Most Specific Consequences in the Description Logic $\mathcal{EL}$

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Abstract

The notion of a most specific consequence with respect to some terminological box is introduced, conditions for its existence in the description logic $\mathcal{EL}$ and its variants are provided, and means for its computation are developed. Algebraic properties of most specific consequences are explored. Furthermore, several applications that make use of this new notion are proposed and, in particular, it is shown how given terminological knowledge can be incorporated in existing approaches for the axiomatization of observations. For instance, a procedure for an incremental learning of concept inclusions from sequences of interpretations is developed.

Keywords: Description logic,
Formal concept analysis, Lattice theory, Most specific consequence, Concept inclusion, Terminological axiom, TBox, Ontology, Interpretation, Model, Closure operator, Axiomatization, Error tolerance, Stream, Machine learning, Inductive learning

1. Introduction

Description logics (abbrv. DLs) (Baader, 1999; Baader, Horrocks, Lutz and Sattler, 2017; Baader, Horrocks and Sattler, 2004, 2009; Baader and Lutz, 2006) are frequently used knowledge representation and reasoning formalisms with a strong logical foundation. In particular, these provide their users with automated inference services that can derive implicit knowledge from the explicitly represented knowledge. Decidability and computational complexity of common reasoning tasks have been widely explored for most DLs. Besides being used in various application domains, their most notable success is the fact that DLs constitute the logical underpinning of the *Web Ontology Language* (abbrv. OWL) (Hitzler, Krötzsch and Rudolph, 2010) and many of its profiles.

Ontologies function as a means for expressing knowledge in terms of description logics. In particular, these are finite set of axioms that can either express *assertional* knowledge, that is, statements about certain individuals, or *terminological* knowledge, that is, statements that simultaneously hold true for all individuals. For instance, an ontology could contain the terminological axiom

$$\mathsf{Elephant} \sqsubseteq \exists{=}2.\,\mathsf{hasParent}.\,\mathsf{Elephant} \sqcap \forall\,\mathsf{hasParent}.\,\mathsf{Elephant},$$

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which states that each elephant has exactly two parents that are elephants as well and further that each of its parents is an elephant. Now further assume that our ontology contains the assertional axiom

$$\mathsf{dumbo} \sqsubseteq \mathsf{Elephant},$$

which expresses that the individual dumbo is an Elephant. It then follows that dumbo has two parents that are elephants, i.e., there must exist individuals mrsJumbo and dumbosFather which are elephants and the parents of dumbo. Continuing the induction of implicit knowledge, we would now infer that also dumbo's parents must have two parents each that are elephants, and so on and so forth.

On the one hand, such ontologies can be constructed manually by experts in the domain of interest and, on the other hand, these can be generated (semi-)automatically from given data and observations. For the latter case, there exist several approaches that can axiomatize observations and yield terminological axioms. Bühmann, Lehmann and Westphal (2016); Lehmann (2009) have developed a framework *DL-Learner* that constructs concept definitions from positive and negative examples. In particular, it is assumed that a concept description is to be learned that has all positive examples as instances, but none of the negative examples. Furthermore, there exist approaches that can axiomatize concept inclusions from observations, that is, given some input data on individuals, all valid concept inclusions can be characterized by a so-called *concept inclusion base* in a sound and complete manner. For instance, Rudolph (2004, 2006) has considered this task for the description logic $\mathcal{FLE}$, and later Baader and Distel (2008, 2009); Borchmann (2014); Borchmann, Distel and Kriegel (2016); Distel (2011) provided refined and adjusted solutions for the description logic $\mathcal{EL}$. However, all of the aforementioned works on the axiomatization of concept inclusions have in common that they have restricted settings for the input, that is, in particular, it is not possible to incorporate existing terminological knowledge that has been learned earlier or hand-crafted by some experts.

Within this document, we shall introduce the notion of *most specific consequences* with respect to sets of terminological axioms, so-called TBoxes, in Section 5. After characterizing existence of most specific consequences and showing how these can be computed in Section 6, we explore algebraic properties of this new notion in Section 7. Then, we provide a number of different applications in Section 8. More specifically, we devise an incremental procedure that can axiomatize concept inclusions from *sequences* of observations in Section 8.4. For instance, if data is collected on a regular basis, then this approach can be utilized to generate subsequent ontologies that represent the terminological knowledge that has been observed so far in a sound and complete manner. We also propose an abstract, but not yet fully developed idea for *merging* knowledge from two sets of terminological axioms in Section 8.3. Similarly, a proposal for an error-tolerant axiomatization of concept inclusions that is guided by some manually verified TBox is presented in Section 8.5 and, eventually, in Section 8.6 we recommend a procedure for axiomatization of observations under *Open World Assumption*, *Unique Name Assumption*, and *Domain Closure Assumption*. Beforehand, Section 2 introduces the description logic $\mathcal{EL}$ and some of its extensions with greatest fixed-point semantics, Section 3 quotes notions from lattice theory, e.g., that of a closure operator, and Section 4 gives an overview on related work that has been briefly mentioned above.

2. The Description Logic $\mathcal{EL}^\bot$

In this section we shall introduce the light-weight description logic $\mathcal{EL}^\bot$.

2.1. Syntax

Throughout the whole document assume that Σ is a *signature*, that is, a disjoint union of a set Σ_C of *concept names*, a set Σ_R of *role names*, and a set Σ_I of *individual names*. An $\mathcal{EL}^\bot$ *concept descriptions* C over Σ is a term that is constructed by means of the following inductive rule, where $A \in \Sigma_\mathsf{C}$ and $r \in \Sigma_\mathsf{R}$.

$$C ::= \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C$$

We call $\bot$ the *bottom concept description*, $\top$ is the *top concept description*, $C \sqcap D$ is the *conjunction* of C and D, and $\exists\, r.\, C$ is the *existential restriction* of C w.r.t. r. We further introduce some syntactic sugar. Firstly, we allow using words of role names within existential restrictions: if $w \in \Sigma_\mathsf{R}^*$ and C is some concept description, then $\exists\, w.\, C$ is a well-formed concept description; it is defined by $\exists\, \varepsilon.\, C \coloneqq C$ and $\exists\, rw.\, C \coloneqq \exists\, r.\, \exists\, w.\, C$. Secondly, we allow conjunctions of any finite number of concept descriptions: if $\mathbf{C}$ is a finite set of concept descriptions, then $\bigsqcap \mathbf{C}$ is a well-formed concept description as well; it is defined by $\bigsqcap \emptyset \coloneqq \top$ and $\bigsqcap \mathbf{C} \coloneqq C \sqcap \bigsqcap(\mathbf{C} \setminus \{C\})$ where C is an arbitrary element of $\mathbf{C}$. The set of all $\mathcal{EL}^\bot$ concept descriptions over Σ is denoted as $\mathcal{EL}^\bot(\Sigma)$. A *concept inclusion* (abbrv. CI) is an expression $C \sqsubseteq D$ where both the *premise* C as well as the *conclusion* D are concept descriptions. A *concept equivalence* is an expression $C \equiv D$ such that C and D are concept descriptions, and furthermore a *concept definition* is a term $A \equiv C$ where A is a concept name and C is a concept description. A *terminological box* (abbrv. TBox) is a finite set of concept inclusions, concept equivalences, and concept definitions. A *concept assertion* is a term $a \mathrel{\varepsilon} C$ where $a \in \Sigma_\mathsf{I}$ is an individual name and C is a concept description, and a *role assertion* is a term $(a, b) \mathrel{\varepsilon} r$ where $a, b \in \Sigma_\mathsf{I}$ are individual names and $r \in \Sigma_\mathsf{R}$ is a role name. An *assertional box* (abbrv. ABox) is a finite set of concept assertions and role assertions; an ABox is called *simple* if all concept descriptions occurring in concept assertions are concept names. Furthermore, an *ontology* $\mathcal{O}$ is a union of an assertional box and a terminological box, and elements that can occur in ontologies are also called *axioms*.

The *role depth* $\mathsf{rd}(C)$ of a concept description C is recursively defined as follows.

$$\begin{aligned}
\mathsf{rd}(A) &\coloneqq 0 \quad \text{if } A \in \Sigma_\mathsf{C} \cup \{\bot, \top\} \\
\mathsf{rd}(C \sqcap D) &\coloneqq \mathsf{rd}(C) \vee \mathsf{rd}(D) \\
\mathsf{rd}(\exists\, r.\, C) &\coloneqq 1 + \mathsf{rd}(C)
\end{aligned}$$

For a role-depth bound $d \in \mathbb{N}$, we define $\mathcal{EL}^\bot_d(\Sigma)$ as the set of all $\mathcal{EL}^\bot$ concept descriptions with a role depth not exceeding d.

The set $\mathsf{Sub}(C)$ of all *subconcepts* of a concept description C is recursively defined as follows.

$$\begin{aligned}
\mathsf{Sub}(A) &\coloneqq \{A\} \quad \text{if } A \in \Sigma_\mathsf{C} \cup \{\bot, \top\} \\
\mathsf{Sub}(C \sqcap D) &\coloneqq \{C \sqcap D\} \cup \mathsf{Sub}(C) \cup \mathsf{Sub}(D) \\
\mathsf{Sub}(\exists\, r.\, C) &\coloneqq \{\exists\, r.\, C\} \cup \mathsf{Sub}(C)
\end{aligned}$$

For an axiom α, we define its set $\mathsf{Sub}(\alpha)$ of subconcepts as follows.

$$\begin{aligned}\mathsf{Sub}(C \sqsubseteq D) &\coloneqq \mathsf{Sub}(C) \cup \mathsf{Sub}(D)\\ \mathsf{Sub}(C \equiv D) &\coloneqq \mathsf{Sub}(C) \cup \mathsf{Sub}(D)\\ \mathsf{Sub}(a \mathrel{\varepsilon} C) &\coloneqq \mathsf{Sub}(C)\end{aligned}$$

Furthermore, the set $\mathsf{Sub}(\mathcal{O})$ of subconcepts of an ontology is defined as

$$\mathsf{Sub}(\mathcal{O}) \coloneqq \bigcup \{\, \mathsf{Sub}(\alpha) \mid \alpha \in \mathcal{O} \,\}.$$

The *size* $|C|$ of a concept description C is the number of nodes in its syntax tree and, more specifically, we recursively define it as follows.

$$\begin{aligned}|A| &\coloneqq 1 \quad \text{ if } A \in \Sigma_{\mathsf{C}} \cup \{\bot, \top\}\\ |C \sqcap D| &\coloneqq |C| + 1 + |D|\\ |\exists\, r.\, C| &\coloneqq 1 + |C|\end{aligned}$$

Then, the *size* $|\alpha|$ of an axiom α is given by the following definitions.

$$\begin{aligned}|C \sqsubseteq D| &\coloneqq |C| + |D|\\ |C \equiv D| &\coloneqq |C| + |D|\\ |a \mathrel{\varepsilon} C| &\coloneqq 1 + |C|\\ |(a, b) \mathrel{\varepsilon} r| &\coloneqq 3\end{aligned}$$

Furthermore, the *size* $|\mathcal{O}|$ of an ontology is defined by

$$|\mathcal{O}| \coloneqq \sum (\, |\alpha| \mid \alpha \in \mathcal{O} \,).$$

2.2. Semantics

An *interpretation* $\mathcal{I} \coloneqq (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ over Σ consists of a non-empty set $\Delta^{\mathcal{I}}$ of *objects*, called the *domain*, and an *extension function* $\cdot^{\mathcal{I}}$ that maps concept names $A \in \Sigma_{\mathsf{C}}$ to subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and that maps role names $r \in \Sigma_{\mathsf{R}}$ to binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Then, the extension function is canonically extended to all $\mathcal{EL}^{\bot}$ concept descriptions by the following recursive definitions.

$$\begin{aligned}\bot^{\mathcal{I}} &\coloneqq \emptyset\\ \top^{\mathcal{I}} &\coloneqq \Delta^{\mathcal{I}}\\ (C \sqcap D)^{\mathcal{I}} &\coloneqq C^{\mathcal{I}} \cap D^{\mathcal{I}}\\ (\exists\, r.\, C)^{\mathcal{I}} &\coloneqq \{\, \delta \mid \delta \in \Delta^{\mathcal{I}},\ (\delta, \epsilon) \in r^{\mathcal{I}},\ \text{and } \epsilon \in C^{\mathcal{I}} \text{ for some } \epsilon \in \Delta^{\mathcal{I}} \,\}\end{aligned}$$

A concept inclusion $C \sqsubseteq D$ is *valid* in $\mathcal{I}$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. A concept equivalence $C \equiv D$ is *valid* in $\mathcal{I}$, denoted by $\mathcal{I} \models C \equiv D$, if $C^{\mathcal{I}} = D^{\mathcal{I}}$. A concept assertion $a \mathrel{\varepsilon} C$ is *valid* in $\mathcal{I}$, symbolized by $\mathcal{I} \models a \mathrel{\varepsilon} C$, if $a^{\mathcal{I}} \in C^{\mathcal{I}}$. A role assertion $(a, b) \mathrel{\varepsilon} r$ is *valid* in $\mathcal{I}$, written $\mathcal{I} \models (a, b) \mathrel{\varepsilon} r$, if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. We also refer to $\mathcal{I}$ as a *model* of the axiom α if $\mathcal{I} \models \alpha$ holds true. Furthermore, $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$, symbolized as $\mathcal{I} \models \mathcal{O}$, if each axiom in $\mathcal{O}$ is valid in $\mathcal{I}$. The entailment relation is lifted to ontologies as follows: an axiom α is *entailed* by an ontology $\mathcal{O}$, denoted as $\mathcal{O} \models \alpha$, if each model of $\mathcal{O}$ is a model of α too. An ontology $\mathcal{O}_1$ *entails* an ontology $\mathcal{O}_2$, symbolized as $\mathcal{O}_1 \models \mathcal{O}_2$, if $\mathcal{O}_1$ entails each axiom in $\mathcal{O}_2$ or, equivalently, if each model of $\mathcal{O}_1$ is also a model of $\mathcal{O}_2$. In case $\mathcal{O} \models C \sqsubseteq D$, we then also say that C is *subsumed* by D with respect to $\mathcal{O}$. Two concept descriptions C and D are *equivalent* with respect to some ontology $\mathcal{O}$ if $\mathcal{O} \models C \equiv D$. For an individual name a and a concept description C, we say that a is an *instance* of C with respect to some ontology $\mathcal{O}$ if $\mathcal{O} \models a \in C$.

Example. *Consider the following ontology $\mathcal{O}$.*

$$\mathcal{O} \coloneqq \left\{ \begin{aligned} \mathsf{Researcher} &\equiv \exists\, \mathsf{has.\,UniversityDegree} \sqcap \exists\, \mathsf{publishes.\,ScientificArticle}, \\ \mathsf{UniversityProfessor} &\equiv \exists\, \mathsf{has.\,DoctoralDegree} \sqcap \exists\, \mathsf{publishes.\,ScientificArticle} \\ &\qquad \sqcap \exists\, \mathsf{teaches.\,UniversityLecture} \sqcap \exists\, \mathsf{publishes.\,TextBook}, \\ \mathsf{DoctoralDegree} &\sqsubseteq \mathsf{UniversityDegree}, \\ \mathsf{somebody} &\in \mathsf{UniversityProfessor} \end{aligned} \right\}$$

As one quickly verifies, $\mathcal{O}$ entails the axioms $\mathsf{UniversityProfessor} \sqsubseteq \mathsf{Researcher}$ *and* $\mathsf{somebody} \in \mathsf{Researcher}$*.*

If $\mathcal{Y}$ is either an interpretation or an ontology and $\leq$ is a suitable relation symbol, e.g., one of $\sqsubseteq$, $\equiv$, $\sqsupseteq$, $\in$, then we may also use the denotation $C \leq_{\mathcal{Y}} D$ instead of $\mathcal{Y} \models C \leq D$ and, analogously, we may write $C \not\leq_{\mathcal{Y}} D$ for $\mathcal{Y} \not\models C \leq D$.

The *active signature* of an interpretation $\mathcal{I}$ is defined as the set $\Sigma^{\mathcal{I}}$ that contains all concept and role names from Σ with a non-empty extension in $\mathcal{I}$, that is, we define $\Sigma^{\mathcal{I}} \coloneqq \{\, \sigma \mid \sigma \in \Sigma \text{ and } \sigma^{\mathcal{I}} \neq \emptyset \,\}$. Furthermore, we call an interpretation $\mathcal{I}$ *finitely representable* if its domain $\Delta^{\mathcal{I}}$ and its active signature $\Sigma^{\mathcal{I}}$ are both finite.

2.3. Complexity

Reasoning in the description logic $\mathcal{EL}^{\bot}$ is *tractable*. More specifically, the *subsumption problem*, which is defined as follows, is decidable in deterministic polynomial time, cf. (Baader, Brandt and Lutz, 2005; Baader, Lutz and Brandt, 2008).

Instance: Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^{\bot}$ TBox.
Question: Is C subsumed by D w.r.t. $\mathcal{T}$?

Since the satisfiability problem in propositional Horn logic is **P**-complete and can be reduced to the subsumption problem for $\mathcal{EL}^{\bot}$, we conclude that the latter is **P**-complete as well.

2.4. Reduced Forms

It is not hard to find $\mathcal{EL}^{\bot}$ concept descriptions which are equivalent w.r.t. $\emptyset$, i.e., have the same extension in *all* interpretations, but are not equal. For instance, consider $C \coloneqq A \sqcap \bot$ and $D \coloneqq \exists\, r.\, \bot$; both concepts must always have an empty extension, and hence both are equivalent to $\bot$. It is therefore helpful for technical details to have a unique *reduced form* for $\mathcal{EL}^{\bot}$ concept descriptions. Let C be an $\mathcal{EL}^{\bot}$ concept description, then its *reduced form* is obtained by exhaustive application of the following *reduction rules* to the subconcepts of C.

$$\begin{aligned} \exists\, r.\, \bot &\mapsto \bot \\ D \sqcap E &\mapsto D \quad \text{if } D \sqsubseteq_\emptyset E \end{aligned}$$

From the definition of $\mathcal{EL}^\bot$ concept descriptions it is immediately clear that each normalized $\mathcal{EL}^\bot$ concept description C is either the bottom concept description $\bot$, or it is an $\mathcal{EL}$ concept description and as such essentially a conjunction of other $\mathcal{EL}$ concept descriptions which are no conjunctions, i.e., C has the form

$$C = \bigsqcap \mathsf{Conj}(C)$$

where $\mathsf{Conj}(C)$ is defined as the set of all top-level conjuncts in C.

Example. *The concept description* $A \sqcap \exists\, r.\, A \sqcap \exists\, s.\, (B \sqcap C) \sqcap \exists\, r.\, (A \sqcap C) \sqcap A$ *has the reduced form* $A \sqcap \exists\, s.\, (B \sqcap C) \sqcap \exists\, r.\, (A \sqcap C)$.

2.5. The Lattice of Concept Descriptions

It is readily verified that, for each TBox $\mathcal{T}$, the *subsumption relation* $\sqsubseteq_\mathcal{T}$ constitutes a quasi-order—a reflexive, transitive binary relation—on the set $\mathcal{EL}^\bot(\Sigma)$ of all $\mathcal{EL}^\bot$ concept descriptions over the signature Σ. Hence, the quotient of $\mathcal{EL}^\bot(\Sigma)$ with respect to the induced *equivalence relation* $\equiv_\mathcal{T}$ is a partially ordered set (abbrv. poset). In the following we will not distinguish between the equivalence classes and their representatives, and we consider only the case $\mathcal{T} = \emptyset$. Furthermore, $\bot$ is the smallest element, $\top$ is the greatest element, and the quotient set $\mathcal{EL}^\bot(\Sigma)/{\equiv_\emptyset}$ is also a lattice. It is easy to verify that the conjunction $\sqcap$ corresponds to the finitary *infimum* operation. In a description logic allowing for disjunction $\sqcup$, it dually holds true that the disjunction $\sqcup$ corresponds to the finitary *supremum* operation. Unfortunately, our considered description logic $\mathcal{EL}^\bot$ does not possess the constructor $\sqcup$. As an obvious solution, we can simply define the lattice-theoretic notion of a *supremum* specifically tailored to the case of $\mathcal{EL}^\bot$ concept descriptions as follows. The *supremum* or *least common subsumer* (abbrv. LCS) of two $\mathcal{EL}^\bot$ concept descriptions C and D is a concept description E such that

1. $C \sqsubseteq_\emptyset E$ as well as $D \sqsubseteq_\emptyset E$, and
2. for each $\mathcal{EL}^\bot$ concept description F, if $C \sqsubseteq_\emptyset F$ and $D \sqsubseteq_\emptyset F$, then $E \sqsubseteq_\emptyset F$.

Since all least common subsumers of C and D are unique up to equivalence, we may denote a representative of the corresponding equivalence class by $C \vee D$. It can also be proven that least common subsumers always exist in $\mathcal{EL}^\bot$; in particular, the least common subsumer $C \vee D$ can be computed, modulo equivalence, by means of the following recursive formula.

$$\begin{aligned} C \vee D = {} & \bigsqcap (\Sigma_\mathsf{C} \cap \mathsf{Conj}(C) \cap \mathsf{Conj}(D)) \\ & \sqcap \bigsqcap \{\, \exists\, r.\, (E \vee F) \mid r \in \Sigma_\mathsf{R},\ \exists\, r.\, E \in \mathsf{Conj}(C), \text{ and } \exists\, r.\, F \in \mathsf{Conj}(D) \,\} \end{aligned}$$

Example. *For the concept descriptions* $A \sqcap B \sqcap \exists\, r.\, (A \sqcap B) \sqcap \exists\, s.\, C$ *and* $B \sqcap C \sqcap \exists\, r.\, A \sqcap \exists\, r.\, (B \sqcap C)$*, the least common subsumer evaluates to* $B \sqcap \exists\, r.\, A \sqcap \exists\, r.\, B$.

Of course, the definition of a least common subsumer can be extended to an arbitrary number of arguments in the obvious way, and we shall then denote the least common subsumer of a set $\mathcal{C}$ of concept descriptions by $\bigvee \mathcal{C}$.

It is easy to see that the equivalence $\equiv_\emptyset$ is compatible with both $\sqcap$ and $\vee$. In the sequel of this document, we shall denote this bounded lattice by $\boldsymbol{\mathcal{EL}^\bot}(\Sigma) := (\mathcal{EL}^\bot(\Sigma), \sqsubseteq_\emptyset)/{\equiv_\emptyset}$, and accordingly $\boldsymbol{\mathcal{EL}^\bot_d}(\Sigma) := (\mathcal{EL}^\bot_d(\Sigma), \sqsubseteq_\emptyset)/{\equiv_\emptyset}$ symbolizes the corresponding bounded lattice of (equivalence classes of) $\mathcal{EL}^\bot$ concept descriptions. Note that $\boldsymbol{\mathcal{EL}^\bot_d}(\Sigma)$ is complete if the underlying signature Σ is finite.

2.6. Simulations and Canonical Models

The semantics of $\mathcal{EL}$ and of its fixed-point extensions, some of which are described in the next section, can be characterized by means of so called simulations. A short overview is given as follows.

