications* during the computation of an implicational base of the induced probabilistic formal context $\mathbb{K}(\mathcal{I})$:

$$\mathcal{S}(\mathcal{I}) \coloneqq \{\, \{C\} \to \{D\} \mid C, D \in M(\mathcal{I}) \text{ and } \emptyset \models C \sqsubseteq D \,\}.$$

For an implication set $\mathcal{B}$ over a set M of $\mathcal{EL}^{\bot}$-concept descriptions we define its *induced TBox* by $\bigsqcap \mathcal{B} \coloneqq \{\, \bigsqcap X \sqsubseteq \bigsqcap Y \mid X \to Y \in \mathcal{B} \,\}$.

Corollary 18. *If $\mathcal{B}$ contains an almost certain implicational base for $\mathbb{K}(\mathcal{I})$, then $\bigsqcap \mathcal{B}$ is complete for the almost certain GCIs of $\mathcal{I}$.*

Proof. We know that a GCI is almost certain in $\mathcal{I}$ if, and only if, it is valid in $\mathcal{I}_\varepsilon^\times$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be an almost certain implicational base for $\mathbb{K}(\mathcal{I})$, i.e., an implicational base for $(\mathbb{K}(\mathcal{I}))_\varepsilon^\times = \mathbb{K}(\mathcal{I}_\varepsilon^\times)$. Then according to Distel (2011, Theorem 5.12) it follows that the TBox $\bigsqcap \mathcal{B}'$ is a base of GCIs for $\mathcal{I}_\varepsilon^\times$, i.e., a base for the almost certain GCIs of $\mathcal{I}$. Consequently, $\bigsqcap \mathcal{B}$ is complete for the almost certain GCIs of $\mathcal{I}$. □

The induced formal context of our exemplary probabilistic interpretation $\mathcal{I}_5$ is shown in Figure 11, and its concept lattice is visualized in Figure 12. Therein, we use the following

$(\mathbb{K}(\mathcal{I}_5)^{\times})^{\partial}$	(d_1, w_2)	(d_1, w_3)	(d_1, w_1)	(d_3, w_1)	(d_2, w_3)	(d_2, w_2)	(d_3, w_3)	(d_2, w_1)	(d_3, w_2)
$\bot$	·	·	·	·	·	·	·	·	·
A_1	×	×	×	·	·	·	·	·	×
A_2	·	·	·	·	×	×	×	×	·
A_3	×	×	×	×	×	·	·	×	×
$\exists\, r_1.\, \top$	×	×	×	·	·	·	·	·	×
$\exists\, r_1.\, A_2$	×	×	×	·	·	·	·	·	×
$\exists\, r_1.\, A_3$	·	×	×	·	·	·	·	·	·
$\exists\, r_1.\, C_1$	·	×	×	·	·	·	·	·	·
$\exists\, r_1.\, C_2$	·	·	·	·	·	·	·	·	·
$\exists\, r_1.\, C_3$	·	·	·	·	·	·	·	·	·
$\exists\, r_1.\, C_4$	·	×	·	·	·	·	·	·	·
$\exists\, r_2.\, \top$	·	·	·	·	×	·	·	·	·
$\exists\, r_2.\, A_2$	·	·	·	·	×	·	·	·	·
$\exists\, r_2.\, A_3$	·	·	·	·	·	·	·	·	·
$\exists\, r_2.\, C_1$	·	·	·	·	·	·	·	·	·
$\exists\, r_2.\, C_2$	·	·	·	·	·	·	·	·	·
$\exists\, r_2.\, C_3$	·	·	·	·	·	·	·	·	·
$\exists\, r_2.\, C_4$	·	·	·	·	·	·	·	·	·

Figure 11. The induced formal context of the (almost) certain scaling of the exemplary probabilistic interpretation from Figure 9

abbreviations in order to increase the readability:

$$\begin{aligned} C_1 &:= A_2 \sqcap A_3 \\ C_2 &:= A_1 \sqcap A_3 \sqcap \exists\, r_1.\, A_2 \\ C_3 &:= A_1 \sqcap A_3 \sqcap \exists\, r_1.\, (A_2 \sqcap A_3) \\ C_4 &:= A_2 \sqcap A_3 \sqcap \exists\, r_2.\, A_2 \end{aligned}$$

For the computation of the canonical base of the induced context $\mathbb{K}(\mathcal{I}_5)$, we first determine the trivial general concept inclusions valid between the concept descriptions which are attributes of $\mathbb{K}(\mathcal{I}_5)$, and then use the corresponding implications as background

