On the Descriptive Complexity of Temporal Constraint Satisfaction Problems*

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Abstract

Finite-domain constraint satisfaction problems are either solvable by Datalog, or not even expressible in fixed-point logic with counting. The border between the two regimes can be described by a universal-algebraic minor condition. For infinite-domain CSPs, the situation is more complicated even if the template structure of the CSP is model-theoretically tame. We prove that there is no Maltsev condition that characterizes Datalog already for the CSPs of first-order reducts of $(\mathbb{Q};<)$; such CSPs are called *temporal CSPs* and are of fundamental importance in infinite-domain constraint satisfaction. Our main result is a complete classification of temporal CSPs that can be expressed in one of the following logical formalisms: Datalog, fixed-point logic (with or without counting), or fixed-point logic with the Boolean rank operator. The classification shows that many of the equivalent conditions in the finite fail to capture expressibility in Datalog or fixed-point logic already for temporal CSPs.

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1 Introduction

The quest for finding a logic capturing Ptime is an ongoing challenge in the field of finite model theory originally motivated by questions from database theory [41]. Ever since its proposal, most candidates are based on various extensions of *fixed-point logic* (FP), for example by *counting* or by *rank operators*. Though not a candidate for capturing Ptime, *Datalog* is perhaps the most studied fragment of FP. Datalog is particularly well-suited for formulating various algorithms for solving *constraint satisfaction problems* (CSPs); examples of famous algorithms that can be formulated in Datalog are the *arc consistency* procedure and the *path consistency* procedure. In general, the expressive power of FP is limited as it fails to express counting properties of finite structures such as even cardinality. However, the combination of a mechanism for iteration and a mechanism for counting provided by *fixed-point logic with counting* (FPC) is strong enough to express most known algorithmic techniques leading to polynomial-time procedures [27,40]. In fact, all known decision problems for finite structures that provably separate FPC from Ptime are at least as hard as deciding solvability of systems of linear equations over a non-trivial finite Abelian group [58]. If we extend FPC further by the *Boolean rank operator* [40], we obtain the logic FPR_2 which is known to capture Ptime for Boolean CSPs [60]. It has recently been announced that the extension of FPC by the even more powerful *uniform rank operator* fails to capture Ptime for finite-domain CSPs [52].

The first inexpressibility result for FPC is due to Cai, Fürer, and Immerman for systems of linear equations over $\mathbb{Z}_2$ [24]. In 2009, this result was extended to arbitrary non-trivial finite Abelian groups by Atserias, Bulatov, and Dawar [1]; their work was formulated purely in the framework of CSPs. At around the same time, Barto and Kozik [5] settled the closely related bounded width conjecture of Larose and Zádori [50]. A combination of both works together with results from [47, 53] yields the following theorem.

Theorem 1.1. *For a finite structure* $\mathbf{B}$*, the following seven statements are equivalent.*

(1) $\text{CSP}(\mathbf{B})$ *is expressible in* Datalog [5].
(2) $\text{CSP}(\mathbf{B})$ *is expressible in* FP [1].
(3) $\text{CSP}(\mathbf{B})$ *is expressible in* FPC [1].
(4) $\mathbf{B}$ *does not pp-construct linear equations over any non-trivial finite Abelian group* [49, 50].
(5) $\mathbf{B}$ *does not pp-construct linear equations over* $\mathbb{Z}_p$ *for any prime* $p \geq 2$ *(see Theorem 47 in* [6]*).*
(6) $\mathbf{B}$ *has weak near-unanimity polymorphisms for all but finitely many arities* [53].
(7) $\mathbf{B}$ *has weak near-unanimity polymorphisms* f, g *that satisfy* $g(x,x,y) \approx f(x,x,x,y)$ [47].

In particular, Datalog, FP, and FPC are equally expressive when it comes to finite-domain CSPs. This observation raises the question whether the above-mentioned fragments and extensions of FP might collapse on CSPs in general. In fact, this question was already answered negatively in 2007 by Bodirsky and Kára in their investigation of the CSPs of first-order reducts of $(\mathbb{Q};<)$, also known as (infinite-domain) *temporal CSPs* [14]; the decision problem $\text{CSP}(\mathbb{Q};\text{R}_{\text{min}})$, where

$$\text{R}_{\text{min}} \coloneqq \{(x,y,z) \in \mathbb{Q}^3 \mid y < x \vee z < x\},$$

is provably not solvable by any Datalog program [15] but it is expressible in FP, as we will see later. Since every CSP represents a class of finite structures whose complement is closed under homomorphisms, this simultaneously yields an alternative proof of a result from [29] stating that the homomorphism preservation theorem fails for FP.

Several famous NP-hard problems such as the *Betweenness* problem or the *Cyclic Ordering* problem are temporal CSPs. Temporal CSPs have been studied for example in artificial intelligence [55], scheduling [15], and approximation [43]. Random instances of temporal CSPs have been studied in [35]. Temporal CSPs fall into the larger class of CSPs of *reducts of finitely bounded homogeneous structures*. It is an open problem whether all CSPs of reducts of finitely bounded homogeneous structures have a complexity dichotomy in the sense that they are in P or NP-complete. In this class, temporal CSPs play a particular role since they are among the few known cases where the important technique of reducing infinite-domain CSPs to finite-domain CSPs from [16] fails to provide any polynomial-time tractability results.

1.1 Contributions

We present a complete classification of temporal CSPs that can be solved in Datalog, FP, FPC, or FPR_2. The classification leads to the following sequence of inclusions for temporal CSPs:

$$\text{Datalog} \subsetneq \text{FP} = \text{FPC} \subsetneq \text{FPR}_2\,.$$

Our results show that the expressibility of temporal CSPs in these logics can be characterised in terms of avoiding pp-constructibility of certain structures, namely $(\mathbb{Q};\mathrm{R_{min}})$, $(\mathbb{Q};\mathrm{X})$ where

$$\mathrm{X} := \{(x,y,z) \in \mathbb{Q}^3 \mid x = y < z \vee y = z < x \vee z = x < y\},$$

and $(\{0,1\};1\mathrm{IN}3)$ where

$$1\mathrm{IN}3 := \{(1,0,0),(0,1,0),(0,0,1)\}.$$

Theorem 1.2. *Let* $\mathbf{B}$ *be a temporal structure. The following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* Datalog.
(2) $\mathbf{B}$ *does not pp-construct* $(\{0,1\};1\mathrm{IN}3)$ *and* $(\mathbb{Q};\mathrm{R_{min}})$.
(3) $\mathbf{B}$ *is preserved by* ll *and* dual ll*, or by a constant operation.*

Theorem 1.3. *Let* $\mathbf{B}$ *be a temporal structure. The following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FP.
(2) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FPC.
(3) $\mathbf{B}$ *does not pp-construct* $(\{0,1\};1\mathrm{IN}3)$ *and* $(\mathbb{Q};\mathrm{X})$.
(4) $\mathbf{B}$ *is preserved by* min*,* mi*,* ll*, the dual of one of these operations, or by a constant operation.*

Theorem 1.4. *Let* $\mathbf{B}$ *be a temporal structure. The following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FPR_2.
(2) $\mathbf{B}$ *does not pp-construct* $(\{0,1\};1\mathrm{IN}3)$.
(3) $\mathbf{B}$ *is preserved by* mx*,* min*,* mi*,* ll*, the dual of one of these operations, or by a constant operation.*

As a byproduct of our classification we get that all polynomial-time algorithms for temporal CSPs from [14] can be implemented in FPR_2. Our results also show that every temporal CSP of a structure that pp-constructs $(\mathbb{Q};\mathrm{X})$ but not $(\{0,1\};1\mathrm{IN}3)$ is solvable in polynomial time, is not expressible in FPC, and cannot encode systems of linear equations over any non-trivial finite Abelian group. Such temporal CSPs are Datalog-interreducible with the following decision problem:

3-ORD-XOR-SAT
INPUT: A finite homogeneous system of linear Boolean equations of length 3.
QUESTION: Does every non-empty subset S of the equations have a solution where at least one variable in an equation from S denotes the value 1?

We have eliminated the following candidates for general algebraic criteria for expressibility of CSPs in FP motivated by the articles [1], [16], and [5], respectively.

Theorem 1.5. *There exist temporal CSPs which are inexpressible in FPC but*

(1) *do not pp-construct linear equations over any non-trivial finite Abelian group,*

(2) *have pseudo-WNU polymorphisms f, g that satisfy $g(x,x,y) \approx f(x,x,x,y)$,*
(3) *have a k-ary pseudo-WNU polymorphism for all but finitely many $k \in \mathbb{N}$.*

We have good news and bad news regarding the existence of general algebraic criteria for expressibility of CSPs in fragments and/or extensions of FP. The bad news is that there is no Maltsev condition that would capture expressibility of temporal CSPs in Datalog (see Theorem 1.6) which carries over to CSPs of reducts of finitely bounded homogeneous structures and more generally to CSPs of *ω-categorical* templates.

Theorem 1.6. *There is no condition preserved by uniformly continuous clone homomorphisms that would capture the expressibility of temporal CSPs in* Datalog.

This is particularly striking because ω-categorical CSPs are otherwise well-behaved when it comes to expressibility in Datalog—every ω-categorical CSP expressible in Datalog admits a *canonical Datalog program* [12]. The good news is that it is not possible to obtain such a result for FP solely through the study of temporal CSPs; we show that the expressibility in FP for finite-domain and temporal CSPs can be characterised by universal-algebraic minor conditions. We introduce a family $\mathcal{E}_{k,n}$ of minor conditions that are similar to the *dissected weak near-unanimity* identities introduced in [4, 33].

Theorem 1.7. *Let* $\mathbf{B}$ *be a finite structure or a temporal structure. The following are equivalent.*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FP / FPC.
(2) $\mathrm{Pol}(\mathbf{B})$ *satisfies $\mathcal{E}_{k,k+1}$ for all but finitely many $k \in \mathbb{N}$.*

The polymorphism clone of every first-order reduct of a finitely bounded homogeneous structure known to the authors satisfies $\mathcal{E}_{k,k+1}$ for all but finitely many k if and only if its CSP is in FP / FPC. This includes in particular all CSPs that are in the complexity class AC_0: all of these CSPs can be expressed as CSPs of reducts of finitely bounded homogeneous structures, by a combination of results of Rossman [59], Cherlin, Shelah, and Shi [26], and Hubicka and Nešetřil [45] (see [10], Section 5.6.1), and their polymorphism clones satisfy $\mathcal{E}_{k,k+1}$ for all but finitely many k. To prove that the polymorphism clone of a given temporal structure does or does not satisfy $\mathcal{E}_{k,k+1}$ we apply a new general characterisation of the satisfaction of minor conditions in polymorphism clones of ω-categorical structures (Theorem 7.10).

1.2 Outline of the article

In Section 2, we introduce various basic concepts from algebra and logic as well as some specific ones for temporal CSPs. In Section 3, we start discussing the descriptive complexity of temporal CSPs by expressing some particularly chosen tractable temporal CSPs in FP. In Section 4, we continue the discussion by showing that $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ is inexpressible in FPC but expressible in FPR_2. At this point we have enough information so that in Section 5 we can classify the temporal CSPs which are expressible in FP / FPC and the temporal CSP which are expressible in FPR_2. In Section 6 we classify the temporal CSPs which are expressible in Datalog. In Section 7 we provide results regarding algebraic criteria for expressibility of finitely bounded homogeneous CSPs in Datalog and in FP based on our investigation of temporal CSPs.

2 Preliminaries

The set $\{1,\dots,n\}$ is denoted by $[n]$. We use the bar notation for tuples; for a tuple $\bar{t}$ indexed by a set I, the value of $\bar{t}$ at the position $i \in I$ is denoted by $\bar{t}[i]$. For a k-ary tuple $\bar{t}$ and $I \subseteq [k]$, we use the notation $\mathrm{proj}_I(\bar{t})$ for the tuple $(\bar{t}[i_1],\dots,\bar{t}[i_\ell])$ where $I = \{i_1,\dots,i_\ell\}$ with $i_1 < \cdots < i_\ell$. For a function $f\colon A^n \to B$ ($n \geq 1$) and k-tuples $\bar{t}_1,\dots,\bar{t}_n \in A^k$, we sometimes use $f(\bar{t}_1,\dots,\bar{t}_n)$ as a shortcut for the k-tuple $(f(\bar{t}_1[1],\dots\bar{t}_n[1]),\dots,f(\bar{t}_1[k],\dots,\bar{t}_n[k]))$.

2.1 Structures and first-order logic

A (*relational*) *signature* τ is a set of *relation symbols*, each $R \in \tau$ with an associated natural number $\mathrm{ar}(R)$ called *arity*. A (*relational*) τ*-structure* $\mathbf{A}$ consists of a set A (the *domain*) together with the relations $R^{\mathbf{A}} \subseteq A^k$ for each relation symbol $R \in \tau$ with arity k. We often describe structures by listing their domain and relations, that is, we write $\mathbf{A} = (A; R_1^{\mathbf{A}}, \dots)$. An *expansion* of $\mathbf{A}$ is a σ-structure $\mathbf{B}$ with $A = B$ such that $\tau \subseteq \sigma$, $R^{\mathbf{B}} = R^{\mathbf{A}}$ for each relation symbol $R \in \tau$. Conversely, we call $\mathbf{A}$ a *reduct* of $\mathbf{B}$. In the context of relational structures, we reserve the notion of a *constant* for singleton unary relations. A *constant symbol* is then a symbol of such a relation.

A *homomorphism* $h\colon \mathbf{A} \to \mathbf{B}$ for τ-structures $\mathbf{A}, \mathbf{B}$ is a mapping $h\colon A \to B$ that *preserves* each relation of $\mathbf{A}$, that is, if $\bar{t} \in R^{\mathbf{A}}$ for some k-ary relation symbol $R \in \tau$, then $h(\bar{t}) \in R^{\mathbf{B}}$. We write $\mathbf{A} \to \mathbf{B}$ if $\mathbf{A}$ maps homomorphically to $\mathbf{B}$ and $\mathbf{A} \not\to \mathbf{B}$ otherwise. We say that $\mathbf{A}$ and $\mathbf{B}$ are *homomorphically equivalent* if $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{A}$. An *endomorphism* is a homomorphism from $\mathbf{A}$ to $\mathbf{A}$. We call a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *strong* if it additionally satisfies the following condition: for every k-ary relation symbol $R \in \tau$ and $\bar{t} \in A^k$ we have $h(\bar{t}) \in R^{\mathbf{B}}$ only if $\bar{t} \in R^{\mathbf{A}}$. An *embedding* is an injective strong homomorphism. We write $\mathbf{A} \hookrightarrow \mathbf{B}$ if $\mathbf{A}$ embeds to $\mathbf{B}$. A *substructure* of $\mathbf{A}$ is a structure $\mathbf{B}$ over $B \subseteq A$ such that the inclusion map $i\colon B \to A$ is an embedding. An *isomorphism* is a surjective embedding. Two structures $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* if there exists an isomorphism from $\mathbf{A}$ to $\mathbf{B}$. An *automorphism* is an isomorphism from $\mathbf{A}$ to $\mathbf{A}$. The set of all automorphisms of $\mathbf{A}$, denoted by $\mathrm{Aut}(\mathbf{A})$, forms a *permutation group* w.r.t. the map composition [44]. The *orbit* of a tuple $\bar{t} \in A^k$ under the *natural action* of $\mathrm{Aut}(\mathbf{A})$ on A^k is the set $\{g(\bar{t}) \mid g \in \mathrm{Aut}(\mathbf{A})\}$.

An n-ary *polymorphism* of a relational structure $\mathbf{A}$ is a mapping $f\colon A^n \to A$ such that, for every k-ary relation symbol $R \in \tau$ and tuples $\bar{t}_1, \dots, \bar{t}_n \in R^{\mathbf{A}}$, we have $f(\bar{t}_1, \dots, \bar{t}_n) \in R^{\mathbf{A}}$. We say that f *preserves* $\mathbf{A}$ to indicate that f is a polymorphism of $\mathbf{A}$. We might also say that an operation *preserves* a relation R over A if it is a polymorphism of $(A; R)$.

We assume that the reader is familiar with classical *first-order* logic (FO); we allow the first-order formulas $x = y$ and $\bot$. The *positive quantifier-free* fragment of FO is abbreviated by pqf. A first-order τ-formula ϕ is *primitive positive* (pp) if it is of the form $\exists x_1, \dots, x_m (\phi_1 \wedge \dots \wedge \phi_n)$, where each ϕ_i is *atomic*, that is, of the form $\bot$, $x_i = x_j$, or $R(x_{i_1}, \dots, x_{i_\ell})$ for some $R \in \tau$. Note that if $\psi_1, \dots, \psi_n$ are primitive positive formulas, then $\exists x_1, \dots, x_m (\psi_1 \wedge \dots \wedge \psi_n)$ can be re-written into an equivalent primitive positive formula, so we sometimes treat such formulas as primitive positive formulas as well. If $\mathbf{A}$ is a τ-structure and $\phi(x_1, \dots, x_n)$ is a τ-formula with free variables $x_1, \dots, x_n$, then the relation $\{\bar{t} \in A^n \mid \mathbf{A} \models \phi(\bar{t})\}$ is called the *relation defined by* ϕ *in* $\mathbf{A}$, and denoted by $\phi^{\mathbf{A}}$. If Θ is a set of τ-formulas, we say that an n-ary relation has a Θ*-definition* in $\mathbf{A}$ if it is of the form $\phi^{\mathbf{A}}$ for some $\phi \in \Theta$. When we work with tuples $\bar{t}$ in a relation defined by a formula $\phi(x_1, \dots, x_n)$, then we sometimes refer to the entries of $\bar{t}$ through the free variables of ϕ, and write $\bar{t}[x_i]$ instead of $\bar{t}[i]$.

Proposition 2.1 (e.g. [9]). *Let* $\mathbf{A}$ *be a relational structure and* R *a relation over* A*.*

(1) *If* R *has a first-order definition in* $\mathbf{A}$*, then it is preserved by all automorphisms of* $\mathbf{A}$*.*
(2) *If* R *has a primitive positive definition in* $\mathbf{A}$*, then it is preserved by all polymorphisms of* $\mathbf{A}$*.*

The main tool for complexity analysis of CSPs is the concept of *primitive positive constructions* (see Proposition 2.8).

Definition 2.2 ([7]). Let $\mathbf{A}$ and $\mathbf{B}$ be relational structures with signatures τ and σ, respectively. We say that $\mathbf{B}$ is a *(d-dimensional) pp-power* of $\mathbf{A}$ if $B = A^d$ for some $d \geq 1$ and, for every $R \in \tau$, the relation

$$\{(\bar{t}_1[1], \dots, \bar{t}_1[d], \dots, \bar{t}_n[1], \dots, \bar{t}_n[d]) \in A^{n \cdot d} \mid (\bar{t}_1, \dots, \bar{t}_n) \in R^{\mathbf{B}}\}$$

has a pp-definition in $\mathbf{A}$. We say that $\mathbf{B}$ is *pp-constructible* from $\mathbf{A}$ if $\mathbf{B}$ is homomorphically equivalent to a pp-power of $\mathbf{A}$. If $\mathbf{B}$ is a 1-dimensional pp-power of $\mathbf{A}$, then we say that $\mathbf{B}$ is *pp-definable* in $\mathbf{A}$.

Primitive positive constructibility, seen as a binary relation, is transitive [7].

A structure is *ω-categorical* if its first-order theory has exactly one countable model up to isomorphism. The theorem of Engeler, Ryll-Nardzewski, and Svenonius (Theorem 6.3.1 in [44]) asserts that the following statements are equivalent for a countably infinite structure $\mathbf{A}$ with countable signature:

- $\mathbf{A}$ is ω-categorical.
- Every relation over A preserved by all automorphisms of $\mathbf{A}$ has a first-order definition in $\mathbf{A}$.
- For every $k \geq 1$, there are finitely many orbits of k-tuples under the natural action of $\mathrm{Aut}(\mathbf{A})$.

A structure $\mathbf{A}$ is *homogeneous* if every isomorphism between finite substructures of $\mathbf{A}$ extends to an automorphism of $\mathbf{A}$. If the signature of $\mathbf{A}$ is finite, then $\mathbf{A}$ is homogeneous if and only if $\mathbf{A}$ is ω-categorical and admits quantifier-elimination [44]. A structure $\mathbf{A}$ is *finitely bounded* if there is a universal first-order sentence ϕ such that a finite structure embeds into $\mathbf{A}$ if and only if it satisfies ϕ. The standard example of a finitely bounded homogeneous structure is $(\mathbb{Q};<)$ [16].

2.2 Finite variable logics and counting

We denote the fragment of FO in which every formula uses only the variables $x_1,\dots,x_k$ by $\mathcal{L}^k$, and its existential positive fragment by $\exists^+\mathcal{L}^k$. By FOC we denote the extension of FO by the counting quantifiers $\exists^i$. If $\mathbf{A}$ is a τ-structure and ϕ a τ-formula with a free variable x, then $\mathbf{A} \models \exists^i x.\phi(x)$ if and only if there exist i distinct $a \in A$ such that $\mathbf{A} \models \phi(a)$. While FOC is not more expressive than FO, the presence of counting quantifiers might affect the number of variables that are necessary to define a particular relation. The k-variable fragment of FOC is denoted by $\mathcal{C}^k$. The infinitary logic $\mathcal{L}^k_{\infty\omega}$ extends $\mathcal{L}^k$ with infinite disjunctions and conjunctions. The extension of $\mathcal{L}^k_{\infty\omega}$ by the counting quantifiers $\exists^i$ is denoted by $\mathcal{C}^k_{\infty\omega}$, and $\mathcal{C}^{\omega}_{\infty\omega}$ stands for $\bigcup_{k\in\mathbb{N}}\mathcal{C}^k_{\infty\omega}$.

We understand the notion of a *logic* as defined in [41]. Given two τ-structures $\mathbf{A}$ and $\mathbf{B}$ and a logic $\mathcal{L}$, we write $\mathbf{A} \equiv_{\mathcal{L}} \mathbf{B}$ to indicate that a τ-sentence from $\mathcal{L}$ holds in $\mathbf{A}$ if and only if it holds in $\mathbf{B}$, and we write $\mathbf{A} \Rightarrow_{\mathcal{L}} \mathbf{B}$ to indicate that every τ-sentence from $\mathcal{L}$ which is true in $\mathbf{A}$ is also true in $\mathbf{B}$. It is known that, for finite τ-structures $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \equiv_{\mathcal{C}^k_{\infty\omega}} \mathbf{B}$ if and only if $\mathbf{A} \equiv_{\mathcal{C}^k} \mathbf{B}$ [38].

2.3 Fixed-point logics

Inflationary fixed-point logic (IFP) is defined by adding formation rules to FO whose semantics is defined with inflationary fixed-points of arbitrary operators, and *least fixed-point logic* (LFP) is defined by adding formation rules to FO whose semantics is defined using least fixed-points of monotone operators. The logics LFP and IFP are *equivalent* in the sense that they define the same relations over the class of all structures [48]. For this reason, they are both commonly referred to as FP (see, e.g., [2]). *Datalog* is usually understood as the existential positive fragment of LFP (see [29]). The existential positive fragments of LFP and IFP are equivalent, because the fixed-point operator induced by a formula from either of the fragments is monotone, which implies that its least and inflationary fixed-point coincide (see Proposition 10.3 in [51]). This means that we can informally identify Datalog with the existential positive fragment of FP. For the definitions of the counting extensions IFPC and LFPC we refer the reader to [37]. One important detail is that the equivalence LFP $\equiv$ IFP extends to LFPC $\equiv$ IFPC (see p. 189 in [37]). Again, we refer to both counting extensions simply as FPC. It is worth mentioning that the extension of Datalog with counting is also equivalent to FPC [39]. All we need to know about FPC in the present article is Theorem 2.3.

Theorem 2.3 (Immerman and Lander [27]). *For every* FPC *τ-sentence ϕ, there exists $k \in \mathbb{N}$ such that $\mathbf{A} \models \phi \Leftrightarrow \mathbf{B} \models \phi$ holds for all finite τ-structures $\mathbf{A}$ and $\mathbf{B}$ which satisfy $\mathbf{A} \equiv_{\mathcal{C}^k} \mathbf{B}$.*

This result follows from the fact that for every FPC formula ϕ there exists k such that, on structures with at most n elements, ϕ is equivalent to a formula of $\mathcal{C}^k$ whose quantifier depth is bounded by a polynomial function of n [27]. Moreover, every formula of FPC is equivalent to a formula of $\mathcal{C}^k_{\infty\omega}$ for some k, that is, FPC forms a fragment of the infinitary logic $\mathcal{C}^\omega_{\infty\omega}$ (Corollary 4.20 in [57]).

The logic FPR_2 extends FPC by the Boolean rank operator making it the most expressive logic explicitly treated in this article. It adds an additional logical constructor that can be used to form a rank term $[\mathsf{rk}_{x,y}\phi(x,y) \bmod 2]$ from a given formula $\phi(x,y)$. The value of $[\mathsf{rk}_{x,y}\phi(x,y) \bmod 2]$ in an input structure $\mathbf{A}$ is the rank of a Boolean matrix specified by $\phi(x,y)$ through its evaluation in $\mathbf{A}$. For instance, $[\mathsf{rk}_{x,y}(x=y)\wedge\psi(x) \bmod 2]$ computes in an input structure $\mathbf{A}$ the number of elements $a\in A$ such that $\mathbf{A}\models\psi(a)$ for a given formula $\psi(x)$ [28]. The satisfiability of a suitably encoded system of Boolean linear equations $M\bar{x}=\bar{v}$ can be tested in FPR_2 by comparing the rank of M with the rank of the extension of M by $\bar{v}$ as a last column. A thorough definition of FPR_2 can be found in [28, 40]; our version below is rather simplified, e.g., we disallow the use of $\leq$ for comparison of numeric terms, and also the use of free variables over the numerical sort.

Let S be a finite set. A *fixed-point* of an operator $F\colon \mathrm{Pow}(S)\to\mathrm{Pow}(S)$ is an element $X\in\mathrm{Pow}(S)$ with $X=F(X)$. A fixed-point X of F is called *inflationary* if it is the limit of the sequence $X_{i+1}:=X_i\cup F(X_i)$ with $X_0=\emptyset$ in which case we write $X=\mathsf{Ifp}(F)$, and *deflationary* if it is the limit of the sequence $X_{i+1}:=X_i\cap F(X_i)$ with $X_0=S$ in which case we write $X=\mathsf{Dfp}(F)$. The members of either of the sequences are called the *stages* of the induction. Clearly, $\mathsf{Ifp}(F)$ and $\mathsf{Dfp}(F)$ exist and are unique for every such operator F.

Let τ be a relational signature. The set of *inflationary fixed-point (IFP) formulas* over τ is defined inductively as follows. Every atomic τ-formula is an IFP τ-formula and formulas built from IFP τ-formulas using the usual first-order constructors are again IFP τ-formulas. If $\phi(\bar{x},\bar{y})$ is an IFP $(\tau\cup\{R\})$-formula for some relation symbol $R\notin\tau$ of arity k, $\bar{x}$ is k-ary, and $\bar{y}$ is ℓ-ary, then $[\mathsf{ifp}_{R,\bar{x}}\phi]$ is an IFP τ-formula with the same set of free variables. The semantics of inflationary fixed-point logic is defined similarly as for first-order logic; we only discuss how to interpret the inflationary fixed point constructor. Let $\mathbf{A}$ be a finite relational τ-structure. For every $\bar{c}\in A^\ell$, we consider the induced operator $\mathsf{Op}^{\mathbf{A}}[\![\phi(\cdot,\bar{c})]\!]\colon \mathrm{Pow}(A^k)\to\mathrm{Pow}(A^k)$, $X\mapsto\{\bar{t}\in A^k \mid (\mathbf{A};X)\models\phi(\bar{t},\bar{c})\}$. Then $\mathbf{A}\models[\mathsf{ifp}_{R,\bar{x}}\phi](\bar{t},\bar{c})$ if and only if $\bar{t}\in\mathsf{Ifp}\,\mathsf{Op}^{\mathbf{A}}[\![\phi(\cdot,\bar{c})]\!]$. To make our IFP formulas more readable, we introduce the expression $[\mathsf{dfp}_{R,\bar{x}}\phi]$ as a shortcut for the IFP formula $\neg[\mathsf{ifp}_{R,\bar{x}}\neg\phi_{R/\neg R}]$ where $\phi_{R/\neg R}$ is obtained from ϕ by replacing every occurrence of R in ϕ with $\neg R$. Note that $\bar{t}\in\mathsf{Dfp}\,\mathsf{Op}^{\mathbf{A}}[\![\phi(\cdot,\bar{c})]\!]$ if and only if $\mathbf{A}\models[\mathsf{dfp}_{R,\bar{x}}\phi](\bar{t},\bar{c})$.

Finally, we present a simplified version of the Boolean rank operator which is, nevertheless, expressive enough for the purpose of capturing those temporal CSPs that are expressible in FPR_2. We define the set of *numeric terms* over τ inductively as follows.

- Every IFP τ-formula is a numeric term taking values in $\{0,1\}$ corresponding to its truth values when evaluated in $\mathbf{A}$.
- Composing numeric terms with the nullary function symbols 0, 1 and the binary function symbols $+,\cdot$, which have the usual interpretation over $\mathbb{N}$, yields numeric terms taking values in $\mathbb{N}$ when evaluated in $\mathbf{A}$.
- Finally, if $f(\bar{x},\bar{y},\bar{z})$ is a numeric term where $\bar{x}$ is k-ary, $\bar{y}$ is ℓ-ary, and $\bar{z}$ is m-ary, then $[\mathsf{rk}_{\bar{x},\bar{y}}f \bmod 2]$ is a numeric term with free variables consisting of the entries of $\bar{z}$.

We use the notation $f^{\mathbf{A}}$ for the evaluation of a numeric term f in $\mathbf{A}$. For $\bar{c}\in A^m$, we write $\mathsf{M}_2^{\mathbf{A}}[\![f(\cdot,\cdot,\bar{c})]\!]\in\{0,1\}^{A^k\times A^\ell}$ for the matrix M whose entry at the coordinate $(\bar{t},\bar{s})\in A^k\times A^\ell$ is $f^{\mathbf{A}}(\bar{t},\bar{s},\bar{c}) \bmod 2$. Then $[\mathsf{rk}_{\bar{x},\bar{y}}f \bmod 2]^{\mathbf{A}}(\bar{c})$ denotes the rank of the matrix M. The value for $[\mathsf{rk}_{\bar{x},\bar{y}}f \bmod 2]$ is well defined because the rank of $\mathsf{M}_2^{\mathbf{A}}[\![f(\cdot,\cdot,\bar{c})]\!]$ does not depend on the ordering of the rows and the columns. Now we can define the set of FPR_2 τ-formulas.

- Every IFP τ-formula is an FPR_2 τ-formula.
- If $f(\bar{x})$ and $g(\bar{y})$ are numeric terms, then $f=g$ is an FPR_2 τ-formula whose free variables are the entries of $\bar{x}$ and $\bar{y}$.

The latter carries the obvious semantics $\mathbf{A} \models (f = g)(\bar{t}, \bar{s})$ iff $f^{\mathbf{A}}(\bar{t}) = g^{\mathbf{A}}(\bar{s})$.

Example 2.4. $\bigwedge_{j=0}^{i-1} \neg([\mathrm{rk}_{x,y}(x = y) \wedge \phi(x) \bmod 2] = j)$ is equivalent to $\exists^i x.\phi(x)$.

2.4 Logical expressibility of constraint satisfaction problems

The *constraint satisfaction problem* CSP($\mathbf{B}$) for a structure $\mathbf{B}$ with a finite relational signature τ is the computational problem of deciding whether a given finite τ-structure $\mathbf{A}$ maps homomorphically to $\mathbf{B}$. By a standard result from database theory, $\mathbf{A}$ maps homomorphically to $\mathbf{B}$ if and only if the *canonical conjunctive query* $Q_{\mathbf{A}}$ is true in $\mathbf{B}$ [25]; $Q_{\mathbf{A}}$ is the pp-sentence whose variables are the domain elements of $\mathbf{A}$ and whose quantifier-free part is the conjunction of all atomic formulas that hold in $\mathbf{A}$. We might occasionally refer to the atomic subformulas of $Q_{\mathbf{A}}$ as *constraints*. We call $\mathbf{B}$ a *template* of CSP($\mathbf{B}$). A *solution* for an instance $\mathbf{A}$ of CSP($\mathbf{B}$) is a homomorphism $\mathbf{A} \to \mathbf{B}$.

Formally, CSP($\mathbf{B}$) stands for the class of all finite τ-structures that homomorphically map to $\mathbf{B}$. Following Feder and Vardi [31], we say that the CSP of a τ-structure $\mathbf{B}$ is *expressible* in a logic $\mathcal{L}$ if there exists a sentence in $\mathcal{L}$ that defines the complementary class co-CSP($\mathbf{B}$) of all finite τ-structures which do not homomorphically map to $\mathbf{B}$.

Example 2.5. $\exists z[\mathrm{ifp}_{T,(x,y)}(x,y) \vee \exists h(x < h \wedge T(h,y))](z,z)$ defines co-CSP($\mathbb{Q}$; $<$).

Definition 2.6 ([1])**.** Let σ, τ be finite relational signatures. Moreover, let Θ be a set of FPR$_2$ σ-formulas. A *Θ-interpretation of τ in σ with p parameters* is a tuple $\mathcal{I}$ of σ-formulas from Θ consisting of a distinguished $(d+p)$-ary *domain* formula $\delta_{\mathcal{I}}(\bar{x}, \bar{y})$ and, for each $R \in \tau$, an $(n \cdot d + p)$-ary formula $\phi_{\mathcal{I},R}(\bar{x}_1, \dots, \bar{x}_n, \bar{y})$ where $n = \mathrm{ar}(R)$. The *image of* $\mathbf{A}$ *under* $\mathcal{I}$ *with parameters* $\bar{c} \in A^p$ is the τ-structure $\mathcal{I}(\mathbf{A}, \bar{c})$ with domain $\{\bar{t} \in A^d \mid \mathbf{A} \models \delta_{\mathcal{I}}(\bar{t}, \bar{c})\}$ and relations

$$R^{\mathcal{I}(\mathbf{A},\bar{c})} = \{(\bar{t}_1, \dots, \bar{t}_n) \in (A^d)^n \mid \mathbf{A} \models \phi_{\mathcal{I},R}(\bar{t}_1, \dots, \bar{t}_n, \bar{c})\}.$$

Let $\mathbf{B}$ be a σ-structure and $\mathbf{C}$ a τ-structure. We write CSP($\mathbf{B}$) $\leq_\Theta$ CSP($\mathbf{C}$) and say that CSP($\mathbf{B}$) *reduces to* CSP($\mathbf{C}$) *under Θ-reducibility* if there exists a Θ-interpretation $\mathcal{I}$ of τ in σ with p parameters such that, for every finite σ-structure $\mathbf{A}$ with $|A| \geq p$, the following are equivalent:

- $\mathbf{A} \to \mathbf{B}$,
- $\mathcal{I}(\mathbf{A}, \bar{c}) \to \mathbf{C}$ for some injective tuple $\bar{c} \in A^p$,
- $\mathcal{I}(\mathbf{A}, \bar{c}) \to \mathbf{C}$ for every injective tuple $\bar{c} \in A^p$.

Seen as a binary relation, Θ-reducibility is transitive if Θ is any of the standard logical fragments or extensions of FO we have mentioned so far. The following reducibility result was obtained in [1] for finite-domain CSPs. A close inspection of the original proof reveals that the statement holds for infinite-domain CSPs as well.

Theorem 2.7 (Atserias, Bulatov, and Dawar [1])**.** *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be structures with finite relational signatures such that* $\mathbf{B}$ *is pp-constructible from* $\mathbf{C}$. *Then* CSP($\mathbf{B}$) $\leq_{\mathrm{Datalog}}$ CSP($\mathbf{C}$).

It is important to note that, for every $\mathcal{L} \in \{\mathrm{Datalog}, \mathrm{FP}, \mathrm{FPC}, \mathrm{FPR}_2\}$, $\leq_{\mathrm{Datalog}}$ preserves the expressibility of CSPs in $\mathcal{L}$. This (not entirely trivial) fact is only mentioned in [1] for $\mathcal{L} = \mathcal{C}^{\omega}_{\infty\omega}$.

Proposition 2.8. *Let* $\mathbf{B}$, $\mathbf{C}$ *be structures with finite relational signatures. If* CSP($\mathbf{B}$) $\leq_{\mathrm{Datalog}}$ CSP($\mathbf{C}$) *and* CSP($\mathbf{C}$) *is expressible in* $\mathcal{L} \in \{\mathrm{Datalog}, \mathrm{FP}, \mathrm{FPC}, \mathrm{FPR}_2\}$, *then* CSP($\mathbf{B}$) *is expressible in* $\mathcal{L}$.

Proof. We only prove the statement in the case of Datalog. The remaining cases are analogous and in fact even simpler, because FP, FPC and FPR$_2$ allow inequalities.

