Description Logics That Count, and What They Can and Cannot Count (Extended Abstract)*

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Simple counting quantifiers that can be used to compare the number of role successors of an individual or the cardinality of a concept with a fixed natural number have been employed in Description Logics (DLs) for more than two decades, under the respective names of *number restrictions* [8,13,12] and *cardinality restrictions on concepts* (CRs, CBoxes) [4,19].

The DL $\mathcal{ALCQ}$ [12] extends the basic DL $\mathcal{ALC}$ with so-called *qualified number restrictions* of the form $(\geq n\, r.C)$ and $(\leq n\, r.C)$, collecting individuals for which the number of r successors belonging to the concept C is bounded from below/above by the natural number n. The computational complexity of concept satisfiability in $\mathcal{ALCQ}$ [12] has been shown to be PSpace-complete without *concept inclusions* (CIs, TBoxes) and ExpTime-complete w.r.t. CIs, independently from the encoding (unary or binary) of the numbers occurring in the restrictions [18,20]. CRs are global constraints of the form $(\geq n\, C)$ and $(\leq n\, C)$, which state a lower/upper bound on how many elements of C a model may contain. By replacing CIs with CRs, the complexity of satisfiability increases to NExpTime-complete if the numbers occurring in the CRs are assumed to be encoded in binary [19]. With unary coding of numbers, it stays ExpTime-complete even w.r.t. CRs [19]. It should be noted that both qualified number restrictions and CRs (which generalize CIs) can be expressed in $\mathcal{C}^2$, the two-variable fragment of first-order logic with counting quantifiers [11,16], whose satisfiability problem is known to be NExpTime-complete [17].

Qualified number restrictions cannot relate cardinalities of different sets of role successors to one another, but can only compare the number of role successors (satisfying certain properties) of an individual against a *fixed* natural number. To overcome this limitation, in [1] we extended $\mathcal{ALCQ}$ by enabling the statement of restrictions on role successors using the quantifier-free fragment of Boolean Algebra with Presburger Arithmetic (QFBAPA) [14], in which one can express Boolean combinations of set constraints and numerical constraints comparing the cardinalities of finite sets. The resulting logic, called $\mathcal{ALCSCC}$, strictly extends the expressive power of $\mathcal{ALCQ}$. In [1] it is shown that the $\mathcal{ALCSCC}$ concept description $succ(|r| = |s|)$, which describes individuals having the same number of r-successors as s-successors, cannot be expressed in $\mathcal{ALCQ}$. In addition, it has been shown in [5] that $succ(|r \cap A| = |r \cap \neg A|)$, the $\mathcal{ALCSCC}$ concept

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describing individuals whose number of r-successors belonging to A is the same as the number of r-successors not belonging to A, is not even expressible in first-order logic. In spite of this considerable increase in expressive power, we were able to show in [1] that there is no increase in complexity: like for $\mathcal{ALCQ}$, the complexity of the satisfiability problem in $\mathcal{ALCSCC}$ is PSpace-complete without CIs and ExpTime-complete w.r.t. CIs. The "in PSpace" result can also be derived from previous work [9] on modal logics with Presburger constraints, while the "in ExpTime" result is a novel contribution of [1].

Just like classical number restrictions, CRs can only relate the cardinality of a concept to a *fixed* number. In [7] we have introduced and investigated a generalization of CRs, which we called *extended cardinality constraints (ECBoxes)*. The main idea was again to use QFBAPA to formulate and combine these constraints. It is shown in [7] that, in the DL $\mathcal{ALC}$, the complexity of reasoning w.r.t. extended cardinality constraints (NExpTime for binary coding of numbers), is the same as for reasoning w.r.t. CRs. Moreover, we identified a special class of ECBoxes called *restricted cardinality constraints (RCBoxes)*, which can express CIs but not CRs, and showed that the complexity of reasoning is lowered to ExpTime if ECBoxes are replaced with RCBoxes. The NExpTime upper bound for the extended case can actually be inferred from the NExpTime upper bound in [21] for a more expressive logic with n-ary relations and function symbols, but the ExpTime upper bound for the restricted case is a novel result.