A *pointed interpretation* is a pair $(\mathcal{I}, \delta)$ consisting of an interpretation $\mathcal{I}$ and an element $\delta \in \Delta^\mathcal{I}$. Now let $(\mathcal{I}, \delta)$ and $(\mathcal{J}, \epsilon)$ be two pointed interpretations, and assume that $\Gamma \subseteq \Sigma$. A Γ-*simulation* from $(\mathcal{I}, \delta)$ to $(\mathcal{J}, \epsilon)$ is a relation $\mathfrak{S} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ that satisfies $(\delta, \epsilon) \in \mathfrak{S}$ as well as the following conditions for all pairs $(\zeta, \eta) \in \mathfrak{S}$.

1. For all concept names $A \in \Gamma_\mathsf{C}$, if $\zeta \in A^\mathcal{I}$, then $\eta \in A^\mathcal{J}$.
2. For all role names $r \in \Gamma_\mathsf{R}$, if there is an element $\theta \in \Delta^\mathcal{I}$ such that $(\zeta, \theta) \in r^\mathcal{I}$, then there is an element $\iota \in \Delta^\mathcal{J}$ such that $(\eta, \iota) \in r^\mathcal{J}$ and $(\theta, \iota) \in \mathfrak{S}$.

We then also write $\mathfrak{S} \colon (\mathcal{I}, \delta) \precsim_\Gamma (\mathcal{J}, \epsilon)$, and to express the mere existence of a Γ-simulation from $(\mathcal{I}, \delta)$ to $(\mathcal{J}, \epsilon)$ we may write $(\mathcal{I}, \delta) \precsim_\Gamma (\mathcal{J}, \epsilon)$. Furthermore, if $\Gamma = \Sigma$, then we speak of *simulations* instead of Γ-simulations, and we leave out the subscript Γ, i.e., we use the symbol $\precsim$ instead of $\precsim_\Gamma$. Two pointed simulations $(\mathcal{I}, \delta)$ and $(\mathcal{J}, \epsilon)$ are *equi-similar* if there is a simulation in each direction.

Assume that $(\mathcal{I}, \delta) \precsim (\mathcal{J}, \epsilon)$. It is easily verified that for all $\mathcal{EL}^\bot$ concept description C, it holds true that $\delta \in C^\mathcal{I}$ only if $\epsilon \in C^\mathcal{J}$. Further important notions and statements related to simulations are cited from Lutz and Wolter (2010) in the following.

(Lutz and Wolter, 2010, Definition 11). *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, and C be an $\mathcal{EL}^\bot$ concept description. The* canonical model $\mathcal{I}_{C,\mathcal{T}}$ *of $\mathcal{T}$ and C consists of the following components.*

$$\Delta^{\mathcal{I}_{C,\mathcal{T}}} := \{C\} \cup \{\, D \mid \exists\, r \in \Sigma_\mathsf{R} \colon \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(C) \,\}$$

$$\cdot^{\mathcal{I}_{C,\mathcal{T}}} \colon \begin{cases} A \mapsto \{\, D \mid D \sqsubseteq_\mathcal{T} A \,\} & \text{for any } A \in \Sigma_\mathsf{C} \\ r \mapsto \left\{ (D, E) \;\middle|\; \begin{array}{l} D \sqsubseteq_\mathcal{T} \exists\, r.\, E \text{ and } \exists\, r.\, E \in \mathsf{Sub}(\mathcal{T}), \\ \text{or } \exists\, r.\, E \in \mathsf{Conj}(D) \end{array} \right\} & \text{for any } r \in \Sigma_\mathsf{R} \end{cases}$$

(Lutz and Wolter, 2010, Lemma 12). *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, and C be an $\mathcal{EL}^\bot$ concept description. Then, the following statements hold true.*

1. $D \in D^{\mathcal{I}_{C,\mathcal{T}}}$ *for all* $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$
2. $\mathcal{I}_{C,\mathcal{T}} \models \mathcal{T}$
3. $(\mathcal{I}_{C,\mathcal{T}}, E) \precsim (\mathcal{I}_{D,\mathcal{T}}, E)$ *for all* $D \in \mathcal{EL}^\bot(\Sigma)$ *and all* $E \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \cap \Delta^{\mathcal{I}_{D,\mathcal{T}}}$

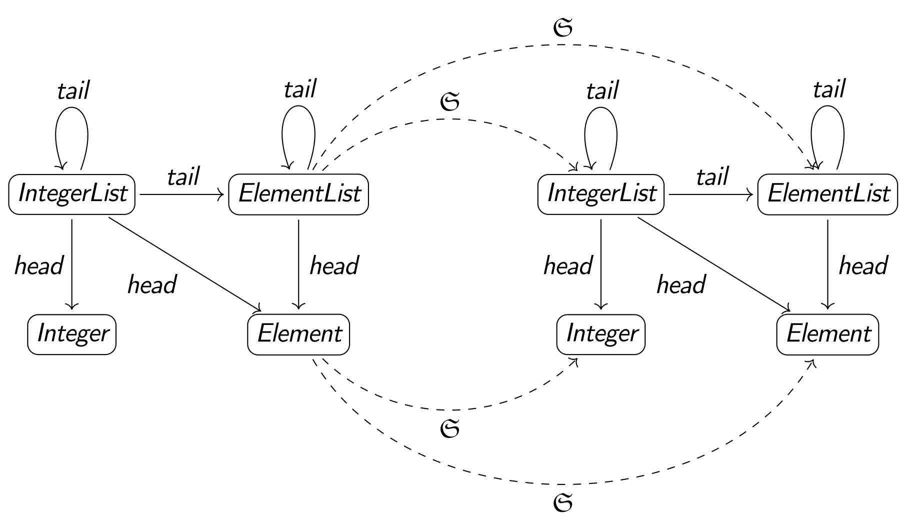
(Lutz and Wolter, 2010, Lemma 13). *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox, and C be an $\mathcal{EL}^{\bot}$ concept description.*

1. *For all models $\mathcal{I}$ of $\mathcal{T}$ and all objects $\delta \in \Delta^{\mathcal{I}}$, the following statements are equivalent.*
   (a) $\delta \in C^{\mathcal{I}}$
   (b) $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$
2. *For all $\mathcal{EL}^{\bot}$ concept descriptions D, the following statements are equivalent.*
   (a) $\mathcal{T} \models C \sqsubseteq D$
   (b) $C \in D^{\mathcal{I}_{C,\mathcal{T}}}$
   (c) $(\mathcal{I}_{D,\mathcal{T}}, D) \precsim (\mathcal{I}_{C,\mathcal{T}}, C)$

Example. *As an example, consider the concept descriptions* $\mathsf{IntegerList}$ *and* $\mathsf{ElementList}$ *as well as the following terminological box $\mathcal{T}$ over the signature Σ*

$$\Sigma_{\mathsf{C}} \coloneqq \{\mathsf{ElementList}, \mathsf{IntegerList}, \mathsf{Element}, \mathsf{Integer}\}$$
$$\Sigma_{\mathsf{R}} \coloneqq \{\mathsf{head}, \mathsf{tail}\}$$
$$\mathcal{T} \coloneqq \left\{ \begin{aligned} \mathsf{ElementList} &\equiv \exists\, \mathsf{head}.\, \mathsf{Element} \sqcap \exists\, \mathsf{tail}.\, \mathsf{ElementList}, \\ \mathsf{IntegerList} &\equiv \exists\, \mathsf{head}.\, \mathsf{Integer} \sqcap \exists\, \mathsf{tail}.\, \mathsf{IntegerList}, \\ \mathsf{Integer} &\sqsubseteq \mathsf{Element} \end{aligned} \right\}$$

It is easy to see that $\mathsf{IntegerList} \sqsubseteq_{\mathcal{T}} \mathsf{ElementList}$ *holds true. Depicted below is an according simulation $\mathfrak{S}$ from the canonical model* $\mathcal{I}_{\mathsf{ElementList},\mathcal{T}}$ *to the canonical model* $\mathcal{I}_{\mathsf{IntegerList},\mathcal{T}}$ *that contains* $(\mathsf{ElementList}, \mathsf{IntegerList})$.



Note that, in order to ease readability, we have not included node labels; to be complete, we list these as follows. Each node C is labeled with C itself, and further $\mathsf{IntegerList}$ *has label* $\mathsf{ElementList}$ *and* $\mathsf{Integer}$ *has label* $\mathsf{Element}$.

It is easy to see that $\precsim$ is a partial ordering relation. Furthermore, infima w.r.t. $\precsim$ always exist and can be characterized by products. The *product* of interpretations $\mathcal{I}$ and $\mathcal{J}$ over the same signature Σ is defined as the interpretation $\mathcal{I} \times \mathcal{J}$ consisting of the following components.

$$\Delta^{\mathcal{I}\times\mathcal{J}} \coloneqq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$$

$$\cdot^{\mathcal{I}\times\mathcal{J}} \colon \begin{cases} A \mapsto \{\, (\delta,\zeta) \mid \delta \in A^{\mathcal{I}} \text{ and } \zeta \in A^{\mathcal{J}} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, ((\delta,\zeta),(\epsilon,\eta)) \mid (\delta,\epsilon) \in r^{\mathcal{I}} \text{ and } (\zeta,\eta) \in r^{\mathcal{J}} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases}$$

Given two pointed interpretations $(\mathcal{I},\delta)$ and $(\mathcal{J},\epsilon)$, their *product* $(\mathcal{I},\delta) \times (\mathcal{J},\epsilon)$ is defined as the pointed interpretation $(\mathcal{I}\times\mathcal{J},(\delta,\epsilon))$. Now, Lutz, Piro and Wolter (2010a, Observation 3) have found that the product operation $\times$ is the infimum operation in the set of (equivalence classes of) pointed interpretations ordered by $\precsim$. It is immediate to extend the notion of a product to an arbitrary number of (pointed) interpretations used as factors, and we shall denote the product of a set $\mathfrak{I}$ of (pointed) interpretations as $\bigtimes \mathfrak{I}$.

2.7. Greatest Fixed-Point Semantics

We cite two description logics introduced by Lutz, Piro and Wolter (2010a) that are extensions of $\mathcal{EL}$ with greatest fixed-point semantics. According to (Lutz, Piro and Wolter, 2010a, Theorem 10) there are polynomial time translations between both, and furthermore reasoning in these extensions remains **P**-complete, cf. (Lutz, Piro and Wolter, 2010a, Theorem 12).

The description logic $\mathcal{EL}_{\mathsf{si}}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$ where $(\mathcal{I},\delta)$ is a pointed interpretation such that $\mathcal{I}$ is finitely representable. The semantics of the additional concept constructor is defined as follows: for each interpretation $\mathcal{J}$ and any object $\epsilon \in \Delta^{\mathcal{J}}$, it holds true that $\epsilon \in (\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}}$ if $(\mathcal{I},\delta) \precsim (\mathcal{J},\epsilon)$. As shown in (Lutz, Piro and Wolter, 2010a, Lemma 7), every $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to a concept description of the form $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$, and furthermore, such an equivalent concept description can be constructed in linear time. Adding the bottom concept description $\bot$ yields the description logic $\mathcal{EL}^{\bot}_{\mathsf{si}}$.

Furthermore, Lutz, Piro and Wolter (2010b, Definition 28) define the *nth characteristic concept description* $\mathsf{X}^n(\mathcal{I},\delta)$ of a pointed interpretation $(\mathcal{I},\delta)$ that has a finite active signature recursively as follows.

$$\mathsf{X}^0(\mathcal{I},\delta) \coloneqq \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \delta \in A^{\mathcal{I}} \,\}$$

$$\mathsf{X}^{n+1}(\mathcal{I},\delta) \coloneqq \mathsf{X}^0(\mathcal{I},\delta) \sqcap \bigsqcap\{\, \exists\, r.\, \mathsf{X}^n(\mathcal{I},\epsilon) \mid r \in \Sigma_{\mathsf{R}} \text{ and } (\delta,\epsilon) \in r^{\mathcal{I}} \,\}$$

For any finitely representable pointed interpretation $(\mathcal{I},\delta)$, the sequence $(\, \mathsf{X}^n(\mathcal{I},\delta) \mid n \in \mathbb{N}\,)$ converges to $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$, that is, it holds true that

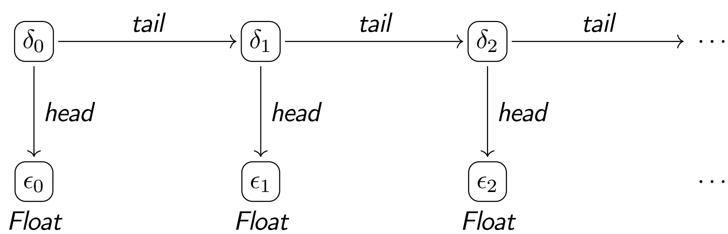
$$(\exists^{\mathsf{sim}}(\mathcal{I},\delta))^{\mathcal{J}} = \bigcap\{\, (\mathsf{X}^n(\mathcal{I},\delta))^{\mathcal{J}} \mid n \in \mathbb{N}\,\}$$

for every interpretation $\mathcal{J}$, and so we also call $\mathsf{X}^n(\mathcal{I},\delta)$ the *nth approximation* of $\exists^{\mathsf{sim}}(\mathcal{I},\delta)$. In general, we shall denote the nth approximation of an $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description C as $C\restriction_n$ where we additionally need to define that $\bot\restriction_n \coloneqq \bot$ for each $n \in \mathbb{N}$. Clearly, if C is an $\mathcal{EL}^{\bot}$ concept description with role depth d, then $C \equiv_{\emptyset} C\restriction_n$ holds true for each $n \geq d$. Alternatively, we may call an nth approximation $C\restriction_n$ also a *restriction* of C to a role depth of n.

Example. *Consider the interpretation $\mathcal{I}$ depicted below.*

The approximations of the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ are then the concept descriptions $\mathsf{X}^n(\mathcal{I}, \delta) = \exists\, r^n.\, \top$.

Example. *As another example, consider the interpretation $\mathcal{I}_{\mathit{List}}$ shown below.*



The finite approximations of $\exists^{\mathsf{sim}}(\mathcal{I}_{\mathit{List}}, \delta_0)$ are the following concept descriptions.

$$\mathsf{X}^0(\mathcal{I}_{\mathit{List}}, \delta_0) = \top$$
$$\mathsf{X}^1(\mathcal{I}_{\mathit{List}}, \delta_0) = \exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, \top$$
$$\mathsf{X}^2(\mathcal{I}_{\mathit{List}}, \delta_0) = \exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, (\exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, \top)$$
$$\mathsf{X}^3(\mathcal{I}_{\mathit{List}}, \delta_0) = \exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, (\exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, (\exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, \top))$$
$$\vdots$$
$$\mathsf{X}^{n+1}(\mathcal{I}_{\mathit{List}}, \delta_k) = \exists\, \mathit{head}.\, \mathit{Float} \sqcap \exists\, \mathit{tail}.\, \mathsf{X}^n(\mathcal{I}_{\mathit{List}}, \delta_{k+1})$$

The description logic $\mathcal{EL}_{\mathsf{st}}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{\mathsf{sim}}\, \Gamma.\, (\mathcal{T}, C)$, where $\Gamma \subseteq \Sigma$ is a finite signature, $\mathcal{T}$ is a TBox, and C is a concept description. More specifically, $\mathcal{EL}_{\mathsf{st}}$ concept descriptions, $\mathcal{EL}_{\mathsf{st}}$ concept inclusions, and $\mathcal{EL}_{\mathsf{st}}$ TBoxes are defined by simultaneous induction as follows.

1. Every $\mathcal{EL}$ concept description, $\mathcal{EL}$ concept inclusion, and $\mathcal{EL}$ TBox, is an $\mathcal{EL}_{\mathsf{st}}$ concept description, $\mathcal{EL}_{\mathsf{st}}$ concept inclusion, and $\mathcal{EL}_{\mathsf{st}}$ TBox, respectively;
2. if $\mathcal{T}$ is an $\mathcal{EL}_{\mathsf{st}}$ TBox, C an $\mathcal{EL}_{\mathsf{st}}$ concept description, and $\Gamma \subseteq \Sigma$ a finite signature, then $\exists^{\mathsf{sim}}\, \Gamma.\, (\mathcal{T}, C)$ is an $\mathcal{EL}_{\mathsf{st}}$ concept description;
3. if C and D are $\mathcal{EL}_{\mathsf{st}}$ concept descriptions, then $C \sqsubseteq D$ is an $\mathcal{EL}_{\mathsf{st}}$ concept inclusion;
4. an $\mathcal{EL}_{\mathsf{st}}$ TBox is a finite set of $\mathcal{EL}_{\mathsf{st}}$ concept inclusions.

The semantics of the additional concept constructor is defined as follows: let $\mathcal{I}$ be an interpretation, then $\delta \in (\exists^{\mathsf{sim}}\, \Gamma.\, (\mathcal{T}, C))^{\mathcal{I}}$ if there exists a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $\mathcal{J}$ is a model of $\mathcal{T}$, $\epsilon \in C^{\mathcal{J}}$, and $(\mathcal{J}, \epsilon) \precsim_{\Sigma \setminus \Gamma} (\mathcal{I}, \delta)$. In case $\Gamma = \emptyset$ we may abbreviate $\exists^{\mathsf{sim}}\, \Gamma.\, (\mathcal{T}, C)$ as $\exists^{\mathsf{sim}}(\mathcal{T}, C)$. Adding the bottom concept description $\bot$ yields the description logic $\mathcal{EL}^{\bot}_{\mathsf{st}}$.

3. Closure Operators in Lattices

In the following text we assume that $\mathbf{M} \coloneqq (M, \leq, \bigwedge, \bigvee, \bot, \top)$ is a complete lattice, i.e., $\leq$ is a partial order relation on M such that, for any subset $X \subseteq M$, the *infimum* $\bigwedge X$ as well as the *supremum* $\bigvee X$ exists, and further it holds true that $\bot$ is the smallest element and $\top$ is the greatest element, cf. (Birkhoff, 1940; Davey and Priestley, 2002; Ganter and Wille, 1999a; Grätzer, 2002). At first we define some basic terms and notions that are used in later sections. We start with the definition of closure operators in $\mathbf{M}$, give some equivalent characterizations, and present the lattice of closure operators in $\mathbf{M}$.

A *closure operator* in $\mathbf{M}$ is a mapping $\phi\colon M \to M$ that satisfies the following properties for all elements $x, y \in M$. Instead of $\phi(x)$ we shall write x^ϕ.

1. $x \leq x^\phi$ (extensive)
2. $x \leq y$ implies $x^\phi \leq y^\phi$ (monotonic)
3. $x^{\phi\phi} = x^\phi$ (idempotent)

An element $x \in M$ that satisfies $x = x^\phi$ is called *closed* with respect to ϕ or, equivalently, a *closure* of ϕ, and $\mathsf{Clo}(\phi)$ denotes the set of all closures of ϕ. The set of all closure operators in $\mathbf{M}$ is denoted by $\mathsf{ClOp}(\mathbf{M})$. Further information on closure operators can be found in (Caspard and Monjardet, 2003; Davey and Priestley, 2002; Higuchi, 1998), and we shall cite some important results in the sequel of this section. For any mapping $\phi\colon M \to M$, the following statements are equivalent.

1. ϕ is a closure operator on $\mathbf{M}$, i.e., $\phi \in \mathsf{ClOp}(\mathbf{M})$
2. $x \leq y^\phi$ if, and only if, $x^\phi \leq y^\phi$ for all $x, y \in M$
3. $x \vee y^{\phi\phi} \leq (x \vee y)^\phi$ for all $x, y \in M$
4. $x \leq x^\phi$ and $(x \vee y)^\phi = (x^\phi \vee y^\phi)^\phi$ for all $x, y \in M$

For every closure operator ϕ in $\mathbf{M}$, the following statements hold true.

1. $(x \wedge y)^\phi \leq x^\phi \wedge y^\phi$ for all $x, y \in M$
2. $(x^\phi \wedge y^\phi)^\phi = x^\phi \wedge y^\phi$ for all $x, y \in M$

A *closure system* in $\mathbf{M}$ is a $\bigwedge$-closed subset $P \subseteq M$, i.e., it holds true that $\bigwedge X \in P$ for each subset $X \subseteq P$. Note that the empty infimum $\bigwedge \emptyset$ in $\mathbf{M}$ equals $\top$, i.e., each closure system in $\mathbf{M}$ contains $\top$. A subset $P \subseteq M$ is a closure system in $\mathbf{M}$ if, and only if, $\{\, p \in P \mid x \leq p \,\}$ has a smallest element for all $x \in M$. There exists a one-to-one-correspondence between closure operators and closure systems as follows. For every closure operator ϕ in $\mathbf{M}$, the set $\mathsf{Clo}(\phi)$ is a closure system in $\mathbf{M}$. For every closure system P in $\mathbf{M}$, the mapping

$$\phi_P\colon x \mapsto \bigwedge\{\, p \mid p \in P \text{ and } x \leq p \,\}$$

is a closure operator in $\mathbf{M}$. Both operations are mutually inverse, i.e., $\phi_{\mathsf{Clo}(\phi)} = \phi$ for all closure operators ϕ, and $\mathsf{Clo}(\phi_P) = P$ for all closure systems P.

Indeed, closure operators can be ordered, cf. (Higuchi, 1998; Rudolph, 2014). For closure operators ϕ and ψ in $\mathbf{M}$, we call ϕ *finer* than ψ and, dually, we call ψ *coarser* than ϕ, denoted as $\phi \trianglelefteq \psi$, if all ψ-closures are also ϕ-closed, that is, if $\mathsf{Clo}(\psi) \subseteq \mathsf{Clo}(\phi)$ holds true. It can be shown that the statements $\phi \trianglelefteq \psi$, $\phi \circ \psi = \psi$, and $\phi \leq \psi$ (pointwise order) are equivalent. As it turns out, the set of all closure operators in $\mathbf{M}$ ordered by $\trianglelefteq$ constitutes a complete lattice

$$\mathbf{ClOp}(\mathbf{M}) := (\mathsf{ClOp}(\mathbf{M}), \trianglelefteq, \bigtriangleup, \bigtriangledown, \underline{\perp}, \overline{\top}).$$

In particular, every set Φ of closure operators in $\mathbf{M}$ has an infimum $\bigtriangleup\Phi$ as well as a supremum $\bigtriangledown\Phi$ and these are given as follows.

$$\bigtriangleup\Phi \colon x \mapsto \bigwedge\{\, x^{\phi} \mid \phi \in \Phi \,\}$$
$$\bigtriangledown\Phi \colon x \mapsto \bigwedge\{\, y \mid x \leq y \text{ and } y = y^{\phi} \text{ for each } \phi \in \Phi \,\}$$

The finest closure operator is the identity mapping $\underline{\perp} \colon x \mapsto x$, and the coarsest closure operator is the constant mapping $\overline{\top} \colon x \mapsto \top$.