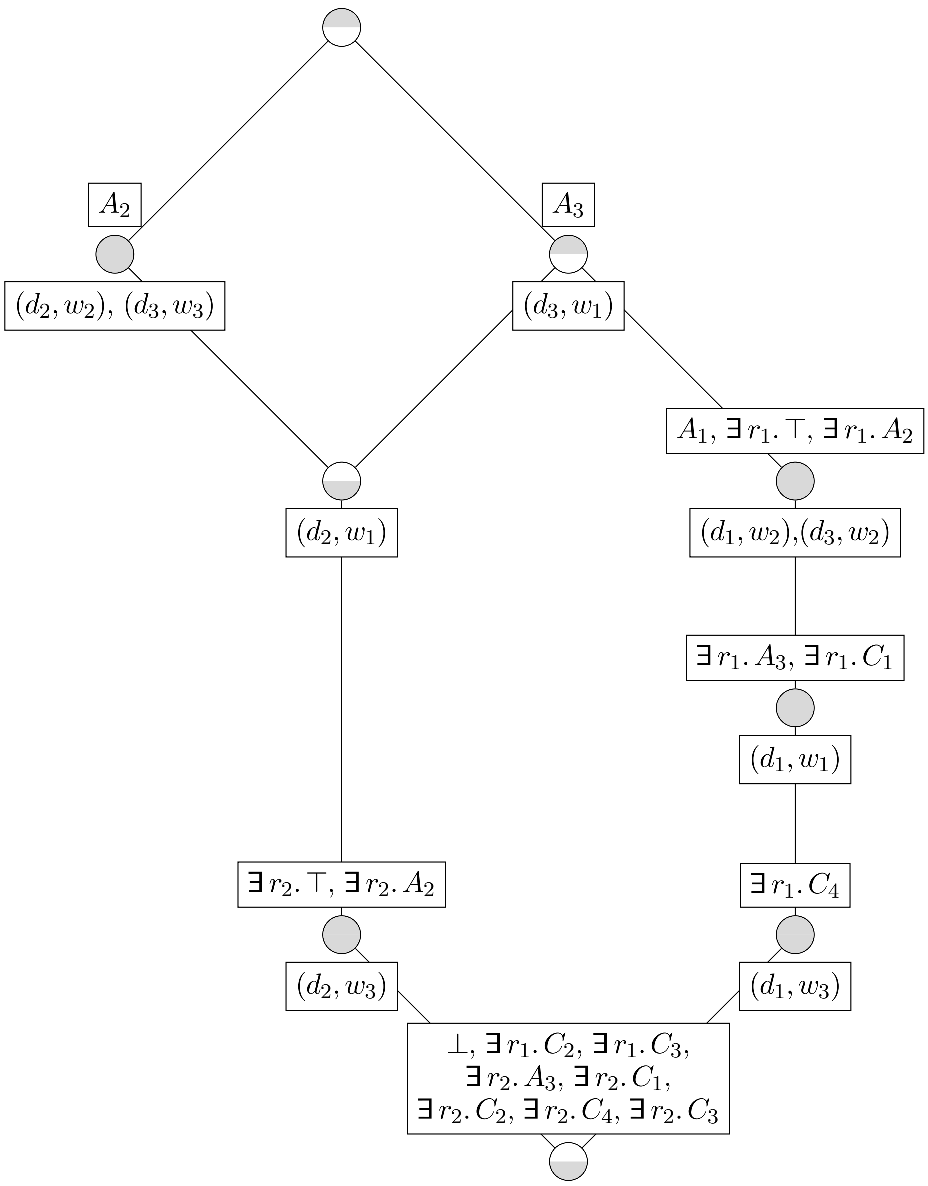
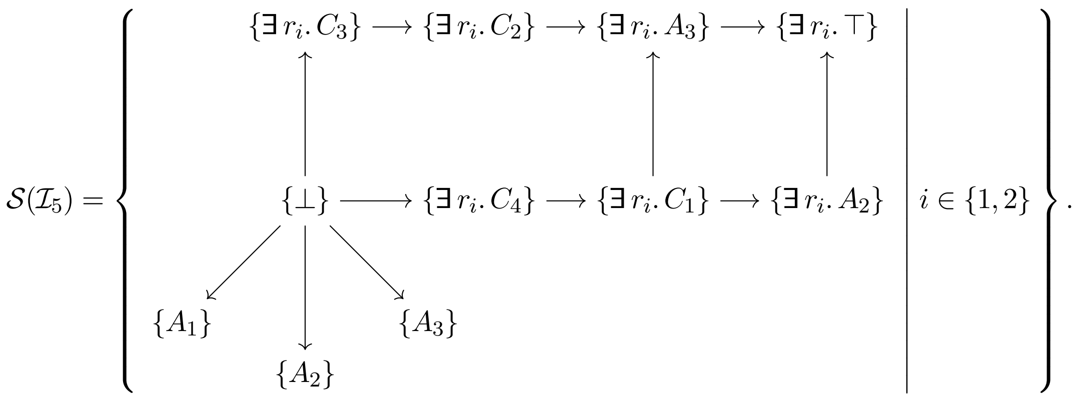


Figure 12. The concept lattice of the induced formal context in Figure 11

knowledge. This background implications are as follows:



Then, the implicational base of $\mathbb{K}(\mathcal{I}_5)^\times$ with respect to the above defined background knowledge $\mathcal{S}(\mathcal{I}_5)$ is constructed as the following set of implications:

$$\mathsf{Can}(\mathbb{K}(\mathcal{I}_5)^\times, \mathcal{S}(\mathcal{I}_5)) = \left\{ \begin{aligned} \{A_1\} &\to \{A_3, \exists\, r_1.\, A_2\}, \\ \{A_1, A_2, A_3, \exists\, r_1.\, A_2\} &\to \{\bot\}, \\ \{A_1, A_3, \exists\, r_1.\, A_2, \exists\, r_1.\, A_3\} &\to \{\exists\, r_1.\, C_1\}, \\ \{A_1, A_3, \exists\, r_1.\, C_2, \exists\, r_1.\, C_1\} &\to \{\bot\}, \\ \{A_2, A_3, \exists\, r_2.\, A_2, \exists\, r_2.\, A_3\} &\to \{\bot\}, \\ \{\exists\, r_1.\, \top\} &\to \{A_1, A_3, \exists\, r_1.\, A_2\}, \\ \{\exists\, r_2.\, \top\} &\to \{A_2, A_3, \exists\, r_2.\, A_2\} \end{aligned} \right\}.$$

Consequently, the following TBox is sound and complete for the certain GCIs of the exemplary probabilistic interpretation $\mathcal{I}_5$:

$$\mathcal{B}(\mathcal{I}_5, 1) = \left\{ \begin{aligned} A_1 &\sqsubseteq A_3 \sqcap \exists\, r_1.\, A_2, \\ A_1 \sqcap A_2 \sqcap A_3 \sqcap \exists\, r_1.\, A_2 &\sqsubseteq \bot, \\ A_1 \sqcap A_3 \sqcap \exists\, r_1.\, A_2 \sqcap \exists\, r_1.\, A_3 &\sqsubseteq \exists\, r_1.\, C_1, \\ A_1 \sqcap A_3 \sqcap \exists\, r_1.\, C_2 \sqcap \exists\, r_1.\, C_1 &\sqsubseteq \bot, \\ A_2 \sqcap A_3 \sqcap \exists\, r_2.\, A_2 \sqcap \exists\, r_2.\, A_3 &\sqsubseteq \bot, \\ \exists\, r_1.\, \top &\sqsubseteq A_1 \sqcap A_3 \sqcap \exists\, r_1.\, A_2, \\ \exists\, r_2.\, \top &\sqsubseteq A_2 \sqcap A_3 \sqcap \exists\, r_2.\, A_2 \end{aligned} \right\}$$

Theorem 19. *Let $\mathcal{I}$ be a probabilistic interpretation, and $p \in [0,1]$ a threshold. If $\mathcal{B}$ is an implicational base for $\mathbb{K}(\mathcal{I})$ and p that contains an almost certain implicational base for $\mathbb{K}(\mathcal{I})$, then $\bigsqcap \mathcal{B}$ is a base of GCIs for $\mathcal{I}$ and p.*

Proof. Consider a GCI $\bigsqcap X \sqsubseteq \bigsqcap Y \in \bigsqcap \mathcal{B}$. Then Lemma 17 yields that the implication $X \to Y$ and the GCI $\bigsqcap X \sqsubseteq \bigsqcap Y$ have the same probability. Since $\mathcal{B}$ is a probabilistic implicational base for $\mathbb{K}(\mathcal{I})$ and p, we conclude that $\mathbb{P}(\bigsqcap X \sqsubseteq \bigsqcap Y) \geq p$ is satisfied.