Let σ be the signature of $\mathbf{B}$, let τ be the signature of $\mathbf{C}$, and let $\phi_{\mathbf{C}}$ be a Datalog τ-sentence that defines co-CSP($\mathbf{C}$). Let $\mathcal{I}$ be an interpretation of τ in σ with p parameters witnessing that

CSP($\mathbf{B}$) $\leq_{\text{Datalog}}$ CSP($\mathbf{C}$). Consider the sentence $\phi'_{\mathbf{B}}$ obtained from $\phi_{\mathbf{C}}$ by the following sequence of syntactic replacements. First, we introduce a fresh p-tuple $\bar{y}$ of existentially quantified variables. Second, we replace each existentially quantified variable x_i in $\phi_{\mathbf{C}}$ by a d-tuple $\bar{x}_i$ of fresh existentially quantified variables and conjoin $\phi_{\mathbf{C}}$ with the formula $\bigwedge_i \delta_{\mathcal{I}}(\bar{x}_i, \bar{y})$. Then, we replace each atomic formula in $\phi_{\mathbf{C}}$ of the form $R(x_{i_1}, \ldots, x_{i_n})$ for $R \in \tau$ by the formula $\phi_{\mathcal{I},R}(\bar{x}_{i_1}, \ldots, \bar{x}_{i_n}, \bar{y})$; we also readjust the arities of the auxiliary relation symbols and the amount of the first-order free variables in each IFP subformula of $\phi_{\mathbf{C}}$. Finally, we conjoin the resulting formula with $\bigwedge_{i \neq j} \bar{y}[i] \neq \bar{y}[j]$. Now, for all σ-structures $\mathbf{A}$ with $|A| \geq p$, we have that $\mathbf{A} \models \phi'_{\mathbf{B}}$ if and only if $\mathcal{I}(\mathbf{A}, \bar{c}) \models \phi_{\mathbf{C}}$ for some injective tuple $\bar{c} \in A^p$. Since $\phi_{\mathbf{C}}$ defines the class of all instances of CSP($\mathbf{C}$) which have no solution, $\phi'_{\mathbf{B}}$ defines the class of all instances of CSP($\mathbf{B}$) with at least p elements which have no solution. Let $\phi''_{\mathbf{B}}$ be the disjunction of the canonical conjunctive queries for all the finitely many instances $\mathbf{A}_1, \ldots, \mathbf{A}_\ell$ of CSP($\mathbf{B}$) with less than p elements which have no solution. Then $\phi''_{\mathbf{B}}$ defines the class of all instances of CSP($\mathbf{B}$) with less than p elements which have no solution. Let $\mathbf{A}$ be a σ-structure with $|A| < p$. If $\mathbf{A} \not\to \mathbf{B}$, then $\mathbf{A} \models Q_{\mathbf{A}_i}$ for some $i \in [\ell]$, which implies $\mathbf{A} \models \phi''_{\mathbf{B}}$. If $\mathbf{A} \to \mathbf{B}$, then $\mathbf{A} \not\models Q_{\mathbf{A}_i}$ for all $i \in [\ell]$, otherwise $\mathbf{A}_i \to \mathbf{A}$ for some $i \in [\ell]$ which yields a contradiction to $\mathbf{A}_i \not\to \mathbf{B}$. Thus $\phi'_{\mathbf{B}} \vee \phi''_{\mathbf{B}}$ defines co-CSP($\mathbf{B}$). We are not finished yet, because $\phi'_{\mathbf{B}} \vee \phi''_{\mathbf{B}}$ is not a valid Datalog sentence. It is, however, a valid sentence in Datalog($\neq$), the expansion of Datalog by inequalities between variables. Note that, if $\mathbf{A} \not\to \mathbf{B}$ and $\mathbf{A} \to \mathbf{A}'$, then $\mathbf{A}' \not\to \mathbf{B}$, i.e., co-CSP($\mathbf{B}$) is a class closed under homomorphisms. Thus, by Theorem 2 in [32], there exists a Datalog sentence $\phi_{\mathbf{B}}$ that defines co-CSP($\mathbf{B}$). We conclude that CSP($\mathbf{B}$) is expressible in Datalog. □

We now introduce a formalism that simplifies the presentation of algorithms for TCSPs.

Definition 2.9 (Projections and contractions of relations)**.** Let $\mathbf{B}$ be a structure with finite relational signature τ. Furthermore, let R be an n-ary symbol from τ. The *projection* of $R^{\mathbf{B}}$ to $I \subseteq [n]$, denoted by $\text{proj}_I(R^{\mathbf{B}})$, is the $|I|$-ary relation defined in $\mathbf{B}$ by the pp-formula $\exists_{j \in [n] \setminus I} x_j . R(x_1, \ldots, x_n)$. We call it *proper* if $I \notin \{\emptyset, [n]\}$, and *trivial* if it equals $B^{|I|}$. The *contraction* of $R^{\mathbf{B}}$ modulo $\sim \subseteq [n]^2$, denoted by $\text{ctrn}_\sim(R^{\mathbf{B}})$, is the n-ary relation defined in $\mathbf{B}$ by the pp-formula $R(x_1, \ldots, x_n) \wedge \big(\bigwedge_{i \sim j} x_i = x_j\big)$.

Whenever it is convenient, we will assume that the set of relations of a temporal structure is closed under projections and contractions and contains the equality relation. Note that adding these relations to a structure does not influence the set of polymorphisms (Proposition 2.1), and it also does not influence the expressibility of its CSP in Datalog, FP, FPC, or FPR_2 (Theorem 2.8).

Definition 2.10 (Projections and contractions of instances)**.** Let $\mathbf{A}$ be an instance of CSP($\mathbf{B}$) for a τ-structure $\mathbf{B}$ that satisfies the assumption below Definition 2.9. For a symbol $R \in \tau$, we write $\text{proj}_I R$ for the symbol which interprets as $\text{proj}_I(R^{\mathbf{B}})$ and $\text{ctrn}_\sim R$ for the symbol which interprets as $\text{ctrn}_\sim(R^{\mathbf{B}})$. The *projection* of $\mathbf{A}$ to $V \subseteq A$ is the τ-structure $\text{proj}_V(\mathbf{A})$ obtained from $\mathbf{A}$ as follows. For every symbol $R \in \tau$ and every tuple $\bar{t} \in R^{\mathbf{A}}$, we remove $\bar{t}$ from $R^{\mathbf{A}}$ and add the tuple $\text{proj}_I(\bar{t})$ to $(\text{proj}_I R)^{\mathbf{A}}$ where I consists of all $i \in [\text{ar}(R)]$ such that $\bar{t}[i] \in V$. Finally, we replace the domain with V. The *contraction* of $\mathbf{A}$ modulo $C \subseteq A^2$ is the τ-structure $\text{ctrn}_C(\mathbf{A})$ obtained from $\mathbf{A}$ as follows. For every symbol $R \in \tau$ and every tuple $\bar{t} \in R^{\mathbf{A}}$, we remove $\bar{t}$ from $R^{\mathbf{A}}$ and add $\bar{t}$ to $(\text{ctrn}_\sim R)^{\mathbf{A}}$ where $i \sim j$ if and only if $(\bar{t}[i], \bar{t}[j]) \in C$. Finally, for every $(a,b) \in C$, we add (a,b) to any relation $R^{\mathbf{A}}$ such that $R \in \tau$ interprets as the equality predicate in $\mathbf{B}$.

2.5 Temporal CSPs

A structure with domain $\mathbb{Q}$ is called *temporal* if its relations are first-order definable in $(\mathbb{Q}; <)$. An important observation is that if $\mathbf{B}$ is a temporal structure and f is an order-preserving map between two finite subsets of $\mathbb{Q}$, then f can be extended to an automorphism of $\mathbf{B}$. This is a consequence of Proposition 2.1 and the fact that $(\mathbb{Q}; <)$ is homogeneous. Relations which are first-order definable in $(\mathbb{Q}; <)$ are called *temporal*. The *dual* of a temporal relation R is defined as $\{(-\bar{t}[1], \ldots, -\bar{t}[\text{ar}(R)]) \mid \bar{t} \in R\}$.

The *dual* of a temporal structure is the temporal structure whose relations are precisely the duals of the relations of the original one. Every temporal structure is homomorphically equivalent to its dual via the map $x \mapsto -x$, which means that both structures have the same CSP. The CSP of a temporal structure is called a *temporal CSP* (TCSP).

Definition 2.11 (Min-sets)**.** The *min-indicator function* $\chi \colon \mathbb{Q}^k \to \{0,1\}^k$ is defined by $\chi(\bar{t})[i] := 1$ if and only if $\bar{t}[i]$ is a minimal entry in $\bar{t}$; we call $\chi(\bar{t}) \in \{0,1\}^k$ the *min-tuple* of $\bar{t} \in \mathbb{Q}^k$. As usual, if $R \subseteq \mathbb{Q}^k$, then $\chi(R)$ denotes $\{\chi(t) \mid t \in R\}$.

For a tuple $\bar{t} \in \mathbb{Q}^k$, we set $\operatorname{argmin}(\bar{t}) := \{i \in [k] \mid \chi(\bar{t})[i] = 1\}$.

Definition 2.12 (Free sets)**.** Let $\mathbf{B}$ be a temporal structure with signature τ and $\mathbf{A}$ an instance of CSP($\mathbf{B}$). A *free set* of $\mathbf{A}$ is a non-empty subset $F \subseteq A$ such that, if $R \in \tau$ is k-ary and $\bar{s} \in R^{\mathbf{A}}$, then either no entry of $\bar{s}$ is contained in F, or there exists a tuple $\bar{t} \in R^{\mathbf{B}}$ such that $\operatorname{argmin}(\bar{t}) = \{i \in [k] \mid \bar{s}[i] \in F\}$. If $R \in \tau$ has arity k and $\bar{s} \in A^k$, we define the *system of min-sets* $\mathrm{SMS}_R(\bar{s})$ as the set of all $M \subseteq \{\bar{s}[1], \ldots, \bar{s}[k]\}$ for which there exists $\bar{t} \in R^{\mathbf{B}}$ such that $\operatorname{argmin}(\bar{t}) = \{i \in [k] \mid \bar{s}[i] \in M\}$. For a subset V of $\{\bar{s}[1], \ldots, \bar{s}[k]\}$, we define $\downarrow_R[\![V]\!](\bar{s})$ as the set of all $M \in \mathrm{SMS}_R(\bar{s})$ such that $M \subseteq V$, and $\uparrow_R[\![V]\!](\bar{s})$ as the set of all $M \in \mathrm{SMS}_R(\bar{s})$ such that $V \subseteq M$.

2.6 Clones

An at least unary operation on a set A is called a *projection onto the i-th coordinate*, and denoted by proj_i, if it returns the i-th argument for each input value. A set of $\mathscr{A}$ operations over a fixed set A is called a *clone* (over A) if it contains all projections and, whenever $f \in \mathscr{A}$ is n-ary and $g_1, \ldots, g_n \in \mathscr{A}$ are m-ary, then $f(g_1, \ldots, g_n) \in \mathscr{A}$. The set of all polymorphisms of a relational structure $\mathbf{A}$, denoted by Pol($\mathbf{A}$), is a clone. For instance, the clone $\mathrm{Pol}(\{0,1\}; 1\mathrm{IN}3)$ consists of all projection maps on $\{0,1\}$, and is called the *projection clone* [21].

Definition 2.13. A map $\xi : \mathscr{A} \to \mathscr{B}$ is called

- a *clone homomorphism* if it preserves arities, projections, and compositions, that is,

$$\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$$

  holds for all n-ary f and m-ary $g_1, \ldots, g_n$ from $\mathscr{A}$,
- a *minion homomorphism* if it preserves arities, projections and those compositions as above where $g_1, \ldots, g_n$ are projections,
- *uniformly continuous* if for every finite $B' \subseteq B$ there exists a finite $A' \subseteq A$ such that if $f, g \in \mathscr{A}$ of the same arity agree on A', then $\xi(f)$ and $\xi(g)$ agree on B'.

The recently closed finite-domain CSPs tractability conjecture can be reformulated as follows: the polymorphism clone of a finite structure $\mathbf{A}$ either admits a minion homomorphism to the projection clone in which case CSP($\mathbf{A}$) is NP-complete, or it does not and CSP($\mathbf{A}$) is polynomial-time tractable [7]. The former is the case if and only if $\mathbf{A}$ pp-constructs all finite structures. For detailed information about clones and clone homomorphisms we refer the reader to [7, 17].

2.7 Polymorphisms of temporal structures

The following notions were used in the P versus NP-complete complexity classification of TCSPs [14]. Let min denote the binary minimum operation on $\mathbb{Q}$. The *dual* of a k-ary operation f on $\mathbb{Q}$ is the map $(x_1, \ldots, x_k) \mapsto -f(-x_1, \ldots, -x_n)$. Let us fix any endomorphisms α, β, γ of $(\mathbb{Q}; <)$ such that $\alpha(x) < \beta(x) < \gamma(x) < \alpha(x + \varepsilon)$ for every $x \in \mathbb{Q}$ and every $\varepsilon \in \mathbb{Q}_{>0}$. Such unary operations can be constructed inductively, see the paragraph below Lemma 26 in [14]. Then mi is the binary operation on $\mathbb{Q}$ defined by

$$\mathrm{mi}(x,y) := \begin{cases} \alpha(\min(x,y)) & \text{if } x = y, \\ \beta(\min(x,y)) & \text{if } x > y, \\ \gamma(\min(x,y)) & \text{if } x < y, \end{cases}$$

and mx is the binary operations on $\mathbb{Q}$ defined by

$$\mathrm{mx}(x,y) := \begin{cases} \alpha(\min(x,y)) & \text{if } x \neq y, \\ \beta(\min(x,y)) & \text{if } x = y. \end{cases}$$

Note that the operations mi and mx both refine the kernel of min. Let ll be an arbitrary binary operation on $\mathbb{Q}$ such that $\mathrm{ll}(a,b) < \mathrm{ll}(a',b')$ if

- $a \leq 0$ and $a < a'$, or
- $a \leq 0$ and $a = a'$ and $b < b'$, or
- $a, a' > 0$ and $b < b'$, or
- $a > 0$ and $b = b'$ and $a < a'$.

Theorem 2.14 (Bodirsky and Kára [14,20])**.** *Let* **B** *be a temporal structure. Either* **B** *is preserved by* min*,* mi*,* mx*,* ll*, the dual of one of these operations, or a constant operation and* $\mathrm{CSP}(\mathbf{B})$ *is in P, or* **B** *pp-constructs* $(\{0,1\}; 1\mathrm{IN}3)$ *and* $\mathrm{CSP}(\mathbf{B})$ *is NP-complete.*

There are two additional operations that appear in soundness proofs of algorithms for TCSPs; pp is an arbitrary binary operation on $\mathbb{Q}$ that satisfies $\mathrm{pp}(a,b) \leq \mathrm{pp}(a',b')$ if and only if

- $a \leq 0$ and $a \leq a'$, or
- $0 < a$, $0 < a'$, and $b \leq b'$,

and lex is an arbitrary binary operation on $\mathbb{Q}$ that satisfies $\mathrm{lex}(a,b) < \mathrm{lex}(a',b')$ if and only if

- $a < a'$, or
- $a = a'$ and $b < b'$.

If a temporal structure is preserved by min, mi, or mx, then it is preserved by pp, and if a temporal structure is preserved by ll, then it is preserved by lex [14].

3 Fixed-point algorithms for TCSPs

In this section, we discuss the expressibility in FP for some particularly chosen TCSPs that are provably in P. By Theorem 2.14, a TCSP is polynomial-time tractable if its template is preserved by one of the operations min, mi, mx, or ll. In the case of min, the known algorithm from [14] can be formulated as an FP algorithm. In the case of mi and ll, the known algorithms from [14, 15] cannot be implemented in FP as they involve choices of arbitrary elements. We show that there exist choiceless versions that can be turned into FP sentences. In the case of mx, the known algorithm from [14] cannot be turned into an FP sentence because it relies on the use of linear algebra. We show in Section 4 that, in general, the CSP of a temporal structure preserved by mx cannot be expressed in FP but it can be expressed in the logic FPR_2.

3.1 A procedure for TCSPs with a template preserved by pp

We first describe a procedure for temporal languages preserved by pp as it appears in [14], and then the choiceless version that is necessary for the translation into an FP sentence.

Let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\mathbf{B})$. The original procedure searches for a non-empty set $S \subseteq A$ for which there exists a solution $\mathbf{A} \to \mathbf{B}$ under the assumption that the projection of $\mathbf{A}$ to $A \setminus S$ has a solution as an instance of $\mathrm{CSP}(\mathbf{B})$. It was shown in [14] that S has this property if it is a free set of $\mathbf{A}$, and that $\mathbf{A} \not\to \mathbf{B}$ if no free set of $\mathbf{A}$ exists. We improve the original result by showing that the same holds if we replace "a free set" in the statement above with "a non-empty union of free sets".

Proposition 3.1. *Let* $\mathbf{A}$ *be an instance of* $\mathrm{CSP}(\mathbf{B})$ *for some temporal structure* $\mathbf{B}$ *preserved by* pp *and let* S *a union of free sets of* $\mathbf{A}$. *Then* $\mathbf{A}$ *has a solution if and only if* $\mathrm{proj}_{A\setminus S}(\mathbf{A})$ *has a solution.*

Proof. Let $F_1,\dots,F_k$ be free sets of $\mathbf{A}$ and set $S := F_1 \cup \dots \cup F_k$. Clearly, if $\mathbf{A}$ has a solution then so has $\mathrm{proj}_{A\setminus S}(\mathbf{A})$. For the converse, suppose that $\mathrm{proj}_{A\setminus S}(\mathbf{A})$ has a solution s. Let $S_j := F_j \setminus (F_1 \cup \dots \cup F_{j-1})$ for every $j \in \{1,\dots,k\}$. We claim that a map $s' : A \to \mathbb{Q}$ is a solution to $\mathbf{A}$ if $s'|_{A\setminus S} = s$, $s'(S_1) < s'(S_2) < \dots < s'(S_k) < s'(A \setminus S)$, and $s'(x) = s'(y)$ whenever there exists $i \in [k]$ such that $x,y \in S_i$. To verify this, let $\bar{s}$ be an arbitrary tuple from $R^{\mathbf{A}} \subseteq A^m$ such that, without loss of generality, $\{\bar{s}[1],\dots,\bar{s}[m]\} \cap S = \{\bar{s}[1],\dots,\bar{s}[\ell]\} \neq \emptyset$. By the definition of $\mathrm{proj}_{A\setminus S}(\mathbf{A})$, there is a tuple $\bar{t} \in R^{\mathbf{B}}$ such that $\bar{t}[i] = s(\bar{s}[i])$ for every $i \in \{\ell+1,\dots,m\}$. Since $F_1,\dots,F_k$ are free, there are tuples $\bar{t}_1,\dots,\bar{t}_k \in R^{\mathbf{B}}$ such that, for every $i \in [k]$ and every $j \in [m]$, we have $j \in \mathrm{argmin}(\bar{t}_i)$ if and only if $\bar{s}[j] \in F_i$. For every $i \in [k]$ let $\alpha_i \in \mathrm{Aut}(\mathbb{Q};<)$ be such that α_i maps the minimal entry of $\bar{t}_i$ to 0. The tuple $\bar{r}_i := \mathrm{pp}(\alpha_i\bar{t}_i,\bar{t})$ is contained in $R^{\mathbf{B}}$ because $R^{\mathbf{B}}$ is preserved by pp. It follows from the definition of pp that, for all $j \in [m]$, $j \in \mathrm{argmin}(\bar{r}_i)$ if and only if $\bar{s}[j] \in F_i$. Moreover, $(\bar{r}_i[\ell+1],\dots,\bar{r}_i[m])$ and $(\bar{t}[\ell+1],\dots,\bar{t}[m])$ lie in the same orbit of $\mathrm{Aut}(\mathbb{Q};<)$. Define $\bar{p}_k,\bar{p}_{k-1},\dots,\bar{p}_1 \in \mathbb{Q}^m$ in this order as follows. Define $\bar{p}_k := \bar{r}_k$ and, for $i \in \{1,\dots,k-1\}$, $\bar{p}_i := \mathrm{pp}(\beta_i\bar{r}_i,\bar{p}_{i+1})$ where $\beta_i \in \mathrm{Aut}(\mathbb{Q};<)$ is chosen such that $\beta_i(\bar{r}_i[j]) = 0$ for all $j \in \mathrm{argmin}(\bar{r}_i)$. We verify by induction that for all $i \in [k]$

(1) $\bar{p}_i$ is contained in $R^{\mathbf{B}}$.
(2) $(\bar{p}_i[\ell+1],\dots,\bar{p}_i[m])$, $(\bar{t}[\ell+1],\dots,\bar{t}[m])$ lie in the same orbit of $\mathrm{Aut}(\mathbb{Q};<)$.
(3) $j \in \mathrm{argmin}(\bar{p}_i)$ if and only if $\bar{s}[j] \in F_i$ for all $j \in [m]$.
(4) $\bar{p}_i[u] = \bar{p}_i[v]$ for all $a \in \{i+1,\dots,k\}$ and $u,v \in [m]$ such that $\bar{s}[u],\bar{s}[v] \in S_a$.
(5) $\bar{p}_i[u] < \bar{p}_i[v]$ for all $a,b \in \{i,i+1,\dots,k\}$ with $a<b$ and $u,v \in [m]$ such that $\bar{s}[u] \in S_a$, $\bar{s}[v] \in S_b$.

For $i = k$, the items (1), (2), and (3) follow from the respective property of $\bar{r}_k$ and items (4) and (5) are trivial. For the induction step and $i \in [k-1]$ we have that $\bar{p}_i = \mathrm{pp}(\beta_i\bar{r}_i,\bar{p}_{i+1})$ satisfies items (1) and (2) because $\bar{p}_{i+1}$ satisfies items (1) and (2) by inductive assumption. For item (3), note that $\mathrm{argmin}(\bar{p}_i) = \mathrm{argmin}(\bar{r}_i)$. Finally, if $\bar{s}[u],\bar{s}[v] \in S_{i+1} \cup \dots \cup S_k$, then $\bar{p}_i[u] \leq \bar{p}_i[v]$ if and only if $\bar{p}_{i+1}[u] \leq \bar{p}_{i+1}[v]$. This implies items (4) and (5) by induction. Note that $(s'(\bar{s}[1]),\dots,s'(\bar{s}[m]))$ lies in the same orbit as $\bar{p}_1$ and hence is contained in $R^{\mathbf{B}}$. ☐

A recursive application of Proposition 3.1 shows the soundness of our choiceless version of the original algorithm which can be found in Figure 1. Its completeness follows from the fact that every instance of a temporal CSP which has a solution must have a free set, namely the set of all variables which denote the minimal value in the solution. Suitable Ptime procedures for finding unions of free sets for TCSPs with a template preserved by min, mi, or mx exist by the results of [14], and they generally exploit the algebraic structure of the CSP that is witnessed by one of these operations. We revisit them in Section 3.2, Section 3.3, and Section 4.1.

Corollary 3.2. *Let* $\mathbf{B}$ *be a temporal structure preserved by* pp. *Let* $\phi(x)$ *be an* FPR_2 *formula in the signature of* $\mathbf{B}$ *extended by a unary symbol* U *such that, for every instance* $\mathbf{A}$ *of* $\mathrm{CSP}(\mathbf{B})$ *and every* $U \subseteq A$, $(\mathbf{A};U) \models \phi(x)$ *iff* x *is not contained in a free set of* $\mathrm{proj}_U(\mathbf{A})$. *Then* $\mathbf{A} \not\to \mathbf{B}$ *iff* $\mathbf{A} \models \exists x[\mathsf{dfp}_{U,x}\phi(x)](x)$.

3.2 An FP algorithm for TCSPs preserved by min

For TCSPs with a template preserved by min, the algorithm in Figure 2 can be used for finding the union of all free sets due to the following lemma. It can be proved by a simple induction using the

Input: An instance $\mathbf{A}$ of CSP($\mathbf{B}$) for a temporal structure $\mathbf{B}$
Output: *true* or *false*
while $\mathbf{A}$ *changes* **do**
    $S \leftarrow$ the union of all free sets of $\mathbf{A}$;
    $\mathbf{A} \leftarrow \text{proj}_{A \setminus S} \mathbf{A}$;
return $A = \emptyset$;

Figure 1: A choiceless algorithm that decides whether an instance of a temporal CSP with a template preserved by pp has a solution using an oracle for testing the containment in a free set.

Input: An instance $\mathbf{A}$ of CSP($\mathbf{B}$) for a temporal structure $\mathbf{B}$
Output: A subset $F \subseteq A$
$F \leftarrow A$;
while F *changes* **do**
    forall $\bar{s} \in R^{\mathbf{A}}$ **do**
        if $\{\bar{s}[1], \ldots, \bar{s}[\text{ar}(R)]\} \cap F \neq \emptyset$ **then**
            $F \leftarrow (F \setminus \{\bar{s}[1], \ldots, \bar{s}[\text{ar}(R)]\}) \cup \bigcup \downarrow_R [\![\{\bar{s}[1], \ldots, \bar{s}[\text{ar}(R)]\} \cap F]\!](\bar{s})$;
return F;

Figure 2: A choiceless algorithm that computes the union of all free sets for temporal CSPs with a template preserved by the operation min.

observation that, for every $\bar{s} \in R^{\mathbf{A}}$ and every $V \subseteq \{\bar{s}[1], \ldots, \bar{s}[\text{ar}(R)]\}$, the set $\downarrow_R [\![V]\!](\bar{s}) \cup \{\emptyset\}$ is closed under taking unions.

Lemma 3.3 ([14]). *Let* $\mathbf{A}$ *be an instance of* CSP($\mathbf{B}$) *for a temporal structure* $\mathbf{B}$ *preserved by a binary operation f such that $f(0,0) = f(0,x) = f(x,0)$ for every $x > 0$. Then the set returned by the algorithm in Figure 2 is the union of all free sets of* $\mathbf{A}$.

The following lemma in combination with Theorem 2.8 shows that instead of presenting an FP algorithm for each TCSP with a template preserved by min, it suffices to present one for CSP($\mathbb{Q}; \text{R}^{\leq}_{\text{min}}, <$) where

$$\text{R}^{\leq}_{\text{min}} := \{(x,y,z) \in \mathbb{Q}^3 \mid y \leq x \vee z \leq x\}.$$

Lemma 3.4. *A temporal relation is preserved by* min *if and only if it is pp-definable in* $(\mathbb{Q}; <, \text{R}^{\leq}_{\text{min}})$.

The following syntactic description of the temporal relations preserved by min is due to Bodirsky, Chen, and Wrona [11].

Proposition 3.5 ([11], page 9). *A temporal relation is preserved by* min *if and only if it can be defined by a conjunction of formulas of the form $z_1 \circ_1 x \vee \cdots \vee z_n \circ_n x$, where $\circ_i \in \{<, \leq\}$.*

Proof of Lemma 3.4. The backward implication is a direct consequence of Proposition 3.5.

For the forward implication, we show that every temporal relation defined by a formula of the form $z_1 \circ_1 x \vee \cdots \vee z_n \circ_n x$, where $\circ_i \in \{<, \leq\}$, has a pp-definition in $(\mathbb{Q}; \text{R}^{\leq}_{\text{min}}, <)$. Then the statement follows from Proposition 3.5. A pp-definition $\phi'_n(x, z_1, \ldots, z_n)$ for the relation defined by $z_1 \leq x \vee \cdots \vee z_n \leq x$ can be obtained by the following simple induction.

In the *base case* $n = 3$ we set $\phi'_3(x, z_1, z_2) := \text{R}^{\leq}_{\text{min}}(x, z_1, z_2)$.

In the *induction step*, we suppose that $n > 3$ and that ϕ'_{n-1} is a pp-definition for the relation defined by $z_1 \leq x \vee \cdots \vee z_{n-1} \leq x$. Then

$$\phi'_n(x, z_1, \ldots, z_n) := \exists h \big(\mathrm{R}^{\leq}_{\min}(x, z_1, h) \wedge \phi'_{n-1}(h, z_2, \ldots, z_n)\big)$$

is a pp-definition of the relation defined by $z_1 \leq x \vee \cdots \vee z_n \leq x$. Finally,

$$\phi_n(x, z_1, \ldots, z_n) = \exists z'_1, \ldots, z'_n (\phi'_n(x, z'_1, \ldots, z'_n) \wedge \bigwedge_{i \in I} z_i < z'_i \wedge \bigwedge_{i \notin I} z'_i = z_i)$$

is a pp-definition of the relation defined by $z_1 \circ_1 x \vee \cdots \vee z_n \circ_n x$ where $\circ_i$ equals $<$ if $i \in I$ and $\leq$ otherwise. □

In the case of $\mathrm{CSP}(\mathbb{Q}; <, \mathrm{R}^{\leq}_{\min})$ a procedure from [14] for finding free sets can be directly implemented in FP.

Proposition 3.6. $\mathrm{CSP}(\mathbb{Q}; <, \mathrm{R}^{\leq}_{\min})$ *is expressible in* FP.

Note that proper projections of all relations of the temporal structure above are trivial, and thus taking a projection of an instance of $\mathrm{CSP}(\mathbb{Q}; \mathrm{R}^{\leq}_{\min}, <)$ is equivalent w.r.t. satisfiability to taking a substructure on the same set. This observation is important for the proof of Proposition 3.6.

Proof. Let $\mathbf{B} := (\mathbb{Q}; \mathrm{R}^{\leq}_{\min}, <)$. Recall that $\mathbf{B}$ is preserved by pp. Our aim is to construct a formula $\phi(x)$ satisfying the requirements of Corollary 3.2 by rewriting the algorithm in Figure 2 in the syntax of FP. In addition to the unary fixed-point variable U coming from Corollary 3.2, we introduce a fresh unary fixed-point variable V for the union F of all free sets of the current projection. The algorithm in Figure 2 computes F using a deflationary induction where parts of the domain which cannot be contained in any free set are gradually cut off. Thus, we may choose $\phi(x)$ to be of the form $\neg[\mathsf{dfp}_{V,x}\psi(x)](x)$ for some formula $\psi(x)$ testing whether whenever the variable x is contained in $\{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\} \cap F$ for some constraint $R(\bar{s})$, then it is also contained in the largest min-set within $\{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\} \cap F$. It is easy to see that

$$\bigcup \downarrow_{<} [\![\{\bar{s}[1], \bar{s}[2]\} \cap F]\!](\bar{s}) = (\{\bar{s}[1], \bar{s}[2]\} \cap F) \setminus \{\bar{s}[1]\} \tag{1}$$

$$\bigcup \downarrow_{\mathrm{R}^{\leq}_{\min}} [\![\{\bar{s}[1], \bar{s}[2], \bar{s}[3]\} \cap F]\!](\bar{s}) = \begin{cases} \{\bar{s}[1], \bar{s}[2], \bar{s}[3]\} \cap F & \text{if } F \cap \{\bar{s}[2], \bar{s}[3]\} \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

This leads to the formula

$$\psi(x) := U(x) \wedge \forall y, z \Big(\overbrace{\big(U(y) \Rightarrow \neg(y < x)\big)}^{(1)} \wedge \overbrace{\big((U(y) \wedge U(z)) \Rightarrow (V(y) \vee V(z) \vee \neg \mathrm{R}^{\leq}_{\min}(x, y, z))\big)}^{(2)}\Big).$$

Therefore, the statement of the proposition follows from Corollary 3.2. □

To increase readability, the formula $\phi(x)$ in the proof of Proposition 3.6 can we rewritten into the following formula, using the conversion rule from dfp to ifp:

$$[\mathsf{ifp}_{V,x} U(x) \Rightarrow \exists y, z \big((U(y) \wedge y < x) \vee (U(y) \wedge U(z) \wedge V(y) \wedge V(z) \wedge \mathrm{R}^{\leq}_{\min}(x, y, z))\big)](x).$$

Input: An instance $\mathbf{A}$ of CSP($\mathbf{B}$) for a temporal structure $\mathbf{B}$
Output: A subset $F \subseteq A$
$F \leftarrow$ the empty unary relation;
forall $x \in A$ **do**
  $F_x \leftarrow \{x\}$;
  while F_x *changes* **do**
    forall $\bar{s} \in R^{\mathbf{A}}$ *such that* $\{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\} \cap F_x \neq \emptyset$ **do**
      if $\uparrow_R [\![F_x \cap \{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\}]\!](\bar{s}) \neq \emptyset$ **then**
        $F_x \leftarrow F_x \cup \bigcap \uparrow_R [\![F_x \cap \{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\}]\!](\bar{s})$;
      else $F_x \leftarrow$ the empty unary relation;
  $F \leftarrow F \cup F_x$;
return F;

Figure 3: A choiceless algorithm that computes the union of all free sets for temporal CSPs with a template preserved by a binary operation f such that $f(0,0) < f(0,x)$ and $f(0,0) < f(x,0)$ for every $x > 0$.

3.3 An FP algorithm for TCSPs preserved by mi

For TCSPs with a template preserved by mi, the algorithm in Figure 2 can be used for finding the union of all free sets due to the following lemma. It can be proved by a simple induction using the observation that, for every $\bar{s} \in R^{\mathbf{A}}$ and every $V \subseteq \{\bar{s}[1], \ldots, \bar{s}[\mathrm{ar}(R)]\}$, the set $\uparrow_R [\![V]\!](\bar{s}) \cup \{\emptyset\}$ is closed under taking intersections.

Lemma 3.7 ([14]). *Let* $\mathbf{A}$ *be an instance of* CSP($\mathbf{B}$) *for a temporal structure* $\mathbf{B}$ *preserved by a binary operation* f *such that* $f(0,0) < f(0,x)$ *and* $f(0,0) < f(x,0)$ *for every* $x > 0$*. Then the set returned by the algorithm in Figure 3 is the union of all free sets of* $\mathbf{A}$.

The following lemma in combination with Theorem 2.8 shows that instead of presenting an FP algorithm for each TCSP with a template preserved by mi, it suffices to present one for CSP($\mathbb{Q}; \mathrm{R_{mi}}, \mathrm{S_{mi}}, \neq$) where

$$\mathrm{R_{mi}} := \{(x,y,z) \in \mathbb{Q}^3 \mid y < x \vee z \leq x\} \quad \text{and} \quad \mathrm{S_{mi}} := \{(x,y,z) \in \mathbb{Q}^3 \mid x \neq y \vee z \leq x\}.$$

Lemma 3.8. *A temporal relation is preserved by* mi *if and only if it is pp-definable in* $(\mathbb{Q}; \mathrm{R_{mi}}, \mathrm{S_{mi}}, \neq)$.

The following syntactic description is due to Michał Wrona.

Proposition 3.9 (see, e.g., [10]). *A temporal relation is preserved by* mi *if and only if it can be defined as conjunction of formulas of the form*

$$z_1 \neq x \vee \cdots \vee z_n \neq x \vee y_1 < x \vee \cdots \vee y_m < x \vee y \leq x \qquad (\dagger)$$

where the last disjunct $y \leq x$ *can be omitted.*

Proof of Lemma 3.8. The backward direction is a direct consequence of Proposition 3.9.

For the forward direction, it suffices by Proposition 3.9 to show that every temporal relation defined by a formula of the form (†), where the last disjunct $y \leq x$ can be omitted, has a pp-definition in $(\mathbb{Q}; <, \mathrm{R}^{\leq}_{\mathrm{min}})$. We prove the statement by induction on m and n. Note that both $\leq$ and $<$ have a pp-definition in $(\mathbb{Q}; \mathrm{R_{mi}}, \mathrm{S_{mi}}, \neq)$. For $m, n \geq 0$, let $R_{m,n}$ denote the $(m+n+2)$-ary temporal relation defined by the formula (†), where we assume that all variables are distinct and in their respective order $x, y_1, \ldots, y_m, z_1, \ldots, z_n, y$.

In the *base case* $m = n = 1$, we set $\phi_{1,0}(x, y_1, y) = \mathrm{R}_{\mathrm{mi}}(x, y_1, y)$ and $\phi_{0,1}(x, z_1, y) = \mathrm{S}_{\mathrm{mi}}(x, z_1, y)$. The *induction step* is divided into three individual claims.

Claim 3.10. *If $\phi_{m-1,0}(x, y_1, \ldots, y_{m-1}, y)$ is a pp-definition of $R_{m-1,0}$, then*

$$\phi_{m,0}(x, y_1, \ldots, y_m, y) := \exists h\big(\phi_{1,0}(h, y_m, y) \wedge \phi_{m-1,0}(x, y_1, \ldots, y_{m-1}, h)\big)$$

is a pp-definition of $R_{m,0}$.

Proof of Claim 3.10. "⇒" Let $\bar{t} \in R_{m,0}$. We have to show that $\bar{t}$ satisfies $\phi_{m,0}$. In case that $\bar{t}[x] > \min(\bar{t}[y_1], \ldots, \bar{t}[y_{m-1}])$ we set $h := \bar{t}[y]$. Otherwise, $\bar{t}[x] > \bar{t}[y_m]$ or $\bar{t}[x] \geq \bar{t}[y]$, in which case we set $h := \bar{t}[x]$.