In [2,3], we have combined the work in [1] and [7] by considering extended cardinality constraints in $\mathcal{ALCSCC}$. This turned out to be non-trivial since the local cardinality constraints of $\mathcal{ALCSCC}$ may interact with the global ones in the extended cardinality constraints. Nevertheless, we were able to show that the complexity results (NExpTime-complete in general, and ExpTime-complete in the restricted case) hold not only for $\mathcal{ALC}$, but also for $\mathcal{ALCSCC}$.

The purpose of the paper [6], whose results this abstract summarizes, is twofold. On the one hand, after giving a compact representation of the known complexity results for the DLs with extended counting facilities mentioned above, we prove that those bounds are preserved in a setting where arbitrary rather than just finite models are considered. On the other hand, we investigate in detail the expressive power of these DLs over arbitrary models.

A first step in this direction had already been made in [5] for number restrictions over role successors. There, to ease the comparison with classical DLs such as $\mathcal{ALCQ}$, where one usually employs a semantics based on arbitrary rather than finite models, we considered variants of QFBAPA and $\mathcal{ALCSCC}$ (called QFBAPA$^\infty$ and $\mathcal{ALCSCC}^\infty$) that allow for possibly infinite sets and interpretations, respectively. After transferring the known complexity results for QFBAPA and $\mathcal{ALCSCC}$ to these variants, we examined their expressive power using appropriate bisimulation relations. Basically, we showed there that $\mathcal{ALCSCC}^\infty$ concepts can go beyond first-order logic (recall the concept description $succ(|r \cap A| = |r \cap \neg A|)$ mentioned earlier) and determined a sub-logic of $\mathcal{ALCSCC}^\infty$, called $\mathcal{ALCCQU}$, that corresponds to the first-order fragment of $\mathcal{ALCSCC}^\infty$. We also proved that $\mathcal{ALCCQU}$ is more expressive than $\mathcal{ALCQ}$ and

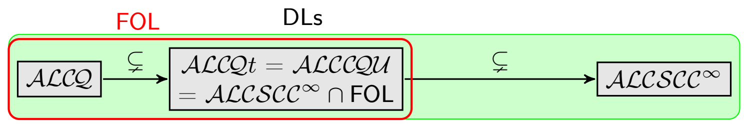

Fig. 1. The relative expressive power of the DLs $\mathcal{ALCQ}$, $\mathcal{ALCQ}t$, $\mathcal{ALCCQU}$, and $\mathcal{ALCSCC}^\infty$. The source of every arrow is a strict sub-logic of the target.

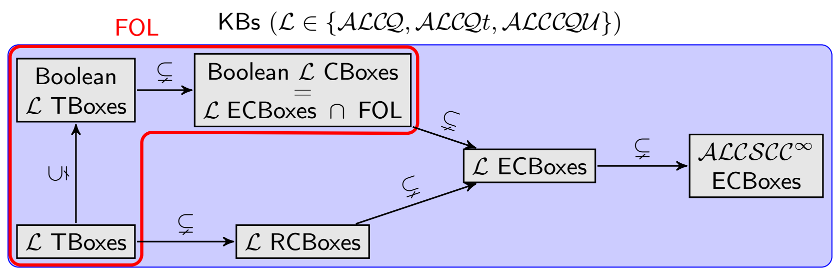

Fig. 2. The relative expressivity of boxes. The source of every arrow is a strict sub-logic of the target.

equivalent to an extension of $\mathcal{ALCQ}$, called $\mathcal{ALCQ}t$, where number restrictions range over *(safe) role types* rather than role names. Figure 1 gives an overview of the expressivity results contained in [5].

In [6], we recall these results, and then extend them to TBoxes, CBoxes, RCBoxes, and ECBoxes, by adapting methods and ideas from [15]. As in [5], we consider the semantic variants QFBAPA$^\infty$ and $\mathcal{ALCSCC}^\infty$, rather than their finite counterparts, and derive the expressivity hierarchy depicted in Figure 2 using suited bisimulation relations and the 0-1 law for first-order sentences [10].

Detailed definitions of the aforementioned formalisms, as well as proofs of the expressivity results mentioned in this abstract, have been published in [6].

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