Assume that $C \sqsubseteq D$ is an arbitrary GCI with probability $\geq p$. We have to show that $\bigsqcap \mathcal{B}$ entails $C \sqsubseteq D$. Let $\mathcal{J}$ be an arbitrary model of $\bigsqcap \mathcal{B}$. Consider an implication $X \to Y \in \mathcal{B}$, then $\bigsqcap X \sqsubseteq \bigsqcap Y \in \bigsqcap \mathcal{B}$, and hence it follows that $(\bigsqcap X)^{\mathcal{J}} \subseteq (\bigsqcap Y)^{\mathcal{J}}$. Consequently, the implication $X \to Y$ is valid in the induced context $\mathbb{K}(\mathcal{J}, M(\mathcal{I}))$. (We here mean the non-probabilistic formal context that is induced by a non-probabilistic interpretation, cf. Distel (2011); Borchmann (2014); Borchmann, Distel, and Kriegel (2016).)

Furthermore, since all model-based most-specific concept descriptions of $\mathcal{I}_\varepsilon^\times$ are expressible in terms of $M(\mathcal{I})$, it follows that $E \equiv \bigsqcap \pi_{M(\mathcal{I})}(E)$ for all mmscs E of $\mathcal{I}_\varepsilon^\times$, cf. Distel (2011); Borchmann (2014); Borchmann, Distel, and Kriegel (2016). Hence, we may conclude that

$$\begin{aligned} \mathbb{P}(C \sqsubseteq D) &= \mathbb{P}(C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times} \sqsubseteq D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}) \\ &= \mathbb{P}(\textstyle\bigsqcap \pi_{M(\mathcal{I})}(C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}) \sqsubseteq \bigsqcap \pi_{M(\mathcal{I})}(D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times})) \\ &= \mathbb{P}(\pi_{M(\mathcal{I})}(C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}) \to \pi_{M(\mathcal{I})}(D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times})). \end{aligned}$$

Consequently, $\mathcal{B}$ entails the implication $\pi_{M(\mathcal{I})}(C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}) \to \pi_{M(\mathcal{I})}(D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times})$, hence it is valid in $\mathbb{K}(\mathcal{J}, M(\mathcal{I}))$, and furthermore the GCI $C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times} \sqsubseteq D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}$ is valid in $\mathcal{J}$. As $\mathcal{J}$ is an arbitrary interpretation, $\bigsqcap \mathcal{B}$ entails $C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times} \sqsubseteq D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}$.

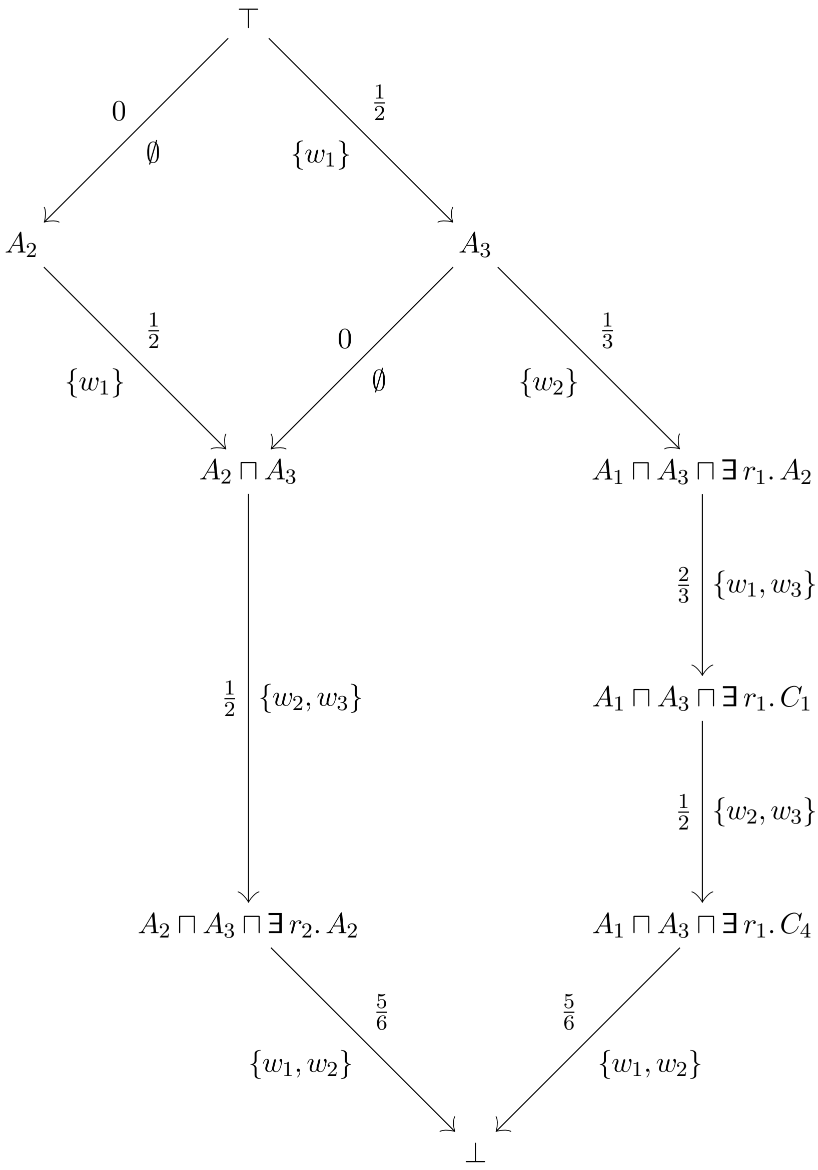


Figure 13. The probabilistic implicational base of the induced formal context $\mathbb{K}(\mathcal{I}_5)$ without the certain part