"⇐" Suppose for contradiction that $\bar{t} \notin R_{m,0}$ satisfies $\phi_{m,0}$ and that this is witnessed by some $h \in \mathbb{Q}$. Since $\bar{t}[x] \leq \min(\bar{t}[y_1], \ldots, \bar{t}[y_{m-1}])$, we must have $\bar{t}[x] \geq h$. But since $\bar{t}[x] \leq \bar{t}[y_m]$ and $\bar{t}[x] < \bar{t}[y]$, we get a contradiction to $\phi_{1,0}(h, \bar{t}[y_m], \bar{t}[y])$ being satisfied. □

Claim 3.11. *If $\phi_{0,n-1}(x, z_1, \ldots, z_{n-1}, y)$ is a pp-definition of $R_{0,n-1}$, then*

$$\phi_{0,n}(x, z_1, \ldots, z_n, y) := \exists h\big(\phi_{0,1}(h, z_n, y) \wedge \phi_{0,n-1}(x, z_1, \ldots, z_{n-1}, h)\big)$$

is a pp-definition of $R_{0,n}$.

The proofs of this claim and the next claim are similar to the proof of the previous claim and omitted.

Claim 3.12. *Let $\phi_{m,0}(x, y_1, \ldots, y_m, y)$ and $\phi_{0,n}(x, z_1, \ldots, z_n, y)$ be pp-definitions of $R_{m,0}$ and $R_{0,n}$, respectively. Then*

$$\phi_{m,n}(x, y_1, \ldots, y_m, z_1, \ldots, z_n, y) = \exists h\big(\phi_{m,0}(x, y_1, \ldots, y_m, h) \wedge \phi_{0,n}(h, y_1, \ldots, y_m, y)\big)$$

is a pp-definition of $R_{m,n}$.

This completes the proof of the lemma because the last clause $y \leq x$ in (†) can be easily eliminated using an additional existentially quantified variable and the relation $<$. □

Proposition 3.13. $\mathrm{CSP}(\mathbb{Q}; \mathrm{R}_{\mathrm{mi}}, \mathrm{S}_{\mathrm{mi}}, \neq)$ *is expressible in* FP.

As in the case of the structure considered in Proposition 3.6, the temporal structure above was particularly chosen so that proper projections of all its relations are trivial. Thus, taking a projection of an instance of $\mathrm{CSP}(\mathbb{Q}; \mathrm{R}_{\mathrm{mi}}, \mathrm{S}_{\mathrm{mi}}, \neq)$ is equivalent w.r.t. satisfiability to taking a substructure on the same set.

Proof. Let $\mathbf{B} := (\mathbb{Q}; \mathrm{R}_{\mathrm{mi}}, \mathrm{S}_{\mathrm{mi}}, \neq)$. The notation $\neq(x, y)$ should not be confused with $\neg(x = y)$: the former is an atomic τ-formula because $\neq$ is part of the signature, while the latter is a valid first-order formula because equality is a built-in part of first-order logic. Recall that $\mathbf{B}$ is preserved by pp. Our aim is to construct a formula $\phi(x)$ satisfying the requirements of Corollary 3.2 by rewriting the algorithm in Figure 3 in the syntax of FP. In addition to the unary fixed-point variable U coming from Corollary 3.2, we introduce a fresh binary fixed-point variable V for the free set propagation relation $\{(x, y) \mid y \in F_x\}$ computed during the algorithm in Figure 2. The computation takes place through inflationary induction where a pair (x, y) is added to the relation if the containment of x in a free set implies the containment of y. The algorithm concludes that a variable x is contained in a free set if there are no $x_1, \ldots, x_k \in F_x$ whose containment in the same free set would lead to a contradiction. Note that $\neq$ is the only relation without a constant polymorphism among the relations of $\mathbf{B}$, i.e., the only relation for which the *if* condition in the algorithm in Figure 3 can evaluate as false. Thus $\phi(x)$ may be chosen to be of the form

$$U(x) \Rightarrow \exists y,z\big([\mathsf{ifp}_{V,x,y}\,\psi(x,y)](x,y) \wedge [\mathsf{ifp}_{V,x,z}\,\psi(x,z)](x,z) \wedge {\neq}(y,z)\big)$$

for some formula $\psi(x,y)$ defining the (transitive) free-set propagation relation. It is easy to see that

$$\bigcap {\uparrow}_{\mathrm{R_{mi}}}[\![F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\}]\!](\bar{s}) = \begin{cases} (F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\}) \cup \{\bar{s}[3]\} & \text{if } F_x \cap \{\bar{s}[1],\bar{s}[3]\} = \{\bar{s}[1]\}, \\ F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\} & \text{otherwise,} \end{cases}$$

$$\bigcap {\uparrow}_{\mathrm{S_{mi}}}[\![F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\}]\!](\bar{s}) = \begin{cases} (F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\}) \cup \{\bar{s}[3]\} & \text{if } F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\} = \{\bar{s}[1],\bar{s}[2]\}, \\ F_x \cap \{\bar{s}[1],\bar{s}[2],\bar{s}[3]\} & \text{otherwise.} \end{cases}$$

This leads to the formula

$$\begin{aligned}\psi(x,y) :=\ & U(x) \wedge U(y) \wedge \big((x=y) \\ & \vee \exists a,b,c\big(U(a) \wedge U(b) \wedge U(c) \wedge V(x,a) \wedge V(x,b) \wedge (\mathrm{R_{mi}}(a,c,y) \vee \mathrm{S_{mi}}(a,b,y))\big)\big)\end{aligned}$$

Now the statement of the proposition follows from Corollary 3.2. ☐

3.4 An FP algorithm for TCSPs preserved by ll

If a temporal structure $\mathbf{B}$ is preserved by ll, then it is also preserved by lex, but not necessarily by pp. In general, the choiceless procedure based on Proposition 3.1 is then not correct for CSP($\mathbf{B}$). We present a modified version of this procedure, motivated by the approach of repeated contractions from [15], and show that this version is correct for CSP($\mathbf{B}$).

Let $\mathbf{A}$ be an instance of CSP($\mathbf{B}$). We repeatedly simulate on $\mathbf{A}$ the choiceless procedure based on Proposition 3.1 and, each time a union S of free sets is computed, we contract in $\mathbf{A}$ all variables in every free set within S that is minimal among all existing free sets in the current projection with respect to set inclusion. This loop terminates when a fixed-point is reached, where $\mathbf{A}$ no longer changes, in which case we accept. The resulting algorithm can be found in Figure 4.

Definition 3.14. A free set (Definition 2.12) of an instance $\mathbf{A}$ of a temporal CSP is called *irreducible* if it does not contain any other free set of $\mathbf{A}$ as a proper subset.

The following proposition shows that we may contract irreducible free sets.

Proposition 3.15. *Let* $\mathbf{B}$ *be a temporal structure preserved by* lex *and* $\mathbf{A}$ *an instance of* CSP($\mathbf{B}$). *Then all variables in an irreducible free set of* $\mathbf{A}$ *denote the same value in every solution for* $\mathbf{A}$.

Proof. Suppose that $\mathbf{A}$ has a solution f. We assume that $|F| > 1$; otherwise, the statement is trivial. Let F' be the set of all elements from F that denote the minimal value in f among all elements from F. Suppose that $F \setminus F'$ is not empty. We show that then F' is a free set that is properly contained in F. Let R be an arbitrary symbol from the signature of $\mathbf{B}$. We set $k := \mathrm{ar}(R)$. Let $\bar{s} \in R^{\mathbf{A}}$ be such that $\{\bar{s}[1],\ldots,\bar{s}[k]\} \cap F' \neq \emptyset$. Without loss of generality, let $1 \leq k_{F'} \leq k_F \leq k$ be such that $\{\bar{s}[1],\ldots,\bar{s}[k_{F'}]\} = \{\bar{s}[1],\ldots,\bar{s}[k]\} \cap F'$ and $\{\bar{s}[1],\ldots,\bar{s}[k_F]\} = \{\bar{s}[1],\ldots,\bar{s}[k]\} \cap F$. There exists a tuple $\bar{t} \in R^{\mathbf{B}}$ such that $\mathrm{argmin}(\bar{t}) = [k_F]$ because F is a free set. Also, by the definition of F', there exists a tuple $\bar{t}' \in R^{\mathbf{B}}$ such that $\mathrm{argmin}\big((\bar{t}'[1],\ldots,\bar{t}'[k_F])\big) = [k_{F'}]$. Let $\bar{t}'' := \mathrm{lex}(\bar{t},\bar{t}')$. It is easy to see that $\mathrm{argmin}(\bar{t}'') = [k_{F'}]$. Since $\bar{s}$ was chosen arbitrarily, we conclude that F' is a free set. This leads to a contradiction to F being irreducible; thus $F' = F$. ☐

Every instance of a temporal CSP which is satisfiable must have a free set, namely the set of all variables which denote the minimal value in the solution. Per definition it must also have a free set

Input: An instance $\mathbf{A}$ of CSP($\mathbf{B}$) for a temporal structure $\mathbf{B}$
Output: *true* or *false*
$C \leftarrow$ the empty binary relation;
while $\mathbf{A}$ *changes* **do**
  $\mathbf{A}' \leftarrow \mathbf{A}$;
  while $\mathbf{A}'$ *changes* **do**
    forall $a, b \in A'$ **do**
      if *a, b are in the same irreducible free set of* $\mathbf{A}'$ **then** $C \leftarrow C \cup \{(a,b)\}$;
    $S \leftarrow$ the union of all irreducible free sets of $\mathbf{A}'$;
    $\mathbf{A}' \leftarrow \mathrm{proj}_{A' \setminus S}(\mathbf{A}')$;
  if $A' \neq \emptyset$ **then return** *false*;
  else $\mathbf{A} \leftarrow \mathrm{ctrn}_C(\mathbf{A})$;
return *true*;

Figure 4: A choiceless algorithm for temporal CSPs with a template preserved by ll using an oracle for the computation of irreducible free sets.

which is irreducible. Moreover, the projection of the given instance to the complement of the union of all irreducible free sets is again an instance which has a solution. This observation combined with Proposition 3.15 shows that the algorithm in Figure 4 is complete. Its soundness can be shown by a recursive application of Lemma 3.16 to the sequence of structures produced during the last run of the inner loop of the algorithm.

Lemma 3.16. *Let* $\mathbf{A}$ *be an instance of* CSP($\mathbf{B}$) *for some temporal structure* $\mathbf{B}$ *preserved by* ll*. Let* $\mathbf{A}$ *be an instance of* CSP($\mathbf{B}$) *for some temporal structure* $\mathbf{B}$ *preserved by* ll*. Let C be a binary relation over A with the property that if a, b are contained in the same irreducible free set of* $\mathrm{ctrn}_C(\mathbf{A})$*, then* $(a,b) \in C$*. Let S be the union of all irreducible free sets of* $\mathrm{ctrn}_C(\mathbf{A})$*. Then* $\mathrm{ctrn}_C(\mathbf{A})$ *has a solution with kernel C if and only if* $\mathrm{proj}_{A \setminus S}(\mathrm{ctrn}_C(\mathbf{A}))$ *has a solution with kernel* $C \cap (A \setminus S)^2$.

To prove Lemma 3.16, we need the following lemma which guarantees that distinct irreducible free sets are disjoint.

Lemma 3.17. *Let* $\mathbf{B}$ *be a temporal structure preserved by* lex*, and* $\mathbf{A}$ *an instance of* CSP($\mathbf{B}$)*. If F, F' are free sets of* $\mathbf{A}$ *such that* $F \cap F' \neq \emptyset$*, then* $F \cap F'$ *is a free set of* $\mathbf{A}$.

Proof. Let F, F' be free sets of $\mathbf{A}$ such that $F \cap F' \neq \emptyset$. Let R be a symbol from the signature of $\mathbf{B}$. We set $k := \mathrm{ar}(R)$. Let $\bar{s} \in R^{\mathbf{A}}$ be such that $\{\bar{s}[1], \ldots, \bar{s}[k]\} \cap F \cap F' \neq \emptyset$. If $\{\bar{s}[1], \ldots, \bar{s}[k]\} \cap (F \setminus F') = \emptyset$ or $\{\bar{s}[1], \ldots, \bar{s}[k]\} \cap (F' \setminus F) = \emptyset$, then there is a tuple $\bar{t} \in R^{\mathbf{B}}$ such that $\mathrm{argmin}(\bar{t}) = \{i \in [k] \mid \bar{s}[i] \in \{\bar{s}[1], \ldots, \bar{s}[k]\} \cap F \cap F'\}$ because F and F' are both free sets. Now suppose that $\{\bar{s}[1], \ldots, \bar{s}[k]\} \cap (F \cup F')$ is contained neither in F nor in F'. Then there are tuples $\bar{t}, \bar{t}' \in R^{\mathbf{B}}$ such that $\mathrm{argmin}(\bar{t}) = \{i \in [k] \mid \bar{s}[i] \in \{\bar{s}[1], \ldots, \bar{s}[k]\} \cap F\}$ and $\mathrm{argmin}(\bar{t}') = \{i \in [k] \mid \bar{s}[i] \in \{\bar{s}[1], \ldots, \bar{s}[k]\} \cap F'\}$ because F and F' are both free sets. Let $\bar{t}'' := \mathrm{lex}(\bar{t}, \bar{t}')$. Then $\mathrm{argmin}(\bar{t}'') = \{i \in [k] \mid \bar{s}[i] \in \{\bar{s}[1], \ldots, \bar{s}[k]\} \cap F \cap F'\}$. Since $\bar{s}$ was chosen arbitrarily, we conclude that $F \cap F'$ is a free set. □

Proof of Lemma 3.16. Let $F_1, \ldots, F_k$ be the irreducible free sets of $\mathbf{A}$ and set $S := F_1 \cup \cdots \cup F_k$. Clearly, if $\mathrm{ctrn}_C(\mathbf{A})$ has a solution with kernel C then so has $\mathrm{proj}_{A \setminus S}(\mathrm{ctrn}_C(\mathbf{A}))$. For the converse, suppose that $\mathrm{proj}_{A \setminus S}(\mathrm{ctrn}_C(\mathbf{A}))$ has a solution $s \colon A \to \mathbb{Q}$ with $\ker s = C \cap (A \setminus S)^2$. Since $F_1, \ldots, F_k$ are irreducible, we have $F_i \cap F_j = \emptyset$ for all distinct $i, j \in [k]$ due to Lemma 3.17. Note that C cannot contain any pair (a,b) with $a \in F_i$ and $b \in F_j$ for $i \neq j$ as this would contradict the fact that F_i and F_j are disjoint free sets of $\mathrm{ctrn}_C(\mathbf{A})$ due to the presence of equality constraints. Let $s' \colon A \to \mathbb{Q}$ be such that $s'|_{A\setminus S} = s$, $s'(F_1) < s'(F_2) < \cdots < s'(F_k) < s'(A \setminus S)$, and $s'(x) = s'(y)$ whenever there exists $i \in [k]$ such that $x, y \in F_i$. We claim that s' is a solution to $\mathrm{ctrn}_C(\mathbf{A})$ with $\ker s' = C$. To verify this, let $\bar{s}$ be an arbitrary tuple from $R^{\mathrm{ctrn}_C(\mathbf{A})} \subseteq A^m$ such that, without loss of generality, $\{\bar{s}[1], \ldots, \bar{s}[m]\} \cap S = \{\bar{s}[1], \ldots, \bar{s}[\ell]\} \neq \emptyset$. By the definition of $\mathrm{proj}_{A\setminus S}(\mathrm{ctrn}_C(\mathbf{A}))$, there is a tuple $\bar{t} \in R^{\mathbf{B}}$ such that $\bar{t}[i] = s(\bar{s}[i])$ for every $i \in \{\ell+1, \ldots, m\}$ and $\bar{t}[u] = \bar{t}[v]$ whenever $(\bar{s}[u], \bar{s}[v]) \in C$. In particular, for $u, v \geq \ell + 1$, we have $\bar{t}[u] = \bar{t}[v]$ if and only if $(\bar{s}[u], \bar{s}[v]) \in C$ because $\ker s = C \cap (A \setminus S)^2$. Since $F_1, \ldots, F_k$ are free sets, there are tuples $\bar{t}_1, \ldots, \bar{t}_k \in R^{\mathbf{B}}$ such that, for every $i \in [k]$ and every $j \in [m]$, we have $j \in \mathrm{argmin}(\bar{t}_i)$ if and only if $\bar{s}[j] \in F_i$. Again, for every $i \in [k]$, we have $\bar{t}_i[u] = \bar{t}_i[v]$ whenever $(\bar{s}[u], \bar{s}[v]) \in C$. For every $i \in [k]$, let $\alpha_i \in \mathrm{Aut}(\mathbb{Q}; <)$ be such that α_i maps the minimal entry of $\bar{t}_i$ to 0. The tuple $\bar{r}_i := \mathrm{ll}(\alpha_i \bar{t}_i, \bar{t})$ is contained in $R^{\mathbf{B}}$ because $R^{\mathbf{B}}$ is preserved by ll. It follows from the definition of ll that, for all $j \in [m]$, $j \in \mathrm{argmin}(\bar{r}_i)$ if and only if $\bar{s}[j] \in F_i$. Moreover, $(\bar{r}_i[\ell+1], \ldots, \bar{r}_i[m])$ and $(\bar{t}[\ell+1], \ldots, \bar{t}[m])$ lie in the same orbit of $\mathrm{Aut}(\mathbb{Q}; <)$, and $\bar{r}_i[u] = \bar{r}_i[v]$ whenever $(\bar{s}[u], \bar{s}[v]) \in C$. Define $\bar{p}_k, \bar{p}_{k-1}, \ldots, \bar{p}_1 \in \mathbb{Q}^m$ in this order as follows. Define $\bar{p}_k := \bar{r}_k$ and, for $i \in \{1, \ldots, k-1\}$, $\bar{p}_i := \mathrm{ll}(\beta_i \bar{r}_i, \bar{p}_{i+1})$ where $\beta_i \in \mathrm{Aut}(\mathbb{Q}; <)$ is chosen such that $\beta_i(\bar{r}_i[j]) = 0$ for all $j \in \mathrm{argmin}(\bar{r}_i)$. We verify by induction that for all $i \in [k]$

(1) $\bar{p}_i$ is contained in $R^{\mathbf{B}}$;
(2) $(\bar{p}_i[\ell+1], \ldots, \bar{p}_i[m])$; $(\bar{t}[\ell+1], \ldots, \bar{t}[m])$ lie in the same orbit of $\mathrm{Aut}(\mathbb{Q}; <)$;
(3) $j \in \mathrm{argmin}(\bar{p}_i)$ if and only if $\bar{s}[j] \in F_i$ for all $j \in [m]$;
(4) $\bar{p}_i[u] = \bar{p}_i[v]$ for all $a \in \{i+1, \ldots, k\}$ and $u, v \in [m]$ such that $\bar{s}[u], \bar{s}[v] \in S_a$;
(5) $\bar{p}_i[u] < \bar{p}_i[v]$ for all $a, b \in \{i, i+1, \ldots, k\}$ with $a < b$ and $u, v \in [m]$ such that $\bar{s}[u] \in F_a$, $\bar{s}[v] \in F_b$.

For $i = k$, the items (1), (2), and (3) follow from the respective property of $\bar{r}_k$ and items (4) and (5) are trivial. For the induction step and $i \in [k-1]$ we have that $\bar{p}_i = \mathrm{ll}(\beta_i \bar{r}_i, \bar{p}_{i+1})$ satisfies items (1) and (2) because $\bar{p}_{i+1}$ satisfies items (1) and (2) by inductive assumption. For item (3), note that $\mathrm{argmin}(\bar{p}_i) = \mathrm{argmin}(\bar{r}_i)$. Finally, if $\bar{s}[u], \bar{s}[v] \in F_{i+1} \cup \cdots \cup F_k$, then $\bar{p}_i[u] \leq \bar{p}_i[v]$ if and only if $\bar{p}_{i+1}[u] \leq \bar{p}_{i+1}[v]$. This implies items (4) and (5) by induction. Note that $(s'(\bar{s}[1]), \ldots, s'(\bar{s}[m]))$ lies in the same orbit as $\bar{p}_1$ and hence is contained in $R^{\mathbf{B}}$. Moreover, it follows from the injectivity of ll that $\bar{p}_1[u] = \bar{p}_1[v]$ if and only if $(\bar{s}[u], \bar{s}[v]) \in C$. ☐

The following lemma in combination with Theorem 2.8 shows that instead of presenting an FP algorithm for each TCSP with a template preserved by ll, it suffices to present one for $\mathrm{CSP}(\mathbb{Q}; \mathrm{R}_{\mathrm{ll}}, \mathrm{S}_{\mathrm{ll}}, \neq)$ where

$$\mathrm{R}_{\mathrm{ll}} := \{(x,y,z) \in \mathbb{Q}^3 \mid y < x \vee z < x \vee x = y = z\}$$
$$\text{and} \quad \mathrm{S}_{\mathrm{ll}} := \{(x,y,u,v) \in \mathbb{Q}^4 \mid x \neq y \vee u \leq v\}.$$

Lemma 3.18. *A temporal relation is preserved by* ll *if and only if it is pp definable in* $(\mathbb{Q}; \mathrm{R}_{\mathrm{ll}}, \mathrm{S}_{\mathrm{ll}}, \neq)$.

The following syntactic description is due to Bodirsky, Kára, and Mottet.

Proposition 3.19 ([9]). *A temporal relation is preserved by* ll *if and only if it can be defined by a conjunction of formulas of the form*

$$x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee z_1 < z \vee \cdots \vee z_n < z \vee (z = z_1 = \cdots = z_n)$$

where the last disjunct $(z = z_1 = \cdots = z_n)$ *can be omitted.*

Proof of Lemma 3.18. The backward implication is a direct consequence of Proposition 3.19.

For the forward implication, we show that every temporal relation defined by a formula of the form $x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee z_1 < z \vee \cdots \vee z_n < z \vee (z = z_1 = \cdots = z_n)$, where the last disjunct $(z = z_1 = \cdots = z_n)$ can be omitted, has a pp-definition in $(\mathbb{Q}; \mathrm{R}_{\mathrm{ll}}, \mathrm{S}_{\mathrm{ll}}, \neq)$. Then the statement follows from Proposition 3.19. We prove the statement by induction on m and n. Note that both $\leq$ and $<$ have a pp-definition in $(\mathbb{Q}; \mathrm{R_{ll}}, \mathrm{S_{ll}}, \neq)$. For $m,n \geq 0$, let $R_{m,n}$ denote the $(2m+n+1)$-ary relation with the syntactic definition by a single formula from Proposition 3.19 where we assume that all variables are distinct, $x_1,\dots,x_m$ refer to the odd entries among $1,\dots,2m$, $y_1,\dots,y_m$ refer to the even entries among $1,\dots,2m$, and $z,z_1,\dots,z_n$ refer to the entries $2m+1,\dots,2m+n+1$. In the *base case* $m=n=1$, we set $\phi_{1,1}(x_1,y_1,z,z_1) \coloneqq \mathrm{S_{ll}}(x_1,y_1,z_1,z)$ and $\phi_{0,2}(z,z_1,z_2) \coloneqq \mathrm{R_{ll}}(z,z_1,z_2)$.

Claim 3.20. *If $\phi_{m-1,1}(x_1,y_1,\dots,x_{m-1},y_{m-1},z,z_1)$ is a pp-definition of $R_{m-1,1}$, then*

$$\begin{aligned}\phi_{m,1}(x_1,y_1,\dots,x_m,y_m,z,z_1) \coloneqq \exists a,b\big(&\phi_{m-1,1}(x_1,y_1,\dots,x_{m-1},y_{m-1},a,b)\\ &\wedge \phi_{1,1}(x_m,y_m,b,a) \wedge \phi_{1,1}(a,b,z,z_1)\big)\end{aligned}$$

is a pp-definition of $R_{m,1}$.

Proof of Claim 3.20. "⇒" Arbitrarily choose $\bar{t} \in R_{m,1}$. We verify that $\bar{t}$ satisfies $\phi_{m,1}$. If $\bar{t}[x_i] \neq \bar{t}[y_i]$ for some $1 \leq i \leq m-1$, then choose any $b > a$. If $\bar{t}[x_m] \neq \bar{t}[y_m]$, then we pick any $a,b \in \mathbb{Q}$ with $a > b$. Otherwise, $\bar{t}[z] \geq \bar{t}[z_1]$ and we pick any $a,b \in \mathbb{Q}$ with $a = b$.

"⇐" Suppose that $\bar{t} \notin R_{m,1}$ satisfies $\phi_{m,1}$ with some witnesses a,b. Since $\bar{t}[x_i] = \bar{t}[y_i]$ for every $1 \leq i \leq m$, we have $a \geq b$ and $b \geq a$, thus $a=b$. But then $\phi_{1,1}(a,b,\bar{t}[z],\bar{t}[z_1])$ cannot hold, a contradiction. □

It is easy to see that $R_{m,0}$ has the pp-definition

$$\phi_{m,0}(x_1,y_1,\dots,x_m,y_m) = \exists a,b\big((b>a) \wedge \phi_{m,1}(x_1,y_1,\dots,x_m,y_m,a,b)\big).$$

Claim 3.21. *If $\phi_{0,n-1}(z,z_1,\dots,z_{n-1})$ is a pp-definition of $R_{0,n-1}$, then*

$$\phi_{0,n}(z,z_1,\dots,z_n) \coloneqq \exists h\big(\phi_{0,2}(h,z_{n-1},z_n) \wedge \phi_{0,n-1}(z,z_1,\dots,z_{n-2},h)\big)$$

is a pp-definition of $R_{0,n}$.

The proofs of this claim and the next claim are similar to the proof of the previous claim and omitted.

Claim 3.22. *Let $\phi_{m,1}(x_1,y_1,\dots,x_m,y_m,z,z_1)$ and $\phi_{0,n}(z,z_1,\dots,z_n)$ be pp-definitions of $R_{m,1}$ and $R_{0,n}$, respectively, then*

$$\phi_{m,n}(x_1,y_1,\dots,x_m,y_m,z,z_1,\dots,z_n) \coloneqq \exists h\big(\phi_{0,n}(h,z_1,\dots,z_n) \wedge \phi_{m,1}(x_1,y_1,\dots,x_m,y_m,z,h)\big)$$

is a pp-definition of $R_{m,n}$.

This completes the proof of the lemma because the part $(z = z_1 = \cdots = z_n)$ in the formula from Proposition 3.19 can be easily eliminated using an additional existentially quantified variable and the relation $<$. □

In the case of $\mathrm{CSP}(\mathbb{Q}; \mathrm{R_{ll}}, \mathrm{S_{ll}}, \neq)$, we can use the same FP procedure for finding free sets from [14] that we use for instances of $\mathrm{CSP}(\mathbb{Q}; \mathrm{R_{mi}}, \mathrm{S_{mi}}, \neq)$.

Proposition 3.23. $\mathrm{CSP}(\mathbb{Q}; \mathrm{R_{ll}}, \mathrm{S_{ll}}, \neq)$ *is expressible in* FP.

As with the structures considered in Proposition 3.6 and Proposition 3.13 we have that proper projections of the relations of the temporal structure above are trivial. However, this is not the case for proper projections of contractions of the relations, e.g., the relation $\mathrm{proj}_{\{3,4\}}(\mathrm{ctrn}_{\{(1,2)\}}\, \mathrm{S_{ll}})$ equals $\leq$. Thus, in the context of this particular CSP, we cannot ignore projections if we want to obtain an FP sentence using the algorithm in Figure 4.

Proof. Let $\mathbf{B} := (\mathbb{Q}; \mathrm{R}_{\mathrm{ll}}, \mathrm{S}_{\mathrm{ll}}, \neq)$, and let $\mathbf{A}$ be an arbitrary instance of $\mathrm{CSP}(\mathbf{B})$. Note that the operation ll satisfies the requirements of Lemma 3.7. Thus, the algorithm in Figure 3 can be used for computation of free sets for instances of $\mathrm{CSP}(\mathbf{B})$. Also note that the algorithm builds free sets from singletons using only necessary conditions for containment. Thus, for every $x \in A$, the set F_x computed during the algorithm in Figure 3 is an irreducible free set iff it is non-empty and does not contain any other non-empty set of the form F_y as a proper subset. It follows that two variables $x, y \in A$ are contained in the same irreducible free set if $x \in F_y$, $y \in F_x$, and whenever $z \in F_x$ for some $z \in A$, then $x \in F_z$.

Now suppose that there exists an FP formula $\phi(x,y)$ in the signature of $\mathbf{B}$ extended by binary symbols E, T such that, for every instance $\mathbf{A}$ of $\mathrm{CSP}(\mathbf{B})$ and all $E, T \subseteq A^2$, $(\mathbf{A}; E, T) \models \phi(x,y)$ iff x, y are contained in the same irreducible free set of $\mathrm{proj}_U(\mathrm{ctrn}_T(\mathbf{A}))$ where $U = A \setminus \{x \in A \mid (x,x) \in E\}$. Then, given T as a parameter, $(\mathbf{A}; T) \models [\mathsf{ifp}_{E,x,y}\phi(x,y)](x,y)$ iff x, y are contained in the same irreducible free set at some point during the iteration of the inner loop of the algorithm in Figure 4. Consequently, $\mathbf{A} \not\to \mathbf{B}$ if and only if $\exists x. \neg[\mathsf{ifp}_{T,x,y}[\mathsf{ifp}_{E,x,y}\phi(x,y)](x,y)](x,x)$, by the soundness and completeness of the algorithm in Figure 4. We can obtain such a formula ϕ by translating the algorithm in Figure 3 into the syntax of FP and applying the reasoning from the first paragraph of this proof:

$$\begin{aligned}\phi(x,y) := \neg E(x,x) \wedge \neg E(y,y) \wedge [\mathsf{ifp}_{V,x,y}\psi(x,y)](x,y) \wedge [\mathsf{ifp}_{V,y,x}\psi(y,x)](y,x) \\ \wedge \forall a, b\big(([\mathsf{ifp}_{V,x,a}\psi(x,a)](x,a) \wedge [\mathsf{ifp}_{V,x,b}\psi(x,b)](x,b)) \Rightarrow \neg{\neq}(a,b)\big) \\ \wedge \forall z\big([\mathsf{ifp}_{V,x,z}\psi(x,z)](x,z) \Rightarrow [\mathsf{ifp}_{V,z,x}\psi(z,x)](z,x)\big)\end{aligned}$$

where ψ can be defined similarly as in Proposition 3.13 except that each subformula of the form $U(x)$ must be replaced with $\neg E(x,x)$, and taking into consideration all projections of contractions of relations with respect to T. □

4 A TCSP in FPR_2 which is not in FP

Let X be the temporal relation as defined in the introduction. In this section, we show that $\mathrm{CSP}(\mathbb{Q}; \mathrm{X})$ is expressible in FPR_2 (Proposition 4.10) but inexpressible in FPC (Theorem 4.21).

4.1 An FPR_2 algorithm for TCSPs preserved by mx

It is straightforward to verify that the relation X is preserved by the operation mx introduced in Section 2.7 [14]. For TCSPs with a template preserved by mx, the algorithm in Figure 5 can be used for finding the union of all free sets due to the following lemma. It can be proved by a simple induction using the observation that, for every $\bar{s} \in R^{\mathbf{A}}$, $\mathrm{SMS}_R(\bar{s}) \cup \{\emptyset\}$ is closed under taking symmetric difference.

Lemma 4.1 ([14])**.** *Let* $\mathbf{B}$ *be a template of a temporal CSP which is preserved by* mx. *Let* $\mathbf{A}$ *be an instance of* $\mathrm{CSP}(\mathbf{B})$. *Then the set returned by the algorithm in Figure 5 is the union of all free sets of* $\mathbf{A}$.

The following lemma in combination with Theorem 2.8 shows that instead of presenting an FPR_2 algorithm for each TCSP with a template preserved by mx, it suffices to present one for $\mathrm{CSP}(\mathbb{Q}; \mathrm{X})$.

Lemma 4.2. *A temporal relation is preserved by* mx *if and only if it has a pp-definition in* $(\mathbb{Q}; \mathrm{X})$.

Recall the min-indicator function χ from Definition 2.11.

Definition 4.3. For a temporal relation R, we set $\chi_{\bar{0}}(R) := \chi(R) \cup \{\bar{0}\}$. If R is preserved by mx, then $\chi_{\bar{0}}(R)$ is closed under the Boolean addition and forms a linear subspace of $\{0,1\}^n$ [14]. A *basic Ord-Xor relation* a temporal relation R for which there exists a homogeneous system $A\bar{x} = \bar{0}$ of Boolean linear equations such that $\chi_{\bar{0}}(R)$ is the solution space of $A\bar{x} = \bar{0}$, and R contains all tuples $\bar{t} \in \mathbb{Q}^n$ with

Input: An instance $\mathbf{A}$ of CSP($\mathbf{B}$) for a temporal structure $\mathbf{B}$
Output: A subset $F \subseteq A$
$E \leftarrow$ the empty set of Boolean linear equations;
forall $\bar{s} \in R^{\mathbf{A}}$ **do**
| $E \leftarrow E \cup \{$the Boolean linear equations for $\{\emptyset\} \cup \mathrm{SMS}_R(\bar{s})\}$;
$F \leftarrow \emptyset$;
forall $x \in A$ **do**
| **if** $E \cup \{x = 1\}$ *has a solution over* $\mathbb{Z}_2$ **then** $F \leftarrow F \cup \{x\}$;
return F;

Figure 5: A choiceless algorithm that computes the union of all free sets for temporal CSPs with a template preserved by mx.

$A\chi(\bar{t}) = \bar{0}$. If the system $A\bar{x} = \bar{0}$ for the relation specifying a basic Ord-Xor relation consists of a single equation $\sum_{i \in I} x_i = 0$ for $I \subseteq [n]$, then we denote this relation by $R^{\mathrm{mx}}_{I,n}$. A *basic Ord-Xor formula* is a $\{<\}$-formula $\phi(x_1,\ldots,x_n)$ that defines a basic Ord-Xor relation. An *Ord-Xor formula* is a conjunction of basic Ord-Xor formulas.

As usual, a solution to a homogeneous system of Boolean linear equations is called *trivial* if all variables take value 0, and *non-trivial* otherwise. The next lemma is a straightforward consequence of Definition 4.3.

Lemma 4.4. *If* $\{\sum_{i \in I_j} x_i = 0 \mid j \in J\}$ *is the homogeneous system of Boolean linear equations for a basic Ord-Xor relation R, then* $R = \bigcap_{j \in J} R^{\mathrm{mx}}_{I_j,n}$.

The following syntactic description is due to Bodirsky, Chen, and Wrona.

Theorem 4.5 ([11], Thm. 6)**.** *A temporal relation can be defined by an Ord-Xor formula if and only if it is preserved by* mx.

Proof of Lemma 4.2. We have already mentioned that X is preserved by mx, and hence all relations that are pp-definable in $(\mathbb{Q};X)$ are preserved by mx as well. For the converse direction, we show that $R^{\mathrm{mx}}_{I,n}$ has a pp-definition in $(\mathbb{Q};\mathrm{X})$ for every $n \in \mathbb{N}_{>0}$ and $I \subseteq [n]$; then the claim follows from Theorem 4.5 together with Lemma 4.4. Note that we trivially have a pp-definition of $<$ in $(\mathbb{Q};\mathrm{X})$ via $\phi^{\mathrm{mx}}_{\{1\},2}(x,y) := \mathrm{X}(x,x,y)$. We first show that the relations

$$R^{\mathrm{mx}}_{\{1\},3} = \mathrm{R}_{\min} = \{\bar{t} \in \mathbb{Q}^3 \mid \bar{t}[2] < \bar{t}[1] \vee \bar{t}[3] < \bar{t}[1]\}$$
$$\text{and} \quad R^{\mathrm{mx}}_{[3],4} = \{\bar{t} \in \mathbb{Q}^4 \mid \bar{t}[4] < \min(\bar{t}[1],\bar{t}[2],\bar{t}[3]) \vee (\bar{t}[1],\bar{t}[2],\bar{t}[3]) \in \mathrm{X}\}$$

have pp-definitions in $(\mathbb{Q};\mathrm{X})$ and then proceed with treating the other relations of the form $R^{\mathrm{mx}}_{I,n}$.

Claim 4.6. *The following primitive positive formula defines* $R^{\mathrm{mx}}_{[3],4}$ *in* $(\mathbb{Q};\mathrm{X})$.

$$\phi^{\mathrm{mx}}_{[3],4}(x_1,x_2,x_3,x_4) := \exists x'_1,x'_2,x'_3,x''_1,x''_2,x''_3 \big(x''_1 > x_4 \wedge x''_2 > x_4 \wedge x''_3 > x_4$$
$$\wedge \mathrm{X}(x'_1,x'_2,x'_3) \wedge \mathrm{X}(x_1,x'_1,x''_1) \wedge \mathrm{X}(x_2,x'_2,x''_2) \wedge \mathrm{X}(x_3,x'_3,x''_3)\big)$$

Proof. "⇒" We first prove that every $\bar{t} \in R^{\mathrm{mx}}_{[3],4}$ satisfies $\phi^{\mathrm{mx}}_{[3],4}$.