An *implication* in $\mathbf{M}$ is an expression $p \to c$ where both the *premise* p as well as the *conclusion* c are elements of M. A *model* of $p \to c$ is an element $m \in M$ such that $p \leq m$ implies $c \leq m$. Then, $p \to c$ is *valid* for a closure operator ϕ in $\mathbf{M}$, written $\phi \models p \to c$ or $p \to_{\phi} c$, if each closure of ϕ is a model of $p \to c$. It is a finger exercise to show that $p \to c$ is valid in ϕ if, and only if, $c \leq p^{\phi}$ holds true, cf. Kriegel (2016b, Section 3). An implication set $\mathcal{L}$ *entails* an implication $p \to c$, denoted as $\mathcal{L} \models p \to c$ or $p \to_{\mathcal{L}} c$, if every model of all implications in $\mathcal{L}$ is also a model of $p \to c$.

Example. *Consider the lattice $\mathbf{M}$ with base set $M := \mathbb{R}^3$ and pointwise ordering $\leq$. Of course, the implication $(5, -9, 37) \to (2, -100, -54)$ is a tautology, since it is valid in any closure operator in $\mathbf{M}$. We now define the closure operator*

$$\phi \colon (x, y, z) \mapsto (x \vee y \vee z,\ x + y + z,\ x \cdot y \cdot z)$$

on $\mathbf{M}$. For instance, the implication $(1, 2, 3) \to (\mathrm{e}, 3, -\pi)$ is valid for ϕ, since $(\mathrm{e}, 3, -\pi) \leq (3, 6, 6)$ holds true. Further consider the implication set $\mathcal{L}$ that is defined as follows.

$$\mathcal{L} := \left\{ \begin{array}{c} (1,2,3) \to (1 \cdot \pi, 2 \cdot \pi, 3 \cdot \pi), \\ (3,6,8) \to (42, 0, 23), \\ (10, 0, 20) \to (100, 200, 400) \end{array} \right\}$$

It is easy to see that the models of $\mathcal{L}$ are the following.

$$(x, y, z) \text{ where } x < 1 \text{ or } y < 2 \text{ or } z < 3$$
$$(x, y, z) \text{ where } x \geq 100 \text{ and } y \geq 200 \text{ and } z \geq 400$$

Thus, $\mathcal{L}$ entails the implication $(2, 2, 3) \to (99, 199, 299)$.

An *implication base* for a closure operator ϕ is an implication set $\mathcal{L}$ that is *sound* for ϕ, that is, each implication in $\mathcal{L}$ is valid for ϕ, and further is *complete* for ϕ, that is, any implication that is valid for ϕ is also entailed by $\mathcal{L}$. A *pseudo-closure* of a closure operator ϕ is an element $p \in M$ that is no closure of ϕ, but satisfies that $q^{\phi} \leq p$ holds true for each pseudo-closure q of ϕ with $q \lneq p$. Then, the following implication set $\mathsf{Can}(\phi)$, called the *canonical base* of ϕ, is an implication base for ϕ.

$$\mathsf{Can}(\phi) \coloneqq \{\, p \to p^{\phi} \mid p \text{ is a pseudo-closure of } \phi \,\}$$

It can be shown that $\mathsf{Can}(\phi)$ is a *minimal* implication base for ϕ, that is, there does not exist any implication base with fewer implications, cf. Kriegel (2016b, Section 3).

Furthermore, an implication is simultaneously valid for two closure operators if, and only if, it is valid for their infimum, cf. (Kriegel, 2016b, Section 3.1). Later in this document, we are going to consider closure operators in $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{st}}(\Sigma)$ and in $\boldsymbol{\mathcal{EL}}^{\bot}(\Sigma)$. Of course, the concept inclusions are exactly the implications in these two lattices, and so it makes sense to also allow for the denotations $\phi \models C \sqsubseteq D$ and $C \sqsubseteq_{\phi} D$ to express that $C \sqsubseteq D$ is valid in ϕ, where ϕ is a closure operator in (the dual of) $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{st}}(\Sigma)$ or in $\boldsymbol{\mathcal{EL}}^{\bot}(\Sigma)$, and where $C \sqsubseteq D$ is an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusion or an $\mathcal{EL}^{\bot}$ concept inclusion.

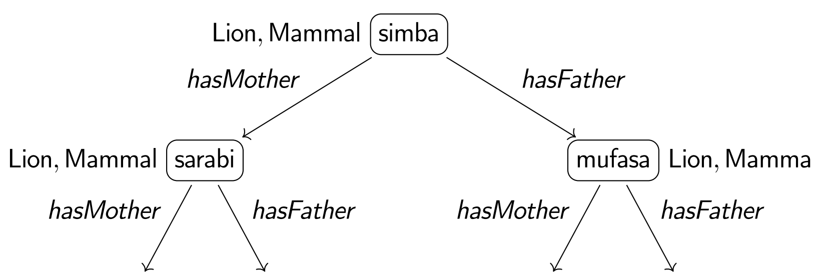
4. Related Work

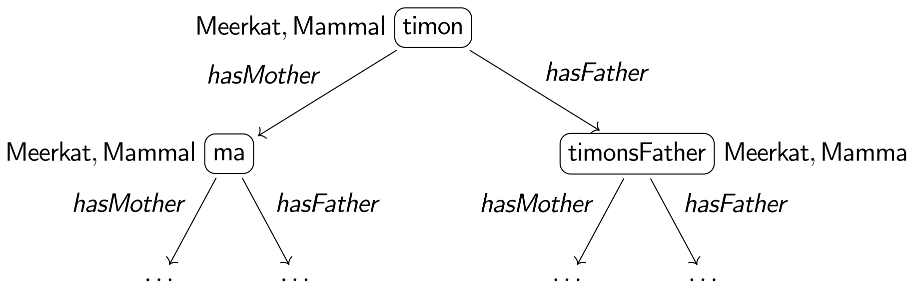
So far, several approaches for axiomatizing concept inclusions in different description logics have been developed, and many of these utilize sophisticated techniques from Formal Concept Analysis (Ganter and Obiedkov, 2016; Ganter and Wille, 1999b): on the one hand, there is the so-called *canonical base*, cf. Guigues and Duquenne (1986), that provides a concise representation of the implicative theory of a formal context in a sound and complete manner and, on the other hand, the interactive algorithm *attribute exploration* exists, which guides an expert through the process of axiomatizing the theory of implications that are valid in a domain of interest, cf. Ganter (1984). In particular, attribute exploration is an interactive variant of an algorithm for computing canonical bases (Ganter, 1984), and it works as follows: the input is a formal context that only partially describes the domain of interest (that is, there may be implications that are not valid, but for which this partial description does not provide a counterexample), and during the run of the exploration process a minimal number of questions is enumerated and posed to the expert (such a question is an implication for which no counterexample has been explored, and the expert can either confirm its validity or provide a suitable counterexample). On termination, a minimal sound and complete representation of the theory of implications that are valid in the considered domain has been generated.

A first pioneering work on axiomatizing concept inclusions in the description logic $\mathcal{FLE}$ has been developed by Rudolph (2006), which allows for the exploration of a concept inclusion base for a given interpretation in a multistep approach such that each step increases the role depth of concept descriptions occurring in the concept inclusions. Later, a refined approach has been developed by Baader and Distel (2008); Distel (2011) for axiomatizing concept inclusion bases in the description logic $\mathcal{EL}^{\bot}$. They found techniques for computing and for exploring such bases that contain a *minimal* number of concept inclusions and that are both sound and complete not only for those valid concept inclusions up to certain role depth but instead for *all* valid ones. However, due to possible presence of cycles in the input interpretation they need to apply greatest fixed-point semantics; luckily, there is a finite *closure ordinal* for any finitely representable interpretation, that is, there is a certain role depth up to which the concept descriptions in the base can be unraveled to obtain a base for *all* valid concept inclusions with respect to the standard semantics. Borchmann, Distel and Kriegel (2016) devised a variant of these techniques that circumvents the use of greatest fixed-point semantics, but which can only compute *minimal* concept inclusion bases that are sound and complete for all concept inclusions up to a set role depth—of course, if one chooses the closure ordinal as role-depth bound, then also these bases are sound and complete for *all* valid concept inclusions w.r.t. standard semantics. Further variants that allow for the incorporation of background knowledge or allow for a more expressive description logic can be found in (Kriegel, 2015, 2016a, 2017).

Since we shall later expand on the aforementioned results for axiomatizing $\mathcal{EL}^\bot$ concept inclusions valid in an interpretation, we briefly introduce these as follows. A *concept inclusion base* for an interpretation $\mathcal{I}$ is a TBox $\mathcal{T}$ such that, for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{I} \models C \sqsubseteq D$ if, and only if, $\mathcal{T} \models C \sqsubseteq D$. For each finite interpretation $\mathcal{I}$ with finite active signature, there is a *canonical base* $\mathsf{Can}(\mathcal{I})$ with respect to greatest fixed-point semantics, which contains a minimal number of concept inclusions among all concept inclusion bases for $\mathcal{I}$, cf. Distel (2011, Corollary 5.13 and Theorem 5.18), and similarly there is a minimal *canonical base* $\mathsf{Can}(\mathcal{I}, d)$ with respect to an upper bound $d \in \mathbb{N}$ on the role depths, cf. Borchmann, Distel and Kriegel (2016, Theorem 4.32). The construction of both canonical bases is built upon the notion of a *model-based most specific concept description* (abbrv. MMSC), which, for an interpretation $\mathcal{I}$ and some subset $\Xi \subseteq \Delta^\mathcal{I}$, is a concept description C such that $\Xi \subseteq C^\mathcal{I}$ and, for each concept description D, it holds true that $\Xi \subseteq D^\mathcal{I}$ implies $\emptyset \models C \sqsubseteq D$. These exist either if greatest fixed-point semantics is applied (in order to be able to express cycles present in $\mathcal{I}$) or if the role depth of C is bounded by some $d \in \mathbb{N}$, and these are then denoted as $\Xi^\mathcal{I}$ or $\Xi^{\mathcal{I}_d}$, respectively. These mappings $\cdot^\mathcal{I} \colon \wp(\Delta^\mathcal{I}) \to \mathcal{EL}^\bot(\Sigma)$ and $\cdot^{\mathcal{I}_d} \colon \wp(\Delta^\mathcal{I}) \to \mathcal{EL}^\bot_d(\Sigma)$ are the adjoints of the extension functions $\cdot^\mathcal{I} \colon \mathcal{EL}^\bot(\Sigma) \to \wp(\Delta^\mathcal{I})$ and $\cdot^\mathcal{I} \colon \mathcal{EL}^\bot_d(\Sigma) \to \wp(\Delta^\mathcal{I})$, respectively, and the respective pair of both constitutes a *Galois connection*, cf. Distel (2011, Lemma 4.1) and Borchmann, Distel and Kriegel (2016, Lemmas 4.3 and 4.4), respectively. As a consequence, we obtain that the mappings $\phi_\mathcal{I} \colon C \mapsto C^{\mathcal{I}\mathcal{I}}$ and $\phi_{\mathcal{I},d} \colon C \mapsto C^{\mathcal{I}\mathcal{I}_d}$ are closure operators. It is straight-forward to verify that, in the description logic $\mathcal{EL}^\bot_{\mathsf{si}}$, $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is the most specific concept description of $\{\delta\}$ in $\mathcal{I}$, and that $\exists^{\mathsf{sim}}(\bigtimes\{\, (\mathcal{I}, \xi) \mid \xi \in \Xi \,\})$ is the most specific concept description of Ξ in $\mathcal{I}$. Analogously in the description logic $\mathcal{EL}^\bot_d$, it holds true that $\mathsf{X}^d(\bigtimes\{\, (\mathcal{I}, \xi) \mid \xi \in \Xi \,\})$ is the most specific concept description of Ξ in $\mathcal{I}$.

Example. *Fix the following interpretation* $\mathcal{I}_{\mathsf{LionKing}}$ *that contains the objects* simba *and* timon *as well as corresponding ancestors.*

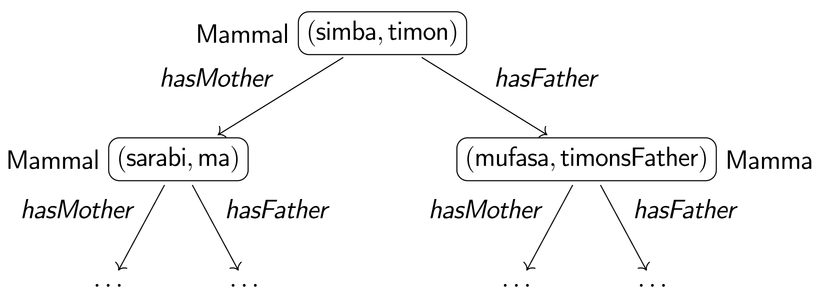




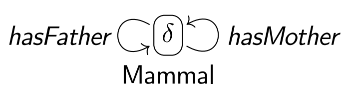
We now want to construct the model-based most specific concept description of $\{\mathsf{simba}, \mathsf{timon}\}$ *in* $\mathcal{I}_{\mathsf{LionKing}}$*, which is equivalent to*

$$\exists^{\mathsf{sim}}\left((\mathcal{I}_{\mathsf{LionKing}}, \mathsf{simba}) \times (\mathcal{I}_{\mathsf{LionKing}}, \mathsf{timon})\right).$$

The product $\mathcal{I}_{\mathsf{LionKing}} \times \mathcal{I}_{\mathsf{LionKing}}$ *is as follows where we only construct objects that are reachable from* $(\mathsf{simba}, \mathsf{timon})$*.*



It is not hard to verify that $(\mathcal{I}_{\mathsf{LionKing}} \times \mathcal{I}_{\mathsf{LionKing}}, (\mathsf{simba}, \mathsf{timon}))$ *is equi-similar to the following pointed interpretation* $(\mathcal{I}, \delta)$*.*



We conclude that $\{\mathsf{simba}, \mathsf{timon}\}^{\mathcal{I}_{\mathsf{LionKing}}}$ *is equivalent to* $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$*.*

Since the $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ concept inclusion base from Distel (2011, Corollary 5.13 and Theorem 5.18) can be unraveled up to a depth of $d_{\mathcal{I}} \coloneqq |\Delta^{\mathcal{I}}|^{|\Delta^{\mathcal{I}}|+1}$ such that adding some further concept inclusions with a role depth $d_{\mathcal{I}}+1$ yields an $\mathcal{EL}^{\bot}$ concept inclusion base for $\mathcal{I}$, cf. Distel (2011, Section 5.3), we conclude that, in order to get a TBox which is sound and complete for *all* valid concept inclusions of $\mathcal{I}$, we may also directly compute an $\mathcal{EL}^{\bot}_{d_{\mathcal{I}}+1}$ concept inclusion base for $\mathcal{I}$ by means of the techniques from Borchmann, Distel and Kriegel (2016). This is formulated in the next lemma in more detail.

Proposition 1. *Let* $\mathcal{I}$ *be a finitely representable interpretation, and define* $d_{\mathcal{I}} \coloneqq |\Delta^{\mathcal{I}}|^{|\Delta^{\mathcal{I}}|+1}$*. Then, each* $\mathcal{EL}^{\bot}_{d_{\mathcal{I}}+1}$ *concept inclusion base for* $\mathcal{I}$ *is also an* $\mathcal{EL}^{\bot}$ *concept inclusion base for* $\mathcal{I}$*.*

Proof. Let $\mathcal{B}$ be a base for the valid concept inclusions of $\mathcal{I}$ for which the role depth is at most $d_{\mathcal{I}} + 1$. By assumption, then $\mathcal{B}$ is sound for $\mathcal{I}$. For proving completeness, it suffices to show that $\mathcal{B}$ entails the base $\mathcal{B}_4$ of Distel (2011, Section 5.3). Since $\mathcal{B}_4$ only contains concept inclusions with a role depth not exceeding $d_{\mathcal{I}} + 1$, we may immediately conclude that $\mathcal{B} \models \mathcal{B}_4$ is indeed satisfied. □

As a variant of these two approaches, Kriegel (2015) presented a method for constructing canonical bases relative to an existing terminological box. If $\mathcal{I}$ is an interpretation and $\mathcal{B}$ is a terminological box such that $\mathcal{I} \models \mathcal{B}$, then a *concept inclusion base* for $\mathcal{I}$ *relative* to $\mathcal{B}$ is a terminological box $\mathcal{T}$ such that, for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{I} \models C \sqsubseteq D$ if, and only if, $\mathcal{T} \cup \mathcal{B} \models C \sqsubseteq D$. The corresponding *canonical base* is denoted as $\mathsf{Can}(\mathcal{I}, \mathcal{B})$, cf. Kriegel (2015, Theorem 1).

So far, the complexity of computing concept inclusion bases in the description logic $\mathcal{EL}^{\bot}$ has not been determined. Using simple arguments, one could only infer that the canonical base $\mathsf{Can}(\mathcal{I})$ can be computed in double exponential time with respect to the cardinality of the domain $\Delta^{\mathcal{I}}$. However, we give an answer to this open question in the following proposition. As it turns out, $\mathsf{Can}(\mathcal{I})$ can always be computed in (single) exponential time w.r.t. $|\Delta^{\mathcal{I}}|$, and further there exist interpretations $\mathcal{I}$ for which all concept inclusion bases must have sizes that are at least exponential w.r.t. $|\Delta^{\mathcal{I}}|$, that is, for which a concept inclusion base cannot be encoded in polynomial space.

Proposition 2. *For each finitely representable interpretation $\mathcal{I}$, its canonical base $\mathsf{Can}(\mathcal{I})$ can be computed in deterministic exponential time with respect to the cardinality of the domain $\Delta^{\mathcal{I}}$. Furthermore, there are finitely representable interpretations $\mathcal{I}$ for which a concept inclusion base cannot be encoded in polynomial space with respect to the cardinality of $\Delta^{\mathcal{I}}$.*

Proof. We start the proof with introducing a few notions from *Formal Concept Analysis* that are needed to understand this proof. A *formal context* $\mathbb{K}$ is a triple (G, M, I) where both G and M are sets and where $I \subseteq G \times M$ is a binary relation. The relation I induces mappings $\cdot^I \colon \wp(G) \to \wp(M)$ and $\cdot^I \colon \wp(M) \to \wp(G)$ by setting $A^I \coloneqq \{\, m \in M \mid (g, m) \in I \text{ for each } g \in A \,\}$ and dually $B^I \coloneqq \{\, g \in G \mid (g, m) \in I \text{ for each } m \in B \,\}$. The composition $\phi_{\mathbb{K}} \colon \wp(M) \to \wp(M)$ where $\phi_{\mathbb{K}}(B) \coloneqq B^{II}$ is a closure operator in M. Now an *intent* of $\mathbb{K}$ is a closure of $\phi_{\mathbb{K}}$ and a *pseudo-intent* of $\mathbb{K}$ is a pseudo-closure of $\phi_{\mathbb{K}}$. Furthermore, $\mathsf{Int}(\mathbb{K})$ denotes the set of all intents of $\mathbb{K}$, and $\mathsf{Can}(\mathbb{K})$ is the canonical base of $\phi_{\mathbb{K}}$.

We continue with citing an important result on sizes of canonical bases of formal contexts: Albano (2017, Theorem 3.2.1) has shown that, for any formal context $\mathbb{K} \coloneqq (G, M, I)$, it holds true that $|\mathsf{Can}(\mathbb{K})| \leq |M| \cdot |\mathsf{Int}(\mathbb{K})|$. Furthermore, Distel (2011, Theorem 5.18) has shown that, for each finitely representable interpretation $\mathcal{I}$, the canonical base $\mathsf{Can}(\mathcal{I})$ from Distel (2011, Corollary 5.13) contains a minimal number of concept inclusions among all concept inclusion bases for $\mathcal{I}$. The premises of the concept inclusions in $\mathsf{Can}(\mathcal{I})$ correspond to the pseudo-intents of the induced context $\mathbb{K}_{\mathcal{I}} \coloneqq (\Delta^{\mathcal{I}}, M_{\mathcal{I}}, I)$, cf. Distel (2011, Definitions 4.2 and 5.3), and, more specifically, for every pseudo-intent P of $\mathbb{K}_{\mathcal{I}}$, its conjunction $\bigsqcap P$ is such a premise of a concept inclusion in $\mathsf{Can}(\mathcal{I})$. It follows that the number of concept inclusions in $\mathsf{Can}(\mathcal{I})$ is bounded by the number of implications in $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$. Applying Albano's result yields that the number of concept inclusions in $\mathsf{Can}(\mathcal{I})$ cannot be greater than $|M_{\mathcal{I}}| \cdot |\mathsf{Int}(\mathbb{K}_{\mathcal{I}})|$. Analyzing the definition of the attribute set $M_{\mathcal{I}}$ shows that its number of elements is bounded by $1 + |\Sigma_{\mathsf{C}}| + |\Sigma_{\mathsf{R}}| \cdot (2^{|\Delta^{\mathcal{I}}|} - 1)$, that is, the cardinality of $M_{\mathcal{I}}$ is at most exponential in the cardinality of the domain $\Delta^{\mathcal{I}}$. Furthermore, for every formal context $\mathbb{K} \coloneqq (G, M, I)$, the number of intents of $\mathbb{K}$ is bounded by the minimum of $2^{|G|}$ and $2^{|M|}$. Consequently, the cardinality of $\mathsf{Int}(\mathbb{K}_{\mathcal{I}})$ cannot exceed $2^{|\Delta^{\mathcal{I}}|}$. Summing up, the cardinality of $\mathsf{Can}(\mathcal{I})$ is at most exponential in $|\Delta^{\mathcal{I}}|$.

According to Distel (2011, Section 4.1), the model-based most specific concept descriptions for $\mathcal{I}$ always have a representative the size of an (efficient) encoding of which is exponential in the size of the domain $\Delta^{\mathcal{I}}$. More specifically, Distel has shown that model-based most specific concept descriptions of singletons can be constructed by a traversal of the graph induced by the considered interpretation, and that these always have a linear size, while model-based most specific concept descriptions of arbitrary subsets of the interpretation's domain can be constructed as least common subsumers of such singleton model-based most specific concept descriptions. Condensing these two computation steps into one yields that the model-based most specific concept description $\Xi^{\mathcal{I}}$ can always be obtained as the $\mathcal{EL}^{\bot}_{\mathsf{gfp}}$ concept description $(\Xi, \mathcal{T}_{\mathcal{I}})$ where the TBox $\mathcal{T}_{\mathcal{I}}$ consists of the concept definitions

$$\begin{aligned} \Upsilon \equiv & \bigsqcap \{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \Upsilon \subseteq A^{\mathcal{I}} \,\} \\ & \sqcap \bigsqcap \{\, \exists\, r.\, \Phi \mid r \in \Sigma_{\mathsf{R}} \text{ and, for each } \upsilon \in \Upsilon, \text{ there is a } \phi \in \Phi \text{ with } (\upsilon, \phi) \in r^{\mathcal{I}} \,\} \end{aligned}$$

for all subsets $\Upsilon \subseteq \Delta^{\mathcal{I}}$, that is, we treat all subsets of the domain also as defined concept names in $\mathcal{T}_{\mathcal{I}}$.