Corollary 18 yields that $\bigsqcap \mathcal{B}$ is complete for the almost certain GCIs of $\mathcal{I}$. In particular, the GCI $C \sqsubseteq C^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}$ is almost certain in $\mathcal{I}$, and hence follows from $\bigsqcap \mathcal{B}$. We conclude that $\bigsqcap \mathcal{B} \models C \sqsubseteq D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times}$. Of course, the GCI $D^{\mathcal{I}_\varepsilon^\times \mathcal{I}_\varepsilon^\times} \sqsubseteq D$ is valid in all interpretations. Finally, we conclude that $\bigsqcap \mathcal{B}$ entails $C \sqsubseteq D$. □

Returning back to our running example, and applying Theorem 19, we infer that the following TBox is a probabilistic base of GCIs for $\mathcal{I}_5$ and $\frac{3}{5}$:

$$\mathcal{B}(\mathcal{I}_5, \tfrac{3}{5}) = \mathcal{B}(\mathcal{I}_5, 1) \cup \left\{ \begin{array}{r} A_1 \sqcap A_3 \sqcap \exists r_1. A_2 \sqsubseteq \exists r_1. (A_2 \sqcap A_3), \\ A_1 \sqcap A_3 \sqcap \exists r_1. (A_2 \sqcap A_3 \sqcap \exists r_2. A_2) \sqsubseteq \bot, \\ A_2 \sqcap A_3 \sqcap \exists r_2. A_2 \sqsubseteq \bot \end{array} \right\}.$$

Corollary 20. *Let $\mathcal{I}$ be a probabilistic interpretation, and $p \in [0, 1]$ a threshold. Then $\mathcal{B}(\mathcal{I}, p) := \bigsqcap \mathcal{B}(\mathbb{K}(\mathcal{I}), p)$ is a base of GCIs for $\mathcal{I}$ and p where $\mathcal{B}(\mathbb{K}(\mathcal{I}), p)$ is defined as in*

Theorem 12.

8. Implications over Probabilistic Attributes

In Section 4, the notion of probability of an implication in a probabilistic formal context has been defined. However, it was not possible to express implications between probabilistically quantified attributes, e.g., we could not state that having attribute m with probability $\frac{1}{3}$ implies having attribute n with probability $\frac{2}{3}$. In this section we will resolve this issue by defining the notion of probabilistic attributes, and considering implications over probabilistic attributes. Furthermore, a technique for the construction of bases of such probabilistic implications is proposed.

Definition 21. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. For object sets $A \subseteq G$ and attribute sets $B \subseteq M$, the* incidence probability *is defined as*

$$\mathbb{P}(A, B) := \mathbb{P}\{\, w \in W \mid A \times B \times \{w\} \subseteq I \,\}.$$

A probabilistic attribute *over M is an expression $\mathsf{d} \geq p.\, B$ where $\emptyset \neq B \subseteq M$, and $p \in (0, 1]$. The set of all probabilistic attributes over M is denoted as $\mathsf{d}(M)$. For a subset $\mathbf{X} \subseteq \mathsf{d}(M)$, its* extension *in $\mathbb{K}$ is given by*

$$\mathbf{X}^I := \{\, g \in G \mid \forall\, \mathsf{d} \geq p.\, B \in \mathbf{X} \colon \mathbb{P}(\{g\}, B) \geq p \,\}.$$

Considering our exemplary probabilistic formal context $\mathbb{K}_2$ from Figure 4, the set $\{\mathsf{d} \geq \frac{1}{2}.\{m_1\}\}$ has the following extension in $\mathbb{K}_2$:

$$\{\mathsf{d} \geq \tfrac{1}{2}.\{m_1\}\}^I = \{\, g \in G \mid \mathbb{P}(\{g\}, \{m_1\}) \geq \tfrac{1}{2} \,\} = \{g_1\},$$

i.e., only the object g_1 has the attribute m_1 with a probability of at least $\frac{1}{2}$.

Definition 22. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. A* probabilistic implication *over M is an implication over $\mathsf{d}(M)$, and the set of all probabilistic implications over M is denoted as $\mathsf{d}\,\mathsf{Imp}(M)$. A probabilistic implication $\mathbf{X} \to \mathbf{Y}$ is* valid *in $\mathbb{K}$ if $\mathbf{X}^I \subseteq \mathbf{Y}^I$ is satisfied, and we shall denote this by $\mathbb{K} \models \mathbf{X} \to \mathbf{Y}$. A* probabilistic implication set *$\mathcal{L}$ is a set of probabilistic implications. $\mathcal{L}$ is* valid *in a probabilistic formal context $\mathbb{K}$ if $\mathbb{K} \models \mathbf{X} \to \mathbf{Y}$ for all $\mathbf{X} \to \mathbf{Y} \in \mathcal{L}$, and we shall denote this as $\mathbb{K} \models \mathcal{L}$. Furthermore, $\mathcal{L}$* entails *$\mathbf{X} \to \mathbf{Y}$, symbolized as $\mathcal{L} \models \mathbf{X} \to \mathbf{Y}$, if for each probabilistic formal context $\mathbb{K}$, $\mathbb{K} \models \mathcal{L}$ implies $\mathbb{K} \models \mathbf{X} \to \mathbf{Y}$.*

An example for a probabilistic implication within the domain of the probabilistic formal context presented in Figure 4 is $\{\mathsf{d} \geq \frac{1}{2}.\{m_1\}\} \to \{\mathsf{d} \geq \frac{2}{3}.\{m_2\}, \mathsf{d} \geq \frac{4}{5}.\{m_3\}\}$. However, it is not valid in $\mathbb{K}$ since the premise's extension $\{g_1\}$ of $\{\mathsf{d} \geq \frac{1}{2}.\{m_1\}\}$ is not a subset of the conclusion's extension

$$\{\mathsf{d} \geq \tfrac{2}{3}.\{m_2\}, \mathsf{d} \geq \tfrac{4}{5}.\{m_3\}\}^I = \{\mathsf{d} \geq \tfrac{2}{3}.\{m_2\}\}^I \cap \{\mathsf{d} \geq \tfrac{4}{5}.\{m_3\}\}^I = \{g_2\} \cap \{g_1, g_3\} = \emptyset.$$

It can be easily verified that the probabilistic implication $\{\mathsf{d} \geq \frac{1}{2}.\{m_1\}\} \to \{\mathsf{d} \geq \frac{4}{5}.\{m_3\}\}$ is valid in $\mathbb{K}$.