Case 1: $(\bar{t}[1],\bar{t}[2],\bar{t}[3]) \in \mathrm{X}$. We choose witnesses for the quantifier-free part of $\phi^{\mathrm{mx}}_{[3],4}$ as follows: $x'_1 := \bar{t}[1]$, $x'_2 := \bar{t}[2]$, $x'_3 := \bar{t}[3]$, and for x''_1,x''_2,x''_3 we choose values arbitrarily such that $\max(\bar{t}[1],\bar{t}[2],\bar{t}[3],\bar{t}[4]) < \min(x''_1,x''_2,x''_3)$. It is easy to see that this choice satisfies the quantifier-free part of $\phi^{\mathrm{mx}}_{[3],4}$.

Case 2: $(\bar{t}[1],\bar{t}[2],\bar{t}[3]) \notin \mathrm{X}$. We have $\bar{t}[4] < \min(\bar{t}[1],\bar{t}[2],\bar{t}[3])$ by the definition of $R^{\mathrm{mx}}_{[3],4}$. By symmetry, it suffices consider the following three subcases.

Subcase 2.i: $\bar{t}[3] < \bar{t}[2] < \bar{t}[1]$. We choose $x_1' = x_2' = x_1'' = x_2'' = x_3'' := \bar{t}[3]$ and $x_3' := \bar{t}[1]$.

Subcase 2.ii: $\bar{t}[3] < \bar{t}[1] = \bar{t}[2]$. We choose the same witnesses as in the previous case.

Subcase 2.iii: $\bar{t}[1] = \bar{t}[2] = \bar{t}[3]$. We choose any combination of x_1',x_2',x_3', x_1'',x_2'',x_3'' that satisfies $\bar{t}[4] < x_1' = x_2' = x_1'' = x_2'' < x_3' = x_3'' < \bar{t}[1]$.

In each of the subcases 2.i-iii above, our choice satisfies the quantifier-free part of $\phi^{\mathrm{mx}}_{[3],4}$.

"$\Leftarrow$" Suppose for contradiction that there exists a tuple $\bar{t} \notin R^{\mathrm{mx}}_{[3],4}$ that satisfies $\phi^{\mathrm{mx}}_{[3],4}$. Then $(\bar{t}[1],\bar{t}[2],\bar{t}[3]) \notin \mathrm{X}$ and $\bar{t}[4] \geq \min(\bar{t}[1],\bar{t}[2],\bar{t}[3])$. Consider the witnesses $x_1',x_2',x_3',x_1'',x_2'',x_3''$ for the fact that $\bar{t}$ satisfies $\phi^{\mathrm{mx}}_{[3],4}$. Without loss of generality, we only have the following three cases.

Case 1: $\bar{t}[1] > \bar{t}[2] > \bar{t}[3]$. We have $x_3' = \bar{t}[3]$ because $(\bar{t}[3],x_3',x_3'') \in \mathrm{X}$ and $x_3'' > \bar{t}[4] \geq \min(\bar{t}[1],\bar{t}[2],\bar{t}[3]) = \bar{t}[3]$.

Subcase 1.i: $x_3' > \min(x_1',x_2')$. We have $x_1' = x_2' < x_3'$, because $(x_1',x_2',x_3') \in \mathrm{X}$. This implies $x_1'' = x_1'$, because $x_1' < x_3' = \bar{t}[3] < \bar{t}[1]$ and $(\bar{t}[1],x_1',x_1'') \in \mathrm{X}$. But then $x_1'' < \bar{t}[3] \leq \bar{t}[4]$, a contradiction.

Subcase 1.ii: $x_3' = \min(x_1',x_2',x_3')$. Either $x_1' = x_3' < x_2'$ or $x_2' = x_3' < x_1'$ because $(x_1',x_2',x_3') \in \mathrm{X}$.

Subcase 1.ii.a: $x_1' = x_3'$. We have $x_1'' = x_1'$ because $x_1' = x_3' = \bar{t}[3] < \bar{t}[1]$ and $(\bar{t}[1],x_1',x_1'') \in \mathrm{X}$. But then $x_1'' = \bar{t}[3] \leq \bar{t}[4]$, a contradiction.

Subcase 1.ii.b: $x_2' = x_3'$. We have $x_2'' = x_2'$ because $x_2' = x_3' = \bar{t}[3] < \bar{t}[2]$ and $(\bar{t}[2],x_2',x_2'') \in \mathrm{X}$. But then $x_2'' = \bar{t}[3] \leq \bar{t}[4]$, a contradiction.

Case 2: $\bar{t}[1] = \bar{t}[2] > \bar{t}[3]$. We obtain a contradiction similarly as in the previous case.

Case 3: $\bar{t}[1] = \bar{t}[2] = \bar{t}[3]$. We must have $x_3' = \bar{t}[3]$, $x_2' = \bar{t}[2]$ and $x_1' = \bar{t}[1]$ because $\min(x_1'',x_2'',x_3'') > \bar{t}[4] \geq \bar{t}[1] = \bar{t}[2] = \bar{t}[3]$. But then $(x_1',x_2',x_3') \notin \mathrm{X}$, a contradiction.

In all three cases above, we get a contradiction which means that there is no tuple $\bar{t} \notin R^{\mathrm{mx}}_{[3],4}$ that satisfies $\phi^{\mathrm{mx}}_{[3],4}(\bar{t}[1],\bar{t}[2],\bar{t}[3],\bar{t}[4])$. □

It is easy to see that the pp-formula

$$\phi^{\mathrm{mx}}_{[2],3}(x_1,x_2,x_3) := \exists h\big(\phi^{\mathrm{mx}}_{[3],4}(x_1,x_2,h,x_3) \wedge (h > x_1)\big)$$

is equivalent to $(x_1 > x_3 \wedge x_2 > x_3) \vee x_1 = x_2$.

Claim 4.7. *The following pp formula defines* $R^{\mathrm{mx}}_{\{1\},3}$.

$$\phi^{\mathrm{mx}}_{\{1\},3}(x_1,x_2,x_3) := \exists h_2,h_3\big(\phi^{\mathrm{mx}}_{[2],3}(x_1,h_2,x_3) \wedge (h_2 > x_2) \wedge \phi^{\mathrm{mx}}_{[2],3}(x_1,h_3,x_2) \wedge (h_3 > x_3)\big)$$

Proof. "$\Rightarrow$" Suppose that $\phi^{\mathrm{mx}}_{\{1\},3}(\bar{t})$ is true for some $\bar{t} \in \mathbb{Q}^3$. If $\bar{t}[1] \leq \bar{t}[2]$ and $\bar{t}[1] \leq \bar{t}[3]$, then $h_2 = \bar{t}[1]$ and $h_3 = \bar{t}[1]$ which contradicts $h_2 > \bar{t}[2]$ and $h_3 > \bar{t}[3]$. Thus $\bar{t} \in R^{\mathrm{mx}}_{\{1\},3}$.

"$\Leftarrow$" Suppose that $\bar{t} \in R^{\mathrm{mx}}_{\{1\},3}$ for some $\bar{t} \in \mathbb{Q}^3$. Without loss of generality, $\bar{t}[1] > \bar{t}[2]$. Then $\phi^{\mathrm{mx}}_{\{1\},3}(\bar{t})$ being true is witnessed by $h_2 := \bar{t}[1]$ and any $h_3 \in \mathbb{Q}$ that satisfies $h_3 > \bar{t}[3]$. □

Since we already have a pp-definition $\phi^{\mathrm{mx}}_{\{1\},3}$ for $R^{\mathrm{mx}}_{\{1\},3}$, we can obtain a pp-definition $\phi^{\mathrm{mx}}_{\{1\},n+1}$ of $R^{\mathrm{mx}}_{\{1\},n+1}$ inductively as in Proposition 3.4. The challenging part is showing the pp-definability of $R^{\mathrm{mx}}_{[k],k+1}$. Note that we have already covered the cases where $k \in [3]$ in the beginning of the proof.

Claim 4.8. *For $k \geq 4$, the relation $R^{\mathrm{mx}}_{[k],k+1}$ can be pp-defined by*

$$\phi^{\mathrm{mx}}_{[k],k+1}(x_1,\ldots,x_k,y) := \exists h_2,\ldots,h_{k-2}\big(\phi^{\mathrm{mx}}_{[3],4}(x_1,x_2,h_2,y) \wedge \phi^{\mathrm{mx}}_{[3],4}(h_{k-2},x_{k-1},x_k,y) \wedge \bigwedge_{i=3}^{k-2} \phi^{\mathrm{mx}}_{[3],4}(h_{i-1},x_i,h_i,y)\big).$$

Proof. Suppose that $\bar{t} \in \mathbb{Q}^{k+1}$ satisfies $\phi^{\text{mx}}_{[k],k+1}$. Let $h_2,\ldots,h_{k-2} \in \mathbb{Q}$ be witnesses of the fact that $\bar{t}$ satisfies $\phi^{\text{mx}}_{[k],k+1}$. If $\bar{t}[y] < m := \min(\bar{t}[x_1],\ldots,\bar{t}[x_k])$ then $\bar{t} \in R^{\text{mx}}_{[k],k+1}$ and we are done, so suppose that $\bar{t}[y] \geq m$. Define $h'_2,\ldots,h'_{k-2} \in \{0,1\}$ by $h'_i := 1$ if $h_i = m$ and $h'_i := 0$ otherwise. Note that if $m < \min(\bar{t}[x_1],\bar{t}[x_2],h_2)$ then $\chi(\bar{t})[x_1] = \chi(\bar{t})[x_2] = h'_2 = 0$. Otherwise, if $m = \min(\bar{t}[x_1],\bar{t}[x_2],h_2)$, then exactly two out of $\chi(\bar{t})[x_1],\chi(\bar{t})[x_2],h'_2$ take the value 1. The same holds for each conjunct of $\phi^{\text{mx}}_{[k],k+1}$, so they imply

$$\begin{aligned}
\chi(\bar{t})[x_1] + \chi(\bar{t})[x_2] + h'_2 &= 0 \mod 2, && \\
h'_{k-2} + \chi(\bar{t})[x_{k-1}] + \chi(\bar{t})[x_k] &= 0 \mod 2, && \text{and} \\
h'_{i-1} + \chi(\bar{t})[x_i] + h'_i &= 0 \mod 2 && \text{for every } i \in \{3,\ldots,k-2\}.
\end{aligned}$$

Summing all these equations we deduce that $\sum_{i=1}^{k} \chi(\bar{t})[x_i] = 0 \mod 2$ and hence $\bar{t} \in R^{\text{mx}}_{[k],k+1}$.

Conversely, suppose that $\bar{t} \in R^{\text{mx}}_{[k],k+1}$. We have to show that $\bar{t}$ satisfies $\phi^{\text{mx}}_{[k],k+1}(x_1,\ldots,x_k,y)$. If $\bar{t}[y] < m := \min(\bar{t}[x_1],\ldots,\bar{t}[x_k])$ then we set all of $h_2,\ldots,h_{k-2}$ to m and all conjuncts of $\phi^{\text{mx}}_{[k],k+1}(x_1,\ldots,x_k,y)$ are satisfied. We may therefore suppose in the following that $\bar{t}[y] \geq m$. Let $s := \max(\bar{t}[x_1],\ldots,\bar{t}[x_{k+1}])$. Define

$$h_2 := \begin{cases} s & \text{if } \bar{t}[x_1] = \bar{t}[x_2] \\ \min(\bar{t}[x_1],\bar{t}[x_2]) & \text{otherwise} \end{cases}$$

and for $i \in \{3,\ldots,k-2\}$ define

$$h_i := \begin{cases} s & \text{if } h_{i-1} = \bar{t}[x_i] \\ \min(h_{i-1},\bar{t}[x_i]) & \text{otherwise.} \end{cases}$$

This clearly satisfies all conjuncts of $\phi^{\text{mx}}_{[k],k+1}$ except for possibly the second. Note that h_i is set to m if and only if $\sum_{j=1}^{i} \chi(\bar{t})[x_j]$ is odd. In particular, h_{k-2} is set to m if and only if $\sum_{j=1}^{k-2} \chi(\bar{t})[x_j]$ is odd. If this is the case, then exactly one of $\bar{t}[x_{k-1}],\bar{t}[x_k]$ equals m because $\sum_{j=1}^{k} \chi(\bar{t})[x_j]$ is even. If h_{k-2} is set to s then for the same reason we must have $\bar{t}[x_{k-1}] = \bar{t}[x_k]$. In both cases the second conjunct of $\phi^{\text{mx}}_{[k],k+1}$ is satisfied as well. $\square$

For the general case, let $k := |I|$. Without loss of generality we may assume that $I = [k]$.

Claim 4.9. *The following pp-formula defines* $R^{\text{mx}}_{[k],n}$.

$$\phi^{\text{mx}}_{[k],n}(x_1,\ldots,x_n) := \exists h\big(\phi^{\text{mx}}_{[k],k+1}(x_1,\ldots,x_k,h) \wedge \phi^{\text{mx}}_{\{1\},n+1}(h,x_1,\ldots,x_n)\big)$$

Proof. "$\Rightarrow$" Let $\bar{t}$ be an arbitrary tuple from $R^{\text{mx}}_{[k],n}$. If $\sum_{i=1}^{k} \chi(\bar{t})[i] = 0$, then $\phi^{\text{mx}}_{[k],k+1}(\bar{t}[1],\ldots,\bar{t}[k],h)$ holds for every $h \in \mathbb{Q}$ and $\phi^{\text{mx}}_{\{1\},n+1}(h,\bar{t}[1],\ldots,\bar{t}[n])$ holds for every $h \in \mathbb{Q}$ with $h > \min(\bar{t}[k+1],\ldots,\bar{t}[n])$. Otherwise we have $\min(\bar{t}[1],\ldots,\bar{t}[k]) > \min(\bar{t}[k+1],\ldots,\bar{t}[n])$. Then $\phi^{\text{mx}}_{\{1\},n+1}(h,\bar{t}[1],\ldots,\bar{t}[n])$ is true for every $h \in \mathbb{Q}$ such that $\min(\bar{t}[1],\ldots,\bar{t}[k]) > h > \min(\bar{t}[k+1],\ldots,\bar{t}[n])$. Thus $\bar{t}$ satisfies $\phi^{\text{mx}}_{[k],n}$.

"$\Leftarrow$" Let $\bar{t}$ be an arbitrary n-tuple over $\mathbb{Q}$ not contained in $R^{\text{mx}}_{[k],n}$. Then $\sum_{i=1}^{k} \chi(\bar{t})[i] \neq 0$, and also $\min(\bar{t}[1],\ldots,\bar{t}[k]) \leq \min(\bar{t}[k+1],\ldots,\bar{t}[n])$. For every witness $h \in \mathbb{Q}$ such that $\phi^{\text{mx}}_{[k],k+1}(\bar{t}[1],\ldots,\bar{t}[k],h)$ is true, we have $\min(\bar{t}[1],\ldots,\bar{t}[k]) > h$. But then no such h can witness $\phi^{\text{mx}}_{\{1\},n+1}(h,\bar{t}[1],\ldots,\bar{t}[n])$ being true. Thus, $\bar{t}$ does not satisfy $\phi^{\text{mx}}_{[k],n}$. $\square$

This completes the proof of Lemma 4.2. $\square$

The expressibility of $\mathrm{CSP}(\mathbb{Q};X)$ in FPR_2 can be shown using the same approach as in the first part of Section 3 via Proposition 3.1 if the suitable procedure from [14] for finding free sets can be implemented in FPR_2. This is possible by encoding systems of Boolean linear equations in FPR_2 similarly as in the case of symmetric reachability in directed graphs in the paragraph above Corollary III.2. in [28].

Proposition 4.10. $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ *is expressible in* FPR_2.

Proof. Recall that $(\mathbb{Q};\mathrm{X})$ is preserved by mi and hence also by pp. Our aim is to construct a $\{X,U\}$-formula $\phi(x)$ satisfying the requirements of Corollary 3.2 by rewriting the algorithm in Figure 5 in the syntax of FPR_2. In the computation of the algorithm in Figure 5, each constraint is of the form $X(x,y,z)$, and hence contributes a single equation to E, namely $x+y+z=0$. A variable x denotes the value 1 in a non-trivial solution of E iff $E\cup\{x=1\}$ has some solution. The algorithm subsequently isolates those variables which denote the value 1 in some non-trivial solution for E. Write E as $M\bar{x}=\bar{v}$. We define two numeric terms f_M and $f_{\bar{v}}$ which encode the matrix and the vector, respectively, of this system.

$$\begin{aligned} f_M(x_1,x_2,x_3,y_1,x) := \big(&X(x_1,x_2,x_3)\wedge U(x_1)\wedge U(x_2)\wedge U(x_3)\wedge(y_1=x_1\vee y_1=x_2\vee y_1=x_3)\big)\\ &\vee(x_1=x_2=x_3=y_1=x)\end{aligned}$$

$$f_{\bar{v}}(x_1,x_2,x_3,y_1,x) := (x_1=x_2=x_3=y_1=x)$$

Let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ and $U\subseteq A$ arbitrary. For every $x\in U$, the matrix $\mathsf{M}_2^{\mathbf{A}}[\![f_M(\cdot,\cdot,x)]\!]\in\{0,1\}^{A^3\times A}$ contains

(1) for each constraint $X(x_1,x_2,x_3)$ of $\mathbf{A}$, where $x_1,x_2,x_3\in U$, three 1s in the (x_1,x_2,x_3)-th row: namely, in the x_1-th, x_2-th, and x_3-th column, and
(2) a single 1 in the (x,x,x)-th row: namely, in the x-th column.

We can test the solvability of $M\bar{x}=\bar{v}$ in FPR_2 by comparing the rank of M with the rank of $(M|\bar{v})$: the system is satisfiable if and only if they have the same rank. The case that $\mathbf{A}$ contains a constraint of the form $X(y,y,y)$ is treated specially; in this case, $\mathbf{A}$ does not have a solution (note that our encoding of $M\bar{x}=\bar{v}$ is incorrect whenever $\mathbf{A}$ contains such a constraint). The formula $\phi(x)$ can be defined as follows.

$$\begin{aligned}\phi(x) := \exists y\big(&X(y,y,y)\vee\neg([\mathsf{rk}_{(x_1,x_2,x_3),y_1}f_A(x_1,x_2,x_3,y_1,x) \bmod 2] =\\ &[\mathsf{rk}_{(x_1,x_2,x_3),(y_1,y_2)}(y_2\neq y)\cdot f_A(x_1,x_2,x_3,y_1,x)+(y_2=y)\cdot f_{\bar{b}}(x_1,x_2,x_3,y_1,x) \bmod 2])\big)\end{aligned}$$

Now the statement of the proposition follows from Corollary 3.2. □

4.2 A proof of inexpressibility in FPC

Interestingly, the inexpressibility of $\mathrm{CSP}(\mathbb{Q};X)$ in FPC cannot be shown by giving a pp-construction of systems of Boolean linear equations and utilizing the inexpressibility result of Atserias, Bulatov, and Dawar [1] (see Corollary 7.21). For this reason we resort to the strategy of showing that $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ has unbounded counting width and then applying Theorem 2.3 [30]. Formally, the *counting width* of $\mathrm{CSP}(\mathbf{B})$ for a τ-structure $\mathbf{B}$ is the function that assigns to each $n\in\mathbb{N}$ the minimum value k for which there is a τ-sentence ϕ in $\mathcal{C}^k$ such that, for every τ-structure $\mathbf{A}$ with $|A|\leq n$, $\mathbf{A}\models\phi$ if and only if $\mathbf{A}\to\mathbf{B}$. In the following we need the reflexive and transitive relation $\Rightarrow_{\exists^+\mathcal{L}^k}$ and the equivalence relation $\equiv_{\mathcal{C}^k}$. We utilize both the *existential k-pebble game* which characterises $\Rightarrow_{\exists^+\mathcal{L}^k}$, and the *bijective k-pebble game* which characterises $\equiv_{\mathcal{C}^k}$. See [2] for details about the approach to these relations via model-theoretic games.

The problem $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ can be reformulated as a decision problem for systems of linear equations over $\mathbb{Z}_2$. Each constraint $X(x,y,z)$ of $\mathbf{A}$ is viewed as a Boolean equation $x+y+z=0$.

Proposition 4.11. *The problem* 3-ORD-XOR-SAT *(defined in the introduction) and* CSP$(\mathbb{Q};\mathrm{X})$ *are the same computational problem.*

Proof. The structure $(\mathbb{Q};\mathrm{X})$ is preserved by mx. Thus, the algorithm in Figure 1 jointly with the algorithm in Figure 5 for computing free sets is sound and complete for CSP$(\mathbb{Q};\mathrm{X})$.

If the equations in $\mathbf{A}$ are a positive instance of 3-ORD-XOR-SAT, then the algorithm finds a free set in every step and accepts $\mathbf{A}$. Since our algorithm is correct for CSP$(\mathbb{Q};\mathrm{X})$, we conclude that $\mathbf{A} \to (\mathbb{Q};X)$. Conversely, suppose that $\mathbf{A} \to (\mathbb{Q};\mathrm{X})$. Then the algorithm described above produces a sequence $A_1,\dots,A_\ell$ of subsets of A where $A_1 := A$, and for every $i < \ell$ the set A_{i+1} is the subset of A_i where we remove all elements which are contained in a free set of the substructure of $\mathbf{A}$ with domain A_i. Moreover, $\mathbf{A}$ contains no Boolean equations on variables from $\mathbf{A}_\ell$. Let F be a non-empty subset of the equations from $\mathbf{A}$ and let B be the variables that appear in the equations from F. Let i be maximal such that $B \subseteq A_i$. Then mapping all variables in $B \cap A_{i+1}$ to 0 and all variables in $B \setminus A_{i+1}$ to 1 is a non-trivial solution to F:

- an even number of variables of each constraint is in $B \setminus A_{i+1}$, by the definition of free sets;
- B cannot be fully contained in $B \setminus A_{i+1}$ because F is non-empty;
- B cannot be fully contained in $B \cap A_{i+1}$ by the maximal choice of i. □

Definition 4.12. The satisfiability problem for systems of linear equations $A\bar{x} = \bar{b}$ over a finite Abelian group $\mathscr{G}$ with at most k variables per equation can be formulated as CSP$(\mathbf{E}_{\mathscr{G},k})$ where $\mathbf{E}_{\mathscr{G},k}$ is the structure over the domain G of $\mathscr{G}$ with the relations

$$eqn_{a,j}^{\mathbf{E}_{\mathscr{G},k}} = \{\bar{t} \in G^j \mid \textstyle\sum_{i\in[j]} \bar{t}[i] = a\}$$

for every $j \leq k$ and $a \in G$. The *homogeneous companion* of an instance $\mathbf{A}$ of CSP$(\mathbf{E}_{\mathscr{G},k})$ is obtained by moving the tuples from each $eqn_{a,j}^{\mathbf{A}}$ to $eqn_{e,j}^{\mathbf{A}}$ where e is the neutral element in $\mathscr{G}$.

According to the discussion above Definition 4.12, we may view CSP$(\mathbb{Q};\mathrm{X})$ as a proper subset of CSP$(\mathbf{E}_{\mathbb{Z}_2,3})$. We use the probabilistic construction of multipedes from [8, 42] as a black box for extracting certain systems of Boolean linear equations that represent instances of CSP$(\mathbb{Q};\mathrm{X})$. More specifically, we use the reduction of the isomorphism problem for multipedes to the satisfiability of a system of linear equations over $\mathbb{Z}_2$ with 3 variables per equation from the proof of Theorem 23 in [8]. The following concepts were introduced in [42]; we mostly follow the terminology in [8].

Definition 4.13. A *multipede* is a finite relational structure $\mathbf{M}$ with the signature $\{<,E,H\}$, where $<,E$ are binary symbols and H is a ternary relation symbol, such that $\mathbf{M}$ satisfies the following axioms. The domain of $\mathbf{M}$ has a partition into *segments* $\mathrm{SG}(\mathbf{M})$ and *feet* $\mathrm{FT}(\mathbf{M})$ such that $<^{\mathbf{M}}$ is a linear order on $\mathrm{SG}(\mathbf{M})$, and $E^{\mathbf{M}}$ is the graph of a surjective function $\mathrm{sg}\colon \mathrm{FT}(\mathbf{M}) \to \mathrm{SG}(\mathbf{M})$ with $|\mathrm{sg}^{-1}(x)| = 2$ for every $x \in \mathrm{SG}(\mathbf{M})$. For every $\bar{t} \in H^{\mathbf{M}}$, either the entries of $\bar{t}$ are contained in $\mathrm{SG}(\mathbf{M})$ and we call $\bar{t}$ a *hyperedge*, or they are contained in $\mathrm{FT}(\mathbf{M})$ and we call $\bar{t}$ a *positive triple*. The relation $H^{\mathbf{M}}$ is *fully symmetric*, that is, closed under all permutations of entries, and only contains triples with pairwise distinct entries. For every positive triple $\bar{t}$, the triple $(\mathrm{sg}(\bar{t}[1]),\mathrm{sg}(\bar{t}[2]),\mathrm{sg}(\bar{t}[3]))$ is a hyperedge. If $\bar{t} \in H^{\mathbf{M}}$ is a hyperedge where $\mathrm{sg}^{-1}(\bar{t}[i]) = \{x_{i,0}, x_{i,1}\}$ for every $i \in [3]$, then we require that exactly 4 elements of the set $\{(x_{1,i},x_{2,j},x_{3,k}) \mid i,j,k \in \{0,1\}\}$ are positive triples. We also require that, for each pair of triples $(x_{1,i},x_{2,j},x_{3,k}),(x_{1,i'},x_{2,j'},x_{3,k'})$ from the set above, we have $(i-i')+(j-j')+(k-k') = 0 \bmod 2$. A multipede $\mathbf{M}$ is *odd* if for each $\emptyset \subsetneq X \subseteq \mathrm{SG}(\mathbf{M})$ there is a hyperedge $\bar{t} \in H^{\mathbf{M}}$ such that $|\{\bar{t}[1],\bar{t}[2],\bar{t}[3]\} \cap X|$ is odd. A multipede $\mathbf{M}$ is *k-meager* if for each $\emptyset \subsetneq X \subseteq \mathrm{SG}(\mathbf{M})$ of size at most $2k$ we have $|X| > 2 \cdot |H^{\mathbf{M}} \cap X^3|$.

The following four statements (Proposition 4.14, Lemma 4.15, Proposition 4.16, and Lemma 4.17) are crucial for our application of multipedes in the context of CSP$(\mathbb{Q};\mathrm{X})$. An automorphism is called *trivial* if it is an identity map, and *non-trivial* otherwise.

Proposition 4.14 ([8], Proposition 17)**.** *Odd multipedes have no non-trivial automorphisms.*

Lemma 4.15 ([42], Lemma 4.5)**.** *For any $k \in \mathbb{N}_{>0}$, let $\mathbf{M}$ be a $2k$-meager multipede. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two expansions of $\mathbf{M}$ obtained by placing a constant on the two different feet of one particular segment, respectively. Then $\mathbf{M}_1 \equiv_{\mathcal{C}^k_{\infty\omega}} \mathbf{M}_2$. The statement even holds for expansions of $\mathbf{M}_1$ and $\mathbf{M}_2$ by constants for all segments.*

Proposition 4.16 ([8], Proposition 18)**.** *For every $k \in \mathbb{N}_{>0}$, there exists an odd k-meager multipede.*

Let $\mathbf{M}$ be a multipede. Consider the matrix $A \in \{0,1\}^{(H^{\mathbf{M}} \cap \mathrm{SG}(\mathbf{M})^3) \times \mathrm{SG}(\mathbf{M})}$ whose value at the coordinate $(\bar{t}, s)$ is 1 iff $s \in \{\bar{t}[1], \bar{t}[2], \bar{t}[3]\}$. Note that A has exactly three non-zero entries per row. Fix an arbitrary bijection $f\colon M \to M$ that preserves $<^{\mathbf{M}}$ and $E^{\mathbf{M}}$. Let $\bar{b} \in \{0,1\}^{H^{\mathbf{M}} \cap \mathrm{SG}(\mathbf{M})^3}$ be the tuple such that $\bar{b}'[\bar{t}] = 0$ iff f preserves positive triples of $\mathbf{M}$ at $\bar{t}$. For $X \subseteq \mathrm{SG}(\mathbf{M})$, let

$$f_X(x) := \begin{cases} f(y) & \text{if } \mathrm{sg}^{-1}(s) = \{x,y\} \text{ for some } s \in X, \\ f(x) & \text{otherwise} \end{cases}$$

be the map obtained from f by transposing the images of the feet at every segment from X. Every automorphism of $\mathbf{M}$ is of the form f_X for some $X \subseteq \mathrm{SG}(\mathbf{M})$ (see the second-but-last paragraph in the proof of Theorem 23 of [8]). For every $X \subseteq \mathrm{SG}(\mathbf{M})$, let $\bar{t}_X \in \{0,1\}^{\mathrm{SG}(\mathbf{M})}$ be the tuple defined by $\bar{t}_X[s] := 1$ if and only if $s \in X$. The following lemma is a simple consequence of the definition of a multipede.

Lemma 4.17 ([8], the proof of Theorem 23)**.** *For every $X \subseteq \mathrm{SG}(\mathbf{M})$, the restriction of f_X to some $Y \cup Z$ where $Y \subseteq \mathrm{SG}(\mathbf{M})$ and $Z := \mathrm{sg}^{-1}(Y)$ is an automorphism of the substructure of $\mathbf{M}$ on $Y \cup Z$ if and only if $\bar{t}_X$ restricted to Y is a solution to the system $A\bar{x} = \bar{b}$ restricted to Y.*

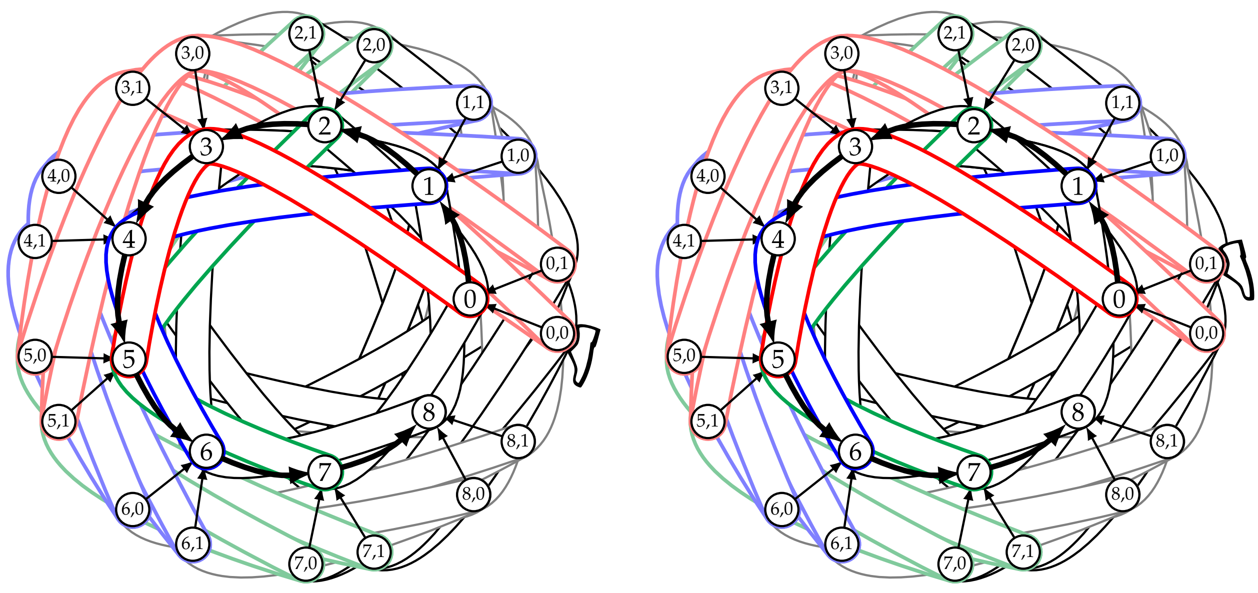

Figure 6: A pair of non-isomorphic expansions of an odd 2-meager multipede by a constant (shoe).

Example 4.18. We now describe the multipede $\mathbf{M}$ from Figure 6 in detail. We have that $\mathrm{SG}(\mathbf{M}) = \mathbb{Z}_9$, $\mathrm{FT}(\mathbf{M}) = \mathbb{Z}_9 \times \mathbb{Z}_2$, $<^{\mathbf{M}}$ is the linear order $0 < \cdots < 8$, and $E^{\mathbf{M}} = \{(\bar{t}, s) \in (\mathbb{Z}_9 \times \mathbb{Z}_2) \times \mathbb{Z}_9 \mid \bar{t}[1] = s\}$. Moreover, we have the following set of hyperedges:

$$H^{\mathbf{M}} \cap \mathrm{SG}(\mathbf{M})^3 = \{\bar{s} \in \mathbb{Z}_9^3 \mid \bar{s}[2] = \bar{s}[1] + 3 \bmod 9 \text{ and } \bar{s}[3] = \bar{s}[1] + 5 \bmod 9\},$$

and the following set of positive triples:

$$H^{\mathbf{M}} \cap \mathrm{FT}(\mathbf{M})^3 = \{(\bar{t}_1,\bar{t}_2,\bar{t}_3) \in (\mathbb{Z}_9 \times \mathbb{Z}_2)^3 \mid (\bar{t}_1[1],\bar{t}_2[1],\bar{t}_3[1]) \in H^{\mathbf{M}} \cap \mathrm{SG}(\mathbf{M})^3 \text{ and } \bar{t}_1[2]+\bar{t}_2[2]+\bar{t}_3[2] = 1 \bmod 2\}.$$

Note that the hyperedges do not overlap on more than one segment, because the minimal distances between two entries of an hyperedge are $2,3$, or 4 mod 9. This directly implies that both multipedes are 2-meager. Using Gaussian elimination, one can check that the system of linear equations $A\bar{x} = \bar{0}$ over $\mathbb{Z}_2$, where A is the incidence matrix of the hyperedges on the segments, only admits the trivial solution. We claim that from this fact it follows that $\mathbf{M}$ is odd. Otherwise, suppose that there exists a non-empty subset X of the hyperedges witnessing that this is not the case. Then $A\bar{x} = \bar{0}$ is satisfied by the non-trivial assignment that maps $\bar{x}[s]$ to 1 if and only if $s \in X$, which yields a contradiction. Thus, by Proposition 4.14 and Lemma 4.17, the expansions $\mathbf{M}_1$ and $\mathbf{M}_2$ of $\mathbf{M}$ obtained by placing a constant on the two different feet of the segment 0 are not isomorphic.

Keeping the construction above Lemma 4.17 in mind, we can derive the following statement about systems of Boolean linear equations.

Proposition 4.19. *For every $k \geq 3$, there exist instances $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathrm{CSP}(\mathbf{E}_{\mathbb{Z}_2,3})$ such that*

(1) *$\mathbf{A}_1$ and $\mathbf{A}_2$ have the same homogeneous companion which only has the trivial solution,*
(2) *$\mathbf{A}_1$ has no solution and $\mathbf{A}_2$ has a solution,*
(3) *$\mathbf{A}_1 \equiv_{\mathcal{C}^k} \mathbf{A}_2$.*

Our proof strategy for Proposition 4.19 is as follows. We first use multipedes to construct instances $\mathbf{A}'_1$ and $\mathbf{A}'_2$ of $\mathrm{CSP}(\mathbf{E}_{\mathbb{Z}_2,3})$ that satisfy item (1) and (2) of the statement. Then we use the following construction of Atserias and Dawar [2] to transform them into instances $\mathbf{A}_1$ and $\mathbf{A}_2$ that additionally satisfy item (3) of the statement. For an instance $\mathbf{A}$ of $\mathrm{CSP}(\mathbf{E}_{\mathbb{Z}_2,3})$, let $G(\mathbf{A})$ be the system that contains for each equation $x_1 + \cdots + x_j = b$ of $\mathbf{A}$ and all $a_1,\ldots,a_j \in \{0,1\}$ the equation

$$x_1^{a_1} + \cdots + x_j^{a_j} = b + a_1 + \cdots + a_j.$$

Lemma 4.20 (Atserias and Dawar [2]). *Let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\mathbf{E}_{\mathbb{Z}_2,3})$ and $k \geq 3$ such that $\mathbf{A} \Rightarrow_{\exists^+\mathcal{L}^k} \mathbf{E}_{\mathbb{Z}_2,3}$. Then $G(\mathbf{A}) \equiv_{\mathcal{C}^k} G(\mathbf{A}_0)$ where $\mathbf{A}_0$ is the homogeneous companion of $\mathbf{A}$.*

Proof. The statement is almost Lemma 2 in Atserias and Dawar [2] with the only difference that they additionally assume that all constraints in the instance are imposed on three distinct variables; however, their winning strategy for Duplicator also works in the more general setting. ☐

Proof of Proposition 4.19. For a given $k \geq 3$, let $\mathbf{M}$ be an odd $6k$-meager multipede whose existence follows from Proposition 4.16. Let f be the identity map on M, and let $A\bar{x} = \bar{0}$ be the system of linear equations over $\mathbb{Z}_2$ derived from $\mathbf{M}$ and f using the construction described in the paragraph above Lemma 4.17. Since $\mathbf{M}$ is odd, by Proposition 4.14 and Lemma 4.17 we have that $A\bar{x} = \bar{0}$ only has the trivial solution. This means that the inhomogeneous system obtained from $A\bar{x} = \bar{0}$ by adding the equation $\bar{x}[s] = 1$, where s is the first segment, has no solution. We refer to this system by $\mathbf{A}'_1$ and to its homogeneous companion by $\mathbf{A}'_2$.