Consequently, there is always an encoding of the attribute set $M_{\mathcal{I}}$ with at most exponential size w.r.t. $|\Delta^{\mathcal{I}}|$. Furthermore, the canonical base of $\mathcal{I}$ consists of at most exponentially many concept inclusions the premises and conclusions of which have at most exponentially many top-level conjuncts, and each of these top-level conjuncts has an exponential size. Thus, $\mathsf{Can}(\mathcal{I})$ has a size that is at most exponential in $\Delta^{\mathcal{I}}$.

We proceed with demonstrating that we can compute the canonical base $\mathsf{Can}(\mathcal{I})$ in exponential time w.r.t. $|\Delta^{\mathcal{I}}|$. We divide this computation task into three steps.

Computing the attribute set $M_{\mathcal{I}}$. We have already argued that each model-based most specific concept description can be computed in exponential time, and since there are at most exponentially many model-based most specific concept descriptions, we conclude that $M_{\mathcal{I}}$ can be computed in exponential time too.

Computing the induced context $\mathbb{K}_{\mathcal{I}}$. It remains to compute the incidence relation of $\mathbb{K}_{\mathcal{I}}$. For that purpose, we consider each object $\delta \in \Delta^{\mathcal{I}}$ and each attribute $C \in M_{\mathcal{I}}$, and check if $\delta \in C^{\mathcal{I}}$ holds true. Since each such check requires time polynomial in $|\Delta^{\mathcal{I}}|+|C|$, that is, time exponential in $|\Delta^{\mathcal{I}}|$, and exponentially many such checks are necessary, we conclude that the incidence relation of the induced context can be computed in exponential time. Including the aforementioned result shows that the induced context can be computed in exponential time.

Computing the canonical base $\mathsf{Can}(\mathcal{I})$. We consider the algorithm *NextClosures* from Kriegel and Borchmann (2015). Since $\mathbb{K}_{\mathcal{I}}$ has at most $2^{|\Delta^{\mathcal{I}}|}$ intents, there are at most $2^{|\Delta^{\mathcal{I}}|} \cdot |M_{\mathcal{I}}|$ fresh candidates during the algorithm's run on $\mathbb{K}_{\mathcal{I}}$ as input. We have already argued that the cardinality of $M_{\mathcal{I}}$ is exponential in $|\Delta^{\mathcal{I}}|$, and it follows that each fresh candidate will at most exponentially many times be closed against $\mathcal{L}^*$ (where $\mathcal{L}$ denotes the approximation of $\mathsf{Can}(\mathbb{K}_{\mathcal{I}})$ during the algorithm's run, which will satisfy $\mathcal{L} = \mathsf{Can}(\mathbb{K}_\mathcal{I})$ after termination, cf. Kriegel and Borchmann (2015)). Computing the closure of a subset $C \subseteq M_\mathcal{I}$ against $\mathcal{L}^*$ takes time bounded by $|\mathcal{L}|^2 \cdot (|M_\mathcal{I}|^2 + |M_\mathcal{I}|)$, since for computing this closure we need to loop at most $|\mathcal{L}|$ times and within each loop iteration it is necessary to check, for each implication $X \to Y \in \mathcal{L}$, whether $X \subsetneq C$ holds true and if so, we add all elements of Y to C. Consequently, closing a subset of $M_\mathcal{I}$ against $\mathcal{L}^*$ requires exponential time with respect to $|\Delta^\mathcal{I}|$.

Summing up, during a run of *NextClosures* on an induced context $\mathbb{K}_\mathcal{I}$ at most exponentially many fresh candidates will be computed, each of these candidates will at most exponentially many times be closed against $\mathcal{L}^*$ for the current $\mathcal{L} \subseteq \mathsf{Can}(\mathbb{K}_\mathcal{I})$, and each of these closures can be computed in exponential time. Consequently, *NextClosures* runs in exponential time on the input $\mathbb{K}_\mathcal{I}$. Eventually, the transformation from $\mathsf{Can}(\mathbb{K}_\mathcal{I})$ to $\mathsf{Can}(\mathcal{I})$ is trivial, cf. Distel (2011, Corollary 5.13), does not notably increase the size of an encoding, and needs only one traversal through $\mathsf{Can}(\mathbb{K}_\mathcal{I})$, that is, $\mathsf{Can}(\mathcal{I})$ can be computed from $\mathsf{Can}(\mathbb{K}_\mathcal{I})$ in exponential time as well.

We conclude that, using *NextClosures*, the canonical base of an interpretation can always be computed in deterministic exponential time.

Kuznetsov and Obiedkov (2008, Theorem 4.1) have shown that the number of implications in the canonical base $\mathsf{Can}(\mathbb{K})$ of a formal context $\mathbb{K} \coloneqq (G, M, I)$ can be exponential in $|G| \cdot |M|$. Their proof shows that we can even ignore the size of the attribute set M, since the considered formal contexts (G, M, I) are such that the size of M is linear in the size of G and the corresponding canonical bases contain exponentially many implications also with respect to the size of the object set G. As a consequence, we obtain that there exist formal contexts (G, M, I) for which an implication base cannot be encoded in space polynomial in $|G|$, as the canonical base of a formal context $\mathbb{K}$ has a minimal number of implications among all implication bases for $\mathbb{K}$. Since each formal context can be treated as an interpretation over a signature without role names, this important result immediately transfers from the Formal Concept Analysis setting to the Description Logic setting, and we conclude that there exist interpretations $\mathcal{I}$ for which a concept inclusion base cannot be encoded in polynomial space with respect to the cardinality of the domain $\Delta^\mathcal{I}$. □

5. Most Specific Consequences

The notion of a *most specific consequence* was introduced by the author in (Kriegel, 2016a). However, no conditions for their existence have been known, and it has been unclear how and whether these could be computed—problems that will be solved in Section 6. For describing the origin of that notion, we first take a short detour to the field of *Formal Concept Analysis* (abbrv. FCA). In what follows we will introduce only necessary details; the interest reader can find thorough overviews published by Ganter and Wille (1999a), and by Ganter and Obiedkov (2016).

The basic data structure is that of a *formal context* $\mathbb{K} \coloneqq (G, M, I)$ consisting of a set G of *objects*, a set M of *attributes*, and an *incidence relation* $I \subseteq G \times M$. If $(g, m) \in I$, then we say that the object g *has* the attribute m. An *implication* is an expression $X \to Y$ where X and Y are subsets of M, and it is *valid* in $\mathbb{K}$ if each object that has all attributes in X also has all attributes in Y. Let $\mathcal{L}$ be a set of implications. Then, $\mathcal{L}$

entails an implication $X \to Y$ if it is valid in all formal contexts in which all implications of $\mathcal{L}$ are valid. Furthermore, we can define a mapping $\phi_{\mathcal{L}}\colon X \mapsto X^{\mathcal{L}}$ as follows.

$$\begin{aligned} X^{\mathcal{L}} &\coloneqq \bigcup \{\, X^{\mathcal{L}_n} \mid n \in \mathbb{N} \,\} \\ X^{\mathcal{L}_0} &\coloneqq X \\ X^{\mathcal{L}_{n+1}} &\coloneqq X^{\mathcal{L}_n} \cup \bigcup \{\, Z \mid \exists\, Y\colon Y \to Z \in \mathcal{L} \text{ and } Y \subseteq X^{\mathcal{L}_n} \,\} \end{aligned}$$

This mapping $\phi_{\mathcal{L}}$ is a *closure operator* in the power-set $\wp(M)$, and an implication $X \to Y$ follows from $\mathcal{L}$ if, and only if, $Y \subseteq X^{\mathcal{L}}$ is satisfied, that is, if $X \to Y$ is valid in $\phi_{\mathcal{L}}$. We conclude that, for each attribute set $X \subseteq M$, the implication $X \to X^{\mathcal{L}}$ follows from $\mathcal{L}$, and furthermore, for each superset $Z \supsetneq X^{\mathcal{L}}$, the implication $X \to Z$ is not entailed by $\mathcal{L}$. Hence, we may also refer to $X^{\mathcal{L}}$ as the *most specific consequence* of X with respect to $\mathcal{L}$.

Example. *Fix the following implication set $\mathcal{L}$ over the attribute set $M \coloneqq \{m, n, o, p, q\}$.*

$$\mathcal{L} \coloneqq \left\{ \begin{array}{c} \{m\} \to \{n\}, \\ \{n, p\} \to \{q\}, \\ \{p, q\} \to \{o\} \end{array} \right\}$$

The smallest model of $\mathcal{L}$ that is a superset of $\{m, p\}$ is $\{m, n, o, p, q\}$. Furthermore, the most specific consequence of $\{m\}$ w.r.t. $\mathcal{L}$ is $\{m, n\}$.

As it turns out, there is no such notion in the field of *Description Logic*. Anyways, it is readily verified that sets of implications correspond to TBoxes, and consequently we can simply define the following.

Definition 3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote description logics, and fix some $\mathcal{L}_1$ TBox $\mathcal{T}$ as well as an $\mathcal{L}_1$ concept description C. Then, an $\mathcal{L}_2$ concept description D is called a* most specific consequence *or* most specific subsumer *(abbrv. MSS) in $\mathcal{L}_2$ of C with respect to $\mathcal{T}$ if it satisfies the following conditions.*

1. *The concept inclusion $C \sqsubseteq D$ follows from $\mathcal{T}$, i.e., $C \sqsubseteq_{\mathcal{T}} D$.*
2. *D is most specific with respect to the property of subsuming C w.r.t. $\mathcal{T}$, that is, for each $\mathcal{L}_2$ concept description E, if $C \sqsubseteq_{\mathcal{T}} E$, then $D \sqsubseteq_{\emptyset} E$.*

Within this document, we only consider the description logics $\mathcal{EL}$ and $\mathcal{EL}^{\bot}$ or its extensions with greatest fixed-point semantics, e.g., $\mathcal{EL}_{\mathsf{st}}$ and $\mathcal{EL}^{\bot}_{\mathsf{st}}$, as possible choices for $\mathcal{L}_1$, and for $\mathcal{L}_2$ we investigate the cases $\mathcal{EL}$, $\mathcal{EL}^{\bot}$, $\mathcal{EL}_{\mathsf{st}}$, $\mathcal{EL}^{\bot}_{\mathsf{st}}$, and $\mathcal{EL}_d$ as well as $\mathcal{EL}^{\bot}_d$ for some $d \in \mathbb{N}$.

As one quickly verifies, *all* most specific consequences of C with respect to $\mathcal{T}$ are unique up to equivalence, and hence we shall denote *the* most specific consequence of C with respect to $\mathcal{T}$ by $C^{\mathcal{T}}$—provided that it exists. Another immediate consequence of Definition 3 is that C and its most specific consequence $C^{\mathcal{T}}$ are equivalent with respect to $\mathcal{T}$, since, on the one hand, $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$, and, on the other hand, C clearly is a consequence of itself w.r.t. $\mathcal{T}$, that is, $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C$. Please note, that writing $C^{\mathcal{T}}$ can cause an abuse of notation, since the target DL $\mathcal{L}_2$ is not specified; however, in this document this will not cause any issues.

Remark that Distel (2011, Chapter 7) has investigated a dual notion, namely that of a *minimal possible consequence*, which he utilized to constitute an algorithm for the exploration of ontologies, called *ABox Exploration*. To emphasize this duality, it is also reasonable to use the name of a *minimal certain consequence* for a most specific consequence.

As an exemplary TBox, consider $\mathcal{T} \coloneqq \{\top \sqsubseteq \exists\, r.\, \top\}$. It can be readily verified that, for each $n \in \mathbb{N}$, the concept description $\exists\, r^n.\, \top$ is a *consequence* (i.e., a subsumer) of $\top$ with respect to $\mathcal{T}$. However, $\exists\, r^{n+1}.\, \top$ is more specific than $\exists\, r^n.\, \top$, and thus a most specific consequence of $\top$ w.r.t. $\mathcal{T}$ does not exist in the description logic $\mathcal{EL}^\bot$ with *descriptive semantics* (the standard semantics as introduced in Section 2). There are two solutions to tackle this problem of existence of most specific consequences. The first one is to use an extension of $\mathcal{EL}^\bot$ with *greatest fixed-point semantics*. Such extensions have been extensively studied (Baader, 2003a,b; Distel, 2011; Lutz, Piro and Wolter, 2010a,b), and in particular it has been shown that these extensions can handle terminological cycles (as present in the given TBox $\mathcal{T}$ above) also within concept descriptions. Put simply, one can think of concept descriptions in standard semantics as finite trees, while concept descriptions in greatest fixed-point semantics can be seen as finite graphs that could possibly contain cycles. It is, thus, straight-forward to claim that most specific consequences always exist in variants of $\mathcal{EL}^\bot$ that are equipped with greatest fixed-point semantics, and we are going to prove this fact in the upcoming Sections 6.1 and 6.2. Another solution for ensuring the existence of most specific consequences is to *restrict the role depth* of the concept descriptions under consideration, as this has been done by Borchmann, Distel and Kriegel (2015) to ensure the existence of model-based most specific concept descriptions in $\mathcal{EL}^\bot$ with descriptive semantics. This approach shall be considered in Section 6.3. Returning to our above example, we can readily verify that, for each role-depth bound $d \in \mathbb{N}$, the $\mathcal{EL}^\bot_d$ concept description $\exists\, r^d.\, \top$ is the most specific consequence of $\top$ w.r.t. $\mathcal{T}$ in $\mathcal{EL}^\bot_d$ (for the standard semantics).

6. Existence and Computation of Most Specific Consequences

Within this section, we shall investigate whether most specific consequences exist in $\mathcal{EL}$ and some of its variants. In particular, we also consider the extension $\mathcal{EL}^\bot$ with the bottom concept description, which can be used to express unsatisfiability, and we consider the variant $\mathcal{EL}^\bot_{\mathsf{st}}$ that is equipped with greatest fixed-point semantics. As we will demonstrate, most specific consequences always exist in $\mathcal{EL}^\bot_{\mathsf{st}}$, most specific consequences always exist in $\mathcal{EL}^\bot$ for so-called cycle-restricted TBoxes, and further most specific consequences always exist in $\mathcal{EL}^\bot_d$ for any $d \in \mathbb{N}$. Additionally, we shall provide means for the computation of most specific consequences, and analyze the complexity of computing these.

6.1. The Unrestricted Case

We start our investigations with the unrestricted case, that is, we do not impose any bound on the role depths. More specifically, we will show that in $\mathcal{EL}_{\mathsf{st}}$ most specific consequences always exist and can be computed in polynomial time. Furthermore, it holds true that most specific consequences need not exist in $\mathcal{EL}$, but we can decide in polynomial time whether these exist in $\mathcal{EL}$. The only reason that prevents the existence of $C^\mathcal{T}$ in $\mathcal{EL}$ is that $\mathcal{T}$ induces a cycle for some subconcept of C or, more generally, for some concept description that is entailed by some subconcept of C w.r.t.

$\mathcal{T}$. By such a cycle we mean a concept description D together with a non-empty word $r_1r_2\ldots r_n$ of role names such that

$$D \sqsubseteq_{\mathcal{T}} \exists\, r_1r_2\ldots r_n.\, D.$$

It turns out that $C^{\mathcal{T}}$ can be constructed from the canonical model $\mathcal{I}_{C,\mathcal{T}}$, and that $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description of $\{C\}$ in $\mathcal{I}_{C,\mathcal{T}}$, which is an $\mathcal{EL}_{\mathsf{st}}$ concept description in general due to the possible presence of cycles in the canonical model. Of course, if $\mathcal{I}_{C,\mathcal{T}}$ does not contain cycles, then $C^{\mathcal{T}}$ is equivalent to an $\mathcal{EL}$ concept description. Thus, in order to check existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$, it suffices to construct the canonical model, which can be done in polynomial time, then compute its reachability relation, i.e., the transitive closure of its set of edges, and finally test if there is some vertex reachable from itself on a path of length at least 1. The task of computing the reachability relation can be solved with the Floyd-Warshall algorithm, which is well-known to run in polynomial time. Furthermore, we shall prove that, for cycle-restricted TBoxes $\mathcal{T}$, all canonical models $\mathcal{I}_{C,\mathcal{T}}$ for arbitrary concept descriptions C are acyclic, which means that most specific consequences with respect to cycle-restricted TBoxes always exist in $\mathcal{EL}$.

The next proposition demonstrates that most specific consequences of $\mathcal{EL}_{\mathsf{st}}$ concept descriptions with respect to $\mathcal{EL}_{\mathsf{st}}$ TBoxes always exist in $\mathcal{EL}_{\mathsf{st}}$.

Proposition 4. *For each $\mathcal{EL}_{\mathsf{st}}$ TBox and each $\mathcal{EL}_{\mathsf{st}}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{st}}$. More specifically, it holds true that*

$$C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}\,(\mathcal{T}, C).$$

Proof. Firstly, we show that $\exists^{\mathsf{sim}}\,(\mathcal{T}, C)$ is a consequence of C with respect to $\mathcal{T}$. Let $\mathcal{I}$ be a model of $\mathcal{T}$ such that $\delta \in C^{\mathcal{I}}$. It is trivial that $(\mathcal{I}, \delta) \precsim (\mathcal{I}, \delta)$, and hence we immediately conclude that $\delta \in (\exists^{\mathsf{sim}}\,(\mathcal{T}, C))^{\mathcal{I}}$.

Secondly, we prove that $\exists^{\mathsf{sim}}\,(\mathcal{T}, C)$ is indeed most specific. For this purpose, consider an $\mathcal{EL}_{\mathsf{st}}$ concept description E such that $C \sqsubseteq_{\mathcal{T}} E$, and let $\mathcal{I}$ be an arbitrary interpretation such that $\delta \in (\exists^{\mathsf{sim}}\,(\mathcal{T}, C))^{\mathcal{I}}$. Of course, then there is a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$, $\epsilon \in C^{\mathcal{J}}$, and $\mathcal{J} \models \mathcal{T}$. We proceed with a case distinction on E. If E is an $\mathcal{EL}$ concept description, then we immediately conclude that $\epsilon \in E^{\mathcal{J}}$, and so $\delta \in E^{\mathcal{I}}$. Otherwise, let $E = \exists^{\mathsf{sim}}\,\Gamma.\,(\mathcal{U}, D)$ be an $\mathcal{EL}_{\mathsf{st}}$ concept description. It then follows that $\epsilon \in (\exists^{\mathsf{sim}}\,\Gamma.\,(\mathcal{U}, D))^{\mathcal{J}}$, and so there is another pointed interpretation $(\mathcal{K}, \zeta)$ with $\mathcal{K} \models \mathcal{U}$, $\zeta \in D^{\mathcal{K}}$, and $(\mathcal{K}, \zeta) \precsim_{\Sigma\setminus\Gamma} (\mathcal{J}, \epsilon)$. We may conclude that $(\mathcal{K}, \zeta) \precsim_{\Sigma\setminus\Gamma} (\mathcal{I}, \delta)$, and consequently $\delta \in (\exists^{\mathsf{sim}}\,\Sigma.\,(\mathcal{U}, D))^{\mathcal{I}}$. □

Since $\mathcal{EL}$ is a sublogic of $\mathcal{EL}_{\mathsf{st}}$, we can immediately draw the following conclusion.

Corollary 5. *For each $\mathcal{EL}$ TBox and each $\mathcal{EL}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{st}}$.*

Furthermore, we can interconnect the notions of most specific consequences and of model-based most specific concept descriptions. In particular, according to the following proposition it holds true that the most specific consequence $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description $\{C\}^{\mathcal{I}_{C,\mathcal{T}}}$. This important result will later be used to analyze the complexity of computing $C^{\mathcal{T}}$ as well as for deciding existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$.

Proposition 6. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox, and C be an $\mathcal{EL}$ concept description. Then the most specific consequence of C with respect to $\mathcal{T}$ is equivalent to the model-based most specific concept description of $\{C\}$ with respect to the canonical model of $\mathcal{T}$ and C, i.e.,*

$$C^{\mathcal{T}} \equiv_{\emptyset} \{C\}^{\mathcal{I}_{C,\mathcal{T}}}.$$

Proof. Remark that the model-based most specific concept description of $\{C\}$ with respect to $\mathcal{I}_{C,\mathcal{T}}$ is described by the $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$. Hence, it suffices to show that the concept descriptions $\exists^{\mathsf{sim}}(\mathcal{T}, C)$ and $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$ are equivalent. For this purpose consider an arbitrary interpretation $\mathcal{I}$ and an element $\delta \in \Delta^{\mathcal{I}}$. By definition of the semantics, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$ if, and only if, there is a pointed interpretation $(\mathcal{J}, \epsilon)$ such that $(\mathcal{J}, \epsilon) \precsim (\mathcal{I}, \delta)$, $\mathcal{J} \models \mathcal{T}$, and $\epsilon \in C^{\mathcal{J}}$. Furthermore, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$ if, and only if, $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$, i.e., if there is a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}, \delta)$.

Firstly, assume that $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$. Then (Lutz and Wolter, 2010, Lemma 13) yields that there is a simulation from $(\mathcal{I}_{C,\mathcal{T}}, C)$ to $(\mathcal{I}, \delta)$, since simulations are closed under composition, and thus $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$.

Vice versa, let $\delta \in (\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C))^{\mathcal{I}}$, i.e., $(\mathcal{I}_{C,\mathcal{T}}, C) \precsim (\mathcal{I}, \delta)$. By (Lutz and Wolter, 2010, Lemma 12) we have that $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$, and that $C \in C^{\mathcal{I}_{C,\mathcal{T}}}$. Consequently, $\delta \in (\exists^{\mathsf{sim}}(\mathcal{T}, C))^{\mathcal{I}}$. □

As already mentioned, the existence of cycles induced by the TBox $\mathcal{T}$ can require that also a description of the most specific consequence $C^{\mathcal{T}}$ must contain a cycle, which can be expressed in $\mathcal{EL}_{\mathsf{st}}$, but not in $\mathcal{EL}$. This observation yields a sufficient condition for the existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$, namely that most specific consequences always exist w.r.t. cycle-restricted TBoxes—a notion that is cited below.

(Baader, Borgwardt and Morawska, 2012a, Definition 2). *An $\mathcal{EL}$ TBox $\mathcal{T}$ is* cycle-restricted *if there does not exist an $\mathcal{EL}$ concept description C and a non-empty role word $w \in \Sigma_{\mathsf{R}}^{+}$ such that $C \sqsubseteq_{\mathcal{T}} \exists\, w.\, C$.*

Example. *The TBox $\mathcal{T}$ defined below is not cycle-restricted, since it entails the cyclic concept inclusion $A \sqsubseteq \exists\, r^3.\, A$.*

$$\mathcal{T} := \left\{ \begin{aligned} A &\sqsubseteq \exists\, rs.\, (B \sqcap \exists\, r.\, B), \\ \exists\, s.\, \exists\, r.\, \top &\sqsubseteq B, \\ \exists\, r.\, B &\sqsubseteq \exists\, rrr.\, A \end{aligned} \right\}$$

In the following, we shall consider the directed graphs $(\Delta^{\mathcal{I}}, \bigcup\{\, r^{\mathcal{I}} \mid r \in \Sigma_{\mathsf{R}} \,\})$ for interpretations $\mathcal{I}$ over Σ. That way, we can utilize graph-theoretic notions when speaking about interpretations.