Lemma 23. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. Then the following probabilistic implications are valid in $\mathbb{K}$.*

(1) $\{\mathsf{d} \geq p.\, X\} \to \{\mathsf{d} \geq q.\, Y\}$ *if $X \supseteq Y$ and $p \geq q$.*
(2) $\{\mathsf{d} \geq p.\, X, \mathsf{d} \geq q.\, Y\} \to \{\mathsf{d} \geq p + q - 1.\, X \cup Y\}$ *if $p + q - 1 > 0$.*

Proof. Clearly, $\mathbb{P}(\{g\}, X) \geq p$ implies that $\mathbb{P}(\{g\}, X) \geq q$. Furthermore, since $X \supseteq Y$ yields that $\{g\} \times X \times \{w\} \supseteq \{g\} \times Y \times \{w\}$, we conclude that $\mathbb{P}(\{g\}, Y) \geq q$.

For the second implication, observe that in case $p + q - 1 > 0$ the intersection of $\{\, w \in W \mid \forall x \in X \colon (g, x, w) \in I \,\}$ and $\{\, w \in W \mid \forall y \in Y \colon (g, y, w) \in I \,\}$ must be non-empty, and in particular must have a $\mathbb{P}$-measure of at least $p + q - 1$. □

Definition 24. *Let $\mathbb{K} = (G, M, W, I, \mathbb{P})$ be a probabilistic formal context. The* probabilistic scaling *of $\mathbb{K}$ is defined as the formal context $\mathsf{d}(\mathbb{K}) := (G, \mathsf{d}(M), I)$ the incidence relation I of which is defined by*

$$(g, \mathsf{d} \geq p.\, B) \in I \text{ if, and only if, } \mathbb{P}(\{g\}, B) \geq p,$$

and by $\mathsf{d}^{\times}(\mathbb{K})$ we denote the subcontext of $\mathsf{d}(\mathbb{K})$ with attribute set

$$\mathsf{d}^{\times}(M) := \{\, \mathsf{d} \geq p.\, B^{I^{\times} I^{\times}} \mid \emptyset \neq B \subseteq M \text{ and } p \in (0, 1] \,\}.$$

Figure 14 shows the probabilistic scaling $\mathsf{d}^{\times}(\mathbb{K}_2)$ the attribute set of which is given by

$$\begin{aligned} &\mathsf{d}^{\times}(\{m_1, m_2, m_3\}) \\ &= \{\, \mathsf{d} \geq p.\, B \mid B \in \mathsf{Int}(\mathbb{K}^{\times}) \text{ and } p \in (0, 1] \,\} \\ &= \left\{ \begin{array}{l} \mathsf{d} \geq p.\{m_2\}, \mathsf{d} \geq p.\{m_3\}, \mathsf{d} \geq p.\{m_1, m_3\}, \\ \mathsf{d} \geq p.\{m_2, m_3\}, \mathsf{d} \geq p.\{m_1, m_2, m_3\} \end{array} \middle| \; p \in (0, 1] \right\}. \end{aligned}$$

Please note that the dual formal context is displayed, i.e., the incidence table is transposed. More formally, for a formal context $\mathbb{K} := (G, M, I)$ its *dual context* is $\mathbb{K}^{\partial} := (M, G, \{\, (m, g) \mid (g, m) \in I \,\})$. Furthermore, the concept lattice of $\mathsf{d}^{\times}(\mathbb{K}_2)$ is presented in Figure 15.

Lemma 25. *Let $\mathbb{K}$ be a probabilistic formal context, and $A \subseteq G$ as well as $B \subseteq M$. Then the following equations are satisfied:*

$$\mathbb{P}(A, B) = \mathbb{P}(A, B^{I^{\times} I^{\times}}) = \mathbb{P}(A, B^{I_{\varepsilon}^{\times} I_{\varepsilon}^{\times}}).$$

Proof. In the proof of Lemma 8 we have shown that $B^{I(w)} = B^{I^{\times} I^{\times} I(w)}$ for all $B \subseteq M$. Hence, we may conclude that

$$A \times B \times \{w\} \subseteq I \Leftrightarrow A \subseteq B^{I(w)} \Leftrightarrow A \subseteq B^{I^{\times} I^{\times} I(w)} \Leftrightarrow A \times B^{I^{\times} I^{\times}} \times \{w\} \subseteq I,$$

and so $\mathbb{P}(A, B) = \mathbb{P}(A, B^{I^{\times} I^{\times}})$. Analogously, $\mathbb{P}(A, B) = \mathbb{P}(A, B^{I_{\varepsilon}^{\times} I_{\varepsilon}^{\times}})$ is satisfied, since $B^{I(w)} = B^{I_{\varepsilon}^{\times} I_{\varepsilon}^{\times} I(w)}$ for all possible worlds $w \in W_{\epsilon}$ and attribute sets $B \subseteq M$. □

Lemma 26. *Let $\mathbb{K}$ be a probabilistic formal context. Then for all probabilistic implications $\{\, \mathsf{d} \geq p_t.\, X_t \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y\}$, the following statements are equivalent:*

(1) $\{\, \mathsf{d} \geq p_t.\, X_t \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y\}$ is valid in $\mathbb{K}$.
(2) $\{\, \mathsf{d} \geq p_t.\, X_t \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y\}$ is valid in $\mathsf{d}(\mathbb{K})$.
(3) $\{\, \mathsf{d} \geq p_t.\, X_t^{I^{\times} I^{\times}} \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y^{I^{\times} I^{\times}}\}$ is valid in $\mathsf{d}^{\times}(\mathbb{K})$.