We claim that also $\mathbf{A}'_1 \Rightarrow_{\exists^+\mathcal{L}^k} \mathbf{E}_{\mathbb{Z}_2,3}$. Without loss of generality, we may assume that $\mathbf{M}$ has its signature expanded by constant symbols for every segment (Lemma 4.15). For convenience, we fix an arbitrary linear order on M which coincides with $<^{\mathbf{M}}$ on $\mathrm{SG}(\mathbf{M})$, and say that x is a *left foot* and y a *right foot* of a segment s with $\mathrm{sg}^{-1}(s) = \{x,y\}$ if x is less than y w.r.t. this order. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the expansions of $\mathbf{M}$ by a constant for the left and the right foot of the first segment, respectively. By Lemma 4.15, we know that Duplicator has a winning strategy in the bijective $3k$-pebble game played on

$\mathbf{M}_1$ and $\mathbf{M}_2$. We use it to construct a winning strategy for Duplicator in the existential k-pebble game played on $\mathbf{A}_1'$ and $\mathbf{E}_{\mathbb{Z}_2,3}$.

Suppose we have a position in the existential k-pebble game with pebbles $1,\dots,k'$ placed on some $x_1,\dots,x_{k'} \in A_1'$ and $v_1,\dots,v_{k'} \in \{0,1\}$ for $k' \leq k$. If Spoiler chooses a pebble $i > k'$ and places it onto some $x_i \in A_1$, then we consider the situation in the bijective $3k$-pebble game played on $\mathbf{M}_1$ and $\mathbf{M}_2$ where Spoiler places, in three succeeding rounds, a pebble i on the corresponding segment x_i of $\mathbf{M}_1$, and two pebbles i_ℓ, i_r on its left and right foot x_{i_ℓ} and x_{i_r}, respectively. Since Duplicator has a winning strategy in this game, she can react by placing the pebbles i, i_ℓ, i_r on some elements y_i, y_{i_ℓ}, y_{i_r} of $\mathbf{M}_2$. Since her placement corresponds to a partial isomorphism and the signature contains constant symbols for every segment, we must have $x_i = y_i$ and $\{x_{i_\ell}, x_{i_r}\} = \{y_{i_\ell}, y_{i_r}\}$. Now if y_{i_ℓ} is the left and y_{i_r} the right foot of y_i, then Duplicator places v_i on 0 in the existential $2k$-pebble game, otherwise on 1. Note that, if x_i is the first segment, then y_{i_ℓ} must be is the right and y_{i_r} the left foot of y_i due to the presence of the additional constant. The case when $i \leq \ell$ corresponds to the situation when pebbles i, i_ℓ, i_r are lifted from both $\mathbf{M}_1$ and $\mathbf{M}_2$. Clearly Duplicator can maintain this condition, and her pebbling specifies a partial homomorphism by Lemma 4.17.

For $i \in \{1,2\}$, let $\mathbf{A}_i$ be $G(\mathbf{A}_i')$. Note that the homogeneous companion $\mathbf{A}$ of $\mathbf{A}_1$ and $\mathbf{A}_2$ is identical and contains a copy of $\mathbf{A}_2'$ with variables x_i^a for both upper indices $a \in \{0,1\}$. Thus, $\mathbf{A}$ only admits the trivial solution, which proves item (1). Also note that $\mathbf{A}_2$ is satisfiable by setting every variable x_i^a to a, and that the unsatisfiability of $\mathbf{A}_1'$ implies the unsatisfiability of $\mathbf{A}_1$, because the variables of the form x_i^0 induce a copy of $\mathbf{A}_1'$ in $\mathbf{A}_1$. This proves item (2). It follows from Lemma 4.20 that $\mathbf{A}_1 \equiv_{\mathcal{C}^k} \mathbf{A}_2$, which proves item (3). □

Theorem 4.21. $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ *is inexpressible in* FPC.

Proof. Our strategy is to pp-define in $(\mathbb{Q};\mathrm{X})$ a temporal structure $\mathbf{B}$ such that from each pair $\mathbf{A}_1$ and $\mathbf{A}_2$ as in Proposition 4.19 we can obtain instances $\mathbf{A}_1'$ and $\mathbf{A}_2'$ of $\mathrm{CSP}(\mathbf{B})$ with

(1) $\mathbf{A}_1' \equiv_{\mathcal{C}^k} \mathbf{A}_2'$,
(2) $\mathbf{A}_1' \not\to \mathbf{B}$, and $\mathbf{A}_2' \to \mathbf{B}$.

The signature of $\mathbf{B}$ is $\{R_2,\dots,R_5\}$, and we set $R_i^{\mathbf{B}} := R^{\mathrm{mx}}_{[i],i}$ for $i \in \{2,\dots,5\}$ (see Definition 4.3). By Lemma 4.2 together with Theorem 4.5, we have that $\mathbf{B}$ is pp-definable in $(\mathbb{Q};\mathrm{X})$. We now uniformly construct $\mathbf{A}_i'$ from $\mathbf{A}_i$ for both $i \in \{1,2\}$. The domain of $\mathbf{A}_i'$ is the domain of $\mathbf{A}_i$ extended by a new element z, and the relations of $\mathbf{A}_i'$ are given as follows:

- for every $(x_1,\dots,x_j) \in eqn^{\mathbf{A}_i}_{j,1}$, the relation $R^{\mathbf{A}_i'}_{j+1}$ contains the tuple $(x_1,\dots,x_j,z)$, and
- for every $(x_1,\dots,x_j) \in eqn^{\mathbf{A}_i}_{j,0}$, the relation $R^{\mathbf{A}_i'}_{j+2}$ contains the tuple $(x_1,\dots,x_j,z,z)$.

We have $\mathbf{A}_1' \equiv_{\mathcal{C}^2} \mathbf{A}_2'$ by taking the extension of the winning strategy for Duplicator in the bijective 2-pebble game played on $\mathbf{A}_1$ and $\mathbf{A}_2$ where the new variable z of $\mathbf{A}_1'$ is always mapped to its counterpart in $\mathbf{A}_2'$. This proves item (1).

We already know from Proposition 4.11 that $\mathrm{CSP}(\mathbb{Q};R^{\mathrm{mx}}_{[3],3}) = \mathrm{CSP}(\mathbb{Q};\mathrm{X})$ can be reformulated as a certain decision problem for linear Boolean equations which we call 3-ORD-XOR-SAT. Note that double occurrences of variables, such as the occurrence of z above, do matter for 3-ORD-XOR-SAT in contrast to plain satisfiability of linear Boolean equations. Also $\mathrm{CSP}(\mathbf{B})$ has a reformulation as a decision problem for linear Boolean equations where each constraint $R_j(x_1,\dots,x_j)$ for $j \in \{2,\dots,5\}$ is interpreted as the homogeneous Boolean equation $x_1 + \cdots + x_j = 0$. The reformulation is as follows and can be obtained as in the proof of Proposition 4.11:

INPUT: A finite homogeneous system of linear Boolean equations of length $\ell \in \{2,\dots,5\}$.
QUESTION: Does every non-empty subset S of the equations have a solution where at least one variable in an equation from S denotes the value 1?

Note that every solution of $\mathbf{A}_2$ viewed as an instance of CSP($\mathbf{E}_{\mathbb{Z}_2,3}$) extended by setting z to 1 restricts to a non-trivial solution to every subset S of the equations of $\mathbf{A}_2'$ with respect to the variables that appear in S, because z occurs in every equation of S. We claim that the system $\mathbf{A}_1'$ only admits the trivial solution. If z assumes the value 0 in a solution of $\mathbf{A}_1'$, then this case reduces to the homogeneous companion of $\mathbf{A}_1$ which has only the trivial solution. If z assumes the value 1 in a solution of $\mathbf{A}_1'$, then this case reduces to $\mathbf{A}_1$ which has no solution at all. This proves item (2).

It now follows from Theorem 2.3 that CSP($\mathbf{B}$) is inexpressible in FPC. Since $\mathbf{B}$ has a pp-definition in $(\mathbb{Q}; \mathrm{X})$, by Theorem 2.7 and Proposition 2.8, also CSP$(\mathbb{Q}; \mathrm{X})$ is inexpressible in FPC. □

5 Classification of TCSPs in FP

In this section we classify CSPs of temporal structures with respect to expressibility in fixed-point logic. We start with the case of a temporal structure $\mathbf{B}$ that is not preserved by any operation mentioned in Theorem 2.14. In general, it is not known whether the NP-completeness of CSP($\mathbf{B}$) is sufficient for obtaining inexpressibility in FP. What is sufficient is the fact that $\mathbf{B}$ pp-constructs $(\{0,1\}; \mathrm{1IN3})$.

Lemma 5.1. *Let* $\mathbf{B}$ *be a relational structure that pp-constructs* $(\{0,1\}; \mathrm{1IN3})$. *Then* CSP($\mathbf{B}$) *is inexpressible in FPC.*

Proof. It is well-known that $(\{0,1\}; \mathrm{1IN3})$ pp-constructs all finite structures. By the transitivity of pp-constructability, $\mathbf{B}$ pp-construct the structure $\mathbf{E}_{\mathbb{Z}_2,3}$ whose CSP is inexpressible in FPC by Theorem 10 in [1]. Thus, CSP($\mathbf{B}$) is inexpressible in FPC by Theorem 2.8. □

We show in Theorem 5.2 that the temporal structures preserved by mx for which we know that their CSP is expressible in FP by the results in Section 3 are precisely the ones unable to pp-define the relation X which we have studied in Section 4.

Theorem 5.2. *Let* $\mathbf{B}$ *be a temporal structure preserved by* mx. *Then either* $\mathbf{B}$ *admits a pp-definition of* X, *or one of the following is true:*

(1) $\mathbf{B}$ *is preserved by a constant operation,*
(2) $\mathbf{B}$ *is preserved by* min.

Proof. If (1) or (2) holds, then X cannot have a pp-definition in $\mathbf{B}$ by Proposition 2.1, because X is neither preserved by a constant operation nor by min.

Suppose that neither (1) nor (2) holds for $\mathbf{B}$, that is, $\mathbf{B}$ contains a relation that is not preserved by any constant operation, and a relation that is not preserved by min. Our goal is to show that X has a pp-definition in $\mathbf{B}$. The proof strategy is as follows. We first show that temporal relations which are preserved by mx and not preserved by a constant operation admit a pp-definition of $<$. Then we analyse the behaviour of projections of temporal relations which are preserved by mx and not preserved by min and use the pp-definability of $<$ to pp-define X.

We need to introduce some additional notation. Recall the definition of $\chi_{\bar{0}}(R)$ for a temporal relation R (Definition 4.3). For every $I \subseteq [n]$, we fix an arbitrary homogeneous system $M_I(R)\bar{x} = \bar{0}$ of linear equations over $\mathbb{Z}_2$ with solution space $\chi_{\bar{0}}(\mathrm{proj}_I(R))$, where the matrix $M_I(R)$ is in *reduced row echelon form*:

- all non-zero rows are above all rows that only contain zeros,
- the leading coefficient of a non-zero row is always strictly to the right of the leading coefficient of the row above it,
- every leading coefficient is the only non-zero entry in its column.

We reorder the columns of $M_I(R)$ such that it takes the form

$$\begin{pmatrix} U_m & * \end{pmatrix} \tag{3}$$

where U_m is the $m \times m$ unit matrix for some $m \leq n$; we also write $m_I(R)$ for m. Without loss of generality, we may also assume that I consists of the first $|I|$ elements of $[n]$. Finally, we define

$$\mathrm{supp}_{I,i}(R) := \{j \in [|I|] \mid M_I(R)[i,j] = 1\}.$$

Claim 5.3. *Let R be a non-empty temporal relation preserved by* mx*. If R is not preserved by a constant operation, then $<$ has a pp-definition in $(\mathbb{Q};R)$.*

Proof. Let n be the arity of R and let $m := m_{[n]}(R)$. Since R is not preserved by a constant operation, we have $\bar{1} \notin \chi(R)$. This means that $|\mathrm{supp}_{[n],i}(R)|$ is odd for some $i \leq n$ which is fixed for the remainder of the proof. Let R' be the contraction of R given by the pp-definition

$$R(x_1,\ldots,x_n) \wedge \bigwedge_{p,q \in [n]\setminus\{1,\ldots,m\}} x_p = x_q.$$

Note that R' is non-empty since R contains a tuple $\bar{t}$ which satisfies $\chi(\bar{t})[x_j] = 1$ if and only if

- $j > m$, or
- $j \leq m$ and $|\mathrm{supp}_{[n],j}(R)|$ is even.

We claim that every $\bar{t} \in R'$ is of this form. If $\chi(\bar{t})[x_j] = 0$ for some $j > m$, then $\chi(\bar{t})[x_\ell] = 0$ for every $\ell > m$ by the definition of R', which implies that $\chi(\bar{t})[x_\ell] = 0$ for every $\ell \leq m$ by a parity argument with the equations of $M_{[n]}(R)\bar{x} = \bar{0}$. But then no entry can be minimal in $\bar{t}$, a contradiction. Hence, $\chi(\bar{t})[x_\ell] = 1$ for every $\ell > m$.

For every $\ell \leq m$ we have $\chi(\bar{t})[\ell] = 1$ if and only if $|\mathrm{supp}_{[n],\ell}(R)|$ is even. Since R is non-empty, there exists an index $k \in \{m+1,\ldots,n\}$. We have $\bar{t}[k] < \bar{t}[i]$ for every $\bar{t} \in R'$ due to our previous argumentation. Hence, the relation $<$ coincides with $\mathrm{proj}_{\{k,i\}}(R')$ and therefore has a pp-definition in $(\mathbb{Q};R)$. $\square$

Claim 5.4. *Let R be an n-ary temporal relation preserved by* mx *such that for every $I \subseteq [n]$, the set $\chi_{\bar{0}}(\mathrm{proj}_I(R))$ is the solution space of a homogeneous system of Boolean linear equations with at most two variables per equation. Then R is preserved by* min.

Proof. We show the claim by induction on n. For $n = 0$, there is nothing to show. Suppose that the statement holds for all relations with arity less than n. For every $I \subseteq [n]$ we fix an arbitrary homogeneous system $M_I(R)\bar{x} = \bar{0}$ of Boolean linear equations with solution space $\chi_{\bar{0}}(\mathrm{proj}_I(R))$ that has at most two variables per equation. Note that $\chi_{\bar{0}}(\mathrm{proj}_I(R))$ is preserved by the Boolean maximum operation max. Moreover, for all $\bar{s},\bar{s}' \in \{0,1\}^{|I|}$ we have $\bar{0} = \max(\bar{s},\bar{s}')$ if and only if $\bar{s} = \bar{s}' = \bar{0}$, which means that $\chi(\mathrm{proj}_I(R))$ itself is preserved by max. Now for every pair $\bar{t},\bar{t}' \in R$ we want to show that $\min(\bar{t},\bar{t}') \in R$. If $\min(\bar{t}) = \min(\bar{t}')$, then $\chi(\min(\bar{t},\bar{t}')) = \max(\chi(\bar{t}),\chi(\bar{t}')) \in \chi(R)$. If $\min(\bar{t}) \neq \min(\bar{t}')$, then $\chi(\min(\bar{t},\bar{t}')) \in \{\chi(\bar{t}),\chi(\bar{t}')\} \subseteq \chi(R)$. Thus, there exists a tuple $\bar{c} \in R$ with $\chi(\bar{c}) = \chi(\min(\bar{t},\bar{t}'))$. We set $I := \mathrm{argmin}(\bar{c})$. Since the statement holds for $\mathrm{proj}_{n\setminus I}(R)$ by induction hypothesis and $\mathrm{proj}_{[n]\setminus I}(\min(\bar{t},\bar{t}')) = \min(\mathrm{proj}_{[n]\setminus I}(\bar{t}),\mathrm{proj}_{[n]\setminus I}(\bar{t}')) \in \mathrm{proj}_{[n]\setminus I}(R)$, there exists $\bar{r} \in R$ with $\mathrm{proj}_{[n]\setminus I}(\min(\bar{t},\bar{t}')) = \mathrm{proj}_{[n]\setminus I}(\bar{r})$. We can apply an automorphism to $\bar{r}$ to obtain a tuple $\bar{r}' \in R$ where all entries are positive. We can also apply an automorphism to obtain a tuple $\bar{c}' \in R$ so that its minimal entries $i \in I$ are equal 0 and for every other entry $i \in [n] \setminus I$ it holds that $\bar{c}'[i] > \bar{r}'[i]$. Then $\mathrm{mx}(\bar{c}',\bar{r}')$ yields a tuple in R which in the same orbit as $\min(\bar{t},\bar{t}')$. Hence, R is preserved by min. $\square$

Claim 5.5. *Let R be a temporal relation preserved by* mx*. If R is not preserved by* min*, then* X *has a pp-definition in $(\mathbb{Q};R,<)$.*

Proof. Let n be the arity of R. Since R is not preserved by min, Claim 5.4 implies that there exists $I \subseteq [n]$ such that $\chi_{\bar{0}}(\text{proj}_I(R))$ is not the solution space of any homogeneous system of Boolean linear equations with at most two variables per equation. Recall that $\chi_{\bar{0}}(\text{proj}_I(R))$ is the solution space of a system $M_I(R)\bar{x} = \bar{0}$ of linear equations over $\mathbb{Z}_2$ where $M_I(R)$ is as in (3). Let $m := m_I(R)$ and fix an arbitrary index $i \in [m]$ with $|\text{supp}_{I,i}(R)| \geq 3$. We also fix an arbitrary pair of distinct indices $k, \ell \in \text{supp}_{I,i}(R) \setminus \{i\}$. Note that $k, \ell \in \{m+1, \dots, |I|\}$ by the shape of the matrix $M_I(R)$. We claim that the formula $\phi(x_i, x_k, x_\ell)$ obtained from the formula

$$R(x_1, \dots, x_n) \wedge \bigwedge_{j \in I \setminus \{k, \ell, 1, \dots, m\}} x_i < x_j \tag{4}$$

by existentially quantifying all variables except for x_i, x_k, x_ℓ is a pp-definition of X.

"$\Rightarrow$" Let $\bar{t} \in$ X. We have to prove that $\bar{t}$ satisfies $\phi(x_i, x_k, x_\ell)$. First, suppose that $\bar{t}[x_i] = \bar{t}[x_k] < \bar{t}[x_\ell]$. Note that $M_I(R)\bar{x} = \bar{0}$ has a solution where

- x_k takes value 1,
- all variables x_j such that the j-th equation contains x_k are also set to 1, and
- all other variables are set to 0.

The reason is that in this way in each equation that contains x_k exactly two variables are set to 1, and in each equation that does not contain k no variable is set to 1. Hence, R contains a tuple $\bar{s}'$ such that $\chi(\text{proj}_I(\bar{s}'))$ corresponds to this solution. Note that $\bar{s}'$ also satisfies (4), and that there exists $\alpha' \in \text{Aut}(\mathbb{Q}; <)$ such that $\bar{t} = (\alpha'\bar{s}'[x_i], \alpha'\bar{s}'[x_k], \alpha'\bar{s}'[x_\ell])$.

The case $\bar{t}[x_i] = \bar{t}[x_\ell] < \bar{t}[x_k]$ can be treated analogously to the first one, using a tuple $\bar{s}'' \in R$ such that for all $j \in I$

$$\chi(\text{proj}_I(\bar{s}''))[x_j] = \begin{cases} 1 & \text{if } \ell = j, \\ 1 & \text{if } \ell \in \text{supp}_{I,j}(R), \\ 0 & \text{otherwise,} \end{cases}$$

and $\alpha'' \in \text{Aut}(\mathbb{Q}; <)$ such that $\bar{t} = (\alpha''\bar{s}''[x_i], \alpha''\bar{s}''[x_k], \alpha''\bar{s}''[x_\ell])$.

Finally, suppose that $\bar{t}[x_k] = \bar{t}[x_\ell] < \bar{t}[x_i]$. Let $\bar{s}', \bar{s}''$ and α', α'' be the auxiliary tuples and automorphisms of $(\mathbb{Q}; <)$, respectively, from the two previous cases. Then $\bar{s} := \text{mx}(\alpha'\bar{s}', \alpha''\bar{s}'') \in R$ is a tuple that satisfies the quantifier-free part of (4), and there exists $\alpha \in \text{Aut}(\mathbb{Q}; <)$ such that $\bar{t} = (\alpha\bar{s}[x_i], \alpha\bar{s}[x_k], \alpha\bar{s}[x_\ell])$.

"$\Leftarrow$" Suppose $\bar{s} \in R$ satisfies (4). We must show $(\text{proj}_I(\bar{s})[x_i], \text{proj}_I(\bar{s})[x_k], \text{proj}_I(\bar{s})[x_\ell]) \in X$. By the final conjuncts of (4) all indices of minimal entries in $\text{proj}_I(\bar{s})$ must be from $x_k, x_\ell, x_1, \dots, x_m$. Let $j \in I$ be the index of a minimal entry in $\text{proj}_I(\bar{s})$. First consider the case $j \in \{i, k, \ell\}$. The shape of $M_I(R)$ implies that the variables of the i-th equation of $M_I(R)\bar{x} = \bar{0}$ must come from $x_i, x_{m+1}, \dots, m_{|I|}$. As mentioned, none of these variables can denote a minimal entry in $\text{proj}_I(\bar{s})$ except for x_i, x_k, and x_ℓ. Hence, the i-th equation implies that $\text{proj}_I(\bar{s})$ takes a minimal value at exactly two of the indices $\{i, k, \ell\}$. So we conclude that $(\text{proj}_I(\bar{s})[x_i], \text{proj}_I(\bar{s})[x_k], \text{proj}_I(\bar{s})[x_\ell]) \in X$.

Otherwise, $j \in \{1, \dots, m\} \setminus \{i\}$. The shape of $M_I(R)$ implies that the variables of the j-th equation of $M_I(R)\bar{x} = \bar{0}$ must come from $x_j, x_{m+1}, \dots, m_{|I|}$. None of these variables can denote a minimal entry in $\text{proj}_I(\bar{s})$ except for x_j, x_k, and x_ℓ. Hence, the j-th equation of $M_I(R)\bar{x} = \bar{0}$ implies that $\text{proj}_I(\bar{s})$ takes a minimal value at exactly two of the indices $\{j, k, \ell\}$. We have thus reduced the situation to the first case. □

The statement of Theorem 5.2 follows from Claim 5.3 and Claim 5.5. □

We are now ready for the proof of our characterisation of temporal CSPs in FP and FPC.

Proof of Theorem 1.3. Let **B** be a temporal structure.

(1)$\Rightarrow$(2): Trivial because FP is a fragment of FPC.

(2)⇒(3): Lemma 5.1 implies that **B** does not pp-construct $(\{0,1\};1\text{IN}3)$; Theorem 4.21 and Theorem 2.8 show that **B** does not pp-construct $(\mathbb{Q};\mathrm{X})$.

(3)⇒(4): Since **B** does not pp-construct $(\{0,1\};1\text{IN}3)$, by Theorem 2.14, **B** is preserved by min, mi, mx, ll, the dual of one of these operations, or by a constant operation. If **B** is preserved by mx but neither by min nor by a constant operation, then **B** pp-defines X by Theorem 5.2, a contradiction to (3). If **B** is preserved by dual mx but neither by max nor by a constant operation, then **B** pp-defines $-X$ by the dual version of Theorem 5.2. Since $(\mathbb{Q};X)$ and $(\mathbb{Q};-X)$ are homomorphically equivalent, we get a contradiction to (3) in this case as well. Thus (4) must hold for **B**.

(4)⇒(1): If **B** has a constant polymorphism, then CSP(**B**) is trivial and thus expressible in FP. If **B** is preserved by min, mi, or ll, then every relation of **B** is pp-definable in $(\mathbb{Q};\mathrm{R}^{\leq}_{\min},<)$ by Lemma 3.4, or in $(\mathbb{Q};\mathrm{R}_{\mathrm{mi}},\mathrm{S}_{\mathrm{mi}},\neq)$ by Lemma 3.8, or in $(\mathbb{Q};\mathrm{R}_{\mathrm{ll}},\mathrm{S}_{\mathrm{ll}},\neq)$ by Lemma 3.18. Thus, CSP(**B**) is expressible in FP by Proposition 3.6, Proposition 3.13, or Proposition 3.23 combined with Theorem 2.8. Each of the previous statements can be dualized to obtain expressibility of CSP(**B**) in FP if **B** is preserved by max, dual mi, or dual ll. □

We finally prove our characterisation of the temporal CSPs in FPR_2.

Proof of Theorem 1.4. If **B** pp-constructs all finite structures, then **B** pp-constructs in particular the structure $\mathbf{E}_{\mathbb{Z}_3,3}$. It follows from work of Grädel and Pakusa [40] that $\text{CSP}(\mathbf{E}_{\mathbb{Z}_3,3})$ is inexpressible in FPR_2 (see the comments after Theorem 6.8 in [36]). Theorem 2.8 then implies that CSP(**B**) is inexpressible in FPR_2 as well by.

For the backward direction suppose that **B** does not pp-construct all finite structures. Then **B** is preserved by one of the operations listed in Theorem 2.14. If **B** is preserved by min, mi, ll, the dual of one of these operations, or by a constant operation, then **B** is expressible in FP by Theorem 1.3 and thus in FPR_2. If **B** has mx as a polymorphism, then every relation of **B** is pp-definable in the structure $(\mathbb{Q};\mathrm{X})$ by Lemma 4.2. Thus, CSP(**B**) is expressible in FPR_2 by Proposition 4.10 combined with Theorem 2.8. Dually, if **B** has the polymorphism dual mx, then CSP(**B**) is expressible in FPR_2 as well. □

6 Classification of TCSPs in Datalog

In this section, we classify temporal CSPs with respect to expressibility in Datalog. In some of our syntactic arguments in this section it will be convenient to work with formulas over the structure $(\mathbb{Q};\leq,\neq)$ instead of the structure $(\mathbb{Q};<)$. A $\{\leq,\neq\}$-formula is called *Ord-Horn if it is a conjunction of clauses of the form* $x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee x \leq y$ where the last disjunct is optional [9]. Nebel and Bürckert [55] showed that satisfiability of Ord-Horn formulas can be decided in polynomial time. Their algorithm shows that if a all relations of a template **B** are definable by Ord-Horn formulas, then CSP(**B**) can be solved by a Datalog program. We prove in Proposition 6.1 that Ord-Horn definability of temporal relations can be characterised in terms of admitting certain polymorphisms. Later we will prove that there is no characterisation of expressibility in Datalog in terms of identities for polymorphism clones (see Proposition 7.6).

Proposition 6.1. *A temporal relation is definable by an Ord-Horn formula if and only if it is preserved by every binary injective operation on* $\mathbb{Q}$ *that preserves* $\leq$.

Proposition 6.1 is proved using the syntactic normal form for temporal relations preserved by pp from [11] and the syntactic normal form for temporal relations preserved by ll from [9].

Proposition 6.2 ([11])**.** *A temporal relation is preserved by* pp *if and only if it can be defined by a conjunction of formulas of the form* $z_1 \circ z_0 \vee \cdots \vee z_n \circ_n z_0$ *where* $\circ_i \in \{\leq,\neq\}$ *for each* $i \in \{1,\ldots,n\}$.

Lemma 6.3. *Every temporal relation preserved by* pp *or* ll *can be defined by a conjunction of formulas of the form* $x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee z_1 \leq z_0 \vee \cdots \vee z_\ell \leq z_0$.

Proof. If R is a temporal relation preserved by pp then the statement follows from Proposition 6.2. Let R be a temporal relation preserved by ll. Then R is definable by a conjunction of clauses $\phi := \bigwedge_i \phi_i$ where each clause ϕ_i is as in Proposition 3.19. For an index i, let ψ_i be obtained from ϕ_i by dropping the inequality disjuncts of the form $x \neq y$. So ψ_i is of the form

(1) $z_1 < z \vee \cdots \vee z_\ell < z$, or of the form
(2) $z_1 < z \vee \cdots \vee z_\ell < z \vee (z = z_1 = \cdots = z_\ell)$.

If ψ_i is of the form (1), then ψ_i is a formula preserved by min (Proposition 3.5). If ψ_i is of the form (2), then it is easy to see that ψ_i is equivalent to

$$\bigwedge_{j \in [\ell]} \left(z_j \leq z \vee \bigvee_{k \in [\ell] \setminus \{j\}} x_k < z \right).$$

which is a formula preserved by min as well (Proposition 3.5). In particular, in both cases ψ_i is preserved by pp. We replace in ϕ_i the disjunct ψ_i by an equivalent conjunction of clauses as in Proposition 6.2. By use of distributivity of $\vee$ and $\wedge$, we can then rewrite ϕ_i into a definition of R that has the desired form. ☐

A $\{\leq, \neq\}$-formula ϕ is said to be in *conjuctive normal form (CNF)* if it is a conjunction of *clauses*; a *clause* if a disjunction of *literals*, i.e., atomic $\{\leq, \neq\}$-formulas or negations of atomic $\{\leq, \neq\}$-formulas. We say that ϕ is *reduced* if for any literal of ϕ, the formula obtained by removing a literal from ϕ is not equivalent to ϕ over $(\mathbb{Q}; \leq, \neq)$. The next lemma is a straightforward but useful observation.

Lemma 6.4. *Every $\{\leq, \neq\}$-formula is equivalent to a formula in reduced CNF. If ϕ is in reduced CNF with a clause γ and a literal α of γ, then there exists a tuple which satisfies $\gamma \wedge \alpha$ but does not satisfy all other literals of γ.*

Definition 6.5. The k-ary operation *lex* on $\mathbb{Q}$ is defined by

$$\text{lex}_k(\bar{t}) := \text{lex}\big(\bar{t}[1], \text{lex}\big(\bar{t}[2], \dots \text{lex}(\bar{t}[k-1], \bar{t}[k]) \dots\big)\big).$$

Proof of Proposition 6.1. The forward direction is straightforward: every clause of an Ord-Horn formula is preserved by every injective operation on $\mathbb{Q}$ that preserves $\leq$. For the backward direction, let R be a temporal relation preserved by every binary injective operation on $\mathbb{Q}$ that preserves $\leq$. In particular, R is preserved by ll and by $f\colon \mathbb{Q}^2 \to \mathbb{Q}$ defined by $f(x,y) := \text{lex}_3(\max(x,y), x, y)$. Let ϕ be a definition of R provided by Lemma 6.3. Note that if we remove literals from ϕ, then the resulting formula is still of the same syntactic form, so by Lemma 6.4 we may assume that ϕ is a reduced CNF definition. Let $\psi = (x_1 \neq y_1 \vee \cdots \vee x_m \neq y_m \vee z_1 \leq z_0 \vee \cdots \vee z_\ell \leq z_0)$ be a conjunct of ϕ. We claim that $\ell \leq 1$. Otherwise, by Lemma 6.4, there exist tuples $\bar{t}_1, \bar{t}_2 \in R$ such that $\bar{t}_1$ does not satisfy all disjuncts of ψ except for $z_1 \leq z_0$ and $\bar{t}_2$ does not satisfy all disjuncts of ψ except for $z_2 \leq z_0$. Without loss of generality, we may assume that $\bar{t}_1[z_0] = \bar{t}_2[z_0]$, because otherwise we may replace $\bar{t}_1$ with $\alpha\bar{t}_1$ for some $\alpha \in \text{Aut}(\mathbb{Q}; <)$ that maps $\bar{t}_1[z_0]$ to $\bar{t}_2[z_0]$. Note that $f(\bar{t}_1, \bar{t}_2)$ does not satisfy $z_i \leq z_0$ for every $i \in \{3, \dots, \ell\}$ because f preserves $<$. Also note that $\bar{t}_1[z_1] \leq \bar{t}_1[z_0]$, $\bar{t}_1[z_0] < \bar{t}_1[z_2]$, $\bar{t}_2[z_0] < \bar{t}_2[z_1]$, and $\bar{t}_2[z_2] \leq \bar{t}_2[z_0]$. Since $\bar{t}_1[z_0] = \bar{t}_2[z_0]$, we have $f(\bar{t}_1, \bar{t}_2)[z_0] < f(\bar{t}_1, \bar{t}_2)[z_1]$ and $f(\bar{t}_1, \bar{t}_2)[z_0] < f(\bar{t}_1, \bar{t}_2)[z_2]$ by the definition of f. But then $f(\bar{t}_1, \bar{t}_2)$ does not satisfy ψ, a contradiction to f being a polymorphism of R. Hence $\ell \leq 1$. Since ψ was chosen arbitrarily, we conclude that ϕ is Ord-Horn. ☐

Let R_{min} be the temporal relation defined by $y < x \vee z < x$ that was already mentioned in the introduction. Recall that $\text{CSP}(\mathbb{Q}; \text{R}_{\text{min}})$ is inexpressible in Datalog [15]. The reason for inexpressibility is not unbounded counting width, but the combination of the two facts that $\text{CSP}(\mathbb{Q}; \text{R}_{\text{min}})$ admits unsatisfiable instances of arbitrarily high girth, and that all proper projections of R_{min} are trivial.

The counting width of $\mathrm{CSP}(\mathbb{Q};\mathrm{R_{min}})$ is bounded because $\text{co-CSP}(\mathbb{Q};\mathrm{R_{min}})$ is definable using the FP sentence $\exists x[\mathsf{dfp}_{U,x}\exists y,z(U(y)\wedge U(z)\wedge \mathrm{R_{min}}(x,y,z))](x)$, see the paragraph below Theorem 2.3. We show in Theorem 6.6 that the inability of a temporal structure with a polynomial-time tractable CSP to pp-define $\mathrm{R_{min}}$ can be characterised in terms of being preserved by a constant operation, or by the operations from Proposition 6.1 which witness Ord-Horn definability.

Theorem 6.6. *Let* $\mathbf{B}$ *be a temporal structure that admits a pp-definition of* $<$. *Then exactly one of the following two statements is true:*

(1) $\mathbf{B}$ *admits a pp-definition of the relation* $\mathrm{R_{min}}$ *or of the relation* $-\mathrm{R_{min}}$,
(2) $\mathbf{B}$ *is preserved by every binary injective operation on* $\mathbb{Q}$ *that preserves* $\leq$.

Proof. If (2) holds, then $\mathbf{B}$ is in particular preserved by ll and dual ll. But then, by Proposition 2.1, neither $\mathrm{R_{min}}$ nor $-\mathrm{R_{min}}$ has a pp-definition in $\mathbf{B}$ because $\mathrm{R_{min}}$ is not preserved by dual ll and $-\mathrm{R_{min}}$ is not preserved by ll [15].

Suppose that (2) does not hold. The *Betweenness* problem mentioned in the introduction can be formulated as the CSP of a temporal structure $(\mathbb{Q};\mathrm{Betw})$ where

$$\mathrm{Betw} := \{(x,y,z)\in\mathbb{Q}^3 \mid x<y<z \text{ of } z<y<x\}.$$

Claim 6.7. $(\mathbb{Q};\mathrm{Betw},<)$ *pp-defines both relations* $\mathrm{R_{min}}$ *and* $-\mathrm{R_{min}}$.

Proof. We show that $\phi(x,y,z) := \exists a,b\big(\mathrm{Betw}(a,x,b)\wedge y<a\wedge z<b\big)$ is a pp-definition of $\mathrm{R_{min}}$. Then it will be clear that $-\mathrm{R_{min}}$ has the pp-definition $\exists a,b\big(\mathrm{Betw}(a,x,b)\wedge(a<y)\wedge(b<z)\big)$.

"$\Rightarrow$" Let $\bar{t}\in\mathrm{R_{min}}$ be arbitrary. We may assume with loss of generality that $\bar{t}[3]<\bar{t}[1]$. Then any $a,b\in\mathbb{Q}$ such that $\bar{t}[3]<a<\bar{t}[1]$ and $\max(\bar{t}[1],\bar{t}[2])<b$ witness that $\bar{t}$ satisfies ϕ.

"$\Leftarrow$" Let $\bar{t}\in\mathbb{Q}^3\setminus\mathrm{R_{min}}$ be arbitrary. Then $\bar{t}[1]\leq\bar{t}[2]$ and $\bar{t}[1]\leq\bar{t}[3]$. Suppose that there exist $a,b\in\mathbb{Q}$ witnessing that $\bar{t}$ satisfies ϕ. Then $\bar{t}[1]\leq\bar{t}[2]<a$ and $\bar{t}[1]\leq\bar{t}[3]<b$, a contradiction to $\mathrm{Betw}(a,\bar{t}[1],b)$. Thus, $\bar{t}$ does not satisfy ϕ. $\square$

Now suppose that $\mathbf{B}$ does not pp-define Betw. Then by Lemma 10 and Lemma 49 in [14], $\mathbf{B}$ is preserved by some operation $g\in\{\mathrm{pp},\mathrm{ll},\mathrm{dual\,pp},\mathrm{dual\,ll}\}$. If $\mathbf{B}$ is preserved by pp or ll then we can apply Lemma 6.3. The case where $\mathbf{B}$ is preserved by dual pp or dual ll can be shown analogously using a dual version of Lemma 6.3. The proof strategy is as follows. We fix any relation R of $\mathbf{B}$ which is not preserved by some binary injective operation f on $\mathbb{Q}$ that preserves $\leq$. Lemma 6.3 implies that R has a definition of a particular form. It turns out that the projection onto a particular set of three entries in R behaves like $\mathrm{R_{min}}$ modulo imposing some additional constraints onto the remaining variables. These additional constraints rely on the pp-definability of $<$.