Proposition 7. *For each $\mathcal{EL}$ TBox $\mathcal{T}$, the following statements are equivalent.*

1. *$\mathcal{T}$ is cycle-restricted.*
2. *For each $\mathcal{EL}$ concept description C, the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is acyclic.*
3. *For each $\mathcal{EL}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}$.*

Proof. We start with demonstrating that Statement 1 implies Statement 2. Consider a TBox $\mathcal{T}$ and a concept description C, and assume that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ is not acyclic. Then, $\mathcal{I}_{C,\mathcal{T}}$ contains some cycle

$$D \xrightarrow{r_1} D_1 \xrightarrow{r_2} D_2 \xrightarrow{r_3} \ldots \xrightarrow{r_n} D.$$

It immediately follows that $D \sqsubseteq_{\mathcal{T}} \exists\, w.\, D$ where $w \coloneqq r_1 r_2 r_3 \ldots r_n \in \Sigma_{\mathsf{R}}^+$, which yields that $\mathcal{T}$ is not cycle-restricted.

We proceed with proving that Statement 2 implies Statement 1. Let $w \in \Sigma_{\mathsf{R}}^+$ and consider some concept description C such that $C \sqsubseteq_{\mathcal{T}} \exists\, w.\, C$. Firstly, assume that the word w has length 1, i.e., $w = r$ for some role name $r \in \Sigma_{\mathsf{R}}$. By the very definition of a canonical model, it is then apparent that $(C, C) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ is a loop in the canonical model $\mathcal{I}_{C,\mathcal{T}}$, that is, $\mathcal{I}_{C,\mathcal{T}}$ is not acyclic. Secondly, assume that w has a length of at least 2, i.e., there are role names $r, s \in \Sigma_{\mathsf{R}}$ and a role word $v \in \Sigma_{\mathsf{R}}^*$ such that $w = rvs$. Our assumption implies that $C \sqsubseteq_{\mathcal{T}} \exists\, r.\, \exists\, v.\, \exists\, s.\, C$, and so (Lutz and Wolter, 2010, Lemma 13) shows that $C \in (\exists\, r.\, \exists\, v.\, \exists\, s.\, C)^{\mathcal{I}_{C,\mathcal{T}}}$, i.e., there is a path $C \xrightarrow{r} D \xrightarrow{v} E \xrightarrow{s} F$ in $\mathcal{I}_{C,\mathcal{T}}$ such that $F \in C^{\mathcal{I}_{C,\mathcal{T}}}$. In particular, we infer that $E \sqsubseteq_{\mathcal{T}} \exists\, s.\, F$ and $F \sqsubseteq_{\mathcal{T}} C$, and thus $E \sqsubseteq_{\mathcal{T}} \exists\, s.\, C$. Consequently, we have found a path $C \xrightarrow{r} D \xrightarrow{v} E \xrightarrow{s} C$ in the canonical model $\mathcal{I}_{C,\mathcal{T}}$, which is hence not acyclic.

Of course, an $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}\,(\mathcal{I}, \delta)$ is equivalent to an $\mathcal{EL}$ concept description if, and only if, the connected component of $\mathcal{I}$ that contains δ is acyclic. Since we have shown in Proposition 6 that $C^{\mathcal{T}} \equiv_{\emptyset} \exists^{\mathsf{sim}}\,(\mathcal{I}_{C,\mathcal{T}}, C)$ holds true, we immediately conclude that Statement 2 implies Statement 3. For the converse direction, assume that Statement 3 is satisfied and consider some $\mathcal{EL}$ concept description C. We then know that, for each concept description $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, the most specific consequence $D^{\mathcal{T}}$ exists, i.e., in $\mathcal{I}_{D,\mathcal{T}}$ the connected component containing D is acyclic. Apparently, the union of all these $\mathcal{I}_{D,\mathcal{T}}$ for $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$ equals the $\mathcal{I}_{C,\mathcal{T}}$, and it follows that all connected components of $\mathcal{I}_{C,\mathcal{T}}$ must be acyclic, i.e., the whole canonical model $\mathcal{I}_{C,\mathcal{T}}$ is acyclic, which yields Statement 2. □

Corollary 8. *The problem whether all most specific consequences with respect to some $\mathcal{EL}$ TBox $\mathcal{T}$ exist in $\mathcal{EL}$ can be decided in deterministic polynomial time.*

Proof. The problem whether an $\mathcal{EL}$ TBox is cycle-restricted can be decided in deterministic polynomial time, cf. (Baader, Borgwardt and Morawska, 2012b, Lemma 21). Thus, the statement follows from Proposition 7. □

However, the condition that $\mathcal{T}$ is cycle-restricted is not necessary for the existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$. To see this, consider the TBox $\mathcal{T} \coloneqq \{A \sqsubseteq \exists\, r.\, A\}$. It is apparent that $\mathcal{T}$ is not cycle-restricted, although the most specific consequence of B w.r.t. $\mathcal{T}$ exists in $\mathcal{EL}$, and is (equivalent to) B. We see that $\mathcal{T}$ induces a cycle which does not affect the concept description B or, more specifically, B does not contain any subconcept that entails A, and so the cycle in $\mathcal{T}$ does not induce a cycle in a description of $B^{\mathcal{T}}$. This idea is utilized in the proof of the upcoming proposition, which shows that existence of $C^{\mathcal{T}}$ in $\mathcal{EL}$ can always be decided in polynomial time.

Proposition 9. *The problem whether the most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}$ concept description C with respect to an $\mathcal{EL}$ TBox $\mathcal{T}$ exists in $\mathcal{EL}$ can be decided in deterministic polynomial time.*

Proof. We have shown in Proposition 6 that the most specific consequence $C^{\mathcal{T}}$ is equivalent to the model-based most specific concept description $\{C\}^{\mathcal{I}_{C,\mathcal{T}}}$, which is equivalent to $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$. Furthermore, an $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is equivalent to an $\mathcal{EL}$ concept description if, and only if, the connected component of $\mathcal{I}$ that contains δ is acyclic. According to Lutz and Wolter (2010), the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be constructed in time polynomial in the size of C and $\mathcal{T}$. By means of the Floyd-Warshall algorithm, the transitive closure E^+ for a given directed graph (V, E) can be computed in deterministic time $\mathcal{O}(|V|^3)$ and in deterministic space $\mathcal{O}(|V|^2)$. It follows that reachability in the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be decided in time polynomial in the size of C and $\mathcal{T}$. We now only need to check whether in $\mathcal{I}_{C,\mathcal{T}}$ there is some object in the connected component containing C that is reachable from itself on a path of length at least 1. Clearly, such an object exists if, and only if, $C^{\mathcal{T}}$ does not exist in $\mathcal{EL}$. □

Eventually, we analyze the complexity of computing $C^{\mathcal{T}}$. This is an easy task, since we have already shown that $C^{\mathcal{T}}$ can be computed as a model-based concept description for the canonical model $\mathcal{I}_{C,\mathcal{T}}$, and since canonical models can be constructed in polynomial time.

Proposition 10. *The most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}$ concept description C with respect to an $\mathcal{EL}$ TBox $\mathcal{T}$ can be computed in deterministic polynomial time, and its size is polynomial in $|C| + |\mathcal{T}|$.*

Proof. The statements are obtained as immediate corollaries from Proposition 6 and the fact that the canonical model $\mathcal{I}_{C,\mathcal{T}}$ can be computed in polynomial time, cf. Lutz and Wolter (2010). □

6.2. The Bottom Concept Description

As next step, we investigate the problems of existence and computation of most specific consequences as well as their complexities when we further incorporate the bottom concept description $\bot$ in our considered description logics $\mathcal{EL}$ and $\mathcal{EL}_{\mathsf{st}}$. Since there has not been published any notion of canonical models for $\mathcal{EL}^{\bot}$ and $\mathcal{EL}^{\bot}_{\mathsf{st}}$, and extending the existing results from $\mathcal{EL}$ and $\mathcal{EL}_{\mathsf{st}}$, respectively, would take plenty of space herein, we are taking the lazy way and rather reduce the mentioned problems to the solved cases in Section 6.1.

We begin with showing an unsurprising result, namely that, for any $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C which is not satisfiable with respect to some $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$, the most specific consequence $C^{\mathcal{T}}$ always exists in $\mathcal{EL}^{\bot}$ and is (equivalent to) the bottom concept description $\bot$. For the remaining cases, we argue that it suffices to consider only the satisfiable part $\mathcal{T}_{\mathsf{sat}}$ of $\mathcal{T}$, i.e., the subset of $\mathcal{T}$ that contains only those concept inclusions the premises of which are satisfiable with respect to $\mathcal{T}$. More specifically, the most specific consequence $C^{\mathcal{T}}$ is then equivalent to $C^{\mathcal{T}_{\mathsf{sat}}}$ if C is satisfiable w.r.t. $\mathcal{T}$.

Lemma 11. *Fix some $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$ and an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C. Then, C is unsatisfiable with respect to $\mathcal{T}$ if, and only if, $\bot$ is the most specific consequence of C with respect to $\mathcal{T}$.*

Proof. If C is not $\mathcal{T}$-satisfiable, then $C \sqsubseteq_{\mathcal{T}} \bot$, i.e., $\bot$ is a consequence of C w.r.t. $\mathcal{T}$. Obviously, there does not exist any more specific consequence, and so $\bot$ is the most specific consequence. Vice versa, let $\bot$ be the most specific consequence of C. It then immediately follows that $C \sqsubseteq_{\mathcal{T}} \bot$, which is equivalent to $\mathcal{T}$-unsatisfiability of C. □

Fix some $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$ and define the following TBox $\mathcal{T}_{\mathsf{sat}}$, which we call the *satisfiable part* of $\mathcal{T}$.

$$\mathcal{T}_{\mathsf{sat}} \coloneqq \{\, C \sqsubseteq D \mid C \sqsubseteq D \in \mathcal{T} \text{ and } C \text{ is satisfiable w.r.t. } \mathcal{T}\,\}$$

It then follows that, for each concept inclusion $C \sqsubseteq D \in \mathcal{T}_{\mathsf{sat}}$, both concept descriptions C and D are satisfiable with respect to $\mathcal{T}$ and are, thus, also satisfiable w.r.t. $\emptyset$. In particular, we infer that $\mathcal{T}_{\mathsf{sat}}$ must be an $\mathcal{EL}_{\mathsf{st}}$ TBox. We continue with demonstrating that, for each $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description which is satisfiable w.r.t. $\mathcal{T}$, its most specific consequences w.r.t. $\mathcal{T}$ and w.r.t. $\mathcal{T}_{\mathsf{sat}}$ are equivalent. That way, we infer that most specific consequences of $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions with respect to $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBoxes always exist in $\mathcal{EL}^{\bot}_{\mathsf{st}}$, and that these can be constructed from the canonical model $\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}$ if C is $\mathcal{T}$-satisfiable.

Piro (2012, Proposition 5.1.13) has shown that $\mathcal{EL}$ is *invariant under direct products*, that is, $C^{\mathcal{I}\times\mathcal{J}} = C^{\mathcal{I}} \times C^{\mathcal{J}}$ holds true for each $\mathcal{EL}$ concept description C. This result immediately extends to $\mathcal{EL}^{\bot}$, since $\bot^{\mathcal{I}\times\mathcal{J}} = \bot^{\mathcal{I}} \times \bot^{\mathcal{J}}$. Furthermore, since the product operation $\times$ is the infimum operation in the set of (equivalence classes of) pointed interpretations ordered by $\precsim$, we can immediately conclude that also $\mathcal{EL}_{\mathsf{si}}$ is invariant under products, that is,

$$(\exists^{\mathsf{sim}}\,(\mathcal{I},\delta))^{\mathcal{J}\times\mathcal{K}} = (\exists^{\mathsf{sim}}\,(\mathcal{I},\delta))^{\mathcal{J}} \times (\exists^{\mathsf{sim}}\,(\mathcal{I},\delta))^{\mathcal{K}}$$

holds true for all finitely representable pointed interpretations $(\mathcal{I},\delta)$ and for all interpretations $\mathcal{J}$ and $\mathcal{K}$. Consequently, each $\mathcal{EL}^{\bot}_{\mathsf{si}}$ concept inclusion $C \sqsubseteq D$ that is valid in both $\mathcal{I}$ and $\mathcal{J}$ is also valid in the direct product $\mathcal{I}\times\mathcal{J}$.

Now we are ready to show that, for each concept description C that is satisfiable w.r.t. $\mathcal{T}$, its most specific consequence $C^{\mathcal{T}}$ exists and can furthermore be constructed from the satisfiable part $\mathcal{T}_{\mathsf{sat}}$, which is an $\mathcal{EL}_{\mathsf{st}}$ TBox. Thus, for the construction of $C^{\mathcal{T}}$ we can utilize our previous results on most specific consequences in $\mathcal{EL}_{\mathsf{st}}$ from Section 6.1. Beforehand, we need the following lemma.

Lemma 12. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox, and consider $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C and D such that C is satisfiable with respect to $\mathcal{T}$. Then, the concept inclusion $C \sqsubseteq D$ is entailed by $\mathcal{T}$ if, and only if, it is entailed by $\mathcal{T}_{\mathsf{sat}}$.*

Proof. Since $\mathcal{T}_{\mathsf{sat}} \subseteq \mathcal{T}$, the *if* direction is trivial. We shall show the contraposition of the *only if* direction; consider a model $\mathcal{I}_{\mathsf{sat}}$ of $\mathcal{T}_{\mathsf{sat}}$ that contains a counterexample against $C \sqsubseteq D$, that is, $\mathcal{I}_{\mathsf{sat}}$ is such that $C^{\mathcal{I}_{\mathsf{sat}}} \setminus D^{\mathcal{I}_{\mathsf{sat}}} \neq \emptyset$. Since C is satisfiable with respect to $\mathcal{T}$, there exists some model $\mathcal{I}_C$ of $\mathcal{T}$ such that $C^{\mathcal{I}_C} \neq \emptyset$. By definition, each premise E of a concept inclusion $E \sqsubseteq F$ in $\mathcal{T} \setminus \mathcal{T}_{\mathsf{sat}}$ is not satisfiable with respect to $\mathcal{T}$ and, thus, we have that $E^{\mathcal{I}_C} = \emptyset$.

Of course, the direct product $\mathcal{I}_{\mathsf{sat}} \times \mathcal{I}_C$ is a model of $\mathcal{T}_{\mathsf{sat}}$. It also follows that $\mathcal{I}_{\mathsf{sat}} \times \mathcal{I}_C$ is a model of $\mathcal{T}$, since $E^{\mathcal{I}_{\mathsf{sat}}\times\mathcal{I}_C} = E^{\mathcal{I}_{\mathsf{sat}}} \times E^{\mathcal{I}_C} = \emptyset$ holds true for each premise E of a concept inclusion $E \sqsubseteq F \in \mathcal{T} \setminus \mathcal{T}_{\mathsf{sat}}$. Additionally, $C^{\mathcal{I}_{\mathsf{sat}}} \setminus D^{\mathcal{I}_{\mathsf{sat}}} \neq \emptyset$ in conjunction with $C^{\mathcal{I}_C} \neq \emptyset$ yields that

$$C^{\mathcal{I}_{\mathsf{sat}}\times\mathcal{I}_C} \setminus D^{\mathcal{I}_{\mathsf{sat}}\times\mathcal{I}_C} = (C^{\mathcal{I}_{\mathsf{sat}}} \times C^{\mathcal{I}_C}) \setminus (D^{\mathcal{I}_{\mathsf{sat}}} \times D^{\mathcal{I}_C}) \neq \emptyset,$$

that is, $\mathcal{I}_{\mathsf{sat}} \times \mathcal{I}_C$ contains a counterexample against $C \sqsubseteq D$ too. Eventually, we conclude that $C \not\sqsubseteq_{\mathcal{T}} D$. ☐

Proposition 13. *If C is satisfiable with respect to $\mathcal{T}$, then the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}_{\mathsf{st}}$ and is equivalent to $C^{\mathcal{T}_{\mathsf{sat}}}$.*

Proof. Since C is satisfiable w.r.t. $\mathcal{T}$, it is satisfiable w.r.t. $\emptyset$, and it follows that C does not contain $\bot$ as a subconcept, that is, C is an $\mathcal{EL}_{\mathsf{st}}$ concept description. Furthermore, Lemma 11 yields that the most specific concept description $C^{\mathcal{T}}$—if it exists—is satisfiable w.r.t. $\emptyset$. In order to prove that $C^{\mathcal{T}_{\mathsf{sat}}}$ is the most specific consequence of C with respect to $\mathcal{T}$, we need to show the following two statements.

- $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}_{\mathsf{sat}}}$
- $C \sqsubseteq_{\mathcal{T}} D$ implies $C^{\mathcal{T}_{\mathsf{sat}}} \sqsubseteq_{\emptyset} D$ or, equivalently, $C \sqsubseteq_{\mathcal{T}_{\mathsf{sat}}} D$ for each $\mathcal{EL}_{\mathsf{st}}$ concept description D.

As $\mathcal{T}_{\mathsf{sat}}$ is a subset of $\mathcal{T}$, the first statement is apparently true. The second statement has been proven in Lemma 12. □

The following statements are immediate consequences of combining the results from Section 6.1 with Lemma 11 and Proposition 13, and provide answers concerning the complexity of deciding existence of $C^{\mathcal{T}}$ in $\mathcal{EL}^{\bot}$ as well as of computing $C^{\mathcal{T}}$. Remark that deciding subsumption w.r.t. a TBox in $\mathcal{EL}^{\bot}$ and $\mathcal{EL}^{\bot}_{\mathsf{st}}$ has polynomial time complexity, and so the satisfiable part $\mathcal{T}_{\mathsf{sat}}$ can be computed in polynomial time as well.

Corollary 14.
1. *For each $\mathcal{EL}^{\bot}$ TBox and each $\mathcal{EL}^{\bot}$ concept description C, the most specific consequence $C^{\mathcal{T}}$ exists in $\mathcal{EL}^{\bot}_{\mathsf{st}}$.*
2. *The problem whether all most specific consequences with respect to some $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ exist in $\mathcal{EL}^{\bot}$ can be decided in deterministic polynomial time.*
3. *The problem whether the most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}^{\bot}$ concept description C with respect to an $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ exists in $\mathcal{EL}^{\bot}$ can be decided in deterministic polynomial time.*
4. *The most specific consequence $C^{\mathcal{T}}$ of an $\mathcal{EL}^{\bot}$ concept description C with respect to an $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$ can be computed in deterministic polynomial time, and its size is polynomial in $|C| + |\mathcal{T}|$.*

6.3. The Role-Depth Bounded Case

We close this section with an investigation of the role-depth bounded case, that is, for any role-depth bound $d \in \mathbb{N}$, we consider the problem whether most specific consequences of $\mathcal{EL}^{\bot}$ concept descriptions with respect to $\mathcal{EL}^{\bot}$ TBoxes exist in $\mathcal{EL}^{\bot}_{d}$ and, if so, how these can be computed. We shall find that existence is always guaranteed, simply because there are only finitely many appropriate candidates and this set of candidates is closed under conjunction. Furthermore, it holds true that role-depth bounded most specific consequences correspond to role-depth bounded most specific concept descriptions with respect to canonical models.

Let C be an $\mathcal{EL}^{\bot}$ concept description, let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox, and consider any role-depth bound $d \in \mathbb{N}$. To avoid confusion with the unrestricted case in Sections 6.1 and 6.2, we shall denote the most specific consequence of C w.r.t. $\mathcal{T}$ in $\mathcal{EL}^{\bot}_{d}$ as $C^{\mathcal{T}_d}$—under the assumption of existence. Apparently, it holds true that $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}_d}$ and, furthermore if $\mathsf{rd}(C) \leq d$, from $C \sqsubseteq_{\mathcal{T}} C$ we conclude that $C^{\mathcal{T}_d} \sqsubseteq_{\emptyset} C$. In summary, $C \equiv_{\mathcal{T}} C^{\mathcal{T}_d}$ for any $C \in \mathcal{EL}^{\bot}_{d}(\Sigma)$.

Lemma 15. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, let C be an $\mathcal{EL}^\bot$ concept description, and assume that $d \in \mathbb{N}$ is a role-depth bound. Then, the most specific consequence of C with respect to $\mathcal{T}$ exists in $\mathcal{EL}^\bot_d$.*

Proof. Consider an $\mathcal{EL}^\bot$ TBox $\mathcal{T}$, an $\mathcal{EL}^\bot$ concept description C, as well as a role-depth bound $d \in \mathbb{N}$. Of course, both $\mathcal{T}$ and C can only contain finitely many symbols from the signature; let $\Sigma_{C,\mathcal{T}}$ be the set of all these symbols from Σ that occur in C or in $\mathcal{T}$. Since every subsumer of C w.r.t. $\mathcal{T}$ can be constructed using only the finitely many symbols in $\Sigma_{C,\mathcal{T}} \cup \{\bot, \top\}$, there are only finitely many such subsumers of a role-depth not exceeding d. Since the restriction of C to a role-depth of d is a consequence of C with respect to $\mathcal{T}$, at least one such consequence exists. Furthermore, if C_1 and C_2 are consequences of C with respect to $\mathcal{T}$, then also their conjunction $C_1 \sqcap C_2$ is a consequence of C with respect to $\mathcal{T}$. Consequently, the conjunction of the (finitely many) consequences of C w.r.t. $\mathcal{T}$ is itself a consequence of C w.r.t. $\mathcal{T}$. We denote this concept description by D, and prove that it is indeed a most specific consequence. If there were a smaller consequence E that satisfies the role-depth bound, i.e., if we had $E \sqsubsetneq_\emptyset D$ and $C \sqsubseteq_\mathcal{T} E$, then E would be contained as a top-level conjunct in D, i.e., we could infer the contradiction $D \sqsubseteq_\emptyset E$. □

Now that we have demonstrated that most specific consequences always exist in $\mathcal{EL}^\bot_d$ for each role-depth bound $d \in \mathbb{N}$, we continue with providing a means for their computation in the following lemma.

Lemma 16. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, C be an $\mathcal{EL}^\bot$ concept description, and let $d \in \mathbb{N}$ be a role-depth bound. If C is not satisfiable with respect to $\mathcal{T}$, then it holds true that $C^{\mathcal{T}_d} \equiv_\emptyset \bot$. Otherwise, the following equivalence is satisfied.*

$$C^{\mathcal{T}_d} \equiv_\emptyset \{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d}$$

Proof. The unsatisfiable case can be handled similarly as in Lemma 11. Otherwise, it suffices to consider only the satisfiable part $\mathcal{T}_{\mathsf{sat}}$, cf. Lemma 12.

We prove the claim by induction on the role-depth bound d. For the base case $d = 0$, the following equivalences hold true.

$$\begin{aligned}\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_0} &\equiv_\emptyset \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \in A^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \,\} \\ &\equiv_\emptyset \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \mathcal{T}_{\mathsf{sat}} \models C \sqsubseteq A \,\}\end{aligned}$$

It is easy to verify that $\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_0}$ is indeed a consequence of C with respect to $\mathcal{T}_{\mathsf{sat}}$, and furthermore has a role depth of 0.