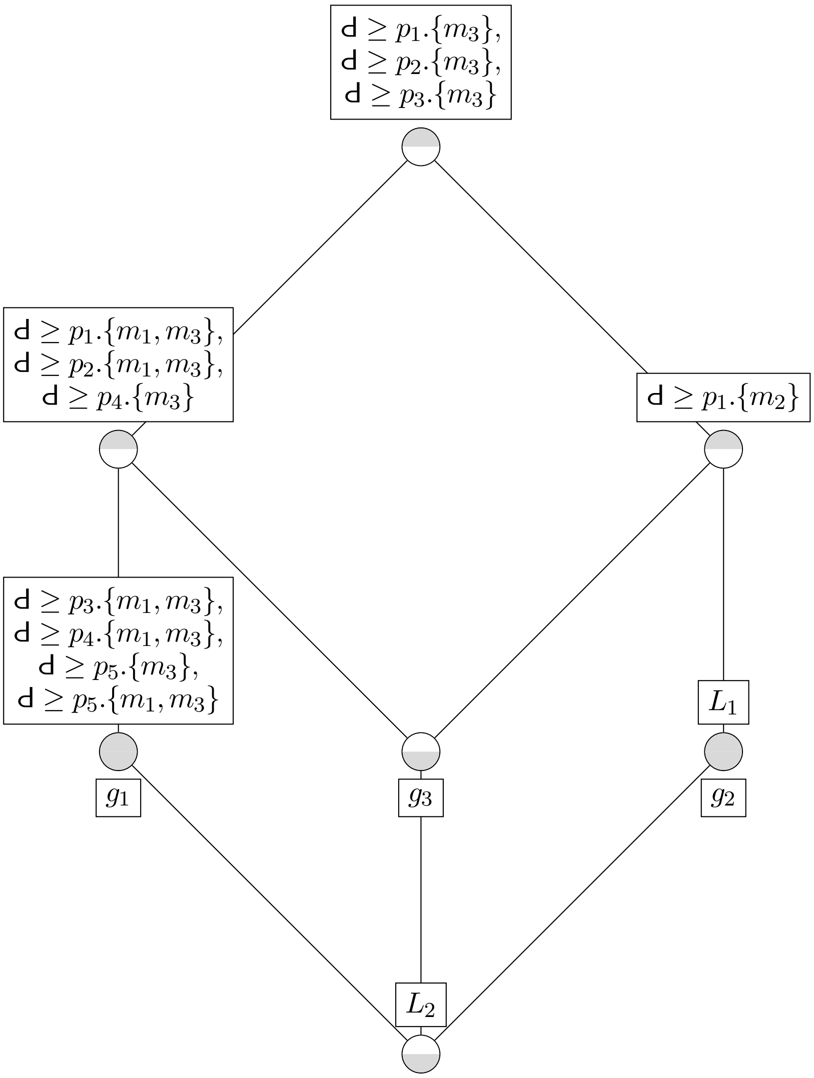
Proof. Statements 1 and 2 are equivalent by Definitions 21 and 24. Furthermore, from Lemma 25 we conclude that $\{\mathsf{d} \geq p.\, X\}^{I} = \{\mathsf{d} \geq p.\, X^{I^{\times} I^{\times}}\}^{I}$ for all probabilistic attributes $\mathsf{d} \geq p.\, X$. The equivalence of Statements 2 and 3 then follows easily. □

	$(\mathsf{d}^{\times}(\mathbb{K}_2))^{\partial}$	g_1	g_2	g_3
	$\mathsf{d} \geq p_1.\{m_2\}$	·	×	×
	$\mathsf{d} \geq p_1.\{m_3\}$	×	×	×
$p_1 \in (0, \frac{1}{6}]$	$\mathsf{d} \geq p_1.\{m_1, m_3\}$	×	·	×
	$\mathsf{d} \geq p_1.\{m_2, m_3\}$	·	×	·
	$\mathsf{d} \geq p_1.\{m_1, m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_2.\{m_2\}$	·	×	·
	$\mathsf{d} \geq p_2.\{m_3\}$	×	×	×
$p_2 \in (\frac{1}{6}, \frac{1}{3}]$	$\mathsf{d} \geq p_2.\{m_1, m_3\}$	×	·	×
	$\mathsf{d} \geq p_2.\{m_2, m_3\}$	·	×	·
	$\mathsf{d} \geq p_2.\{m_1, m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_3.\{m_2\}$	·	×	·
	$\mathsf{d} \geq p_3.\{m_3\}$	×	×	×
$p_3 \in (\frac{1}{3}, \frac{2}{3}]$	$\mathsf{d} \geq p_3.\{m_1, m_3\}$	×	·	·
	$\mathsf{d} \geq p_3.\{m_2, m_3\}$	·	×	·
	$\mathsf{d} \geq p_3.\{m_1, m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_4.\{m_2\}$	·	×	·
	$\mathsf{d} \geq p_4.\{m_3\}$	×	·	×
$p_4 \in (\frac{2}{3}, \frac{5}{6}]$	$\mathsf{d} \geq p_4.\{m_1, m_3\}$	×	·	·
	$\mathsf{d} \geq p_4.\{m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_4.\{m_1, m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_5.\{m_2\}$	·	×	·
	$\mathsf{d} \geq p_5.\{m_3\}$	×	·	·
$p_5 \in (\frac{5}{6}, 1]$	$\mathsf{d} \geq p_5.\{m_1, m_3\}$	×	·	·
	$\mathsf{d} \geq p_5.\{m_2, m_3\}$	·	·	·
	$\mathsf{d} \geq p_5.\{m_1, m_2, m_3\}$	·	·	·

Figure 14. The probabilistic scaling of the probabilistic formal context given in Figure 4

Definition 27. *Let $\mathbb{K} \coloneqq (G, M, W, I, \mathbb{P})$ be a probabilistic formal context, and assume that $\mathcal{L} \subseteq \mathsf{Imp}(M)$ is a set of implications and $\mathcal{P} \subseteq \mathsf{d}\,\mathsf{Imp}(M)$ is a set of probabilistic implications. Then $(\mathcal{L}, \mathcal{P})$ is* valid *in $\mathbb{K}$ if all implications in $\mathcal{L}$ are certain in $\mathbb{K}$ and $\mathcal{P}$ is valid in $\mathbb{K}$, and we shall denote this as $\mathbb{K} \models (\mathcal{L}, \mathcal{P})$. A probabilistic implication $\mathbf{X} \to \mathbf{Y}$ is* entailed *by $(\mathcal{L}, \mathcal{P})$ if $\mathbf{X} \to \mathbf{Y}$ is valid in all probabilistic formal contexts in which $(\mathcal{L}, \mathcal{P})$ is valid, and this is symbolized as $(\mathcal{L}, \mathcal{P}) \models \mathbf{X} \to \mathbf{Y}$.*