Let R be a relation of arity n with a primitive positive definition in $\mathbf{B}$ such that R is not preserved by a binary operation f on $\mathbb{Q}$ preserving $\leq$. Let $\phi(u_1,\dots,u_n)$ be a definition of R of the form as described in Lemma 6.3; by Lemma 6.4, we may assume that ϕ is reduced. Then ϕ must have a conjunct ψ of the form

$$x_1\neq y_1\vee\cdots\vee x_m\neq y_m\vee z_1\leq z_0\vee\cdots\vee z_\ell\leq z_0$$

that is not preserved by f. Since f is injective and preserves $\leq$, it preserves all Ord-Horn formulas. Hence, $\ell\geq 2$. Since ϕ is reduced, there are tuples $\bar{t}_1$ and $\bar{t}_2$ such that for $i\in\{1,2\}$

- $\bar{t}_i$ satisfies the disjunct $z_i\leq z_0$ of ψ;
- $\bar{t}_i$ does not satisfy all other disjuncts of ψ.

Let $\psi_{\mathrm{R_{min}}}(z_0,v_1,v_2)$ be the formula obtained from existentially quantifying all variables except for z_0, v_1, and v_2 in the following formula

$$v_1<z_1\wedge v_2<z_2\wedge R(u_1,\dots,u_n)\wedge\bigwedge_{j\in\{3,\dots,\ell\}} z_0<z_j\wedge\bigwedge_{i\in[m]} x_i=y_i. \tag{5}$$

We claim that $\psi_{\mathrm{R_{min}}}$ is a pp-definition of $R_{\min}$ over $(\mathbb{Q};<,R)$. For the forward direction let $\bar{t} \in \mathrm{R_{min}}$. First suppose that $\bar{t}[v_1] < \bar{t}[z_0]$ Let α be any automorphism of $(\mathbb{Q};<)$ that sends

- $\bar{t}_2[z_0]$ to $\bar{t}[z_0]$, and
- $\bar{t}_2[z_1]$ to some rational number q with $\bar{t}[v_1] < q \leq \bar{t}[z_0]$.

Then $\alpha(\bar{t}_2)$ provides witnesses for the variables $u_1,\dots,u_n$ in (5) showing that $\bar{t}$ satisfies $\psi_{\mathrm{R_{min}}}$. The case where $\bar{t}[z_0] \leq \bar{t}[v_1]$ and $\bar{t}[z_0] > \bar{t}[v_2]$ can be treated analogously, using $\bar{t}_1$ instead of $\bar{t}_2$.

For the backward direction, suppose that $s \in \mathbb{Q}^{n+2}$ satisfies (5). In particular, $\bar{s}[z_0] < \bar{s}[z_j]$ for every $j \in \{3,\dots,\ell\}$ and $\bar{s}[x_i] = \bar{s}[y_i]$ for every $i \in \{1,\dots,m\}$, and hence $\bar{s}[z_1] \leq \bar{s}[z_0]$ or $\bar{s}[z_2] \leq \bar{s}[z_0]$ because s satisfies ψ. If $\bar{s}[z_1] \leq \bar{s}[z_0]$ then $\bar{s}[v_1] < \bar{s}[z_1] \leq \bar{s}[z_0]$ and hence $(\bar{s}[z_0],\bar{s}[z_1],\bar{s}[z_2]) \in \mathrm{R_{min}}$. Similarly, if $\bar{s}[z_2] \leq \bar{s}[z_0]$ then $\bar{s}[v_2] < \bar{s}[z_2] \leq \bar{s}[z_0]$ and again $(\bar{s}[z_0],\bar{s}[z_1],\bar{s}[z_2]) \in \mathrm{R_{min}}$. □

We are ready for the proof of our second classification result; it combines Theorem 2.8, Proposition 6.1, Theorem 6.6, and results from previous sections.

Proof of Theorem 1.2. Let **B** be a temporal structure.

(1)⇒(2): If CSP(**B**) is expressible in Datalog, then **B** does not pp-construct $(\{0,1\};\mathrm{1IN3})$; otherwise we get a contradiction to the expressibility of CSP(**B**) in Datalog by Theorem 1.3, because Datalog is a fragment of FP. Moreover, **B** does not pp-construct $(\mathbb{Q},\mathrm{R_{min}})$; otherwise we get a contradiction to the inexpressibility of CSP$(\mathbb{Q},\mathrm{R_{min}})$ in Datalog (Theorem 5.2 in [15]) through Theorem 2.8.

(2)⇒(3): Since **B** does not pp-construct $(\{0,1\};\mathrm{1IN3})$, Theorem 2.14 implies that **B** is preserved by min, mi, mx, ll, the dual of one of these operations, or by a constant operation. In the case where **B** is preserved by a constant operation we are done, so suppose that **B** is not preserved by a constant operation. First consider the case that $<$ is pp definable in **B**. Since $\mathrm{R_{min}}$ and $-\mathrm{R_{min}}$ are not pp-definable in **B**, Theorem 6.6 shows that **B** is preserved by every binary injective operation on $\mathbb{Q}$ preserving $\leq$. In particular, **B** is preserved by ll and dual ll.

Now consider the case that $<$ is not pp definable. Since none of the temporal relations Cycl, Betw, Sep listed in Theorem 10.3.2 in [9] is preserved by any of the operations min, mi, mx, ll, or their duals, the theorem implies that Aut(**B**) contains all permutations of $\mathbb{Q}$. This means that **B** is an *equality constraint language* as defined in [13]. The structure **B** has a polymorphism which depends on two arguments but it does not have a constant polymorphism. Therefore, **B** has a binary injective polymorphism, by Theorem 4 in [13]. Since **B** is an equality constraint language with a binary injective polymorphism, by Lemma 2 in [13], **B** is preserved by every binary injection on $\mathbb{Q}$. In particular, **B** is preserved by ll and dual ll.

(3)⇒(1): If **B** has a constant polymorphism, then CSP(**B**) is trivial and thus expressible in Datalog. Otherwise, **B** is preserved by both ll and dual ll. Then also the expansion **B**′ of **B** by $<$ is preserved by both ll and dual ll. The latter implies that **B**′ cannot pp-define $\mathrm{R_{min}}$. Thus, Theorem 6.6 implies that **B**′ is preserved by every binary injective operation on $\mathbb{Q}$ that preserves $\leq$. Then Proposition 6.1 then shows that all relations of **B**′ and in particular of **B** are Ord-Horn definable. Therefore, CSP(**B**) is expressible in Datalog by Theorem 22 in [55]. □

7 Algebraic conditions for temporal CSPs

In this section, we consider several candidates for general algebraic criteria for expressibility of CSPs in FP and Datalog stemming from the well-developed theory of finite-domain CSPs. Some of these criteria have already been displayed in Theorem 1.1; several other conditions that are equivalent over finite structures will be discussed in this section.

Our results imply that none of them can be used to characterise expressibility of temporal CSPs in FP or in Datalog. However, we also present a new simple algebraic condition which characterises expressibility of both finite-domain and temporal CSPs in FP, proving Theorem 1.7. We assume basic knowledge of universal algebra; see, e.g., the textbook of Burris and Sankappanavar [54].

Definition 7.1. An *identity* is a formal expression $s(x_1,\dots,x_n) = t(y_1,\dots,y_m)$ where s and t are terms built from function symbols and the variables $x_1,\dots,x_n$ and $y_1,\dots,y_m$, respectively. An *(equational) condition* is a set of identities. Let $\mathscr{A}$ be a set of operations on a fixed set A. For a set $F \subseteq A$, a condition $\mathcal{E}$ is *satisfied in* $\mathscr{A}$ *on* F if the function symbols of $\mathcal{E}$ can be assigned functions in $\mathscr{A}$ in such a way that all identities of $\mathcal{E}$ become true for all possible values of their variables in F. If $F = A$, then we simply say that $\mathcal{E}$ *is satisfied in* $\mathscr{A}$.

7.1 Failures of known equational conditions

If we add to the assumptions of Theorem 1.1 that all polymorphisms f of $\mathbf{B}$ are *idempotent*, i.e., satisfy $f(x,\dots,x) \approx x$, then the list of equivalent items can be prolonged further. In this setting, a prominent condition is that the variety of $\mathrm{Pol}(\mathbf{B})$ is *congruence meet-semidistributive*, short SD($\wedge$), which can also be studied over infinite domains. By Theorem 1.7 in [56], in general SD($\wedge$) is equivalent to the existence of so-called $(3+n)$-polymorphisms for some n; these are idempotent operations f, g_1, g_2 where g_i is m_i-ary and f is m_1+m_2-ary, that satisfy

$$f(x,\dots,x,\overset{i}{y},x,\dots,x) \approx g_1(x,\dots,x,\overset{i}{y},x,\dots,x) \qquad \text{for every } i \leq m_1,$$
$$f(x,\dots,x,\overset{m_1+i}{y},x,\dots,x) \approx g_2(x,\dots,x,\overset{i}{y},x,\dots,x) \qquad \text{for every } i \leq m_2.$$

Proposition 7.2 below implies that the correspondence between SD($\wedge$) and expressibility in Datalog / FP / FPC fails for temporal CSPs. A set of identities $\mathcal{E}$ is called

- *idempotent* if, for each operation symbol f appearing in the condition, $f(x,\dots,x) \approx x$ is a consequence of $\mathcal{E}$, and
- *trivial* if $\mathcal{E}$ can be satisfied by projections over a set A with $|A| \geq 2$, and *non-trivial* otherwise.

An operation $f\colon \mathbb{Q}^n \to \mathbb{Q}$ *depends* on the i-th argument if there exist $a_1,\dots,a_n,a \in \mathbb{Q}$ with $a_i \neq a$ and $f(a_1,\dots,a_n) \neq f(a_1,\dots,a_{i-1},a,a_{i+1},\dots,a_n)$. Let $\{i_1,\dots,i_m\} \subseteq [n]$ be the set of all indices $i \in [n]$ such that f depends on the i-th argument. We define the *essential part* of an operation f as the map

$$f^{ess}\colon \mathbb{Q}^m \to \mathbb{Q}, \quad (x_1,\dots,x_m) \mapsto f(x_{\mu_f(1)},\dots,x_{\mu_f(n)})$$

where $\mu_f\colon [n] \to [m]$ is any map that satisfies $\mu_f(i_\ell) = \ell$ for each $\ell \in [m]$. Proposition 6.1.4 in [9] states that every $f \in \mathrm{Pol}(\mathbb{Q}; I_4)$ for $I_4 := \{\bar t \in \mathbb{Q}^4 \mid \bar t[1] = \bar t[2] \Rightarrow \bar t[3] = \bar t[4]\}$ that depends on all of its arguments is injective.

Proposition 7.2. *The polymorphism clone of* $(\mathbb{Q}; \neq, \mathrm{S_{ll}})$ *does not satisfy any non-trivial idempotent condition (but* $\mathrm{CSP}(\mathbb{Q}, \neq, S_{ll})$ *is expressible in* FP*).*

Proof. Suppose, on the contrary, that $\mathrm{Pol}(\mathbb{Q}; \mathrm{S_{ll}}, \neq)$ satisfies a non-trivial condition $\mathcal{E}$ witnessed by some idempotent operations f_i. For each i, consider the idempotent operation f_i^{ess}; it is at least unary because $\neq$ has no constant polymorphism. It is also injective because the relation I_4 has a pp definition in $(\mathbb{Q}, \neq, S_{ll})$. Suppose for contradiction that f_i^{ess} has more than one argument. Let $c := f_i^{ess}(0,\dots,0,1)$. Note that $f_i^{ess}(c,\dots,c) = c$ by the idempotence of f_i^{ess}, and hence $f_i^{ess}(0,\dots,0,1) = f_i^{ess}(c,\dots,c)$, contradicting the injectivity of f_i^{ess}. Thus, f_i^{ess} must be unary. But the idempotence of f_i implies that each f_i is a projection, in contradiction to the non-triviality of $\mathcal{E}$. □

Simply dropping idempotence in the definition of $3+n$ terms does not provide a characterisation of FP either, as the following proposition shows.

Proposition 7.3. $\mathrm{Pol}(\mathbb{Q};\mathrm{X})$ *contains not necessarily idempotent* $(3+3)$*-operations (but* $\mathrm{CSP}(\mathbb{Q};X)$ *is not in FP).*

Proof. Consider the operations

$$\begin{aligned} \tilde{g}_2(x_1,x_2,x_3) = \tilde{g}_1(x_1,x_2,x_3) &\coloneqq \mathrm{mx}(\mathrm{mx}(x_1,x_2),\mathrm{mx}(x_2,x_3)) \\ \tilde{f}(x_1,x_2,x_3,x_4,x_5,x_6) &\coloneqq \mathrm{mx}\left(\tilde{g}_1(x_1,x_2,x_3),\tilde{g}_1(x_4,x_5,x_6)\right). \end{aligned}$$

For every finite $S \subseteq \mathbb{Q}$, there exists $\alpha \in \mathrm{Aut}(\mathbb{Q};<)$ such that

$$\begin{aligned} \alpha \circ \tilde{g}_1(y,x,x) &= \tilde{f}(y,x,x,x,x,x) \\ \alpha \circ \tilde{g}_1(x,y,x) &= \tilde{f}(x,y,x,x,x,x) \\ \alpha \circ \tilde{g}_1(x,x,y) &= \tilde{f}(x,x,y,x,x,x) \end{aligned}$$

holds for all $x,y \in S$. An analogous statement holds for $\tilde{g}_2$. Then Lemma 4.4 in [4] yields functions f, g_1, and g_2 which are $(3+3)$-operations in $\mathrm{Pol}(\mathbb{Q};\mathrm{X})$. □

The requirement of the existence of $3+n$-operations is an example of a so-called *minor condition*, which is a set of identities of the special form

$$f_1(x^1_1,\dots,x^1_{n_1}) \approx \dots \approx f_k(x^1_k,\dots,x^k_{n_k}).$$

(such identities are sometimes also called *height-one identities* [7][1]). Another example of minor conditions can be found in item (5) and (6) of Theorem 1.1: an at least binary operation f is called a *weak near-unanimity* (WNU) if it satisfies

$$f(y,x,\dots,x) \approx f(x,y,x,\dots,x) \approx \dots \approx f(x,\dots,x,y).$$

Despite their success in the setting of finite-domain CSPs, finite minor conditions such as item (6) in Theorem 1.1 are insufficient for classification purposes in the context of ω-categorical CSPs.

Proposition 7.4. *Let* $\mathcal{L}$ *be any logic at least as expressive as the existential positive fragment of* FO. *Then there is no finite minor condition that would capture the expressibility of the CSPs of reducts of finitely bounded homogeneous structures in* $\mathcal{L}$.

Proposition 7.4 is a consequence of the proof of Theorem 1.3 in [17]. Both statements rely on the following result.

Theorem 7.5 ([26, 45]). *For every finite set* $\mathcal{F}$ *of finite connected structures with a finite signature* τ*, there exists a* τ*-reduct* $\mathrm{CSS}(\mathcal{F})$ *of a finitely bounded homogeneous structure such that* $\mathrm{CSS}(\mathcal{F})$ *embeds precisely those finite* τ*-structures which do not contain a homomorphic image of any member of* $\mathcal{F}$.

For ω-categorical structures, a statement that is stronger than in the conclusion of Proposition 7.4 has been shown in [33]; however, the proof in [33] does not apply to reducts of finitely bounded homogeneous structures in general.

Proof of Proposition 7.4. Suppose, on the contrary, that there exists such a condition $\mathcal{E}$. By the proof of Theorem 1.3 in [17], there exists a finite family $\mathcal{F}$ of finite connected structures with a finite signature τ such that $\mathrm{Pol}(\mathrm{CSS}(\mathcal{F}))$ does not satisfy $\mathcal{E}$. The existential positive sentence $\phi_{\mathrm{CSS}(\mathcal{F})} \coloneqq \bigvee_{\mathbf{A}\in\mathcal{F}} Q_{\mathbf{A}}$ defines the complement of $\mathrm{CSP}(\mathrm{CSS}(\mathcal{F}))$. But then $\mathrm{CSP}(\mathrm{CSS}(\mathcal{F}))$ is expressible in $\mathcal{L}$, a contradiction. □

[1] *Linear identities* are defined similarly, but also allow that the terms in the identities consist of a single variable, which is more general. A finite minor condition therefore is a special case of what has been called a *strong linear Maltsev condition* in the universal algebra literature).

The satisfiability of minor conditions in polymorphism clones is preserved under minion homomorphisms, and the satisfiability of sets of arbitrary identities in polymorphism clones is preserved under clone homomorphisms [7]. In Theorem 1.6 we use the latter to show that, for Datalog, Proposition 7.4 can be strengthened to sets of arbitrary identities. We hereby give a negative answer to a question from [17] concerning the existence of a fixed set of identities that would capture Datalog expressibility for ω-categorical CSPs. This question was in fact already answered negatively in [34], however, our result also provably applies in the setting of finitely bounded homogeneous structures. Recall the relation S_{ll} defined before Lemma 3.18.

Proposition 7.6.

(1) $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ *does not pp-construct* $(\mathbb{Q};\mathrm{R_{min}})$.

(2) *There exists a uniformly continuous clone homomorphism from* $\mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ *to* $\mathrm{Pol}(\mathbb{Q};\mathrm{R_{min}})$.

Proof. For (1), suppose on the contrary that $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ pp-constructs $(\mathbb{Q};\mathrm{R_{min}})$. Since $\neq$ and $\mathrm{S_{ll}}$ are Ord-Horn definable, $\mathrm{CSP}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ is expressible in Datalog by Theorem 1.2. Then, by Theorem 2.8, $\mathrm{CSP}(\mathbb{Q};\mathrm{R_{min}})$ is expressible in Datalog, which contradicts the fact that $\mathrm{CSP}(\mathbb{Q};\mathrm{R_{min}})$ is inexpressible in Datalog by Theorem 5.2 in [15]. Thus $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ does not pp-construct $(\mathbb{Q};\mathrm{R_{min}})$.

For (2), we define $\xi\colon \mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}}) \to \mathrm{Pol}(\mathbb{Q};\mathrm{R_{min}})$ as follows. Let $f \in \mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ be arbitrary and let $\{i_1,\dots,i_m\} \subseteq [n]$ be the set of all indices $i \in [n]$, where n is the arity of f, such that f depends on the i-th argument. We have $m \geq 1$ since $\neq$ is not preserved by constant operations. Then $\xi(f)$ is defined to be the map $(x_1,\dots,x_n) \mapsto \min\{x_{\mu_f(1)},\dots,x_{\mu_f(n)}\}$ where $\mu_f\colon [n] \to [m]$ is any map that satisfies $\mu_f(i_\ell) = \ell$ for each $\ell \in [m]$. We claim that ξ is a clone homomorphism. Clearly, ξ preserves arities and projections. The essential part f^{ess} is injective because the relation I_4 is pp-definable in $(\mathbb{Q};\neq,\mathrm{S_{ll}})$. Hence, $f(g_1,\dots,g_n)^{ess} = f\big(g^{ess}_{\mu_f(1)},\dots,g^{ess}_{\mu_f(n)}\big)$. Now the claim that ξ is a clone homomorphism follows from the simple fact that

$$\min\{x_{1,1},\dots,x_{\ell,k_\ell}\} = \min\{\min\{x_{i,1},\dots,x_{i,k_i}\} \mid i \in [\ell]\}$$

holds for all $\ell,k_1,\dots,k_\ell \geq 1$. Finally, ξ defined this way is uniformly continuous (see Definition 2.13 and choose $A' := B'$). This concludes the proof of (2). □

Proof of Theorem 1.6. Suppose, on the contrary, that there is a condition $\mathcal{E}$ which is preserved by uniformly continuous clone homomorphisms and captures expressibility of temporal CSPs in Datalog. $\mathrm{CSP}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ is expressible in Datalog, $\mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ must satisfy $\mathcal{E}$. By Item (2) in Proposition 7.6, $\mathrm{Pol}(\mathbb{Q};\mathrm{R_{min}})$ satisfies $\mathcal{E}$ as well. By assumption, $\mathrm{CSP}(\mathbb{Q};\mathrm{R_{min}})$ must be expressible in Datalog, a contradiction to Theorem 5.2 in [15]. □

7.2 New minor conditions

The expressibility of temporal and finite-domain CSPs in FP / FPC can be characterised by a family of minor conditions (Theorem 1.7). Inspired by [3, 4], we introduce general terminology to conveniently reason with minor conditions.

Definition 7.7. Let $\mathbf{A}_1,\mathbf{A}_2$ be relational structures with a finite signature τ. For every $R \in \tau$ and every $\bar{t} \in R^{\mathbf{A}_2}$, we introduce a unique $|R^{\mathbf{A}_1}|$-ary function symbol $g^R_{\bar{t}}$. Also, for every $R \in \tau$, we fix an arbitrary enumeration of $R^{\mathbf{A}_1}$. The elements of A_1 will be used as names for variables in the following. We define the minor condition $\mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$ as follows. Let $R,S \in \tau$, $k := \mathrm{ar}(R)$, $\ell := \mathrm{ar}(S)$, $i \in [k]$, and $j \in [\ell]$. If $\bar{r} \in R^{\mathbf{A}_2}$ and $\bar{s} \in S^{\mathbf{A}_2}$ are such that $\bar{r}[i] = \bar{s}[j]$, then $\mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$ contains the equation $g^R_{\bar{r}}(x_{i,1},\dots,x_{i,m}) \approx g^S_{\bar{s}}(y_{j,1},\dots,y_{j,n})$ where $(x_{1,1},\dots,x_{k,1}),\dots,(x_{1,m},\dots,x_{k,m})$ and $(y_{1,1},\dots,y_{n,1}),\dots,(y_{1,n},\dots,y_{\ell,n})$ are the fixed enumerations of $R^{\mathbf{A}_1}$ and $S^{\mathbf{A}_1}$, respectively. If a tuple $\bar{t}$ only appears in a single relation $R^{\mathbf{A}_2}$, then we write $g_{\bar{t}}$ instead of $g^R_{\bar{t}}$.

The relation 1IN3 defined in the introduction can be generalised to

$$1\mathrm{IN}k := \{\bar{t} \in \{0,1\}^k \mid \bar{t}[i] = 1 \text{ for exactly one } i \in [k]\}.$$

Definition 7.8. We write $\mathcal{E}_{k,n}$ for $\mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ if

$$\begin{aligned} &\mathbf{A}_1 := (\{0,1\}; 1\mathrm{IN}k) \\ \text{and} \quad &\mathbf{A}_2 := ([n]; \{\bar{t} \in [n]^k \mid \bar{t}[1] < \cdots < \bar{t}[k]\}). \end{aligned}$$

The minor condition $\mathcal{E}_{k,n}$ is properly contained in a minor condition called *dissected WNUs* in [33].

Example 7.9. The minor condition $\mathcal{E}_{3,4}$ equals

$$\begin{aligned} g_{(1,3,4)}(y,x,x) &\approx g_{(1,2,4)}(y,x,x) \approx g_{(1,2,3)}(y,x,x), \\ g_{(2,3,4)}(y,x,x) &\approx g_{(1,2,4)}(x,y,x) \approx g_{(1,2,3)}(x,y,x), \\ g_{(2,3,4)}(x,y,x) &\approx g_{(1,3,4)}(x,y,x) \approx g_{(1,2,3)}(x,x,y), \\ g_{(2,3,4)}(x,x,y) &\approx g_{(1,3,4)}(x,x,y) \approx g_{(1,2,4)}(x,x,y). \end{aligned}$$

Note that $\mathcal{E}_{k,n}$ is implied by the existence of a single k-ary WNU operation. Also note that $\mathcal{E}_{k,n}$ implies $\mathcal{E}_{k,k+1}$ for all $n > k > 1$. We first present important general facts about the conditions $\mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ and then restrict our attention to the family $(\mathcal{E}_{k,n})$. We believe that the following theorem is of independent interest in universal algebra; it is similar in spirit to the results obtained in [23] but, to the best of our knowledge, it has not been published in this exact formulation. A *minion* is a set of functions which is closed under compositions of a single function with projections; in particular, clones are minions.

Theorem 7.10. *For every finite minor condition $\mathcal{E}$, there exist finite τ-structures $\mathbf{A}_1, \mathbf{A}_2$ such that $\mathcal{E}$ and $\mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ are equivalent with respect to satisfiability in minions. For each such pair $\mathbf{A}_1, \mathbf{A}_2$ and every countable ω-categorical structure $\mathbf{B}$, the following are equivalent:*

(1) $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}$.
(2) *For every pp-power $\mathbf{C}$ of $\mathbf{B}$, if $\mathbf{A}_1 \to \mathbf{C}$, then $\mathbf{A}_2 \to \mathbf{C}$.*

To prove the equivalence of (1) and (2) in Theorem 7.10, we need the following lemma.

Lemma 7.11.

(1) *For any structure $\mathbf{D}$, if $\mathrm{Pol}(\mathbf{D})$ satisfies $\mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on the image of a homomorphism from $\mathbf{A}_1$ to $\mathbf{D}$, then there exists a homomorphism from $\mathbf{A}_2$ to $\mathbf{D}$.*
(2) *For every countable ω-categorical structure $\mathbf{B}$, every finite $F \subseteq B$, and every finite structure $\mathbf{A}_1$, there exists an $|F|^{|A_1|}$-dimensional pp-power $\mathbf{B}_F(\mathbf{A}_1)$ of $\mathbf{B}$ such that $\mathbf{A}_1 \to \mathbf{B}_F(\mathbf{A}_1)$ and*

$$\mathbf{A}_2 \to \mathbf{B}_F(\mathbf{A}_1) \quad \textit{if and only if} \quad \mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2) \textit{ on } F.$$

The proof of Lemma 7.11(1) is similar to the proof of Lemma 3.14(2) in [3], and the proof of Lemma 7.11(2) is similar to the proof of Theorem 3.12(1) in [3].

Proof. For (1), let $f\colon \mathbf{A}_1 \to \mathbf{D}$ be a homomorphism such that $\mathrm{Pol}(\mathbf{D}) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on $f(A_1)$. Consider the map $h\colon A_2 \to D$ defined as follows. If $x \in A_2$ does not appear in any tuple from a relation of $\mathbf{A}_2$, then we set $h(x)$ to be an arbitrary element of D. If there exists $\bar{r} \in R^{\mathbf{A}_2}$ such that $x = \bar{r}[i]$ for some $i \in [k]$ where $k := \mathrm{ar}(R)$, then we take the operation $g^R_{\bar{r}} \in \mathrm{Pol}(\mathbf{D})$ witnessing $\mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on $f(A_1)$ and set $h(x) := g^R_{\bar{t}}(f(x_{i,1}), \ldots, f(x_{i,m}))$ where $(x_{1,1}, \ldots, x_{k,1}), \ldots, (x_{1,m}, \ldots, x_{k,m})$ is the fixed enumeration of $R^{\mathbf{A}_1}$ from Definition 7.7. The map h is well-defined: if $x = \bar{r}[i] = \bar{s}[j]$ for some $\bar{r} \in R^{\mathbf{A}_2}$ and $\bar{s} \in S^{\mathbf{A}_2}$, then

$$g^R_{\bar{r}}(f(x_{i,1}), \ldots, f(x_{i,m})) = g^S_{\bar{s}}(f(y_{j,1}), \ldots, f(y_{j,n}))$$

by the definition of $\mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$. It remains to show that h is a homomorphism. Let $R \in \tau$ and $\bar{r} \in R^{\mathbf{A}_2}$ be arbitrary. Since f is a homomorphism, we have $(f(x_{1,i}),\dots,f(x_{k,i})) \in R^{\mathbf{D}}$ for every $i \in [m]$. Since $g_{\bar{r}}^{R}$ is a polymorphism of $\mathbf{D}$, we have

$$(h(\bar{r}[1]),\dots,h(\bar{r}[k])) = \big(g_{\bar{r}}^{R}(f(x_{1,1}),\dots,f(x_{1,m})),\dots,g_{\bar{r}}^{R}(f(x_{k,1}),\dots,f(x_{k,m}))\big) \in R^{\mathbf{D}}.$$

For (2), we fix a finite $F \subseteq B$ and a finite τ-structure $\mathbf{A}_1$. Let $f_1,\dots,f_d$ be an arbitrary fixed enumeration of all possible maps from A_1 to F. Let R be an arbitrary symbol from τ and set $k := \mathrm{ar}(R)$. We fix an arbitrary enumeration $(x_{1,1},\dots,x_{k,1}),\dots,(x_{1,m},\dots,x_{k,m})$ of $R^{\mathbf{A}_1}$. Consider the $(k\cdot d)$-ary relation R' over B consisting of all tuples of the form

$$\begin{pmatrix} g\big(f_1(x_{1,1}),\dots,f_1(x_{1,m})\big) \\ \vdots \\ g\big(f_1(x_{k,1}),\dots,f_1(x_{k,m})\big) \\ \vdots \\ g\big(f_d(x_{1,1}),\dots,f_d(x_{1,m})\big) \\ \vdots \\ g\big(f_d(x_{k,1}),\dots,f_d(x_{k,m})\big) \end{pmatrix}$$

where g is an m-ary operation from $\mathrm{Pol}(\mathbf{B})$. We show that R' has a pp-definition in $\mathbf{B}$. By Theorem 4 in [18], it is enough to show that R' is preserved by every polymorphism of $\mathbf{B}$. Let g be an arbitrary operation from $\mathrm{Pol}(\mathbf{B})$ and let ℓ be its arity. If there exist $g_1,\dots,g_\ell \in \mathrm{Pol}(\mathbf{B})$ such that

$$\begin{pmatrix} g_i\big(f_1(x_{1,1}),\dots,f_1(x_{1,m})\big) \\ \vdots \\ g_i\big(f_1(x_{k,1}),\dots,f_1(x_{k,m})\big) \\ \vdots \\ g_i\big(f_d(x_{1,1}),\dots,f_d(x_{1,m})\big) \\ \vdots \\ g_i\big(f_d(x_{k,1}),\dots,f_d(x_{k,m})\big) \end{pmatrix} \in R'$$

for every $i \in [\ell]$, then

$$\begin{pmatrix} g\big(g_1\big(f_1(x_{1,1}),\dots,f_1(x_{1,m})\big),\dots,g_\ell\big(f_1(x_{1,1}),\dots,f_1(x_{1,m})\big)\big) \\ \vdots \\ g\big(g_1\big(f_1(x_{k,1}),\dots,f_1(x_{k,m})\big),\dots,g_\ell\big(f_1(x_{k,1}),\dots,f_1(x_{k,m})\big)\big) \\ \vdots \\ g\big(g_1\big(f_d(x_{1,1}),\dots,f_d(x_{1,m})\big),\dots,g_\ell\big(f_d(x_{1,1}),\dots,f_d(x_{1,m})\big)\big) \\ \vdots \\ g\big(g_1\big(f_d(x_{k,1}),\dots,f_d(x_{k,m})\big),\dots,g_\ell\big(f_d(x_{k,1}),\dots,f_d(x_{k,m})\big)\big) \end{pmatrix} \in R'$$

because $(x_1,\dots,x_n) \mapsto g\big(g_1(x_1,\dots,x_m),\dots,g_\ell(x_1,\dots,x_m)\big)$ is an m-ary polymorphism of $\mathbf{B}$. Now we can define the structure $\mathbf{B}_F(\mathbf{A}_1)$. The domain of $\mathbf{B}_F(\mathbf{A}_1)$ is B^d, and, for every $R \in \tau$ with $k = \mathrm{ar}(R)$, we set

$$R^{\mathbf{B}_F(\mathbf{A}_1)} := \{(\bar{t}_1,\dots,\bar{t}_k) \in (B^d)^k \mid (\bar{t}_1[1],\dots,\bar{t}_k[1],\dots,\bar{t}_1[d],\dots,\bar{t}_k[d]) \in R'\}.$$

Since each R' has a pp-definition in $\mathbf{B}$, we have that $\mathbf{B}_F(\mathbf{A}_1)$ is a $|F|^{|A_1|}$-dimensional pp-power of $\mathbf{B}$.

Next we show that the map $h\colon A_1 \to B^d$ defined by $h(x) := (f_1(x), \dots, f_d(x))$ is a homomorphism from $\mathbf{A}_1$ to $\mathbf{B}_F(\mathbf{A}_1)$. Let $\bar{t}$ be an arbitrary tuple from $R^{\mathbf{A}_1}$ for some $R \in \tau$, and let k be the arity of R. Let $j \in [m]$ be such that $\bar{t} = (x_{1,j}, \dots, x_{k,j})$ for the fixed enumeration of $R^{\mathbf{A}_1}$ from the definition of R'. Then

$$(h(\bar{t}[1]), \dots, h(\bar{t}[k])) = \big((f_1(\bar{t}[1]), \dots, f_d(\bar{t}[1])), \dots, (f_1(\bar{t}[k]), \dots, f_d(\bar{t}[k]))\big) \in R^{\mathbf{B}_F(\mathbf{A}_1)}$$

because

$$\begin{pmatrix} f_1(\bar{t}[1]) \\ \vdots \\ f_1(\bar{t}[k]) \\ \vdots \\ f_d(\bar{t}[1]) \\ \vdots \\ f_d(\bar{t}[k]) \end{pmatrix} = \begin{pmatrix} \mathrm{proj}_j\left(f_1(x_{1,1}), \dots, f_1(x_{1,m})\right) \\ \vdots \\ \mathrm{proj}_j\left(f_1(x_{k,1}), \dots, f_1(x_{k,m})\right) \\ \vdots \\ \mathrm{proj}_j\left(f_d(x_{1,1}), \dots, f_d(x_{1,m})\right) \\ \vdots \\ \mathrm{proj}_j\left(f_d(x_{k,1}), \dots, f_d(x_{k,m})\right) \end{pmatrix} \in R'.$$

It remains to show that $\mathbf{A}_2 \to \mathbf{B}_F(\mathbf{A}_1)$ if and only if $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on F.

"$\Leftarrow$" Suppose that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on F. Note that each m-ary polymorphism f of $\mathbf{B}$ induces an m-ary polymorphism $\tilde{f}$ of $\mathbf{B}_F(\mathbf{A}_1)$ via

$$\tilde{f}(\bar{t}_1, \dots, \bar{t}_m) := \big(f(\bar{t}_1[1], \dots, \bar{t}_m[1]), \dots, f(\bar{t}_1[d], \dots, \bar{t}_m[d])\big).$$

This follows directly from the fact that, for every $R \in \tau$, R' is preserved by $\mathrm{Pol}(\mathbf{B})$. In particular, if

$$g^R_{\bar{r}}(x_{i,1}, \dots, x_{i,m}) \approx g^S_{\bar{s}}(y_{j,1}, \dots, y_{j,n})$$

holds in $\mathrm{Pol}(\mathbf{B})$ on F, then

$$\tilde{g}^R_{\bar{r}}(x_{i,1}, \dots, x_{i,m}) \approx \tilde{g}^S_{\bar{s}}(y_{j,1}, \dots, y_{j,n})$$

holds in $\mathrm{Pol}(\mathbf{B}_F(\mathbf{A}_1))$ on $h(F)$ where $h\colon \mathbf{A}_1 \to \mathbf{B}_F(\mathbf{A}_1), x \mapsto (f_1(x), \dots, f_d(x))$ is the homomorphism from the previous paragraph. This means that $\mathrm{Pol}(\mathbf{B}_F(\mathbf{A}_1)) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on $h(F)$ and thus the requirements in (1) are satisfied. It now follows from (1) that $\mathbf{A}_2 \to \mathbf{B}_F(\mathbf{A}_1)$.