For the inductive case, the special case of a result of Kriegel (2017, Theorem 8.3) for $\mathcal{EL}^\bot$ together with the induction hypothesis yields the following.

$$\begin{aligned}
\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_{d+1}} &\equiv_\emptyset \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \in A^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \,\} \\
&\qquad \sqcap \bigsqcap\{\, \exists\, r.\, \{D\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \mid r \in \Sigma_{\mathsf{R}} \text{ and } (C,D) \in r^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \,\} \\
&\overset{(*)}{\equiv_\emptyset} \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \in A^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \,\} \\
&\qquad \sqcap \bigsqcap\{\, \exists\, r.\, \{D\}^{(\mathcal{I}_{D,\mathcal{T}_{\mathsf{sat}}})_d} \mid r \in \Sigma_{\mathsf{R}} \text{ and } (C,D) \in r^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \,\} \\
&\overset{\text{I.H.}}{\equiv_\emptyset} \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \mathcal{T}_{\mathsf{sat}} \models C \sqsubseteq A \,\} \\
&\qquad \sqcap \bigsqcap\left\{\, \exists\, r.\, D^{\mathcal{T}_d} \;\middle|\; \begin{array}{l} \mathcal{T}_{\mathsf{sat}} \models C \sqsubseteq \exists\, r.\, D \text{ and } \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}_{\mathsf{sat}}), \\ \text{or } \exists\, r.\, D \in \mathsf{Conj}(C) \end{array} \right\}
\end{aligned}$$

The equivalence $(*)$ is valid, since (Lutz and Wolter, 2010, Lemma 12) states that the pointed interpretations $(\mathcal{I}_{C,\mathcal{T}}, D)$ and $(\mathcal{I}_{D,\mathcal{T}}, D)$ are equi-similar due to the fact that $D \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \cap \Delta^{\mathcal{I}_{D,\mathcal{T}}}$. Obviously, $\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_{d+1}}$ has a role depth not exceeding $d+1$, and it is easy to verify that it is a consequence of C with respect to $\mathcal{T}_{\mathsf{sat}}$.

For both the base case and the inductive case, it remains to prove that $\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d}$ is subsumed (w.r.t. $\emptyset$) by every other $\mathcal{T}_{\mathsf{sat}}$-consequence of C with a role depth of at most d. Hence, consider such a consequence E, i.e., it holds true that $C \sqsubseteq_{\mathcal{T}_{\mathsf{sat}}} E$ as well as $\mathsf{rd}(E) \leq d$. Then, (Lutz and Wolter, 2010, Lemma 13) yields that $C \in E^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$, and so we may immediately conclude that $\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \sqsubseteq_\emptyset E$, cf. (Borchmann, Distel and Kriegel, 2016, Lemma 4.3). □

As a last result in this section, we shall prove that each most specific consequence $C^{\mathcal{T}_d}$ can also be constructed as the dth approximation of the most specific consequence $C^{\mathcal{T}}$. This especially shows that we can approximate $C^{\mathcal{T}}$ with arbitrary precision, and that the sequence $(\, C^{\mathcal{T}_d} \mid d \in \mathbb{N} \,)$ converges to $C^{\mathcal{T}}$.

Corollary 17. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, let C be an $\mathcal{EL}^\bot$ concept description, and assume that $d \in \mathbb{N}$ is a role-depth bound. Then, the dth approximation of the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}^\bot_{\mathsf{si}}$ is the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}^\bot_{\mathrm{d}}$, that is, the following holds true.*

$$C^{\mathcal{T}_d} \equiv_\emptyset C^{\mathcal{T}}\!\restriction_d$$

Proof. If C is not satisfiable with respect to $\mathcal{T}$, then we immediately conclude that both $C^{\mathcal{T}_d}$ and $C^{\mathcal{T}}$ are equivalent to $\bot$, cf. Lemmas 11 and 16. The equivalence $\bot \equiv_\emptyset \bot\!\restriction_d$ is trivial, and so it follows that $C^{\mathcal{T}_d} \equiv_\emptyset C^{\mathcal{T}}\!\restriction_d$ is indeed satisfied.

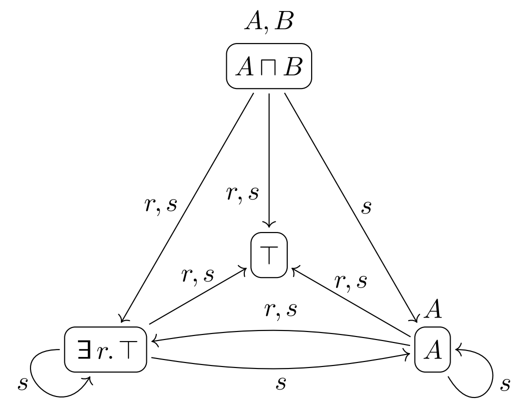
Otherwise, according to Proposition 6, Lemma 16, (Distel, 2011, Lemma 4.5), and (Borchmann, Distel and Kriegel, 2016, Theorem 4.17) the following equivalences hold true.

$$C^{\mathcal{T}_d} \equiv_\emptyset \{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \equiv_\emptyset \{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})}\!\restriction_d \equiv_\emptyset C^{\mathcal{T}}\!\restriction_d \qquad \square$$

Example. *For illustrating the computation of most specific consequences, we consider the exemplary TBox*

$$\mathcal{T} \coloneqq \left\{ \begin{array}{r} A \sqsubseteq \exists\, rr.\, \top \\ \exists\, r.\, \top \sqsubseteq \exists\, s.\, A \end{array} \right\}$$

and the concept description $C \coloneqq A \sqcap B$. The canonical model $\mathcal{I}_{C,\mathcal{T}}$ is shown below.



Now the most specific consequence $C^{\mathcal{T}}$ can be read off the canonical model $\mathcal{I}_{C,\mathcal{T}}$ as the model-based most specific concept description of $\{C\}$. As there is a cycle reachable from C, the most specific consequence does not exist in $\mathcal{EL}$, but only in $\mathcal{EL}_{\mathsf{si}}$ or in $\mathcal{EL}_d$ for each role-depth bound $d \in \mathbb{N}$. Of course, $C^{\mathcal{T}}$ is equivalent to the $\mathcal{EL}_{\mathsf{si}}$ concept description $\exists^{\mathsf{sim}}(\mathcal{I}_{C,\mathcal{T}}, C)$, and we further list the first three approximations in the following, which are the most specific consequences in $\mathcal{EL}_0$, in $\mathcal{EL}_1$, and in $\mathcal{EL}_2$, respectively.

$$C^{\mathcal{T}_0} \equiv_\emptyset A \sqcap B$$
$$C^{\mathcal{T}_1} \equiv_\emptyset A \sqcap B \sqcap \exists\, r. \top \sqcap \exists\, s. A$$
$$C^{\mathcal{T}_2} \equiv_\emptyset A \sqcap B \sqcap \exists\, r. (\exists\, r. \top \sqcap \exists\, s. A) \sqcap \exists\, s. (A \sqcap \exists\, r. \top \sqcap \exists\, s. A)$$

7. Algebraic Properties of Most Specific Consequences

This section's aim is to explore algebraic properties of most specific consequences. In particular, we shall connect Sections 3, 5 and 6. For instance, the mappings $C \mapsto C^{\mathcal{T}}$ and $C \mapsto C^{\mathcal{T}_d}$ for each $d \in \mathbb{N}$ constitute closure operators in the lattice of concept descriptions, which immediately implies a series of mathematical laws and properties. Recursion formulae that are satisfied by most specific consequences are also provided within this section. We split our exploration in two cases: firstly, we consider the unrestricted case, and secondly, we investigate the role-depth bounded case.

7.1. The Unrestricted Case

As announced, we shall start with the unrestricted case. The next lemma formulates that, for any TBox $\mathcal{T}$, the function which maps concept descriptions to their most specific consequence with respect to $\mathcal{T}$ constitutes a closure operator. Then, the following corollary shows some statements that immediately follow from the fact that the most specific consequence mapping is a closure operator.

Lemma 18. *For any $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$, the mapping $\phi_{\mathcal{T}} \colon C \mapsto C^{\mathcal{T}}$ is a closure operator in the dual of $\boldsymbol{\mathcal{EL}}^{\bot}_{\mathsf{st}}(\Sigma)$, i.e., for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C and D, the following conditions are satisfied.*

1. $C^{\mathcal{T}} \sqsubseteq_\emptyset C$ *(extensive)*

2. $C \sqsubseteq_\emptyset D$ *implies* $C^{\mathcal{T}} \sqsubseteq_\emptyset D^{\mathcal{T}}$ *(monotonic)*

3. $C^{\mathcal{T}} \equiv_\emptyset C^{\mathcal{T}\mathcal{T}}$ *(idempotent)*

Proof. Since C is a consequence of itself with respect to $\mathcal{T}$, it follows by Definition 3 that $C^{\mathcal{T}} \sqsubseteq_\emptyset C$.

Of course, $C^{\mathcal{T}} \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ is trivially valid, and so it follows that $C^{\mathcal{T}\mathcal{T}} \sqsubseteq_\emptyset C^{\mathcal{T}}$. Furthermore, it holds true that $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}} \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}\mathcal{T}}$, that is, $C^{\mathcal{T}\mathcal{T}}$ is a consequence of C with respect to $\mathcal{T}$. Since $C^{\mathcal{T}}$ is most specific, we conclude that $C^{\mathcal{T}} \sqsubseteq_\emptyset C^{\mathcal{T}\mathcal{T}}$.

Eventually, assume that $C \sqsubseteq_\emptyset D$. Since $D \sqsubseteq_{\mathcal{T}} D^{\mathcal{T}}$, it follows that $C \sqsubseteq_{\mathcal{T}} D^{\mathcal{T}}$, i.e., $D^{\mathcal{T}}$ is a consequence of C w.r.t. $\mathcal{T}$. Since $C^{\mathcal{T}}$ is most specific, we infer that $C^{\mathcal{T}} \sqsubseteq_\emptyset D^{\mathcal{T}}$. □

Corollary 19. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot_{\mathsf{st}}$ TBox, and assume that C as well as D are $\mathcal{EL}^\bot_{\mathsf{st}}$ concept descriptions. Then, the following statements hold true.*

1. $(C \sqcap D)^{\mathcal{T}} \sqsubseteq_\emptyset C \sqcap D^{\mathcal{T}\mathcal{T}}$

2. $(C \sqcap D)^{\mathcal{T}} \equiv_\emptyset (C^{\mathcal{T}} \sqcap D^{\mathcal{T}})^{\mathcal{T}}$

3. $C^{\mathcal{T}} \vee D^{\mathcal{T}} \sqsubseteq_\emptyset (C \vee D)^{\mathcal{T}}$

4. $C^{\mathcal{T}} \vee D^{\mathcal{T}} \equiv_\emptyset (C^{\mathcal{T}} \vee D^{\mathcal{T}})^{\mathcal{T}}$

Proof. The statements are obtained as corollaries of Lemma 18 and Section 3. □

Each TBox $\mathcal{T}$ can be normalized by means of the closure operator $\phi_{\mathcal{T}}$ in the sense that there is a TBox which is equivalent to $\mathcal{T}$ and only contains concept inclusions of the form $C \sqsubseteq C^{\mathcal{T}}$. On the one hand, this holds true for the TBox that contains all these concept inclusions for all concept descriptions C and, on the other hand, it suffices to only take those concept descriptions C that occur as a premise in $\mathcal{T}$. A more sophisticated characterization is provided in the following lemma.

Lemma 20. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot_{\mathsf{st}}$ TBox. We define $\mathsf{Prem}(\mathcal{T})$ as the set of all premises of concept inclusions in $\mathcal{T}$, i.e., we set $\mathsf{Prem}(\mathcal{T}) \coloneqq \{\, C \mid \exists\, D \colon C \sqsubseteq D \in \mathcal{T} \,\}$. Then, the following sets of concept inclusions are both equivalent to $\mathcal{T}$.*[1]

$$\mathcal{T}^\circ \coloneqq \{\, C \sqsubseteq C^{\mathcal{T}} \mid C \in \mathcal{EL}^\bot_{\mathsf{st}}(\Sigma) \,\}$$
$$\mathcal{T}^* \coloneqq \{\, C \sqsubseteq C^{\mathcal{T}} \mid C \in \mathsf{Prem}(\mathcal{T}) \,\}$$

Proof. Since $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$ for all $\mathcal{EL}^\bot_{\mathsf{st}}$ concept descriptions C, it immediately follows that $\mathcal{T}$ entails both $\mathcal{T}^\circ$ and $\mathcal{T}^*$. Furthermore, since $\mathcal{T}^\circ \supseteq \mathcal{T}^*$ and hence $\mathcal{T}^\circ \models \mathcal{T}^*$, it suffices to show that $\mathcal{T}^* \models \mathcal{T}$. Consider a concept inclusion $C \sqsubseteq D \in \mathcal{T}$, then $C \sqsubseteq C^{\mathcal{T}} \in \mathcal{T}^*$. Since D is a consequence of C with respect to $\mathcal{T}$, we infer that $C^{\mathcal{T}} \sqsubseteq_\emptyset D$, and as a consequence it then follows that $C \sqsubseteq_{\mathcal{T}^*} D$. Since $C \sqsubseteq D$ is an arbitrary concept inclusion from $\mathcal{T}$, we have just proven that $\mathcal{T}^* \models \mathcal{T}$. □

[1] Note that since $\mathcal{T}$ is a TBox and hence finite, also $\mathcal{T}^*$ is a finite set of concept inclusions, i.e., a TBox.

As a further important result, we shall show that entailment with respect to some TBox $\mathcal{T}$ and validity for the associated closure operator $\phi_{\mathcal{T}}$ is equivalent for any concept inclusion. It also holds true that subsumption w.r.t. a TBox is equivalent to subsumption of the corresponding most specific consequences w.r.t. $\emptyset$. In particular, subsumption reasoning in $\mathcal{EL}^{\bot}$ with respect to cycle-restricted TBoxes can, thus, be reduced to the simpler task of subsumption reasoning in $\mathcal{EL}^{\bot}$ with respect to the empty TBox where in the reduction the most specific consequence of the premise needs to be computed.

Lemma 21. *For each $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T} \cup \{C \sqsubseteq D\}$, the following statements are equivalent.*

1. $C \sqsubseteq_{\mathcal{T}} D$
2. $C \sqsubseteq_{\mathcal{T}^{\circ}} D$
3. $C \sqsubseteq_{\mathcal{T}^{*}} D$
4. $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$
5. $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D^{\mathcal{T}}$
6. $E^{\mathcal{T}} \sqsubseteq_{\emptyset} C$ *implies* $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ *for each* $\mathcal{EL}^{\bot}_{\mathsf{st}}$ *concept description* E.

Proof. The equivalence of the Statements 1 to 3 follows from Lemma 20.

Statements 1 and 4 are equivalent by the following observations. If $C \sqsubseteq_{\mathcal{T}} D$, then D is a consequence of C with respect to $\mathcal{T}$, and consequently $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ by Definition 3. Vice versa, if $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$, then since $C \sqsubseteq_{\mathcal{T}} C^{\mathcal{T}}$, we infer that $C \sqsubseteq_{\mathcal{T}} D$.

Lemma 18 and Section 3 yield the equivalence of Statements 4 and 5.

Eventually, we demonstrate that Statement 6 is equivalent to the other statements. If $C \sqsubseteq_{\mathcal{T}} D$ and the empty TBox $\emptyset$ entails $E^{\mathcal{T}} \sqsubseteq C$, then it follows that $\mathcal{T}$ entails $E \sqsubseteq C$, and hence $E \sqsubseteq_{\mathcal{T}} D$. Consequently, $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ as claimed.

Vice versa, assume that $E^{\mathcal{T}} \sqsubseteq_{\emptyset} C$ implies $E^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions E. Of course, then $E \sqsubseteq_{\mathcal{T}} C$ only if $E \sqsubseteq_{\mathcal{T}} D$. Since it trivially holds true that $C \sqsubseteq_{\mathcal{T}} C$, we immediately conclude that $C \sqsubseteq_{\mathcal{T}} D$. □

The following two corollaries collect previous results and further connect these to notions from the theory of closure operators.

Corollary 22. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox and assume that $C \sqsubseteq D$ is an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusion. Then, the following statements are equivalent.*

1. $\mathcal{T} \models C \sqsubseteq D$
2. $\phi_{\mathcal{T}} \models C \sqsubseteq D$
3. $\mathcal{I} \models \mathcal{T}$ *implies* $\mathcal{I} \models C \sqsubseteq D$ *for any interpretation* $\mathcal{I}$.
4. $\mathcal{I} \models \mathcal{T}$ *implies* $\phi_{\mathcal{I}} \models C \sqsubseteq D$ *for each interpretation* $\mathcal{I}$.
5. $\bigtriangleup\{\, \phi_{\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\} \models C \sqsubseteq D$
6. $\emptyset \models \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\} \sqsubseteq D$
7. $\emptyset \models C^{\mathcal{T}} \sqsubseteq D$

Proof. The equivalence of Statements 1, 2, and 7 has just been shown in Lemma 21. By the very definition of the semantics, also Statements 1 and 3 are equivalent. Since Distel (2011, Lemma 4.1) has shown that $X \subseteq C^{\mathcal{I}}$ is equivalent to $X^{\mathcal{I}} \sqsubseteq_{\emptyset} C$, we conclude that $C \sqsubseteq_{\mathcal{I}} D$ is equivalent to $C^{\mathcal{I}\mathcal{I}} \sqsubseteq_{\emptyset} D$, which is equivalent to the validity of $C \sqsubseteq D$ for the closure operator $\phi_{\mathcal{I}}$. Consequently, Statements 3 and 4 are equivalent too. Eventually, Section 3 provides the equivalence of Statements 4 to 6. □

Corollary 23. *Consider an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$ as well as an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C. If C is satisfiable w.r.t. $\mathcal{T}$, then the following equivalences hold true.*

$$\begin{aligned} C^{\mathcal{T}} &\equiv_{\emptyset} C^{\mathcal{T}_{\mathsf{sat}}} \equiv_{\emptyset} \{C\}^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \equiv_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \\ &\equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\} \end{aligned}$$

Otherwise, if C is not satisfiable w.r.t. $\mathcal{T}$, then $C^{\mathcal{T}} \equiv_{\emptyset} \bot \equiv_{\emptyset} \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$.

Proof. Let C be $\mathcal{T}$-satisfiable. The first equivalence is proven in Proposition 13 and the second equivalence has been shown in Proposition 6. The equivalence of $C^{\mathcal{T}_{\mathsf{sat}}}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\}$ as well as of $C^{\mathcal{T}}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$ with respect to the empty TBox follows from the equivalence of Statements 6 and 7 in Corollary 22. Since the canonical model $\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}$ is a model of $\mathcal{T}_{\mathsf{sat}}$, we can infer that $\bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \sqsupseteq_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$. Furthermore, $C \in C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$ implies that $\{C\}^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}} \sqsubseteq_{\emptyset} C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$. The case where C is unsatisfiable is obvious. □

Eventually, we formulate a recursive characterization of most specific consequences. It is readily verified that it follows from Proposition 6 and the fact that, in $\mathcal{EL}_{\mathsf{si}}$, the concept description $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ is, on the one hand, the model-based most specific concept description of $\{\delta\}$ in $\mathcal{I}$ and, on the other hand, satisfies the following recursion.

$$\begin{aligned} \exists^{\mathsf{sim}}(\mathcal{I}, \delta) \equiv_{\emptyset} &\bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } \delta \in A^{\mathcal{I}} \,\} \\ &\sqcap \bigsqcap\{\, \exists\, r.\, \exists^{\mathsf{sim}}(\mathcal{I}, \epsilon) \mid r \in \Sigma_{\mathsf{R}} \text{ and } (\delta, \epsilon) \in r^{\mathcal{I}} \,\} \end{aligned}$$

Corollary 24. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox, and C be an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description. If C is satisfiable with respect to $\mathcal{T}$, then the following recursion formula for the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}^{\bot}_{\mathsf{st}}$ holds true. Otherwise, we have $C^{\mathcal{T}} \equiv_{\emptyset} \bot$.*

$$\begin{aligned} C^{\mathcal{T}} \equiv_{\emptyset} &\bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \textit{ and } C \sqsubseteq_{\mathcal{T}} A \,\} \\ &\sqcap \bigsqcap\{\, \exists\, r.\, D^{\mathcal{T}} \mid C \sqsubseteq_{\mathcal{T}} \exists\, r.\, D \textit{ and } \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}_{\mathsf{sat}}), \textit{ or } \exists\, r.\, D \in \mathsf{Conj}(C) \,\} \end{aligned}$$

7.2. The Role-Depth Bounded Case

In this section, we shall continue with our investigations on algebraic properties of most specific consequences for the role-depth bounded case. It is no surprise that we find similar results as in the unrestricted case. We again get a closure operator, and can immediately conclude that general properties of closure operators from Section 3 can be specifically tailored for it.

Lemma 25. *Let $\mathcal{T}$ be an $\mathcal{EL}^{\bot}$ TBox and consider some role-depth bound $d \in \mathbb{N}$. Then, the mapping $\phi_{\mathcal{T},d} \colon C \mapsto C^{\mathcal{T}_d}$ is a closure operator in the dual of $\boldsymbol{\mathcal{EL}^{\bot}_{d}}(\Sigma)$, i.e., for all $\mathcal{EL}^{\bot}_{d}$ concept descriptions C and D, the following conditions are satisfied.*

1. $C^{\mathcal{T}_d} \sqsubseteq_\emptyset C$ *(extensive)*

2. $C \sqsubseteq_\emptyset D$ *implies* $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{T}_d}$ *(monotonic)*

3. $C^{\mathcal{T}_d} \equiv_\emptyset C^{\mathcal{T}_d\mathcal{T}_d}$ *(idempotent)*

Proof. Since subsumption is reflexive, C is always a consequence of itself, and so C subsumes its most specific consequence.

If $C \sqsubseteq_\emptyset D$, then in particular $C \sqsubseteq_\mathcal{T} D$, i.e., D is a consequence of C with respect to $\mathcal{T}$. Now Definition 3 yields $D \sqsubseteq_\mathcal{T} D^{\mathcal{T}_d}$, and it follows that $C \sqsubseteq_\mathcal{T} D^{\mathcal{T}_d}$ or, equivalently, $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{T}_d}$. This proves the monotonicity.