Furthermore, we call $(\mathcal{L}, \mathcal{P})$ a probabilistic implicational base *for $\mathbb{K}$ if for all*



$$\text{where } L_1 \coloneqq \left\{ \begin{array}{l} \mathsf{d} \geq p_1.\{m_2, m_3\}, \mathsf{d} \geq p_2.\{m_2\}, \mathsf{d} \geq p_2.\{m_2, m_3\}, \\ \mathsf{d} \geq p_3.\{m_2\}, \mathsf{d} \geq p_3.\{m_2, m_3\}, \mathsf{d} \geq p_4.\{m_2\}, \mathsf{d} \geq p_5.\{m_2\} \end{array} \right\},$$

$$\text{and } L_2 \coloneqq \left\{ \begin{array}{l} \mathsf{d} \geq p_1.\{m_1, m_2, m_3\}, \mathsf{d} \geq p_2.\{m_1, m_2, m_3\}, \mathsf{d} \geq p_3.\{m_1, m_2, m_3\}, \\ \mathsf{d} \geq p_4.\{m_2, m_3\}, \mathsf{d} \geq p_4.\{m_1, m_2, m_3\}, \\ \mathsf{d} \geq p_5.\{m_2, m_3\}, \mathsf{d} \geq p_5.\{m_1, m_2, m_3\} \end{array} \right\}$$

Figure 15. The concept lattice of the probabilistic scaling in Figure 14

probabilistic implications $\mathbf{X} \to \mathbf{Y}$ *over* M*, the following equivalence is satisfied:*

$$\mathbb{K} \models \mathbf{X} \to \mathbf{Y} \Leftrightarrow (\mathcal{L}, \mathcal{P}) \models \mathbf{X} \to \mathbf{Y}.$$

Corollary 28. *Let* $\mathbb{K}$ *be a probabilistic formal context. Then for each implicational base* $\mathcal{P}$ *of* $\mathsf{d}(\mathbb{K})$*, the pair* $(\emptyset, \mathcal{P})$ *is a probabilistic implicational base of* $\mathbb{K}$*.*

Lemma 29. *Let* $\mathbb{K} \coloneqq (G, M, W, I, \mathbb{P})$ *be a probabilistic formal context. If an implication* $X \to Y$ *is certain in* $\mathbb{K}$*, then the probabilistic implication* $\{\mathsf{d} \geq p.\, X\} \to \{\mathsf{d} \geq p.\, Y\}$ *is valid in* $\mathbb{K}$ *for all* $p \in [0, 1]$*.*

Proof. Assume that $X \to Y$ is certain in $\mathbb{K}$. Then Lemma 5 yields that $X \to Y$ is valid in the certain scaling $\mathbb{K}_\varepsilon^\times$, i.e. $Y \subseteq X^{I^\times I^\times}$ holds true. Consequently, $\mathbb{P}(A, Y) \geq \mathbb{P}(A, X^{I^\times I^\times})$ for all $A \subseteq G$, and Lemma 25 implies $\mathbb{P}(A, Y) \geq \mathbb{P}(A, X)$ for all $A \subseteq G$. In particular then $\mathbb{P}(\{g\}, X) \geq p$ implies $\mathbb{P}(\{g\}, Y) \geq p$ for all $g \in G$ and all $p \in [0, 1]$, i.e. the probabilistic implication $\{\mathsf{d} \geq p.\, X\} \to \{\mathsf{d} \geq p.\, Y\}$ is valid in $\mathbb{K}$. □

Proposition 30. *Let $\mathbb{K}$ be a probabilistic formal context. Then $(\mathcal{B}^\times(\mathbb{K}), \mathcal{B}^{\mathsf{d}}(\mathbb{K}))$ is a probabilistic implicational base for $\mathbb{K}$ where*

$$\mathcal{B}^\times(\mathbb{K}) \coloneqq \{\, P \to P^{I^\times I^\times} \mid P \in \mathsf{PsInt}(\mathbb{K}^\times) \,\},$$

$$\textit{and} \quad \mathcal{B}^{\mathsf{d}}(\mathbb{K}) \coloneqq \{\, \mathbf{P} \to \mathbf{P}^{II} \mid \mathbf{P} \in \mathsf{PsInt}(\mathsf{d}^\times(\mathbb{K})) \,\}.$$

Proof. We start by proving soundness of $\mathcal{B}(\mathbb{K})$. Lemma 26 yields that each implication from $\mathcal{B}^{\mathsf{d}}(\mathbb{K})$ is valid in $\mathbb{K}$. Furthermore, Lemma 5 justifies that all implications in $\mathcal{B}^\times(\mathbb{K})$ are certain in $\mathbb{K}$.

We proceed by showing completeness. Assume that $\mathbb{K} \models \{\, \mathsf{d} \geq p_t.\, X_t \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y\}$. Lemma 26 yields that $\{\, \mathsf{d} \geq p_t.\, X_t^{I^\times I^\times} \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y^{I^\times I^\times}\}$ is valid in $\mathsf{d}^\times(\mathbb{K})$, and thus is entailed by $\mathcal{B}^{\mathsf{d}}(\mathbb{K})$, since by construction, $\mathcal{B}^{\mathsf{d}}(\mathbb{K})$ is complete for the implications that are valid in $\mathsf{d}^\times(\mathbb{K})$.

Since $X_t \to X_t^{I^\times I^\times}$ is certain in $\mathbb{K}$ for each $t \in T$, it is valid in the certain scaling $\mathbb{K}^\times$ and is thus entailed by $\mathcal{B}^\times(\mathbb{K})$. An application of Lemma 29 yields that for each $t \in T$, the probabilistic implication $\{\mathsf{d} \geq p_t.\, X_t\} \to \{\mathsf{d} \geq p_t.\, X_t^{I^\times I^\times}\}$ entailed by $\mathcal{B}^\times(\mathbb{K})$.

In summary, then

$$\begin{aligned}
\mathcal{B}^\times(\mathbb{K}) \cup \mathcal{B}^{\mathsf{d}}(\mathbb{K}) &\models \{\, \{\mathsf{d} \geq p_t.\, X_t\} \to \{\mathsf{d} \geq p_t.\, X_t^{I^\times I^\times}\} \mid t \in T \,\} \\
&\qquad \cup \{\{\, \mathsf{d} \geq p_t.\, X_t^{I^\times I^\times} \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y^{I^\times I^\times}\}\} \\
&\qquad \cup \{\{\mathsf{d} \geq q.\, Y^{I^\times I^\times}\} \to \{\mathsf{d} \geq q.\, Y\}\} \\
&\models \{\, \mathsf{d} \geq p_t.\, X_t \mid t \in T \,\} \to \{\mathsf{d} \geq q.\, Y\},
\end{aligned}$$