"$\Rightarrow$" Suppose that there exists a homomorphism $h\colon \mathbf{A}_2 \to \mathbf{B}_F(\mathbf{A}_1)$. Then, for every $R \in \tau$ and every $\bar{t} \in R^{\mathbf{A}_2}$, by the definition of R', there exists an m-ary operation $g^R_{\bar{t}} \in \mathrm{Pol}(\mathbf{B})$ such that

$$\begin{pmatrix} h(\bar{t}[1])[1] \\ \vdots \\ h(\bar{t}[k])[1] \\ \vdots \\ h(\bar{t}[1])[d] \\ \vdots \\ h(\bar{t}[k])[d] \end{pmatrix} = \begin{pmatrix} g^R_{\bar{t}}\left(f_1(x_{1,1}), \dots, f_1(x_{1,m})\right) \\ \vdots \\ g^R_{\bar{t}}\left(f_1(x_{k,1}), \dots, f_1(x_{k,m})\right) \\ \vdots \\ g^R_{\bar{t}}\left(f_d(x_{1,1}), \dots, f_d(x_{1,m})\right) \\ \vdots \\ g^R_{\bar{t}}\left(f_d(x_{k,1}), \dots, f_d(x_{k,m})\right) \end{pmatrix}.$$

It is straightforward to check that these operations witness that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$ on F. □

Proof of Theorem 7.10. Let $\mathcal{E}$ be an arbitrary finite minor condition. We define A_1 as the set of all variables occurring in $\mathcal{E}$. Fix an arbitrary function symbol f occurring in $\mathcal{E}$, and let $f(x_{1,1}, \dots, x_{1,m}), \dots, f(x_{k,1}, \dots, x_{k,m})$ be all the f-terms occurring in $\mathcal{E}$, listed in any particular but fixed order. Without loss of generality, we may assume that $(x_{1,1}, \dots, x_{k,1}), \dots, (x_{1,m}, \dots, x_{k,m})$ are pairwise distinct tuples, because otherwise we can reduce the arity of f without changing the satisfiability of $\mathcal{E}$ in minions. Now, for every such f, we require that τ contains an m-ary symbol R_f and that

$$R_f^{\mathbf{A}_1} = \{(x_{1,1}, \dots, x_{k,1}), \dots, (x_{1,m}, \dots, x_{k,m})\}.$$

Let $\sim$ be the smallest equivalence relation on the terms which occur in $\mathcal{E}$ given by the identities in $\mathcal{E}$. We define A_2 as the set of all equivalence classes of $\sim$. For every function symbol f which occurs in $\mathcal{E}$, the relation $R_f^{\mathbf{A}_2}$ consists of a single tuple $\mathbf{t}_f$ of equivalence classes of $\sim$ of all f-terms which occur in $\mathcal{E}$, these equivalence classes appear in $\mathbf{t}_f$ in the fixed order from above. It is easy to check that $\mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$ and $\mathcal{E}$ are identical up to reduction of arities through removal of non-essential arguments and adding additional identities which are implied by $\mathcal{E}$. Thus, $\mathcal{E}$ and $\mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$ are equivalent.

(1)⇒(2): Suppose that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$. If $\mathbf{C}$ is a pp-power of $\mathbf{B}$, then it follows from the results in [7] that there exists a minion homomorphism from $\mathrm{Pol}(\mathbf{B})$ to $\mathrm{Pol}(\mathbf{C})$, which means that $\mathrm{Pol}(\mathbf{C}) \models \mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$. If additionally $\mathbf{A}_1 \to \mathbf{C}$, then it follows from Lemma 7.11(1) that $\mathbf{A}_2 \to \mathbf{C}$.

(2)⇒(1): For every finite $F \subseteq B$, the structure $\mathbf{B}_F(\mathbf{A}_1)$ from Lemma 7.11(2) is a pp-power of $\mathbf{B}$. Also, $\mathbf{A}_1 \to \mathbf{B}_F(\mathbf{A}_1)$. By our assumption, we have that $\mathbf{A}_2 \to \mathbf{B}_F(\mathbf{A}_1)$ for every finite $F \subseteq B$. Using Lemma 7.11(2), we conclude that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$ on F for every finite $F \subseteq B$. By a compactness argument, e.g., Lemma 9.6.10 in [10], we have that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}(\mathbf{A}_1,\mathbf{A}_2)$, which finishes the proof. □

Example 7.12. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be the structures from the definition of $\mathcal{E}_{k,n}$ (Definition 7.8). Then Theorem 7.10 implies that $\mathcal{E}_{k,n}$ is non-trivial: indeed, first note that there is no homomorphism from $\mathbf{A}_2$ to $\mathbf{A}_1$. Choose $\mathbf{C} := \mathbf{B} := \mathbf{A}_1$; then trivially $\mathbf{A}_1 \to \mathbf{C}$ but $\mathbf{A}_2 \not\to \mathbf{C}$, and hence $\mathrm{Pol}(\mathbf{B})$, which only contains the projections, does not satisfy $\mathcal{E}_{k,n}$.

Example 7.13. We claim that the structures $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ and $(\mathbb{Q};\mathrm{R_{min}})$ are incomparable w.r.t. pp-constructibility. We already know from Proposition 7.6(1) that $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ does not pp-construct $(\mathbb{Q};\mathrm{R_{min}})$. The reason there was that $\mathrm{CSP}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ is expressible in Datalog whereas $\mathrm{CSP}(\mathbb{Q};\mathrm{R_{min}})$ is not, and that pp-constructions preserve the expressibility of CSPs in Datalog. This argument clearly cannot be used the other way around. However, note that $\mathrm{Pol}(\mathbb{Q};\mathrm{R_{min}})$ contains the ternary WNU operation $(x,y,z) \mapsto \min(x,y,z)$. By Theorem 7.10, $\mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}})$ does not contain a ternary WNU operation if and only if there exists a pp-power $\mathbf{C}$ of $(\mathbb{Q};\neq,\mathrm{S_{ll}})$ such that $\mathbf{A}_1 = (\{0,1\};1\mathrm{IN}3) \to \mathbf{C}$ and $\mathbf{A}_2 = (\{a\};\{(a,a,a)\}) \not\to \mathbf{C}$. And indeed, such a pp-power exists: the structure $\mathbf{C} := (\mathbb{Q};\{(x,y,z) \mid x \neq y \vee x < z\})$ is even pp-definable in $\mathrm{Pol}(\mathbb{Q};\neq,\mathrm{S_{ll}})$. Now it follows from the results in [7] that $(\mathbb{Q};\mathrm{R_{min}})$ does not pp-construct $(\mathbb{Q};\neq,\mathrm{S_{ll}})$.

Recall the structures $\mathbf{E}_{\mathscr{G},k}$ from Definition 4.12.

Lemma 7.14. *For $n \geq 2$, $\mathrm{Pol}(\mathbf{E}_{\mathbb{Z}_n,3})$ satisfies $\mathcal{E}_{k,k+1}$ if and only if $\gcd(k,n) = 1$.*

Proof. First suppose that $\gcd(k,n) = 1$. There exists $\lambda \in \mathbb{Z}_n$ such that $k\lambda = 1 \bmod n$. We write g_i instead of $g_{\bar{t}}$ for $\bar{t} \in [k+1]^k$ with $\bar{t}[1] < \cdots < \bar{t}[k]$ that omits i as an entry. Then $\mathcal{E}_{k,k+1}$ is witnessed by a set of k-ary WNU operations $g_1,\dots,g_{k+1}$ given by the affine combinations $g_j(x_1,\dots,x_k) := \sum_{i=1}^k \lambda x_i$.

Next, suppose that $\mathrm{Pol}(\mathbf{E}_{\mathbb{Z}_n,3})$ satisfies $\mathcal{E}_{k,k+1}$. It is well-known that, for every $k \geq 1$, the structure $\mathbf{E}_{\mathbb{Z}_n,k}$ has a pp-definition in $\mathbf{E}_{\mathbb{Z}_n,3}$. In particular, the structure $\mathbf{C} := (\mathbb{Z}_n;R)$ where $R := \{\bar{t} \in (\mathbb{Z}_n)^k \mid \sum_{i=1}^k \bar{t}[i] = 1 \bmod n\}$ has a pp-definition in $\mathbf{E}_{\mathbb{Z}_n,3}$. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be as in Definition 7.8. Clearly $\mathbf{A}_1 \to \mathbf{C}$. By Theorem 7.10, we have that $\mathbf{A}_2 \to \mathbf{C}$. This means that the inhomogeneous system of $k+1$ Boolean linear equations of the form $\sum_{j \in [k+1]\setminus\{i\}} x_j = 1 \bmod n$ has a solution. By summing up the equations and subtracting k on both sides, we get that $kx_1 + \cdots + kx_{k+1} - k = k(x_1 + \cdots + x_{k+1} - 1) = 1 \bmod n$. This is the case if and only if $\gcd(k,n) = 1$. □

Our next goal is the proof of Theorem 1.7, which states that for temporal CSPs and finite-domain CSPs, the condition $\mathcal{E}_{k,k+1}$ characterises expressibility in FP. For the proof of we need to introduce some new polymorphisms of temporal structures.

Definition 7.15. Let $k \in \mathbb{N}_{\geq 2}$. The following definitions specify k-ary operations on $\mathbb{Q}$:

$$\begin{aligned}
\min_k(\bar{t}) &:= \min\{\bar{t}[1], \ldots, \bar{t}[k]\}, \\
\mathrm{med}_k(\bar{t}) &:= \max\left\{\min\{\bar{t}[i] \mid i \in I\} \;\middle|\; I \in \tbinom{[k]}{k-1}\right\}, \\
\mathrm{mi}_k(\bar{t}) &:= \mathrm{lex}_{k+2}\big(\min_k(\bar{t}), \mathrm{med}_k(-\chi(\bar{t})), -\chi(\bar{t})\big), \\
\mathrm{ll}_k(\bar{t}) &:= \mathrm{lex}_{k+2}\big(\min_k(\bar{t}), \mathrm{med}_k(\bar{t}), \bar{t}\big), \\
\mathrm{mx}_k(\bar{t}) &:= \begin{cases} \mathrm{mx}(\bar{t}) & \text{if } k = 2, \\ \mathrm{mx}\big(\mathrm{mx}_{k-1}(\bar{t}[1], \ldots, \bar{t}[k-1]), \mathrm{mx}_{k-1}(\bar{t}[2], \ldots, \bar{t}[k])\big) & \text{if } k > 2. \end{cases}
\end{aligned}$$

Proposition 7.16. *Let* $\mathbf{B}$ *be a temporal structure that is preserved by* min*,* mi*,* mx*, or* ll*. Then* $\mathbf{B}$ *is preserved by* $\min_k$*,* mi_k*,* mx_k*, or* ll_k*, respectively, for all* $k \geq 2$.

Proof. For tuples $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^n$ and $i \in [n]$, we set $\bar{t}^i := (\bar{t}_1[i], \ldots, \bar{t}_k[i])$. It follows directly from the definitions that $\min_k$ preserves $(\mathbb{Q}; \mathrm{R}^{\leq}_{\min}, <)$ and mx_k preserves $(\mathbb{Q}; X)$. To prove the statement for mi_k it suffices to prove that mi_k preserves $(\mathbb{Q}; \mathrm{R}_{\mathrm{mi}}, \mathrm{S}_{\mathrm{mi}}, \neq)$ by Lemma 3.8.

Suppose that there exist $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^2$ such that $\mathrm{mi}_k(\bar{t}^1) = \mathrm{mi}_k(\bar{t}^2)$. Then we have $\min_k(\bar{t}^1) = \min_k(\bar{t}^2)$ and also $\chi(\bar{t}^1) = \chi(\bar{t}^2)$, because $\mathrm{lex}(-\chi(\bar{t}^1)) = \mathrm{lex}(-\chi(\bar{t}^2))$. Hence, $\bar{t}^1[\ell] = \bar{t}^2[\ell]$ for every $\ell \leq k$ such that the ℓ-th entry of $\bar{t}^1$ is minimal and mi_k preserves $\neq$.

To prove that mi preserves S_{mi}, let $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^3$ be such that $\mathrm{mi}_k(\bar{t}_1, \ldots, \bar{t}_k) \notin \mathrm{S}_{\mathrm{mi}}$. Then in particular we have $\mathrm{mi}_k(\bar{t}^1) = \mathrm{mi}_k(\bar{t}^2)$ which implies $\min_k(\bar{t}^1) = \min_k(\bar{t}^2)$ and $\chi(\bar{t}^1) = \chi(\bar{t}^2)$. We also have $\mathrm{mi}_k(\bar{t}^1) < \mathrm{mi}_k(\bar{t}^3)$ and distinguish the following three exhaustive cases.

Case 1: $\min_k(\bar{t}^1) < \min_k(\bar{t}^3)$. We have $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{mi}}$ for every $\ell \in [k]$ such that the ℓ-th entry is minimal in $\bar{t}^1$. Then $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{mi}}$.

Case 2: $\min_k(\bar{t}^1) = \min_k(\bar{t}^3)$ *and* $\mathrm{med}_k(-\chi(\bar{t}^1)) < \mathrm{med}_k(-\chi(\bar{t}^3))$. Since $\chi(\bar{t}^1) = \chi(\bar{t}^2)$, there exists an $\ell \in [k]$ such that the ℓ-th entry is minimal in of $\bar{t}^1$ and in $\bar{t}^2$, but not in $\bar{t}^3$. Then $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{mi}}$.

Case 3: $\min_k(\bar{t}^1) = \min_k(\bar{t}^3)$*,* $\mathrm{med}_k(-\chi(\bar{t}^1)) = \mathrm{med}_k(-\chi(\bar{t}^3))$*, and* $\mathrm{lex}_k(-\chi(\bar{t}^1)) < \mathrm{lex}_k(-\chi(\bar{t}^3))$. Let ℓ be the leftmost index on which $-\chi(\bar{t}^1)$ is pointwise smaller than $-\chi(\bar{t}^3)$. Since $\chi(\bar{t}^1) = \chi(\bar{t}^2)$, the ℓ-th entry is minimal in $\bar{t}^1$ and in $\bar{t}^2$, but not in $\bar{t}^3$. Again, $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{mi}}$.

We conclude that mi_k preserves S_{mi}.

To show that mi_k preserves R_{mi}, let $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^3$ be such that $\mathrm{mi}_k(\bar{t}_1, \ldots, \bar{t}_k) \notin \mathrm{R}_{\mathrm{mi}}$. Then we have $\mathrm{mi}_k(\bar{t}^1) \leq \mathrm{mi}_k(\bar{t}^2)$ and $\mathrm{mi}_k(\bar{t}^1) < \mathrm{mi}_k(\bar{t}^3)$. The former implies $\min_k(\bar{t}^1) \leq \min_k(\bar{t}^2)$. We distinguish the following three exhaustive cases for $\bar{t}^1$ and $\bar{t}^3$.

Case 1: $\min_k(\bar{t}^1) < \min_k(\bar{t}^3)$. Then $\bar{t}_\ell \notin \mathrm{R}_{\mathrm{mi}}$ for every $\ell \in [k]$ such that the ℓ-th entry is minimal in $\bar{t}^1$.

Case 2: $\min_k(\bar{t}^1) = \min_k(\bar{t}^3)$ *and* $\mathrm{med}_k(-\chi(\bar{t}^1)) < \mathrm{med}_k(-\chi(\bar{t}^3))$. Then there exists $\ell \in [k]$ such that the ℓ-th entry is minimal in $\bar{t}^1$ but not in $\bar{t}^3$. We also have $\bar{t}_\ell[1] \leq \bar{t}_\ell[2]$, because $\min_k(\bar{t}^1) \leq \min_k(\bar{t}^2)$, and hence $\bar{t}_\ell \notin \mathrm{R}_{\mathrm{mi}}$.

Case 3: $\min_k(\bar{t}^1) = \min_k(\bar{t}^3)$*,* $\mathrm{med}_k(-\chi(\bar{t}^1)) = \mathrm{med}_k(-\chi(\bar{t}^3))$*, and* $\mathrm{lex}_k(-\chi(\bar{t}^1)) < \mathrm{lex}_k(-\chi(\bar{t}^3))$. Let ℓ be the leftmost index on which $-\chi(\bar{t}^1)$ is pointwise smaller than $-\chi(\bar{t}^3)$. We have $\bar{t}_\ell[1] \leq \bar{t}_\ell[2]$, because $\min_k(\bar{t}^1) \leq \min_k(\bar{t}^2)$. Again $\bar{t}_\ell \notin \mathrm{R}_{\mathrm{mi}}$.

Thus, mi_k preserves R_{mi}.

To prove the statement for ll_k is suffices to show that ll_k preserves $(\mathbb{Q}; \mathrm{R}_{\mathrm{ll}}, \mathrm{S}_{\mathrm{ll}}, \neq)$ by Lemma 3.18. Let $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^2$ be such that $\mathrm{ll}_k(\bar{t}^1) = \mathrm{ll}_k(\bar{t}^2)$. Then $\bar{t}^1 = \bar{t}^2$, since ll_k also compares both tuples lexicographically.

Thus, ll_k preserves $\neq$. To show that ll_k preserves S_{ll}, let $\bar{t}_1, \ldots, \bar{t}_k \in \mathbb{Q}^4$ be such that $\mathrm{ll}_k(\bar{t}_1, \ldots, \bar{t}_k) \notin \mathrm{S}_{\mathrm{ll}}$. In particular, $\mathrm{ll}_k(\bar{t}^1) = \mathrm{ll}_k(\bar{t}^2)$ which implies $\bar{t}^1 = \bar{t}^2$, and $\mathrm{ll}_k(\bar{t}^3) < \mathrm{ll}_k(\bar{t}^4)$. We distinguish the following three exhaustive cases for $\bar{t}^3$ and $\bar{t}^4$.

Case 1: $\min_k(\bar{t}^3) < \min_k(\bar{t}^4)$. Then there exists an index ℓ with $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{ll}}$.

Case 2: $\min_k(\bar{t}^3) = \min_k(\bar{t}^4)$ *and* $\mathrm{med}_k(\bar{t}^3) < \mathrm{med}_k(\bar{t}^4)$. Then there exists $I_4 \subset [k]$ of cardinality $k-1$ such that $\min\{\bar{t}_i[3] \mid i \in I_4\} < \min\{\bar{t}_i[4] \mid i \in I_4\}$. Let $\ell \in [k]$ be such that the ℓ-th entry of $\bar{t}^3$ is minimal among those entries of $\bar{t}^3$ with an index from I_4. Then $\bar{t}_\ell \notin \mathrm{S}_{\mathrm{ll}}$.

Case 3: $\min_k(\bar{t}^3) = \min_k(\bar{t}^4)$, $\mathrm{med}_k(\bar{t}^3) = \mathrm{med}_k(\bar{t}^4)$*, and* $\mathrm{lex}_k(\bar{t}^3) < \mathrm{lex}_k(\bar{t}^4)$. Let ℓ be the leftmost index on which $\bar{t}^1$ is pointwise smaller than $\bar{t}^3$. We have $\bar{t}_\ell \notin \mathrm{S_{ll}}$.

Thus, ll_k preserves $\mathrm{S_{ll}}$. To show that ll_k preserves $\mathrm{R_{ll}}$, let $\bar{t}_1, \dots, \bar{t}_k \in \mathbb{Q}^3$ be such that $\mathrm{ll}_k(\bar{t}_1, \dots, \bar{t}_k) \notin \mathrm{R_{ll}}$. We may assume without loss of generality that $\mathrm{ll}_k(\bar{t}^1) \leq \mathrm{ll}_k(\bar{t}^2)$ and $\mathrm{ll}_k(\bar{t}^1) < \mathrm{ll}_k(\bar{t}^3)$. The case where $\mathrm{ll}_k(\bar{t}^1) = \mathrm{ll}_k(\bar{t}^2)$ follows by a similar argument as when we dealt with $\mathrm{S_{ll}}$, thus we even assume that $\mathrm{ll}_k(\bar{t}^1) < \mathrm{ll}_k(\bar{t}^2)$. We distinguish the following three exhaustive cases for $\bar{t}^1$ and $\bar{t}^2$.

Case 1: $\min_k(\bar{t}^1) < \min_k(\bar{t}^2)$. Since $\mathrm{ll}_k(\bar{t}^1) < \mathrm{ll}_k(\bar{t}^3)$, we have that $\min_k(\bar{t}^1) \leq \min_k(\bar{t}^3)$. Hence, there exists $\ell \in [k]$ with $\bar{t}_\ell \notin \mathrm{R_{ll}}$.

Case 2: $\min_k(\bar{t}^1) = \min_k(\bar{t}^2)$ *and* $\mathrm{med}_k(\bar{t}^1) < \mathrm{med}_k(\bar{t}^2)$. We may assume that $\min_k(\bar{t}^1) = \min_k(\bar{t}^3)$ holds as well, as otherwise we could apply the reasoning from the previous case. Consider the subcase where $\mathrm{med}_k(\bar{t}^1) < \mathrm{med}_k(\bar{t}^3)$. There are two index sets $I_2, I_3 \subset [k]$ of cardinality $k-1$ such that $\min\{\bar{t}_i[1] \mid i \in I_2\} < \min\{\bar{t}_i[2] \mid i \in I_2\}$ and $\min\{\bar{t}_i[1] \mid i \in I_3\} < \min\{\bar{t}_i[3] \mid i \in I_3\}$. Let $\ell_2, \ell_3 \in [k]$ be such that the ℓ_i-th entry of $\bar{t}^1$ is minimal among those entries of $\bar{t}^1$ with an index from I_i, for $i \in \{2,3\}$. We have $\bar{t}^1[\ell_2] < \bar{t}^2[\ell_2]$ and $\bar{t}^1[\ell_3] < \bar{t}^3[\ell_3]$ by the definition of med_k. If $\ell_2 = \ell_3$, then $\bar{t}_{\ell_2} = \bar{t}_{\ell_3} \notin \mathrm{R_{ll}}$. Otherwise, the ℓ_2-th entry or the ℓ_3-th entry is minimal in $\bar{t}^1$; suppose that it is the ℓ_2-th entry. But then $\bar{t}^3[\ell_2] \geq \bar{t}^1[\ell_2]$ which implies that again $\bar{t}_{\ell_2} \notin \mathrm{R_{ll}}$. Next, consider the subcase where $\mathrm{med}_k(\bar{t}^1) = \mathrm{med}_k(\bar{t}^3)$ and $\mathrm{lex}_k(\bar{t}^1) < \mathrm{lex}_k(\bar{t}^3)$. Again, there is an index set $I_2 \subset [k]$ of cardinality $k-1$ such that $\min\{\bar{t}_i[1] \mid i \in I_2\} < \min\{\bar{t}_i[2] \mid i \in I_2\}$. Let $\ell_2 \in [k]$ be such that the ℓ_2-th entry of $\bar{t}^1$ is minimal among those enries with an index from I_2. We have $\bar{t}^1[\ell_2] < \bar{t}^2[\ell_2]$. If $\bar{t}^1[\ell_2] = \min_k(\bar{t}^1)$, then $\bar{t}^3[\ell_2] \geq \bar{t}^1[\ell_2]$ which implies $\bar{t}_{\ell_2} \notin \mathrm{R_{ll}}$. Otherwise, the entry of the remaining index $\ell_3 \in [k] \setminus I_2$ is minimal in $\bar{t}^1$ and in $\bar{t}^1[\ell_2] = \mathrm{med}_k(\bar{t}^1)$. We must have $\bar{t}^1[\ell_3] = \bar{t}^2[\ell_3] = \bar{t}^3[\ell_3]$ so that $\bar{t}_{\ell_3} \in \mathrm{R_{ll}}$. But then $\bar{t}^3[\ell_2]$ cannot be strictly less than $\bar{t}^1[\ell_2]$, because $\mathrm{med}_k(\bar{t}^1) = \mathrm{med}_k(\bar{t}^3)$. Hence, $\bar{t}_{\ell_2} \notin \mathrm{R_{ll}}$.

Case 3: $\min_k(\bar{t}^1) = \min_k(\bar{t}^2) = \min_k(\bar{t}^3)$, $\mathrm{med}_k(\bar{t}^1) = \mathrm{med}_k(\bar{t}^2) = \mathrm{med}_k(\bar{t}^3)$, $\mathrm{lex}_k(\bar{t}^1) < \mathrm{lex}_k(\bar{t}^2)$*, and* $\mathrm{lex}_k(\bar{t}^1) < \mathrm{lex}_k(\bar{t}^3)$. Let ℓ_2 and ℓ_3 be the leftmost indices on which $\bar{t}^1$ is pointwise smaller than $\bar{t}^2$ and $\bar{t}^3$, respectively. Without loss of generality $\ell_2 < \ell_3$. Then $\bar{t}_{\ell_2} \notin \mathrm{R_{ll}}$.

We conclude that ll_k preserves $\mathrm{R_{ll}}$. □

Proof of Theorem 1.7. We start with the case where **B** is a temporal structure. Suppose that **B** is neither preserved by min, mi, mx, ll, the dual of one of these operations, nor by a constant operation. Then **B** pp-constructs all finite structures by Theorem 2.14 and in particular the structure $(\{0,1\}; \mathrm{1IN3})$. By Theorem 1.8 in [7], there exists a uniformly continuous minion homomorphism from $\mathrm{Pol}(\mathbf{B})$ to $\mathrm{Pol}(\{0,1\}; \mathrm{1IN3})$, the projection clone. By Theorem 7.10, for every $k \geq 2$, the condition $\mathcal{E}_{k,k+1}$ is non-trivial (see Example 7.12). Since minion homomorphisms preserve minor conditions such as $\mathcal{E}_{k,k+1}$ it follows that $\mathrm{Pol}(\mathbf{B})$ cannot satisfy $\mathcal{E}_{k,k+1}$. Next, we distinguish the subcases where **B** is a temporal structure preserved by one of the operations listed above.

Case 1: **B** *is preserved by a constant operation.* Clearly, $\mathcal{E}_{k,k+1}$ is witnessed by a set of k-ary constant operations for every $k \geq 1$.

Case 2: **B** *is preserved by* min. Then $\mathcal{E}_{k,k+1}$ is witnessed by a set of k-ary minimum operations for every $k \geq 2$.

Case 3: **B** *is preserved by* mx. By Theorem 5.2, either **B** is preserved by min or by a constant operation, which are cases that we have already treated, or otherwise **B** admits a pp-definition of X. We claim that $\mathrm{Pol}(\mathbb{Q}; \mathrm{X})$ does not satisfy $\mathcal{E}_{k,k+1}$ for every odd $k > 1$. By Theorem 4.5, the temporal relation $R^{\mathrm{mx}}_{[k],k} := \{\bar{t} \in \mathbb{Q}^k \mid \sum_{\ell=1}^{k} \chi(\bar{t})[\ell] = 0 \bmod 2\}$ is preserved by mx. By Lemma 4.2, $R^{\mathrm{mx}}_{[k],k}$ is pp-definable in $(\mathbb{Q}; \mathrm{X})$. Since k is odd, there exists a homomorphism from $\mathbf{A}_1 := (\{0,1\}; \mathrm{1INk})$ to $(\mathbb{Q}; R^{\mathrm{mx}}_{[k],k})$. However, there exists no homomorphism from $\mathbf{A}_2 := ([k+1]; \{\bar{t} \in [k+1]^k \mid \bar{t}[1] < \cdots < \bar{t}[k]\})$ to $(\mathbb{Q}; R^{\mathrm{mx}}_{[k],k})$. This is because the homogeneous system of $k+1$ Boolean linear equations of the form $\sum_{j \in [k+1] \setminus \{i\}} x_j = 0 \bmod 2$ has no non-trivial solution, which means that $\mathbf{A}_2$ has no free set by Lemma 4.1. Hence, Theorem 7.10 implies that $\mathrm{Pol}(\mathbb{Q}; \mathrm{X})$ does not satisfy $\mathcal{E}_{k,k+1} = \mathcal{E}(\mathbf{A}_1, \mathbf{A}_2)$.

Case 4: **B** *has* mi *as a polymorphism.* We proceed similarly as in the proof of Proposition 4.10

in [4], but using our Theorem 7.10. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be as in the previous case for a fixed $k \geq 3$. Let $\mathbf{C}$ be an arbitrary d-dimensional pp-power of $\mathbf{B}$ with the same signature as $\mathbf{A}_1$ for which there exists a homomorphism $h\colon \mathbf{A}_1 \to \mathbf{C}$. We denote its unique relation (of arity k) by R. Since $\mathbf{B}$ is preserved by mi, it is also preserved by mi_k, by Proposition 7.16. For $i \in [k]$, let $\bar{t}_i^k \in C^k$ be of the form $(h(0),\dots,h(0),h(1),h(0),\dots,h(0))$ where $h(1)$ appears in the i-th entry. For $i \in [k+1]$, the tuples $\bar{t}_i^{k+1}$ are defined in the same way with the only difference that they have arity $k+1$. Since $\mathbf{C}$ is a pp-power of $\mathbf{B}$ and $\{\bar{t}_1^k,\dots,\bar{t}_k^k\} \subseteq R$, it follows from Proposition 2.1 that $(\mathrm{mi}_k(\bar{t}_1^k),\dots,\mathrm{mi}_k(\bar{t}_k^k)) \in R$. Moreover, it follows easily from the definition of mi_k that, for every $i \in [k+1]$, the substructure of $(\mathbb{Q};<)$ on

$$\{\mathrm{mi}_k(\bar{t}_1^k)[1],\dots,\mathrm{mi}_k(\bar{t}_1^k)[d],\mathrm{mi}_k(\bar{t}_2^k)[1],\dots,\mathrm{mi}_k(\bar{t}_2^k)[d],\dots,\mathrm{mi}_k(\bar{t}_k^k)[1],\dots,\mathrm{mi}_k(\bar{t}_k^k)[d]\}$$

can be mapped to the substructure of $(\mathbb{Q};<)$ on

$$\{\mathrm{mi}_{k+1}(\bar{t}_1^{k+1})[1],\dots,\mathrm{mi}_{k+1}(\bar{t}_1^{k+1})[d],\dots,\mathrm{mi}_{k+1}(\bar{t}_{i-1}^{k+1})[1],\dots,\mathrm{mi}_{k+1}(\bar{t}_{i+1}^{k+1})[1],\dots,\mathrm{mi}_{k+1}(\bar{t}_{k+1}^{k+1})[1],\dots\}$$

through an isomorphism given by the order in which the elements of both sets are listed. By the homogeneity of $(\mathbb{Q};<)$ there exist $\alpha_1,\dots,\alpha_{k+1} \in \mathrm{Aut}(\mathbb{Q};<)$ such that for every $i \in [k+1]$

$$\big(\mathrm{mi}_{k+1}(\bar{t}_1^{k+1}),\dots,\mathrm{mi}_{k+1}(\bar{t}_{i-1}^{k+1}),\mathrm{mi}_{k+1}(\bar{t}_{i+1}^{k+1}),\dots,\mathrm{mi}_k(\bar{t}_{k+1}^{k+1})\big) = \big(\alpha_i\,\mathrm{mi}_k(\bar{t}_1^k),\dots,\alpha_i\,\mathrm{mi}_k(\bar{t}_k^k)\big) \in R.$$

But then $h(i) := \mathrm{mi}_{k+1}(\bar{t}_i^{k+1})$ for $i \in [k+1]$ describes a homomorphism from $\mathbf{A}_2$ to $\mathbf{C}$. Since $\mathbf{C}$ was chosen arbitrarily, it follows from Theorem 7.10 that $\mathrm{Pol}(\mathbf{B}) \models \mathcal{E}_{k,k+1}$ for all $k \geq 3$.

Case 5: $\mathbf{B}$ *has* ll *as a polymorphism.* We repeat the strategy above using ll_k instead of mi_k.

The cases 2-5 can be dualized in order to obtain witnesses for $\mathcal{E}_{k,k+1}$ for $k \geq 3$ in the cases where $\mathbf{B}$ is preserved by max, dual mi, dual ll, and show that $\mathrm{Pol}(\mathbf{B})$ does not satisfy $\mathcal{E}_{k,k+1}$ for odd $k > 1$ if it admits a pp-definition of $-\mathrm{X}$.

If $\mathbf{B}$ is a finite structure, then $\mathrm{CSP}(\mathbf{B})$ is in FP / FPC if and only if $\mathbf{B}$ does not pp-construct $\mathbf{E}_{\mathbb{Z}_n,3}$ for every $n \geq 2$ by Theorem 1.1. Then the claim follows from Lemma 7.14. ☐

We can confirm the condition for expressibility in FP from Theorem 1.7 also for the structures $\mathrm{CSS}(\mathcal{F})$ from Theorem 7.5.

Theorem 7.17. *Let $\mathcal{F}$ be a finite set of finite connected structures with a fixed finite signature, and let* $\mathbf{B} := \mathrm{CSS}(\mathcal{F})$. *Then*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FP / FPC, *and*
(2) $\mathrm{Pol}(\mathbf{B})$ *satisfies $\mathcal{E}_{k,k+1}$ for all but finitely many $k \in \mathbb{N}$.*

Proof. $\mathrm{CSP}(\mathrm{CSS}(\mathcal{F}))$ is expressible in FP because it is even expressible in existential positive first-order logic. $\mathrm{Pol}(\mathrm{CSS}(\mathcal{F}))$ satisfies $\mathcal{E}_{k,k+1}$ for all but finitely many arities, because it contains WNU operations for all but finitely many arities by Lemma 5.4 in [17]. ☐

7.3 Failures of known pseudo minor conditions

In the context of infinite-domain ω-categorical CSPs, most classification results are formulated using *pseudo minor conditions* [4] which extend minor conditions by outer unary operations, i.e., they are of the form

$$e_1 \circ f_1(x_1^1,\dots,x_{n_1}^1) \approx \cdots \approx e_k \circ f_k(x_k^1,\dots,x_{n_k}^k).$$

For instance, the following generalization of a WNU operation was used in [9] to give an alternative classification of the computational complexity of TCSPs. An at least binary operation $f \in \mathrm{Pol}(\mathbf{B})$ is called *pseudo weak near-unanimity* (pseudo-WNU) if there exist $e_1,\dots,e_n \in \mathrm{End}(\mathbf{B})$ such that

$$e_1 \circ f(x,\dots,x,y) \approx e_2 \circ f(x,\dots,x,y,x) \approx \cdots \approx e_n \circ f(y,x,\dots,x).$$

Theorem 7.18 ([9]). *Let* $\mathbf{B}$ *be a temporal structure. Then either* $\mathbf{B}$ *has a pseudo-WNU polymorphism and* $\mathrm{CSP}(\mathbf{B})$ *is in P, or* $\mathbf{B}$ *pp-constructs all finite structures and* $\mathrm{CSP}(\mathbf{B})$ *is NP-complete.*

It is natural to ask whether pseudo minor conditions can be used to formulate a generalization of the 3-4 WNU condition from item 7 of Theorem 1.1 that would capture the expressibility in FP for the CSPs of reducts of finitely bounded homogeneous structures. One such generalization was considered in [16]. Proposition 7.19 shows that the criterion provided by Theorem 8 in [16] is insufficient in general.

Proposition 7.19. *There exist pseudo-WNU polymorphisms* f,g *of* $(\mathbb{Q};\mathrm{X})$ *that satisfy*

$$f(x,x,y) \approx g(x,x,x,y).$$

In the proof of Proposition 7.19 we need the following result.

Lemma 7.20 (see Lemma 3 in [22]). *Let* $\mathbf{B}$ *be an* ω*-categorical structure and* $f_1,g_1,\dots,f_n,g_n \in \mathrm{Pol}(\mathbf{B})$ *where* f_i *and* g_i *have the same arity* k_i*. If for every* $i \in \{1,\dots,n\}$ *and all finite* $F \subseteq B^{k_i}$ *we have* $f_i(\bar{t}) = g_i(\bar{t})$ *for all* $\bar{t} \in F$*, then there are* $e,e_1,\dots,e_n \in \mathrm{End}(\mathbf{B})$ *such that* $\mathrm{Pol}(\mathbf{B})$ *satisfies*

$$e(f_i(x_1,\dots,x_{k_i})) \approx e_i(g_i(x_1,\dots,x_{k_i})).$$

Proof of Proposition 7.19. Consider the terms

$$\begin{aligned} f(x_1,x_2,x_3) &:= \mathrm{mx}(\mathrm{mx}(x_1,x_2),\mathrm{mx}(x_2,x_3)), \\ g(x_1,x_2,x_3,x_4) &:= \mathrm{mx}(\mathrm{mx}(x_1,x_2),\mathrm{mx}(x_3,x_4)). \end{aligned}$$

We claim that, for all distinct $x,y \in \mathbb{Q}$, we have

$$\begin{aligned} f(x,x,y) = f(y,x,x) &= \alpha^2(\min(x,y)), \\ f(x,y,x) &= \beta(\alpha(\min(x,y))), \\ g(x,x,x,y) = \cdots = g(y,x,x,x) &= \alpha^2(\min(x,y)), \end{aligned}$$

where α,β are as in the definition of mx. If $x<y$, then $\alpha(x) < \beta(x)$ which means that

$$\begin{aligned} &f(y,x,x) = \mathrm{mx}(\alpha(x),\beta(x)) = \alpha^2(x), && g(y,x,x,x) = \mathrm{mx}(\alpha(x),\beta(x)) = \alpha^2(x), \\ &f(x,y,x) = \mathrm{mx}(\alpha(x),\alpha(x)) = \beta(\alpha(x)), && g(x,y,x,x) = \mathrm{mx}(\alpha(x),\beta(x)) = \alpha^2(x), \\ &f(x,x,y) = \mathrm{mx}(\beta(x),\alpha(x)) = \alpha^2(x), && g(x,x,y,x) = \mathrm{mx}(\beta(x),\alpha(x)) = \alpha^2(x), \\ & && g(x,x,x,y) = \mathrm{mx}(\beta(x),\alpha(x)) = \alpha^2(x). \end{aligned}$$

If $x>y$, then $\alpha(y) < \beta(x)$ which means that

$$\begin{aligned} &f(y,x,x) = \mathrm{mx}(\alpha(y),\beta(x)) = \alpha^2(y), && g(y,x,x,x) = \mathrm{mx}(\alpha(y),\beta(x)) = \alpha^2(y), \\ &f(x,y,x) = \mathrm{mx}(\alpha(y),\alpha(y)) = \beta(\alpha(y)), && g(x,y,x,x) = \mathrm{mx}(\alpha(y),\beta(x)) = \alpha^2(y), \\ &f(x,x,y) = \mathrm{mx}(\beta(x),\alpha(y)) = \alpha^2(y), && g(x,x,y,x) = \mathrm{mx}(\beta(x),\alpha(y)) = \alpha^2(y), \\ & && g(x,x,x,y) = \mathrm{mx}(\beta(x),\alpha(y)) = \alpha^2(y). \end{aligned}$$

We also have $f(x,x,x) = \beta^2(x) = g(x,x,x,x)$ for all $x \in \mathbb{Q}$. By the computation above, for all $x,y,x',y' \in \mathbb{Q}$, we have

$$f(x,x,y) = f(y,x,x) < f(x',x',y') = f(y',x',x') \text{ if and only if } f(x,y,x) < f(x',y',x').$$

This means that for every finite $S \subseteq \mathbb{Q}$, the finite substructures of $(\mathbb{Q};<)$ on the images of $f(x,x,y)$, $f(x,y,x)$ and $f(y,x,x)$ with inputs restricted to S^3 are isomorphic. Since $(\mathbb{Q};<)$ is homogeneous, there exist $\alpha',\beta',\gamma' \in \mathrm{Aut}(\mathbb{Q};<)$ such that for all $x,y \in S$

$$\alpha' \circ f(x,x,y) = \beta' \circ f(x,y,x) = \gamma' \circ f(y,x,x).$$

Lemma 7.20 then implies that f is a pseudo-WNU. Clearly, g is a WNU, and $f(x,x,y) = g(x,x,x,y)$, which completes the proof. □

Proposition 7.19 implies that another well-known characterisation of solvability of finite-domain CSPs in FP, namely the inability to express systems of linear equations over finite non-trivial Abelian groups, fails for temporal CSPs (Corollary 7.21). The present proof of Corollary 7.21 is an interesting application of the fact that the automorphism group of any *ordered homogeneous Ramsey structure* [20] is *extremely amenable* [46], and inspired by [4]. Consider the structure $\mathbf{E}_{\mathscr{G},3}$ defined in Section 4 for every finite Abelian group $\mathscr{G}$.