Finally, it remains to prove that the mapping is idempotent. From $C^{\mathcal{T}_d} \sqsubseteq_\mathcal{T} C^{\mathcal{T}_d}$ we infer that $C^{\mathcal{T}_d\mathcal{T}_d} \sqsubseteq_\emptyset C^{\mathcal{T}_d}$. It further holds true that $C \sqsubseteq_\mathcal{T} C^{\mathcal{T}_d} \sqsubseteq_\mathcal{T} C^{\mathcal{T}_d\mathcal{T}_d}$, and so we can conclude that $C^{\mathcal{T}_d} \sqsubseteq_\emptyset C^{\mathcal{T}_d\mathcal{T}_d}$. □

Please note that, if C has a role depth exceeding d, then it may not follow that $C^{\mathcal{T}_d} \sqsubseteq_\emptyset C$. It is readily verified that, for any $\mathcal{EL}^\bot$ concept description C, the most specific consequence $C^{\emptyset_d}$ and the dth approximation $C\restriction_d$ coincide—thus, we have that $(\exists\, r.\, A)^{\emptyset_0} \equiv_\emptyset \top \not\sqsubseteq_\emptyset \exists\, r.\, A$. However, the above proof shows that monotonicity is ensured even if C and D are $\mathcal{EL}^\bot$ concept descriptions such that the role depth of D does not exceed d, whereas idempotency is satisfied for all $\mathcal{EL}^\bot$ concept descriptions C. More generally, $C \sqsubseteq_\emptyset D$ implies $C^\mathcal{T} \sqsubseteq_\emptyset D^\mathcal{T}$, cf. Lemma 18, which yields that $C^\mathcal{T}\restriction_d \sqsubseteq_\emptyset D^\mathcal{T}\restriction_d$, and finally an application of Corollary 17 shows $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{T}_d}$. In summary, we obtain that the extension of $\phi_{\mathcal{T},d}$ to the domain $\mathcal{EL}^\bot(\Sigma)$, or to $\mathcal{EL}^\bot_{\mathsf{st}}(\Sigma)$, is a monotonic, idempotent mapping, but it is not extensive.

Corollary 26. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, fix some role-depth bound $d \in \mathbb{N}$, and assume that C as well as D are $\mathcal{EL}^\bot_d$ concept descriptions. Then, the following statements hold true.*

1. $(C \sqcap D)^{\mathcal{T}_d} \sqsubseteq_\emptyset C \sqcap D^{\mathcal{T}_d\mathcal{T}_d}$

2. $(C \sqcap D)^{\mathcal{T}_d} \equiv_\emptyset (C^{\mathcal{T}_d} \sqcap D^{\mathcal{T}_d})^{\mathcal{T}_d}$

3. $C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d} \sqsubseteq_\emptyset (C \vee D)^{\mathcal{T}_d}$

4. $C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d} \equiv_\emptyset (C^{\mathcal{T}_d} \vee D^{\mathcal{T}_d})^{\mathcal{T}_d}$

Proof. The statements are obtained as corollaries of Lemma 25 and Section 3. □

Analogously to the unrestricted case, there exist normalizations of a TBox $\mathcal{T}$ that are equivalent to $\mathcal{T}$ and in which all conclusions are most specific consequences. We further show that, for all $\mathcal{EL}^\bot_d$ concept inclusions, entailment w.r.t. some TBox can be reduced to entailment w.r.t. $\emptyset$ by simply replacing the premise of the concept inclusion in question by its most specific consequence, and we demonstrate that entailment w.r.t. a TBox $\mathcal{T}$ is equivalent to validity in the induced closure operator $\phi_{\mathcal{T},d}$.

Lemma 27. *Let $\mathcal{T} \cup \{C \sqsubseteq D\}$ be an $\mathcal{EL}^\bot$ TBox such that D has a role depth of at most d. Then the following statements are equivalent.*

1. $C \sqsubseteq_\mathcal{T} D$

2. $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D$

3. $C \sqsubseteq_{\mathcal{T}_d^\circ} D$ *where* $\mathcal{T}_d^\circ \coloneqq \{\, E \sqsubseteq E^{\mathcal{T}_d} \mid E \in \mathcal{EL}_d^\bot(\Sigma) \,\}$

If all conclusions of concept inclusions in $\mathcal{T}$ *have role depths not exceeding* d*, then furthermore the following statement is equivalent to Statements 1 to 3.*

4. $C \sqsubseteq_{\mathcal{T}_d^*} D$ *where* $\mathcal{T}_d^* \coloneqq \{\, E \sqsubseteq E^{\mathcal{T}_d} \mid E \in \mathsf{Prem}(\mathcal{T}) \,\}$

If the concept description C *has a role depth not exceeding* d*, then the following statement is equivalent to Statements 1 to 3, too.*

5. $E^{\mathcal{T}_d} \sqsubseteq_\emptyset C$ *implies* $E^{\mathcal{T}_d} \sqsubseteq_\emptyset D$ *for each* $\mathcal{EL}_d^\bot$ *concept description* E*.*

Proof. Firstly, we prove the equivalence of Statement 1 and Statement 2. If $C \sqsubseteq_\mathcal{T} D$, i.e., if D is a consequence of C with respect to $\mathcal{T}$, then the very definition of a most specific consequence yields that D must subsume the most specific consequence $C^{\mathcal{T}_d}$ with respect to the empty TBox $\emptyset$. Vice versa, $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D$ and $C \sqsubseteq_\mathcal{T} C^{\mathcal{T}_d}$ immediately implies that $C \sqsubseteq_\mathcal{T} D$.

According to Definition 3, we have that $\mathcal{T} \models \mathcal{T}_d^\circ$. Consequently, Statement 3 implies Statement 1. Furthermore, Statement 2 implies Statement 3 as follows. Clearly, $C \sqsubseteq_{\mathcal{T}_d^\circ} C^{\mathcal{T}_d}$, and since $C^{\mathcal{T}_d} \sqsubseteq_\emptyset D$, we conclude that $C \sqsubseteq_{\mathcal{T}_d^\circ} D$.

Eventually, consider the *d-normalization* $\mathcal{T}_d^*$ of $\mathcal{T}$. It is easy to verify that Definition 3 yields $\mathcal{T} \models \mathcal{T}_d^*$, and hence Statement 4 implies Statement 1. Vice versa, we assume that each conclusion in $\mathcal{T}$ has a role depth of at most d, and we shall prove that $\mathcal{T}_d^* \models \mathcal{T}$. Let $E \sqsubseteq F \in \mathcal{T}$ be a concept inclusion. By construction of $\mathcal{T}_d^*$, then $E \sqsubseteq E^{\mathcal{T}_d} \in \mathcal{T}_d^*$ holds true. We further conclude that $E^{\mathcal{T}_d} \sqsubseteq_\emptyset F$, and thus $E \sqsubseteq_{\mathcal{T}_d^*} F$. Eventually, the TBoxes $\mathcal{T}$ and $\mathcal{T}_d^*$ are equivalent.

We proceed with demonstrating that Statement 1 implies Statement 5. If the empty TBox $\emptyset$ entails $E^{\mathcal{T}_d} \sqsubseteq C$, then it follows that $\mathcal{T}$ entails $E \sqsubseteq C$, and hence $E \sqsubseteq_\mathcal{T} D$. This now yields that $E^{\mathcal{T}_d} \sqsubseteq_\emptyset D$, as claimed. Vice versa, assume that $E^{\mathcal{T}_d} \sqsubseteq_\emptyset C$ implies $E^{\mathcal{T}_d} \sqsubseteq_\emptyset D$ for all $\mathcal{EL}_d^\bot$ concept descriptions E, that is, $E \sqsubseteq_\mathcal{T} C$ holds true only if $E \sqsubseteq_\mathcal{T} D$. Of course, it is trivial that $C \sqsubseteq_\mathcal{T} C$, and so we immediately conclude that $C \sqsubseteq_\mathcal{T} D$. □

In order to collect previous results and connect these to the notions from Section 3, we formulate the following two corollaries.

Corollary 28. *Let* $\mathcal{T} \cup \{C \sqsubseteq D\}$ *be an* $\mathcal{EL}^\bot$ *TBox such that the role depths of* C *and of* D *do not exceed* d*. Then, the following statements are equivalent.*

1. $\mathcal{T} \models C \sqsubseteq D$
2. $\phi_{\mathcal{T},d} \models C \sqsubseteq D$.
3. $\mathcal{I} \models \mathcal{T}$ *implies* $\mathcal{I} \models C \sqsubseteq D$ *for every interpretation* $\mathcal{I}$*.*
4. $\mathcal{I} \models \mathcal{T}$ *implies* $\phi_{\mathcal{I},d} \models C \sqsubseteq D$ *for any interpretation* $\mathcal{I}$*.*
5. $\bigtriangleup\{\, \phi_{\mathcal{I},d} \mid \mathcal{I} \models \mathcal{T} \,\} \models C \sqsubseteq D$
6. $\emptyset \models \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T} \,\} \sqsubseteq D$
7. $\emptyset \models C^{\mathcal{T}_d} \sqsubseteq D$.

Proof. The equivalence of Statements 1, 2, and 7 has just been shown in Lemma 27. By the very definition of the semantics, also Statements 1 and 3 are equivalent. Since Borchmann, Distel and Kriegel (2016, Lemma 4.3) have shown that $X \subseteq C^{\mathcal{I}}$ is equivalent to $X^{\mathcal{I}_d} \sqsubseteq_\emptyset C$, we conclude that $C \sqsubseteq_{\mathcal{I}} D$ is equivalent to $C^{\mathcal{I}\mathcal{I}_d} \sqsubseteq_\emptyset D$, which is equivalent to the validity of $C \sqsubseteq D$ for the closure operator $\phi_{\mathcal{I},d}$. Consequently, Statements 3 and 4 are equivalent too. Eventually, Section 3 provides the equivalence of Statements 4 to 6. $\square$

Corollary 29. *Consider an $\mathcal{EL}^\bot$ TBox $\mathcal{T}$, an $\mathcal{EL}^\bot$ concept description C, and some role-depth bound $d \in \mathbb{N}$. If C is satisfiable w.r.t. $\mathcal{T}$, then the following equivalences hold true.*

$$C^{\mathcal{T}_d} \equiv_\emptyset C^{(\mathcal{T}_{\mathsf{sat}})_d} \equiv_\emptyset \{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \equiv_\emptyset C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d}$$
$$\equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T} \,\}$$

Otherwise, if C is not satisfiable w.r.t. $\mathcal{T}$, then $C^{\mathcal{T}_d} \equiv_\emptyset \bot \equiv_\emptyset \bigvee\{\, C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models \mathcal{T} \,\}$.

Proof. Let C be $\mathcal{T}$-satisfiable. The first equivalence follows from Proposition 13 and Corollary 17 and the second equivalence is proven in Lemma 16. The equivalence of $C^{\mathcal{T}_d}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T} \,\}$ as well as of $C^{(\mathcal{T}_{\mathsf{sat}})_d}$ and $\bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\}$ with respect to the empty TBox follows from the equivalence of Statements 6 and 7 in Corollary 28. Since the canonical model $\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}$ is a model of $\mathcal{T}_{\mathsf{sat}}$, we can infer that $\bigvee\{\, C^{\mathcal{I}\mathcal{I}_d} \mid \mathcal{I} \models \mathcal{T}_{\mathsf{sat}} \,\} \sqsupseteq_\emptyset C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d}$. Furthermore, $C \in C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}}$ implies that $\{C\}^{(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d} \sqsubseteq_\emptyset C^{\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}}(\mathcal{I}_{C,\mathcal{T}_{\mathsf{sat}}})_d}$. The case where C is unsatisfiable is obvious. $\square$

We close this section with a recursive characterization of most specific consequences in the role-depth bounded case. These are obtained as consequences of Lemma 16 in conjunction with the special case of a result from Kriegel (2017, Theorem 8.3) for $\mathcal{EL}^\bot$.

Lemma 30. *Let $\mathcal{T}$ be an $\mathcal{EL}^\bot$ TBox, and C be an $\mathcal{EL}^\bot$ concept description. If C is satisfiable with respect to $\mathcal{T}$, then the following recursion formulae for the most specific consequence of C with respect to $\mathcal{T}$ in $\mathcal{EL}^\bot_d$ hold true. Otherwise, we have $C^{\mathcal{T}_d} \equiv_\emptyset \bot$ for any $d \in \mathbb{N}$.*

$$C^{\mathcal{T}_0} \equiv_\emptyset \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \sqsubseteq_{\mathcal{T}} A \,\}$$
$$C^{\mathcal{T}_{d+1}} \equiv_\emptyset \bigsqcap\{\, A \mid A \in \Sigma_{\mathsf{C}} \text{ and } C \sqsubseteq_{\mathcal{T}} A \,\}$$
$$\sqcap \bigsqcap\{\, \exists\, r.\, D^{\mathcal{T}_d} \mid C \sqsubseteq_{\mathcal{T}} \exists\, r.\, D \text{ and } \exists\, r.\, D \in \mathsf{Sub}(\mathcal{T}_{\mathsf{sat}}), \text{ or } \exists\, r.\, D \in \mathsf{Conj}(C) \,\}$$

8. Applications

Now, we shall present some applications in the field of *Description Logic* that use the notion of most specific consequences, the corresponding closure operators, as well as the operations in the lattice of closure operators. We start with providing a characterization of entailment between TBoxes in Section 8.1 and a characterization of soundness and completeness of TBoxes for interpretations in Section 8.2. Then, we suggest four techniques for the axiomatization of concept inclusions under different assumptions. In particular, we show in Section 8.3 how a *merging* of two TBoxes can be constructed, that is, how we can axiomatize all concept inclusions that are simultaneously entailed by two TBoxes. Furthermore, a technique for the axiomatization of concept inclusions from streams of interpretations is given in Section 8.4, we introduce in Section 8.5 some method for an *error-tolerant* axiomatization of concept inclusions from interpretations where some TBox is used for detecting and filtering out errors in the input interpretation, and in Section 8.6 we present a technique for the axiomatization of concept inclusions from ABoxes in a restricted setting.

8.1. A Characterization of Entailment

We have seen in Section 7.1 that each TBox $\mathcal{T}$ induces a closure operator $\phi_{\mathcal{T}}$ in a way such that any concept inclusion is entailed by $\mathcal{T}$ if, and only if, it is valid for $\phi_{\mathcal{T}}$. In the following lemma, we shall use these closure operators to provide a characterization of entailment between two TBoxes. A similar result can, of course, be found for the role-depth bounded case too using our results from Section 7.2.

Lemma 31. *Let $\mathcal{T}_1 \cup \mathcal{T}_2$ be an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox. Then, the following statements are equivalent.*

1. $\mathcal{T}_1 \models \mathcal{T}_2$.
2. *$C \sqsubseteq_{\mathcal{T}_2} D$ implies $C \sqsubseteq_{\mathcal{T}_1} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusions $C \sqsubseteq D$.*
3. *$C^{\mathcal{T}_1} \sqsubseteq_{\emptyset} C^{\mathcal{T}_2}$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C.*
4. *$C^{\mathcal{T}_1\mathcal{T}_2} \equiv_{\emptyset} C^{\mathcal{T}_1}$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C.*
5. *Each most specific consequence of $\mathcal{T}_1$ is a most specific consequence of $\mathcal{T}_2$, modulo equivalence with respect to the empty TBox $\emptyset$.*
6. $\phi_{\mathcal{T}_1} \trianglerighteq \phi_{\mathcal{T}_2}$

Proof. We start with demonstrating the equivalence of Statements 1 and 3. Assume that $\mathcal{T}_1 \models \mathcal{T}_2$ and consider an arbitrary $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C. By Definition 3 it holds true that $C \sqsubseteq_{\mathcal{T}_2} C^{\mathcal{T}_2}$, and consequently $C \sqsubseteq_{\mathcal{T}_1} C^{\mathcal{T}_2}$. An application of Lemma 21 then yields $C^{\mathcal{T}_1} \sqsubseteq_{\emptyset} C^{\mathcal{T}_2}$. Conversely, let $C^{\mathcal{T}_1} \sqsubseteq_{\emptyset} C^{\mathcal{T}_2}$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C. Of course, Lemma 21 implies $C \sqsubseteq_{\mathcal{T}_1} C^{\mathcal{T}_2}$ for all $C \in \mathcal{EL}^{\bot}_{\mathsf{st}}(\Sigma)$. Now consider a concept inclusion $C \sqsubseteq D \in \mathcal{T}_2$. It is immediately clear that then D is a consequence of C with respect to $\mathcal{T}_2$, and hence $C^{\mathcal{T}_2} \sqsubseteq_{\emptyset} D$. It follows that $C^{\mathcal{T}_1} \sqsubseteq_{\emptyset} D$, and thus $C \sqsubseteq_{\mathcal{T}_1} D$. Since $C \sqsubseteq D$ is an arbitrary concept inclusion from $\mathcal{T}_2$, we have just demonstrated that $\mathcal{T}_1 \models \mathcal{T}_2$.

Furthermore, it is readily verified that Statements 1 and 2 are equivalent. Eventually, Section 3 implies the equivalence of Statements 3 to 6. □

8.2. A Characterization of Soundness and Completeness

As we have already mentioned, there are several works on the axiomatization of concept inclusions from interpretations, e.g., Distel (2011); Borchmann, Distel and Kriegel (2016); and Kriegel (2017). In particular, these approaches can be used to compute so-called concept inclusion bases for interpretations, and a TBox $\mathcal{T}$ is such a concept inclusion base for some interpretation $\mathcal{I}$ if $\mathcal{T}$ is *sound* for $\mathcal{I}$, that is, $\mathcal{I} \models \mathcal{T}$, and is *complete* for $\mathcal{I}$, that is, $C \sqsubseteq_{\mathcal{I}} D$ implies $C \sqsubseteq_{\mathcal{T}} D$ for every concept inclusion $C \sqsubseteq D$. The aim of this section is to characterize these two notions of soundness and completeness using the notions of most specific consequences and of model-based most specific concept inclusions as well as their induced closure operators.

Lemma 32. *Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox. Then, the following statements are equivalent.*

1. *$\mathcal{T}$ is* sound *for $\mathcal{I}$.*
2. *$\mathcal{I} \models \mathcal{T}$*
3. *$C \sqsubseteq_{\mathcal{T}} D$ implies $C \sqsubseteq_{\mathcal{I}} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusions $C \sqsubseteq D$.*
4. *$C^{\mathcal{II}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C.*
5. *Each model-based most specific concept description of $\mathcal{I}$ is a most specific consequence of $\mathcal{T}$, modulo equivalence with respect to the empty TBox $\emptyset$.*
6. *$\phi_{\mathcal{I}} \trianglerighteq \phi_{\mathcal{T}}$*

Proof. The equivalence of Statements 1 to 3 is either true by definition or trivial.

By Lemma 21 we have that $C \sqsubseteq_{\mathcal{T}} D$ is equivalent to $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$. Furthermore, we know that $C \sqsubseteq_{\mathcal{I}} D$ if, and only if, $C^{\mathcal{II}} \sqsubseteq_{\emptyset} D$.

We are now going to show that Statement 4 implies Statement 3. Therefore assume that $\mathcal{T}$ entails $C \sqsubseteq D$, i.e., the concept inclusion $C^{\mathcal{T}} \sqsubseteq D$ is valid in all interpretations. Of course, then $C^{\mathcal{II}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$ yields that also the concept inclusion $C^{\mathcal{II}} \sqsubseteq D$ is valid in all interpretations, and consequently $C \sqsubseteq D$ is valid in $\mathcal{I}$. Vice versa, if $C \sqsubseteq_{\mathcal{T}} D$ implies $C \sqsubseteq_{\mathcal{I}} D$, then $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ implies $C^{\mathcal{II}} \sqsubseteq_{\emptyset} D$. It readily verified that then $C^{\mathcal{II}} \sqsubseteq_{\emptyset} C^{\mathcal{T}}$.

Eventually, the equivalence of Statements 4 and 5 is an immediate consequence of Section 3. □

Lemma 33. *Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox. Then, the following statements are equivalent.*

1. *$\mathcal{T}$ is* complete *for $\mathcal{I}$.*
2. *$C \sqsubseteq_{\mathcal{I}} D$ implies $C \sqsubseteq_{\mathcal{T}} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusions $C \sqsubseteq D$.*
3. *$C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{II}}$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept descriptions C.*
4. *Each most specific consequence of $\mathcal{T}$ is a model-based most specific concept description of $\mathcal{I}$, modulo equivalence with respect to the empty TBox $\emptyset$.*
5. *$\phi_{\mathcal{T}} \trianglerighteq \phi_{\mathcal{I}}$*

Proof. Statements 1 and 2 are equivalent just by definition, and the equivalence of Statements 3 and 4 follows from Section 3. It remains to prove, e.g., that Statements 2 and 3 are equivalent. Hence, assume that $C^{\mathcal{II}} \sqsubseteq_{\emptyset} D$ implies $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$ for all $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept inclusions $C \sqsubseteq D$. Of course, it easily follows that $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{II}}$. For the converse direction, let $C^{\mathcal{II}} \sqsubseteq_{\emptyset} D$. Then $C^{\mathcal{T}} \sqsubseteq_{\emptyset} C^{\mathcal{II}}$ implies $C^{\mathcal{T}} \sqsubseteq_{\emptyset} D$. □

Summing up Lemmas 32 and 33 yields the following corollary.

Corollary 34. *Let $\mathcal{I}$ be an interpretation, and assume that $\mathcal{T}$ is an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox. Then, the following statements are equivalent.*

1. *$\mathcal{T}$ is a base of concept inclusions for $\mathcal{I}$.*

2. $\mathcal{T}$ *is sound as well as complete for* $\mathcal{I}$.

3. $C \sqsubseteq_{\mathcal{T}} D$ *if, and only if,* $C \sqsubseteq_{\mathcal{I}} D$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{st}}$ *concept inclusions* $C \sqsubseteq D$.

4. $C^{\mathcal{T}} \equiv_{\emptyset} C^{\mathcal{II}}$ *for all* $\mathcal{EL}^{\bot}_{\mathsf{st}}$ *concept descriptions* C.

5. *The most specific consequences of* $\mathcal{T}$ *are exactly the model-based most specific concept descriptions of* $\mathcal{I}$*, modulo equivalence with respect to the empty TBox* $\emptyset$.

6. $\phi_{\mathcal{T}} = \phi_{\mathcal{I}}$

8.3. Merging Terminological Boxes

This rather short section provides a characterization of simultaneous entailment of a concept inclusion by two TBoxes. In particular, we shall demonstrate that a concept inclusion is simultaneously entailed by two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ if, and only if, it is valid for the infimum $\phi_{\mathcal{T}_1} \vartriangle \phi_{\mathcal{T}_2}$ of the corresponding closure operators. We leave it open for future research how an effective procedure for computing such a concept inclusion base for $\phi_{\mathcal{T}_1} \vartriangle \phi_{\mathcal{T}_2}$ can be constructed.