i.e., $\mathcal{B}(\mathbb{K})$ is complete for $\mathbb{K}$. □

Returning back to our running example $\mathbb{K}_2$ from Figure 4, we now construct its probabilistic implicational base $\mathcal{B}(\mathbb{K}_2)$. The canonical base of the certain scaling $(\mathbb{K}_2)^\times$ was computed as $\{\{m_1\} \to \{m_3\}\}$. Consequently, we get that the first part of the probabilistic implicational base of $\mathbb{K}_2$ is

$$\mathcal{B}^\times(\mathbb{K}_2) = \{\{m_1\} \to \{m_3\}\}.$$

For the computation of the second part $\mathcal{B}^{\mathsf{d}}(\mathbb{K}_2)$, we consider the probabilistic scaling of $\mathbb{K}_2$. In order to avoid the axiomatization of trivial implications, we construct the implicational base of $\mathsf{d}^\times(\mathbb{K}_2)$ relative to the implication set containing all those probabilistic implications which are described in Lemma 23. This set of background knowledge contains, among

$$\mathcal{B}^{\mathsf{d}}(\mathbb{K}_2) = \left\{\begin{array}{l}
\quad \emptyset \\
\to \{\mathsf{d} \geq \frac{2}{3}.\{m_3\}\}, \\
\quad \{\mathsf{d} \geq 1.\{m_1, m_3\}, \mathsf{d} \geq \frac{1}{6}.\{m_2\}\} \\
\to \{\mathsf{d} \geq 1.\{m_1, m_2, m_3\}\}, \\
\quad \{\mathsf{d} \geq \frac{2}{3}.\{m_2, m_3\}, \mathsf{d} \geq \frac{5}{6}.\{m_3\}, \mathsf{d} \geq 1.\{m_2\}, \mathsf{d} \geq \frac{1}{3}.\{m_1, m_3\}\} \\
\to \{\mathsf{d} \geq 1.\{m_1, m_2, m_3\}\}, \\
\quad \{\mathsf{d} \geq 1.\{m_3\}, \mathsf{d} \geq \frac{1}{3}.\{m_1, m_3\}\} \\
\to \{\mathsf{d} \geq 1.\{m_1, m_3\}\}, \\
\quad \{\mathsf{d} \geq \frac{2}{3}.\{m_1, m_3\}, \mathsf{d} \geq \frac{5}{6}.\{m_3\}\} \\
\to \{\mathsf{d} \geq 1.\{m_1, m_3\}\}, \\
\quad \{\mathsf{d} \geq \frac{2}{3}.\{m_3\}, \mathsf{d} \geq \frac{1}{6}.\{m_2, m_3\}\} \\
\to \{\mathsf{d} \geq \frac{2}{3}.\{m_2, m_3\}, \mathsf{d} \geq 1.\{m_2\}\}, \\
\quad \{\mathsf{d} \geq \frac{2}{3}.\{m_3\}, \mathsf{d} \geq \frac{1}{3}.\{m_2\}\} \\
\to \{\mathsf{d} \geq \frac{2}{3}.\{m_2, m_3\}, \mathsf{d} \geq 1.\{m_2\}\}, \\
\quad \{\mathsf{d} \geq \frac{5}{6}.\{m_3\}\} \\
\to \{\mathsf{d} \geq \frac{1}{3}.\{m_1, m_3\}\}, \\
\quad \{\mathsf{d} \geq \frac{2}{3}.\{m_3\}, \mathsf{d} \geq \frac{1}{6}.\{m_1, m_3\}\} \\
\to \{\mathsf{d} \geq \frac{5}{6}.\{m_3\}, \mathsf{d} \geq \frac{1}{3}.\{m_1, m_3\}\}
\end{array}\right\}$$

Figure 16. The implicational base of $\mathsf{d}^{\times}(\mathbb{K}_2)$ with respect to the background implications that are described in Lemma 23.

others, the following implications:

$$\begin{aligned}
\{\mathsf{d} \geq \tfrac{5}{6}.\{m_1, m_3\}\} &\to \{\mathsf{d} \geq \tfrac{1}{3}.\{m_1\}\} \\
\{\mathsf{d} \geq \tfrac{5}{6}.\{m_1, m_3\}\} &\to \{\mathsf{d} \geq \tfrac{2}{3}.\{m_3\}\} \\
&\vdots \\
\{\mathsf{d} \geq \tfrac{2}{3}.\{m_2\}, \mathsf{d} \geq \tfrac{2}{3}.\{m_3\}\} &\to \{\mathsf{d} \geq \tfrac{1}{3}.\{m_2, m_3\}\} \\
\{\mathsf{d} \geq \tfrac{5}{6}.\{m_1, m_3\}, \mathsf{d} \geq \tfrac{5}{6}.\{m_2, m_3\}\} &\to \{\mathsf{d} \geq \tfrac{2}{3}.\{m_1, m_2, m_3\}\} \\
&\vdots
\end{aligned}$$

The resulting probabilistic implication set is presented in Figure 16.

9. Conclusion

We have introduced the notion of a probabilistic formal context as a triadic context the third dimension of which is a set of worlds equipped with a probability measure. Then the probability of implications in such probabilistic formal contexts was defined, and a construction of a base of implications the probabilities of which exceeds a given threshold was proposed, and its correctness was verified. Furthermore, the results were applied to the light-weight description logic $\mathcal{EL}^{\bot}$ with probabilistic interpretations, and so we formulated a method for the computation of a base of general concept inclusions the probabilities of which satisfies a given lower threshold.

As another approach for combining Formal Concept Analysis and probabilities, we defined the notion of a probabilistic attribute, and provided a method for the axiomatization of implications of probabilistic attributes from probabilistic formal contexts. In particular, this technique allows for a more fine-grained analysis of the probabilistic input data. As a future step, it is planned to extend or apply the results from the last section on probabilistic attributes to the field of Description Logics, and at first to the description logic $\mathcal{EL}^{\bot}$ and its probabilistic variant.

For finite input data sets all of the provided constructions are computable. In particular, Distel (2011); Borchmann, Distel, and Kriegel (2016) provide methods for the computation of model-based most-specific concept descriptions, and the algorithms of Ganter (2010); Kriegel (2015b); Kriegel and Borchmann (2015); Kriegel (2016a) can be utilized to compute concept lattices and canonical implicational bases (or bases of GCIs, respectively).

Acknowledgements

The author thanks Sebastian Rudolph for proof reading and a fruitful discussion, and furthermore the anonymous reviewers for their constructive comments.

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