Corollary 7.21. $(\mathbb{Q};\mathrm{X})$ *does not pp-construct* $\mathbf{E}_{\mathscr{G},3}$ *for any finite non-trivial Abelian group* $\mathscr{G}$.

Proof. Suppose, on the contrary, that $(\mathbb{Q};\mathrm{X})$ does pp-construct $\mathbf{E}_{\mathscr{G},3}$. By Theorem 1.8 in [7], there exists a uniformly continuous minion homomorphism $\xi\colon \mathrm{Pol}(\mathbb{Q};\mathrm{X}) \to \mathrm{Pol}(\mathbf{E}_{\mathscr{G},3})$. Since $(\mathbb{Q};<)$ is an ordered homogeneous Ramsey structure and $\mathbf{E}_{\mathscr{G},3}$ is finite, by the second proof of Theorem 1.9 in [4], there exists a uniformly continuous minion homomorphism $\xi'\colon \mathrm{Pol}(\mathbb{Q};\mathrm{X}) \to \mathrm{Pol}(\mathbf{E}_{\mathscr{G},3})$ that preserves all pseudo minor conditions with outer embeddings which hold in $\mathrm{Pol}(\mathbb{Q};\mathrm{X})$ for at most 4-ary operations. Since every endomorphism of $(\mathbb{Q};\mathrm{X})$ is an embedding, the 3-4 equation for pseudo-WNUs from Proposition 7.19 is such a condition. Thus, it must also be satisfied in $\mathrm{Pol}(\mathbf{E}_{\mathscr{G},3})$. But the only endomorphism of $\mathbf{E}_{\mathscr{G},3}$ is the identity, which means that $\mathrm{Pol}(\mathbf{E}_{\mathscr{G},3})$ would have to satisfy the 3-4 equation for WNUs. This implies that item (6) in Theorem 1.1 holds, while item (4) clearly does not hold, a contradiction. □

Another characterisation of finite-domain CSPs in FP that fails for temporal CSPs is the existence of pseudo-WNU polymorphisms for all but finitely many arities (Proposition 7.22).

Proposition 7.22. *For every* $k \geq 3$, min_k, mx_k, mi_k, *and* ll_k *are pseudo-WNU operations.*

Proof. The statement trivially holds for min_k. To show the statement for the operation mx_k, we first prove the following. Let α,β be the self-embeddings of $(\mathbb{Q};<)$ from the definition of mx.

Claim 7.23. *Let* $m_1,n_1,m_2,n_2 \in \mathbb{N}$ *be such that* $m_1+n_1 = m_2+n_2$ *and* $\beta^{m_1}\circ\alpha^{n_1} = \beta^{m_2}\circ\alpha^{n_2}$. *Then* $m_1 = m_2$ *and* $n_1 = n_2$.

Proof of Claim 7.23. Suppose for contradiction that $m_1 \neq m_2$. We may assume $m_1 > m_2$ without loss of generality. Since α and β are both injective, we have

$$\beta^{m_1}(\alpha^{n_1}(0)) = \beta^{m_2}(\alpha^{n_2}(0)) \text{ if and only if } \beta^{m_1-m_2}(\alpha^{n_1}(0)) = \alpha^{n_2}(0).$$

But this cannot be the case because

$$\alpha^{n_2}(0) < \beta(\alpha^{n_2-n_1-1}(0)) < \cdots < \beta^{m_1-m_2}(\alpha^{n_1}(0)).$$ □

We show by induction on k that for every $k \geq 2$ and every position of y in a tuple of variables of the form $(x,\ldots,x,y,x,\ldots,x)$ there exists a unique $m \geq 0$ and $n > 0$ satisfying $m+n = k-1$ such that

(1) for all distinct $x,y \in \mathbb{Q}$ we have $\mathrm{mx}_k(x,\ldots,x,y,x,\ldots,x) = \beta^m\circ\alpha^n(\min(x,y))$,
(2) if $m > 0$ and $\mathrm{mx}_k(x,\ldots,x,y,x,\ldots,x) = \mathrm{mx}(f'(x,y),f''(x,y))$ for some operations f',f'' given by

$$\mathrm{mx}_{k-1}(x,\ldots,x,y,x,\ldots,x) \quad \text{or by} \quad \mathrm{mx}_{k-1}(x,\ldots,x),$$

then $f' = f''$.

Also note that $\mathrm{mx}_k(x,\ldots,x) \approx \beta^{k-1}(x)$ holds for every $k \geq 2$ and that $\alpha(\min(x,y)) < \beta(x)$ holds whenever $x \neq y$. Together with item (1) this implies that we can use Lemma 7.20 similarly as in the proof of Proposition 7.19 to conclude that mx_k is a pseudo-WNU. The base case $k = 2$ follows directly from the definition of mx, and the case $k = 3$ follows from the proof of Proposition 7.19.

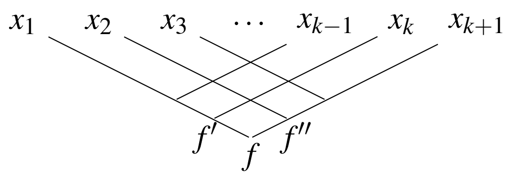

Figure 7: An illustration of the term $f = \text{mx}_{k+1}(x_1, \dots, x_{k+1})$ and its subterms.

In the inductive step we want to show that the statement holds for $k+1$, assuming that it holds for k. By the definition of mx_{k+1} we have

$$\text{mx}_{k+1}(x, \dots, x, y, x, \dots, x) = \text{mx}(g', g'')$$

for some operations g', g'' given by $\text{mx}_k(x, \dots, x, y, x, \dots, x)$ or by $\text{mx}_k(x, \dots, x)$ (see Figure 7). First we consider the case that both g' and g'' depend on y. By (1) in the inductive hypothesis there exist $m', m'' \geq 0$ and $n', n'' > 0$ satisfying $m' + n' = k-1$ and $m'' + n'' = k-1$ such that for all distinct $x, y \in \mathbb{Q}$

$$g'(x,y) = \beta^{m'}(\alpha^{n'}(\min(x,y))) \tag{6}$$

$$\text{and } g''(x,y) = \beta^{m''}(\alpha^{n''}(\min(x,y))). \tag{7}$$

Part (1) of the inductive statement holds if $m' = m'' = 0$: in this case $n' = n'' = k-1$, and hence for all distinct $x, y \in \mathbb{Q}$

$$\begin{aligned} \text{mx}_k(x, \dots, x, y, x, \dots, x) &= \text{mx}_k(g', g'') \\ &= \beta(\min(\alpha^{n'}(\min(x,y)), \alpha^{n'}(\min(x,y)))) \\ &= \beta(\alpha^{n'}(\min(x,y)) \end{aligned}$$

which proves part (1) of the inductive statement for $m = 1$ and $n = n'$; the uniqueness of m and n follows from the claim above. Also note that we have also proven part (2), because the argument above would have applied to any choice of f', f'' of the form $\text{mx}_k(x, \dots, x, y, x, \dots, x)$ or of the form $\text{mx}_k(x, \dots, x)$ instead of g', g''.

Next, suppose that $m' > 0$ and $m'' > 0$. By the definition of mx_k and another application of part (1) of the inductive hypothesis, for every $i \in \{1,2,3\}$ there exist $m_i \geq 0$ and $n_i > 0$ with $m_i + n_i = k-2$ such that for all distinct $x, y \in \mathbb{Q}$

$$\begin{aligned} g'(x,y) &= \text{mx}(\beta^{m_1} \circ \alpha^{n_1}(\min(x,y)), \beta^{m_2} \circ \alpha^{n_2}(\min(x,y))) \\ \text{and} \quad g''(x,y) &= \text{mx}(\beta^{m_2} \circ \alpha^{n_2}(\min(x,y)), \beta^{m_3} \circ \alpha^{n_3}(\min(x,y))). \end{aligned}$$

Since $m' > 0$ and $m'' > 0$, by (2) in the induction hypothesis, for all distinct $x, y \in \mathbb{Q}$

$$\begin{aligned} \beta^{m_1} \circ \alpha^{n_1}(\min(x,y)) &= \beta^{m_2} \circ \alpha^{n_2}(\min(x,y)) \\ \text{and } \beta^{m_2} \circ \alpha^{n_2}(\min(x,y)) &= \beta^{m_3} \circ \alpha^{n_3}(\min(x,y)). \end{aligned}$$

Note that $m_1 + n_1 = m_2 + n_2 = m_3 + n_3 = k-2$. Thus, the claim above implies that $m_1 = m_2$ and that $m_2 = m_3$. So we have that $m_1 = m_2 = m_3$ and $n_1 = n_2 = n_3$ and by the definition of mx

$$\text{mx}_{k+1}(x, \dots, x, y, x, \dots, x) = \text{mx}(g', g'') = \beta^{m'+1}(\alpha^{n'}(\min(x,y))).$$

This shows item (1), and again it also shows item (2) of the statement since we may apply the same reasoning for any such choice of f', f'' instead of g', g''.

Suppose next that $m' = 0$ and $m'' > 0$. Then $\beta^{m'}(\alpha^{n'}(x)) < \beta^{m''}(\alpha^{n''}(x))$ for all $x \in \mathbb{Q}$, and hence by (6) and (7) for all distinct $x,y \in \mathbb{Q}$

$$\mathrm{mx}_{k+1}(x,\dots,x,y,x,\dots,x) = \mathrm{mx}(g',g'') = \alpha(\beta^{m'}(\alpha^{n'}(\min(x,y)))) = \alpha^{n'+1}(\min(x,y)).$$

This again proves part (1) of the inductive statement for $m = 0$ and $n = n'+1$. Note that in this case, part (2) of the inductive statement holds trivially. The case that $m' > 0$ and $m'' = 0$ is can be shown similarly.

Finally, we need to treat the case that one of g' and g'' does not depend on y. Note that one of g' and g'' must depend on y. We may therefore suppose without loss of generality that g' depends on y but g'' does not. By (1) in the induction hypothesis, there exist $m' \geq 0$ and $n' > 0$ with $m'+n' = k-1$ such that for all distinct $x,y \in \mathbb{Q}$

$$g'(x,y) = \beta^{m'}\left(\alpha^{n'}(\min(x,y))\right).$$

Also, $g''(x,y) = \beta^{k-1}(x)$ holds for all $x,y \in \mathbb{Q}$ by the definition of mx_k. Moreover,

$$g'(x,y) = \mathrm{mx}\left(\mathrm{mx}_{k-1}(y,x,\dots,x), \mathrm{mx}_{k-1}(x,\dots,x)\right).$$

By (1) in the induction hypothesis, there exist $m' \geq 0$ and $n'' > 0$ with $m''+n'' = k-2$ such that

$$\mathrm{mx}_{k-1}(y,x,\dots,x) = \beta^{m''}\left(\alpha^{n''}(\min(x,y))\right).$$

Since $n'' > 0$, we have for all distinct $x,y \in \mathbb{Q}$

$$\mathrm{mx}_{k-1}(y,x,\dots,x) = \beta^{m''}\left(\alpha^{n''}(\min(x,y))\right) < \beta^{k-2}(x) = \mathrm{mx}_{k-1}(x,\dots,x).$$

Thus, $m' = 0$ because of (2) in the induction hypothesis for f'. It follows that for all distinct $x,y \in \mathbb{Q}$

$$\mathrm{mx}_{k+1}(x,\dots,x,y,x,\dots,x) = \mathrm{mx}(g',g'') = \mathrm{mx}(\alpha^{k-1}(\min(x,y)),\beta^{k-1}(x)) = \alpha^{k}(\min(x,y)).$$

This proves part (1) of the inductive statement; part (2) is in this case again trivial. This concludes the proof that mx_k is a WNU operation.

Next, we consider the operation ll_k. We claim that, for every $k \geq 3$, there exist self-embeddings α,β of $(\mathbb{Q};<)$ such that for all $x,y \in \mathbb{Q}$

$$\alpha\left(\mathrm{ll}_k(x,\dots,x,y,x,\dots,x)\right) = \beta\left(\mathrm{ll}_k(y,x,\dots,x)\right).$$

To see this, consider all cases in which $\mathrm{ll}_k(x,\dots,x,y,x,\dots,x) < \mathrm{ll}_k(x',\dots,x',y',x',\dots,x')$ holds for some $x,y,x',y' \in \mathbb{Q}$ where y and y' appear at the same argument position. We need to distinguish the following cases.

Case 1: $\min(x,y) < \min(x',y')$.

Case 2: $\min(x,y) = \min(x',y')$ *and* $x < x'$.

Case 3: $\min(x,y) = \min(x',y')$, $x = x'$, *and* $y < y'$.

In all three cases, we have $\mathrm{ll}_k(y,x,\dots,x) < \mathrm{ll}_k(y',x',\dots,x')$ as well. The statement then follows from Lemma 7.20 similarly as in the proof of Proposition 7.19.

The same argument can be applied for the operation mi_k. We have

$$\mathrm{mi}_k(x,\dots,x,y,x,\dots,x) < \mathrm{mi}_k(x',\dots,x',y',x',\dots,x')$$

for some $x,y,x',y' \in \mathbb{Q}$ where y and y' appear at the same argument position if and only if

Case 1: $\min(x,y) < \min(x',y')$.

Case 2: $\min(x,y) = \min(x',y')$, $x \leq y$, *and* $x' > y'$.

In both cases, $\mathrm{mi}_k(y,x,\dots,x) < \mathrm{mi}_k(y',x',\dots,x')$ holds as well. □

7.4 New pseudo minor conditions

We present a new candidate for an algebraic condition given by pseudo minor identities that could capture the expressibility in FP for CSPs of reducts of finitely bounded homogeneous structures. Let $\mathcal{E}'_{k,k+1}$ be the pseudo minor condition obtained from $\mathcal{E}_{k,k+1}$ by replacing each $g_{\bar{t}}$ in $\mathcal{E}_{k,k+1}$ with $e_{\bar{t}} \circ g$ where $e_{\bar{t}}$ is unary and g has arity k. For instance, up to further renaming the function symbols, $\mathcal{E}'_{3,4}$ is the following condition:

$$
\begin{aligned}
b \circ g(y,x,x) &\approx c \circ g(y,x,x) \approx d \circ g(y,x,x), \\
a \circ g(y,x,x) &\approx c \circ g(x,y,x) \approx d \circ g(x,y,x), \\
a \circ g(x,y,x) &\approx b \circ g(x,y,x) \approx d \circ g(x,x,y), \\
a \circ g(x,x,y) &\approx b \circ g(x,x,y) \approx c \circ g(x,x,y).
\end{aligned}
$$

Note that $\mathcal{E}'_{k,k+1}$ implies $\mathcal{E}_{k,k+1}$. Also note that the existence of a k-ary WNU operation implies $\mathcal{E}'_{k,k+1}$. However, $\mathcal{E}'_{k,k+1}$ is in general not implied by the existence of a k-ary pseudo-WNU operation: $\mathrm{Pol}(\mathbb{Q};\mathrm{X})$ contains a k-ary pseudo-WNU operation for every $k \geq 2$ (Proposition 7.22) but does not satisfy $\mathcal{E}_{k,k+1}$ for every odd $k \geq 3$. The latter statement follows from Theorem 7.17, because $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ is not in FP (Theorem 4.21). The proof of Theorem 7.17 shows that the statement of the theorem remains true if we replace $\mathcal{E}_{k,k+1}$ with $\mathcal{E}'_{k,k+1}$.

Corollary 7.24. *Let* $\mathbf{B}$ *be as in Theorem 7.17. The following are equivalent.*

(1) $\mathrm{CSP}(\mathbf{B})$ *is expressible in* FP / FPC.
(2) $\mathrm{Pol}(\mathbf{B})$ *satisfies* $\mathcal{E}'_{k,k+1}$ *for all but finitely many* $k \in \mathbb{N}$.

8 Open questions

We have completely classified expressibility of temporal CSPs in the logics FP, FPC, and Datalog. Our results show that all of the characterisations known for finite-domain CSPs fail for temporal CSPs. However, we have also seen new universal-algebraic conditions that characterise expressibility in FP simultaneously for finite-domain CSPs and for temporal CSPs. It is an open problem to find such conditions for reducts of finitely bounded homogeneous structures $\mathbf{B}$ in general. In this context, we ask the following questions.

(1) Is $\mathrm{CSP}(\mathbf{B})$ inexpressible in FP whenever $\mathrm{Pol}(\mathbf{B})$ does not satisfy the minor condition $\mathcal{E}_{k,k+1}$ for all but finitely many $k \geq 2$?
(2) We ask the previous question for the pseudo-minor condition $\mathcal{E}'_{k,k+1}$ instead of $\mathcal{E}_{k,k+1}$.
(3) If $\mathrm{CSP}(\mathbf{B})$ is in FPC, is it also in FP? To the best of our knowledge, this could hold for CSPs in general, even without the assumption that $\mathbf{B}$ is a reduct of a finitely bounded homogeneous structure.

An important candidate for a logic for PTIME is *choiceless polynomial time (CPT) with counting* [8].

(4) Does CPT with counting capture PTIME for CSPs of reducts of finitely bounded homogeneous structures?

This question is also open if $\mathbf{B}$ is finite. We also include the following more specific questions which we believe might be easier to approach.

(5) Is $\mathrm{CSP}(\mathbb{Q};\mathrm{X})$ expressible in CPT with counting or even without counting?
(6) Is $\mathrm{CSP}(\{0,1\};\mathrm{1IN3})$ inexpressible in CPT with counting?

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A Missing Proofs

In this appendix we present a compact proof of Theorem 2.7. The proof closely follows the proof from [1] which was presented only for structures with finite domains. The appendix is meant as a service to the referees, to substantiate our claim that the proof from [1] applies in our setting, and it is not intended for publication.

A.1 A proof of Theorem 2.7

Theorem 2.7. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be structures with finite relational signatures such that* $\mathbf{B}$ *is pp-constructible from* $\mathbf{C}$*. Then* $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{Datalog}} \mathrm{CSP}(\mathbf{C})$.

We start with an auxiliary lemma.

Lemma A.1. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be structures with finite relational signatures such that* $\mathbf{B}$ *is pp-definable in* $\mathbf{C}$*. Then* $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{Datalog}} \mathrm{CSP}(\mathbf{C})$.

Proof of Lemma A.1. Let τ and σ be the signatures of $\mathbf{B}$ and $\mathbf{C}$, respectively. For each $R \in \tau$, we define the following equivalence relation $\sim_R$ on $[n]$, where $n := \mathrm{ar}(R)$. For $i, j \in [n]$, we set $i \sim_R j$ if and only if $\bar{t}[i] = \bar{t}[j]$ for all $\bar{t} \in R$. Without loss of generality, we may assume that the first $\ell \leq n$ entries in R represent the distinct equivalence classes of $\sim_R$. This convention is fixed for the remainder of the proof. We define a new signature $\tilde{\tau}$ that contains, for every $R \in \tau$, a symbol $\mathrm{proj}_{[\ell]} R$ of arity ℓ derived from $\sim_R$ as above. Let $\tilde{\mathbf{B}}$ be the $\tilde{\tau}$-structure over B where each $\mathrm{proj}_{[\ell]} R$ interprets as $\mathrm{proj}_{[\ell]}(R^{\mathbf{B}})$. We denote the *positive quantifier-free* fragment of first-order logic by pqf.

Claim A.2 ([1], Lemma 13)**.** $\mathrm{CSP}(\tilde{\mathbf{B}}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\mathbf{B})$ *and* $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{Datalog}} \mathrm{CSP}(\tilde{\mathbf{B}})$.

Proof of Claim A.2. For an instance $\mathbf{A}$ of $\mathrm{CSP}(\tilde{\mathbf{B}})$, let $\mathcal{I}(\mathbf{A})$ be the τ-structure over A where each $R \in \tau$ with arity n and $\ell \leq n$ equivalence classes w.r.t. $\sim_R$ interprets as the relation with the pqf-definition

$$\psi_R(x_1,\dots,x_n) := \tilde{R}(x_1,\dots,x_\ell) \wedge \big(\bigwedge_{i \sim_R j} x_i = x_j\big)$$

in $\mathbf{A}$. It is clear that $\mathbf{A} \to \tilde{\mathbf{B}}$ if and only if $\mathcal{I}(\mathbf{A}) \to \mathbf{B}$, which shows that $\mathrm{CSP}(\tilde{\mathbf{B}}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\mathbf{B})$.

Now let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\mathbf{B})$. We define $\mathcal{I}(\mathbf{A})$ to be the $\tilde{\tau}$-structure over A where each $\mathrm{proj}_{[\ell]} R \in \tilde{\tau}$ interprets as the relation with the Datalog-definition

$$\psi_{\mathrm{proj}_{[\ell]} R}(x_1,\dots,x_\ell) := \exists y_1,\dots,y_n,x_{\ell+1},\dots,x_n \Big(R(y_1,\dots,y_n) \wedge \bigwedge_{i=1}^{n} [\mathrm{ifp}_{E,(x,y)}\, \psi_E(x,y)](x_i,y_i)\Big)$$

in $\mathbf{A}$; $\psi_E(x,y)$ is the disjunction of the following formulas:

- $E(y,x)$ (symmetry),
- $\exists z(E(x,y) \wedge E(y,z))$ (transitivity) and,
- $\exists x_1,\dots,x_n\big(S(x_1,\dots,x_n) \wedge (x = x_i) \wedge (y = x_j)\big)$ for every $S \in \tau$ and $i \sim_S j$.

Note that $C := \mathsf{lfp}(\mathsf{Op}^{\mathbf{A}}[\![\psi_E]\!])$ is an equivalence relation on A. Moreover, for every $h \colon \mathbf{A} \to \mathbf{B}$, we have $C \subseteq \ker h$. This can be proven by a simple induction over the inflationary stages of the fixed-point operator $\mathsf{Op}^{\mathbf{A}}[\![\psi_E]\!]$. Let $h \colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. We claim that h is also a homomorphism from $\mathcal{I}(\mathbf{A})$ to $\tilde{\mathbf{B}}$. For every $\bar{t} \in (\mathrm{proj}_{[\ell]} R)^{\mathcal{I}(\mathbf{A})}$ there exist $\bar{t}' \in A^n$ and $\bar{t}'' \in R^{\mathbf{A}}$ such that $\mathrm{proj}_{[\ell]}(\bar{t}') = \bar{t}$ and $(\bar{t}'[i], \bar{t}''[i]) \in C$ for every $i \in [n]$. Since h is a homomorphism, we have $h(\bar{t}') = h(\bar{t}'') \in R^{\mathbf{B}}$. Then

$$h(\bar{t}) = h(\mathrm{proj}_{[\ell]}(\bar{t}')) = \mathrm{proj}_{[\ell]}(h(\bar{t}')) = \mathrm{proj}_{[\ell]}(h(\bar{t}'')) \in \mathrm{proj}_{[\ell]}(R^{\mathbf{B}}) = (\mathrm{proj}_{[\ell]} R)^{\tilde{\mathbf{B}}}.$$

Now let $h\colon \mathcal{I}(\mathbf{A}) \to \tilde{\mathbf{B}}$ be a homomorphism. We define a new mapping g from A to B by setting $g(a) := h(a_C)$, where a_C is some fixed representative of the equivalence class of a w.r.t. C. For a tuple $\bar{t} \in A^n$ we set $\bar{t}_C := (\bar{t}[1]_C, \dots, \bar{t}[n]_C)$. Then, for every $\bar{t} \in R^{\mathbf{A}}$, we have $\mathrm{proj}_\ell(\bar{t}_C) \in (\mathrm{proj}_{[\ell]} R)^{\mathcal{I}(\mathbf{A})}$ and thus

$$\mathrm{proj}_{[\ell]}(g(\bar{t})) = \mathrm{proj}_{[\ell]}(h(\bar{t}_C)) = h(\mathrm{proj}_{[\ell]}(\bar{t}_C)) \in (\mathrm{proj}_{[\ell]} R)^{\tilde{\mathbf{B}}} = \mathrm{proj}_{[\ell]}(R^{\mathbf{B}}).$$

Then $g(\bar{t}) \in R^{\mathbf{B}}$ by the definition of $\sim_R$. □

Let $\tilde{\mathbf{C}}$ be the expansion of $\mathbf{C}$ with the relations of $\tilde{\mathbf{B}}$. Note that these new relations are all pp-definable in $\mathbf{C}$. The following statement is trivial.

Claim A.3 ([1], Lemma 11)**.** $\mathrm{CSP}(\tilde{\mathbf{B}}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\tilde{\mathbf{C}})$.

We need one additional reduction step.

Claim A.4 ([1], Lemma 12)**.** $\mathrm{CSP}(\tilde{\mathbf{C}}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\mathbf{C})$.

Proof of Claim A.4. Without loss of generality, $\tilde{\mathbf{C}}$ has only one additional relation $(\mathrm{proj}_{[\ell]} R)^{\tilde{\mathbf{C}}}$. Let $\theta_{\mathrm{proj}_{[\ell]} R}(x_1, \dots, x_\ell)$ be its pp-definition in $\mathbf{C}$; it can be chosen so that it is of the form

$$\exists x_{\ell+1}, \dots, x_m \Big(\bigwedge_{j=1}^{n_1} R_1(\bar{x}^1_j) \wedge \dots \wedge \bigwedge_{j=1}^{n_s} R_s(\bar{x}^s_j) \Big)$$

where $\bar{x}^1_1, \dots, \bar{x}^1_{n_1}, \dots, \bar{x}^s_1, \dots, \bar{x}^s_{n_1}$ are tuples over $\{x_1, \dots, x_m\}$ and $\{R_1, \dots, R_s\} = \sigma$. We assume that all variables $x_{\ell+1}, \dots, x_m$ are distinct and disjoint from $x_1, \dots, x_\ell$. By the construction of $\tilde{\mathbf{B}}$, we may also assume that $x_1, \dots, x_\ell$ are distinct. Let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\tilde{\mathbf{C}})$. We first define the corresponding instance $\mathcal{I}(\mathbf{A}, \bar{c})$ of $\mathrm{CSP}(\mathbf{C})$ abstractly, and then provide a pqf-interpretation with parameters. We set the domain of $\mathcal{I}(\mathbf{A}, \bar{c})$ to be $A \cup (R^{\mathbf{A}} \times \{x_{\ell+1}, \dots, x_m\})$. We assume that $\{x_{\ell+1}, \dots, x_m\} \cap A = \emptyset$. The interpretation of each R_i in $\mathcal{I}(\mathbf{A}, \bar{c})$ contains: every tuple from $R_i^{\mathbf{A}}$ and, for every $\bar{t}' \in (\mathrm{proj}_{[\ell]} R)^{\mathbf{A}}$ and every $\bar{x}^i_j$ from $\theta_{\mathrm{proj}_{[\ell]} R}$, the tuple $\bar{t}$ defined by

- $\bar{t}[k] := \bar{t}'[i_k]$ where $x_{i_k} = \bar{x}^i_j[k]$ if $\bar{x}^i_j[k]$ is a free variable of $\theta_{\mathrm{proj}_{[\ell]} R}$, and
- $\bar{t}[k] := (\bar{t}'[1], \dots, \bar{t}'[\ell], \bar{x}^i_j[k])$ if $\bar{x}^i_j[k]$ is a bound variable of $\theta_{\mathrm{proj}_{[\ell]} R}$.

Suppose that there exists $h\colon \mathcal{I}(\mathbf{A}, \bar{c}) \to \mathbf{C}$. We claim that the restriction of h to A is a homomorphism from $\mathbf{A}$ to $\tilde{\mathbf{C}}$. Recall that for every $R_i \in \sigma$ we have $R_i^{\mathbf{A}} \subseteq R_i^{\mathcal{I}(\mathbf{A}, \bar{c})}$ and $R_i^{\tilde{\mathbf{C}}} = R_i^{\mathbf{C}}$. Since h is a homomorphism, the latter implies that $h(R_i^{\mathbf{A}}) \subseteq R_i^{\tilde{\mathbf{C}}}$. We claim that also $h((\mathrm{proj}_{[\ell]} R)^{\mathbf{A}}) \subseteq (\mathrm{proj}_{[\ell]} R)^{\tilde{\mathbf{C}}}$. Let $\bar{t}' \in (\mathrm{proj}_{[\ell]} R)^{\mathbf{A}}$ be arbitrary. By the definition of $\mathcal{I}(\mathbf{A}, \bar{c})$, for every $\bar{x}^i_j$, the tuple $\bar{t}$ defined as above is contained in $R_i^{\mathcal{I}(\mathbf{A}, \bar{c})}$. Since $x_1, \dots, x_\ell$ are distinct, the tuples of the form $h(\bar{t})$ witness that $\mathbf{C} \models \theta_{\mathrm{proj}_{[\ell]} R}(h(\bar{t}'))$.

Suppose that there exists $h\colon \mathbf{A} \to \tilde{\mathbf{C}}$. We define a map g from the domain of $\mathcal{I}(\mathbf{A}, \bar{c})$ to C as follows. On A, g coincides with h. Now fix a tuple $\bar{t}' \in (\mathrm{proj}_{[\ell]} R)^{\mathbf{A}}$. We have $h(\bar{t}') \in (\mathrm{proj}_{[\ell]} R)^{\tilde{\mathbf{C}}}$ because h is a homomorphism, and thus $\mathbf{C} \models \theta_{\mathrm{proj}_{[\ell]} R}(h(\bar{t}'))$. Let $c_{\ell+1}, \dots, c_m \in C$ be witnesses for the quantified variables in $\theta_{\mathrm{proj}_{[\ell]} R}$. We define $g(\bar{t}'[1], \dots, \bar{t}'[\ell], x_i) := a_i$ for every $\ell + 1 \leq i \leq m$. It follows directly from the definitions that g is a homomorphism from $\mathcal{I}(\mathbf{A}, \bar{c})$ to $\mathbf{C}$.

Now we provide a formal pqf-interpretation with parameters. Fix a pair p_0, p_1 of distinct parameters that will play the role of binary counters and set $u := m - \ell$, $v := \lfloor \log_2 u \rfloor + 1$. We view the domain of $\mathcal{I}(\mathbf{A}, \bar{c})$ as the subset of $A^{\ell+v+3}$ defined by

$$\begin{aligned}\delta_{\mathcal{I}}(y_0, \dots, y_{\ell+v}, p_0, p_1) := &\big(y_0 = p_0 \wedge y_1 = \dots = y_{\ell+v}\big)\\ &\vee \big(y_0 = p_1 \wedge R(y_1, \dots, y_\ell) \wedge \psi(y_{\ell+1}, \dots, y_{\ell+v})\big),\end{aligned}$$

where $\psi(y_{\ell+1},\dots,y_{\ell+v},p_0,p_1)$ is a formula satisfied by $\{0,\dots,u-1\}$ when encoded in binary; the bits are encoded by $y_{r+b}=p_0$ or $y_{r+b}=p_1$. If u is a power of two, we might take

$$\psi(y_{\ell+1},\dots,y_{\ell+v},p_0,p_1) := \bigwedge_{b=0}^{v-1} \big(y_{\ell+1+b}=p_0 \vee y_{\ell+1+b}=p_1\big),$$

otherwise we simply add dummy variables. The rest of the formal pqf-interpretation with parameters is easy to work out. $\square$

Now the statement follows by composing the reductions obtained in Claim A.2, Claim A.3 and Claim A.4. $\square$

Now we prove the general statement for pp-constructions.

Proof of Theorem 2.7. Let τ and σ be the signatures of $\mathbf{B}$ and $\mathbf{C}$, respectively. Let $\mathbf{P}$ be a pp-power of $\mathbf{C}$ such that $\mathbf{B}$ is homomorphically equivalent to $\mathbf{P}$. Then we clearly have $\mathrm{CSP}(\mathbf{B}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\mathbf{P})$. We define a new signature τ_f that contains, for every $R\in\tau$ of arity n, the symbol R_f of arity $d{\cdot}n$. Let $\mathbf{P}_f$ be the τ_f-structure over A with relations given by

$$R^{\mathbf{P}_f} := \big\{(\bar t_1[1],\dots,\bar t_1[d],\dots,\bar t_n[1],\dots,\bar t_n[d]) \in C^{n\cdot d} \mid (\bar t_1,\dots,\bar t_n)\in R^{\mathbf{P}}\big\}$$

Claim A.5 ([1], Lemma 17)**.** $\mathrm{CSP}(\mathbf{P}) \leq_{\mathrm{pqf}} \mathrm{CSP}(\mathbf{P}_f)$.

Proof of Claim A.5. Let $\mathbf{A}$ be an instance of $\mathrm{CSP}(\mathbf{P})$. We first define the corresponding instance $\mathcal{I}(\mathbf{A},\bar c)$ of $\mathrm{CSP}(\mathbf{P}_f)$ abstractly. The domain of $\mathcal{I}(\mathbf{A},\bar c)$ is $[d]\times A$ and, for every $R\in\tau$ with $n=\mathrm{ar}(R)$,

$$R_f^{\mathcal{I}(\mathbf{A},\bar c)} := \big\{\big((1,\bar t[1]),\dots,(d,\bar t[1]),\dots,(1,\bar t[n]),\dots,(d,\bar t[n])\big)\in([d]\times A)^n \mid \bar t\in R^{\mathbf{A}}\big\}.$$

Now, if $h\colon \mathbf{A}\to\mathbf{P}$, then clearly the mapping $g\colon \mathcal{I}(\mathbf{A},\bar c)\to\mathbf{P}_f$ defined by $g(i,x):=h(x)[i]$ is a homomorphism. If $h\colon \mathcal{I}(\mathbf{A},\bar c)\to\mathbf{P}_f$, then the mapping $g\colon\mathbf{A}\to\mathbf{P}$ given by $g(x):=\big(h(1,x),\dots,h(d,x)\big)$ is a homomorphism as well.

Now describe how one can define the pqf-reduction from above formally using a pqf-interpretation $\mathcal{I}$ with two distinguished parameters p_0 and p_1. Fix a pair p_0,p_1 of distinct parameters that will play the role of binary counters and set $v:=\lfloor\log_2 d\rfloor+1$. The domain formula $\delta_{\mathcal{I}}(\bar y,p_0,p_1)$ is satisfied precisely by those tuples $\bar t\in A^{v+3}$ for which $(\bar t[1],\dots,\bar t[v])$ encodes a number from $\{0,\dots,d-1\}$ in binary; the bits are encoded by $\bar y[b]=p_0$ or $\bar y[b]=p_1$ for $1\leq b\leq v$. The interpretation of each R_f is given by

$$\begin{aligned}\psi_{R_f}(\bar y_1,\dots,\bar y_{d\cdot n},p_0,p_1) = {}& R(\bar y_1[v+1],\bar y_{1+n}[v+1],\dots,\bar y_{1+(n-1)\cdot d}[v+1])\\ &\wedge \bigwedge_{j=1}^{d}\bigwedge_{i=0}^{n-1}(\bar y_{id+j}[1]=b_1\wedge\dots\wedge\bar y_{id+j}[v]=b_v),\end{aligned}$$

where $b_1\cdots b_v$ is the binary representation of $j-1$ using p_0 and p_1. $\square$

Since the relations of $\mathbf{P}_f$ are pp-definable in $\mathbf{C}$, the statement now follows by composing the reductions $\mathrm{CSP}(\mathbf{B})\leq_{\mathrm{pqf}}\mathrm{CSP}(\mathbf{P})$ and $\mathrm{CSP}(\mathbf{P})\leq_{\mathrm{pqf}}\mathrm{CSP}(\mathbf{P}_f)$ together with the one from Lemma A.1. $\square$