Lemma 35. *Let* $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \{C \sqsubseteq D\}$ *be an* $\mathcal{EL}^{\bot}_{\mathsf{st}}$ *TBox. Then, the following statements are equivalent.*

1. $C \sqsubseteq_{\mathcal{T}_1} D$ *and* $C \sqsubseteq_{\mathcal{T}_2} D$

2. $C^{\mathcal{T}_1} \sqsubseteq_{\emptyset} D$ *and* $C^{\mathcal{T}_2} \sqsubseteq_{\emptyset} D$

3. $C^{\mathcal{T}_1} \vee C^{\mathcal{T}_2} \sqsubseteq_{\emptyset} D$

4. $\phi_{\mathcal{T}_1} \models C \sqsubseteq D$ *and* $\phi_{\mathcal{T}_2} \models C \sqsubseteq D$

5. $\phi_{\mathcal{T}_1} \vartriangle \phi_{\mathcal{T}_2} \models C \sqsubseteq D$

Proof. Statements 1 and 2 are equivalent by Lemma 21. The very definition of least common subsumers yields that Statements 2 and 3 are equivalent. Furthermore, Corollary 22 implies the equivalence of Statements 1 and 4. Eventually, Section 3, or alternatively (Kriegel, 2016b, Section 3.1), shows the equivalence of Statements 4 and 5. □

8.4. Axiomatization of Concept Inclusions from Sequences of Interpretations

Consider a setting where a sequence $(\mathcal{I}_n \mid n \in \mathbb{N})$ of interpretations can be observed and, for each time point $n \in \mathbb{N}$, a terminological box $\mathcal{T}_n$ shall be constructed that entails exactly those concept inclusions which are simultaneously valid in all previously observed interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$, that is, such that for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{T}_n \models C \sqsubseteq D$ if, and only if, $\mathcal{I}_k \models C \sqsubseteq D$ for all previous time points $k \leq n$. For the initial moment $n = 0$, we can simply compute $\mathcal{T}_0$ as a concept inclusion base for $\mathcal{I}_0$ utilizing the approaches from Distel (2011); or from Borchmann, Distel and Kriegel (2016). Of course, for the following moments $n \geq 1$, we could construct a concept inclusion base for the disjoint union of the interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$. However, since the aforementioned methods require the construction of so-called *induced contexts* the size of which may be exponential in the cardinality of the interpretation's domain, this technique could possibly be infeasible for late time points. Furthermore, it would require the storing of all interpretations observed so far. We shall present another technique for solving the above mentioned task. Please note that this problem has has already been addressed by Kriegel (2015) for the case where $\mathcal{I}_{n+1} \models \mathcal{T}_n$ for all time points $n \in \mathbb{N}$. Herein, we propose a solution that circumvents this rather restrictive precondition.

The following lemma states that the concept inclusions that are both valid in an interpretation $\mathcal{I}$ and entailed by some TBox $\mathcal{T}$ are exactly those which are valid for the infimum $\phi_{\mathcal{I}} \curlywedge \phi_{\mathcal{T}}$ of the induced closure operators. In what follows, we shall also develop an effective procedure for computing concept inclusion bases of such infima. Unfortunately, it holds true that, as one quickly verifies, the infimum $\phi_{\mathcal{I}} \curlywedge \phi_{\mathcal{T}}$ has infinitely many closures, which makes it hard at first sight to work out a terminating procedure. In contrast, during the axiomatization of finite interpretations there are only finitely many closures of $\phi_{\mathcal{I}}$, namely all those of the form $X^{\mathcal{I}}$ for some $X \subseteq \Delta^{\mathcal{I}}$—so it is not immediately clear whether and how the procedure from Distel (2011) can be suitably generalized. As a practical solution to this, we restrict the role depths of the concept inclusions to be axiomatized as done by Borchmann, Distel and Kriegel (2016) for the non-incremental case without any TBoxes, that is, we consider the closure operators $\phi_{\mathcal{I},d} \curlywedge \phi_{\mathcal{T},d}$ instead. It is then ensured that only finitely many closures exist. Thus, the next lemma is formulated for the role-depth bounded case.

Lemma 36. *Let $\mathcal{I}$ be an interpretation, $\mathcal{T}$ an $\mathcal{EL}^{\bot}$ TBox, and $C \sqsubseteq D$ a concept inclusion such that both its premise and its conclusion have role depths not exceeding d. Then, the following statements are equivalent:*

1. $C \sqsubseteq_{\mathcal{I}} D$ *and* $C \sqsubseteq_{\mathcal{T}} D$
2. $C^{\mathcal{I}\mathcal{I}_d} \sqsubseteq_{\emptyset} D$ *and* $C^{\mathcal{T}_d} \sqsubseteq_{\emptyset} D$
3. $C^{\mathcal{I}\mathcal{I}_d} \vee C^{\mathcal{T}_d} \sqsubseteq_{\emptyset} D$
4. $\phi_{\mathcal{I},d} \models C \sqsubseteq D$ *and* $\phi_{\mathcal{T},d} \models C \sqsubseteq D$
5. $\phi_{\mathcal{I},d} \curlywedge \phi_{\mathcal{T},d} \models C \sqsubseteq D$

Proof. The proof is similar to the proof of Lemma 35. □

Consequently, we can outline the following incremental procedure for computing concept inclusion bases from sequences of interpretations. For that purpose, fix some role-depth bound $d \in \mathbb{N}$.

1. Upon availability of the first observed interpretation $\mathcal{I}_0$, compute its canonical base $\mathcal{T}_0 \coloneqq \mathsf{Can}(\mathcal{I}_0, d)$ using the results of Borchmann, Distel and Kriegel (2016, Theorem 4.32).
2. For each newly observed interpretation $\mathcal{I}_{n+1}$, compute the canonical base $\mathcal{T}_{n+1} \coloneqq \mathsf{Can}(\mathcal{I}_{n+1}, \mathcal{T}_n, d)$ as described later in Corollary 42.

It is readily verified that—by construction—for each time point $n \in \mathbb{N}$, the TBox $\mathcal{T}_n$ entails an $\mathcal{EL}^{\bot}_d$ concept inclusion $C \sqsubseteq D$ if, and only if, $C \sqsubseteq D$ is valid in all interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n$.

In the sequel of this subsection, we devise a suitable generalization of the procedure from Borchmann, Distel and Kriegel (2016). In particular, we show how the problem of constructing a concept inclusion base for $\phi_{\mathcal{I},d} \curlywedge \phi_{\mathcal{T},d}$ can be reduced to the problem of constructing an implication base for $\phi_{\mathbb{K}} \curlywedge \phi_{\mathcal{L}}$ where $\mathbb{K}$ is a suitable formal context and $\mathcal{L}$ is a suitable implication set. Using the parallel algorithm *NextClosures* (Kriegel, 2016b) a canonical base for $\phi_{\mathbb{K}} \vartriangle \phi_{\mathcal{L}}$ can be computed.

We start with proving that, in a concept inclusion base for the infimum $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$, it suffices that all conclusions are closures of this infimum.

Lemma 37. *For any finitely representable interpretation $\mathcal{I}$ and for any TBox $\mathcal{T}$, the following TBox is sound and complete for the concept inclusions that are both valid in $\mathcal{I}$ as well as entailed by $\mathcal{T}$ and further have a role depth not exceeding d.*

$$\mathcal{B}_1 := \{ C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \mid C \in \mathcal{EL}^{\bot}_{d}(\Sigma) \}$$

Proof. Consider a concept inclusion $C \sqsubseteq D$ such that $\mathsf{rd}(C) \leq d$, $\mathsf{rd}(D) \leq d$, $C \sqsubseteq_{\mathcal{I}} D$, and $C \sqsubseteq_{\mathcal{T}} D$. We infer that $C^{\mathcal{I}\mathcal{I}_d} \sqsubseteq_{\emptyset} D$ as well as $C^{\mathcal{T}_d} \sqsubseteq_{\emptyset} D$, and thus $C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} D$. Consequently, $C \sqsubseteq D$ is entailed by the considered TBox $\mathcal{B}_1$, which hence is complete. Soundness is obvious. □

As we want to emulate the problem of computing a concept inclusion base for $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$ in Formal Concept Analysis, we now define the following set of FCA attributes.

$$M_{\mathcal{I},\mathcal{T},d} := \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{ \exists\, r.\, C^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}} \mid r \in \Sigma_{\mathsf{R}} \text{ and } C \in \mathcal{EL}(\Sigma)\restriction_{d-1} \}$$

If $\mathcal{I}$, $\mathcal{T}$, and d are clear from the context, then we may also write M instead of $M_{\mathcal{I},\mathcal{T},d}$ in the following.

As we will infer from the next lemma, all closures of $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$ are expressible in terms of M, that is, for each concept description $C \in \mathcal{EL}^{\bot}(\Sigma)$, there is some subset $U \subseteq M$ such that $C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \equiv_{\emptyset} \bigsqcap U$ holds true. Before we start with proving this fact, we define the *approximation* $\lfloor C \rfloor_{\mathcal{I},\mathcal{T},d}$ of a concept description $C \in \mathcal{EL}(\Sigma)\restriction_d$ as follows.

$$\lfloor C \rfloor_{\mathcal{I},\mathcal{T},d} := \bigsqcap(\mathsf{Conj}(C) \cap \Sigma_{\mathsf{C}}) \sqcap \bigsqcap\{ \exists\, r.\, (D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}) \mid \exists\, r.\, D \in \mathsf{Conj}(C) \}$$

If $\mathcal{I}$, $\mathcal{T}$, and d are clear from the context, then we may also write $\lfloor C \rfloor$ instead of $\lfloor C \rfloor_{\mathcal{I},\mathcal{T},d}$ in the following.

Lemma 38. *For any $\mathcal{EL}^{\bot}_{d}$ concept description C, the following subsumptions are valid.*

$$C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} \lfloor C \rfloor_{\mathcal{I},\mathcal{T},d} \sqsubseteq_{\emptyset} C$$

Proof. Since $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$ is a closure operator, the second subsumption is obvious. We proceed with demonstrating the validity of the first subsumption. We do this by proving that $C \sqsubseteq_{\mathcal{I}} \lfloor C \rfloor$ and $C \sqsubseteq_{\mathcal{T}} \lfloor C \rfloor$. Fix some existential restriction $\exists\, r.\, D$ on the top level conjunction of C. On the one hand, it holds true that $D \sqsubseteq_{\mathcal{I}} D^{\mathcal{I}\mathcal{I}_{d-1}}$ and $D^{\mathcal{I}\mathcal{I}_{d-1}} \sqsubseteq_{\emptyset} D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}$, and thus $\exists\, r.\, D \sqsubseteq_{\mathcal{I}} \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}$. On the other hand, it holds true that $D \sqsubseteq_{\mathcal{T}} D^{\mathcal{T}_{d-1}}$ and $D^{\mathcal{T}_{d-1}} \sqsubseteq_{\emptyset} D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}$, and consequently $\exists\, r.\, D \sqsubseteq_{\mathcal{T}} \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}$. Summing up, we have that

$$C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} (\exists\, r.\, D)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \vartriangle \mathcal{T}_{d-1}}$$

for each existential restriction $\exists\, r.\, D \in \mathsf{Conj}(C)$, and so $C^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} \lfloor C \rfloor$. □

As a side note, we make clear how the set $M_{\mathcal{I},\mathcal{T},d}$ can be enumerated.

1. Initially, output $\bot$ and output each concept name $A \in \Sigma_{\mathsf{C}}$.
2. Set $C \coloneqq \top$.
3. If the role depth of C does not exceed $d-1$, compute the closure $C \coloneqq C^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}}$, and then output $\exists\, r.\, C$ for each role name $r \in \Sigma_{\mathsf{R}}$.
4. Compute the set of lower neighbors of C, e.g., by means of (Kriegel, 2018a, Proposition 5) or (Kriegel, 2018b, Corollarium 3.1.3.6), and for each such lower neighbor L, set $C \coloneqq L$ and go to Statement 3.

As a next step we show that, in a concept inclusion base for $\phi_{\mathcal{I},d} \triangle \phi_{\mathcal{T},d}$, it suffices that all premises are conjunctions of subsets of $M_{\mathcal{I},\mathcal{T},d}$.

Proposition 39. *Let $\mathcal{I}$ be a finitely representable interpretation and assume that $\mathcal{T}$ is a TBox. Then, the following TBox is sound and complete for the concept inclusions that are both valid in $\mathcal{I}$ as well as entailed by $\mathcal{T}$ and further have a role depth not exceeding d.*

$$\mathcal{B}_2 \coloneqq \{\, \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \mid U \subseteq M_{\mathcal{I},\mathcal{T},d} \,\}$$

Proof. Soundness is easy. Completeness is demonstrated using structural induction by showing that the defined TBox $\mathcal{B}_2$ entails the TBox $\mathcal{B}_1$ from Lemma 37. Thus, fix some interpretation $\mathcal{J}$ that is a model of $\mathcal{B}_2$. We show that, for each concept description $C \in \mathcal{EL}^{\bot}_{d}(\Sigma)$, it holds true that $\mathcal{J}$ is a model of the concept inclusion $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$.

The base case where $C = \bot$ is trivial. The base case where $C = \top$ follows from $\top \equiv_\emptyset \bigsqcap \emptyset$ and $\bigsqcap \emptyset \sqsubseteq (\bigsqcap \emptyset)^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \in \mathcal{B}_2$. The base case where $C = A$ for some concept name $A \in \Sigma_{\mathsf{C}}$ is also obvious since $A \in M$ and, thus, $A \sqsubseteq A^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \in \mathcal{B}_2$.

The inductive case where $C = D \sqcap E$ can be proven as follows. By induction hypothesis it holds true that $D \sqsubseteq_{\mathcal{J}} D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$ and $E \sqsubseteq_{\mathcal{J}} E^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$. Furthermore, both closures $D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$ and $E^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$ are expressible in terms of M, that is, there exist subsets $U, V \subseteq M$ such that $D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \equiv_\emptyset \bigsqcap U$ and $E^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \equiv_\emptyset \bigsqcap V$. Consequently, it holds true that

$$D \sqcap E \sqsubseteq_{\mathcal{J}} \bigsqcap (U \cup V).$$

Of course, $\mathcal{B}_2$ contains the concept inclusion $\bigsqcap(U \cup V) \sqsubseteq (\bigsqcap(U \cup V))^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$, and since $\mathcal{J} \models \mathcal{B}_2$ we conclude that

$$D \sqcap E \sqsubseteq_{\mathcal{J}} (D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \sqcap E^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d})^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \equiv_\emptyset (D \sqcap E)^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}.$$

Eventually, we consider the remaining inductive case where $C = \exists\, r.\, D$. The induction hypothesis yields that $D \sqsubseteq_{\mathcal{J}} D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$. It is further trivial that $D^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d} \sqsubseteq_\emptyset D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}}$. We conclude that

$$\exists\, r.\, D \sqsubseteq_{\mathcal{J}} \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}}.$$

As $\exists\, r.\, (D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}})$ is an element of M, we infer that the concept inclusion $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}} \sqsubseteq (\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}})^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}$ is valid in $\mathcal{J}$, and since $D^{\mathcal{I}\mathcal{I}_{d-1} \triangle \mathcal{T}_{d-1}} \sqsubseteq_\emptyset D$ we conclude that

$$\exists\, r.\, D \sqsubseteq_{\mathcal{J}} (\exists\, r.\, D)^{\mathcal{I}\mathcal{I}_d \triangle \mathcal{T}_d}. \qquad \square$$

Fix some finitely representable interpretation $\mathcal{I}$, an $\mathcal{EL}^{\bot}$ TBox $\mathcal{T}$, and some role-depth bound $d \in \mathbb{N}$. For translating our problem setting into notions of FCA, we further define the *induced context*

$$\mathbb{K}_{\mathcal{I},\mathcal{T},d} \coloneqq (\Delta^{\mathcal{I}}, M_{\mathcal{I},\mathcal{T},d}, I_{\mathcal{I},\mathcal{T},d}) \text{ where } I_{\mathcal{I},\mathcal{T},d} \coloneqq \{\, (\delta, C) \mid \delta \in C^{\mathcal{I}} \,\},$$

and we define the *induced implication set*

$$\mathcal{L}_{\mathcal{I},\mathcal{T},d} \coloneqq \{\, U \to V \mid U, V \subseteq M_{\mathcal{I},\mathcal{T},d} \text{ and } \textstyle\bigsqcap U \sqsubseteq_{\mathcal{T}} \bigsqcap V \,\}.$$

If $\mathcal{I}$, $\mathcal{T}$, and d are clear from the context, then we may also write $\mathbb{K}$ instead of $\mathbb{K}_{\mathcal{I},\mathcal{T},d}$, I instead of $I_{\mathcal{I},\mathcal{T},d}$, and $\mathcal{L}$ instead of $\mathcal{L}_{\mathcal{I},\mathcal{T},d}$ in the following.

The next lemma shows that the closure operators $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$ and $\phi_{\mathbb{K}_{\mathcal{I},\mathcal{T},d}} \vartriangle \phi_{\mathcal{L}_{\mathcal{I},\mathcal{T},d}}$ are closely related—a fact that will later be used to translate implication bases of the latter to concept inclusion bases of the former.

Lemma 40. *For any subset $U \subseteq M$, the following holds true.*

$$(\textstyle\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \equiv_{\emptyset} \bigsqcap U^{II \vartriangle \mathcal{L}}$$

Proof. Firstly, we define the *projection* onto M as the mapping $\pi \colon \mathcal{EL}^{\bot}(\Sigma) \to \wp(M)$ where $\pi(C) \coloneqq \{\, D \in M \mid C \sqsubseteq_{\emptyset} D \,\}$. Then, we observe that the following equations are valid.

$$\begin{aligned}
&\pi((\textstyle\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d}) \\
&= \{\, C \mid C \in M \text{ and } (\textstyle\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \sqsubseteq_{\emptyset} C \,\} \\
&= \{\, C \mid C \in M,\ (\textstyle\bigsqcap U)^{\mathcal{I}\mathcal{I}_d} \sqsubseteq_{\emptyset} C, \text{ and } (\textstyle\bigsqcap U)^{\mathcal{T}_d} \sqsubseteq_{\emptyset} C \,\} \\
&= \{\, C \mid C \in M,\ (\textstyle\bigsqcap U)^{\mathcal{I}} \subseteq C^{\mathcal{I}}, \text{ and } \textstyle\bigsqcap U \sqsubseteq_{\mathcal{T}} C \,\} \\
&= \{\, C \mid C \in M,\ U^{I} \subseteq \{C\}^{I}, \text{ and } U \to_{\mathcal{L}} \{C\} \,\} \\
&= \{\, C \mid C \in M,\ C \in U^{II}, \text{ and } C \in U^{\mathcal{L}} \,\} \\
&= \{\, C \mid C \in M \text{ and } C \in U^{II \vartriangle \mathcal{L}} \,\} \\
&= U^{II \vartriangle \mathcal{L}}
\end{aligned}$$

As a consequence, we obtain that $\bigsqcap \pi((\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d}) \equiv_{\emptyset} \bigsqcap U^{II \vartriangle \mathcal{L}}$. Secondly, we have that the pair $(\bigsqcap, \pi)$ is a Galois connection between $(\wp(M), \subseteq)$ and $(\mathcal{EL}(\Sigma), \sqsubseteq_{\emptyset})$, cf. the $\mathcal{EL}^{\bot}$ case of (Kriegel, 2017, Lemma 10.1), and consequently it holds true that $C \equiv_{\emptyset} \bigsqcap \pi(C)$ for each concept description C that is expressible in terms of M. We conclude that $\bigsqcap \pi((\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d}) \equiv_{\emptyset} (\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d}$. □

As a final step we show in the next proposition that, in a concept inclusion base for $\phi_{\mathcal{I},d} \vartriangle \phi_{\mathcal{T},d}$, it is enough that all premises are conjunctions of premises of an implication base for $\phi_{\mathbb{K}} \vartriangle \phi_{\mathcal{L}}$.

Proposition 41. *Let $\mathcal{B}$ be an implication base for $\phi_{\mathbb{K}_{\mathcal{I},\mathcal{T},d}} \vartriangle \phi_{\mathcal{L}_{\mathcal{I},\mathcal{T},d}}$ with respect to the background knowledge $\{\, U \to V \mid \bigsqcap U \sqsubseteq_{\emptyset} \bigsqcap V \,\}$. Then, the TBox*

$$\mathcal{B}_3 \coloneqq \{\, \textstyle\bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}_d \vartriangle \mathcal{T}_d} \mid \exists\, V \colon U \to V \in \mathcal{B} \,\}$$

is sound and complete for those concept inclusions that are both valid in $\mathcal{I}$ and entailed by $\mathcal{T}$ and that furthermore have a role depth not exceeding d.

Proof. Soundness is obvious. We proceed with proving completeness; for this purpose consider a model $\mathcal{J}$ of $\mathcal{B}_3$. We show that $\mathcal{J}$ is a model of the TBox $\mathcal{B}_2$ from Proposition 39.

We define the formal context $\mathbb{K}_{\mathcal{J}} := (\Delta^{\mathcal{J}}, M_{\mathcal{I},\mathcal{T},d}, J)$ where $(\delta, C) \in J$ if $\delta \in C^{\mathcal{J}}$. We begin with proving that $\mathbb{K}_{\mathcal{J}} \models \mathcal{B}$. Fix some implication $U \to V \in \mathcal{B}$. Without loss of generality we assume that $V = U^{II\triangle\mathcal{L}}$. Then, $\mathcal{B}_3$ contains the concept inclusion $\bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}_d\triangle\mathcal{T}_d}$, which is, hence, valid in $\mathcal{J}$. We proceed with demonstrating that the implication $U \to V$ is valid in $\mathbb{K}_{\mathcal{J}}$. Note that according to Lemma 40 it holds true that $(\bigsqcap U)^{\mathcal{I}\mathcal{I}_d\triangle\mathcal{T}_d}$ and $\bigsqcap U^{II\triangle\mathcal{L}}$ are equivalent with respect to the empty TBox. Thus, we have the following equivalences.

$$\begin{aligned}
&\bigsqcap U \sqsubseteq_{\mathcal{J}} (\bigsqcap U)^{\mathcal{I}\mathcal{I}_d\triangle\mathcal{T}_d} \\
\text{if, and only if, } &\delta \in (\bigsqcap U)^{\mathcal{J}} \text{ implies } \delta \in (\bigsqcap U)^{(\mathcal{I}\mathcal{I}_d\triangle\mathcal{T}_d)\mathcal{J}} \text{ for each } \delta \in \Delta^{\mathcal{J}} \\
\text{if, and only if, } &\delta \in (\bigsqcap U)^{\mathcal{J}} \text{ implies } \delta \in (\bigsqcap U^{II\triangle\mathcal{L}})^{\mathcal{J}} \text{ for each } \delta \in \Delta^{\mathcal{J}} \\
\text{if, and only if, } &\delta \in U^{J} \text{ implies } \delta \in U^{(II\triangle\mathcal{L})J} \text{ for each } \delta \in \Delta^{\mathcal{J}} \\
\text{if, and only if, } &\mathbb{K}_{\mathcal{J}} \models U \to U^{II\triangle\mathcal{L}}
\end{aligned}$$

Let now $V \subseteq M$ be an arbitrary subset. Of course, then $\phi_{\mathbb{K}} \triangle \phi_{\mathcal{L}} \models V \to V^{II\triangle\mathcal{L}}$, and so $\mathcal{B} \models V \to V^{II\triangle\mathcal{L}}$, since $\mathcal{B}$ is complete for $\phi_{\mathbb{K}} \triangle \phi_{\mathcal{L}}$. As a consequence we obtain that $\mathbb{K}_{\mathcal{J}} \models V \to V^{II\triangle\mathcal{L}}$, and thus

$$\bigsqcap V \sqsubseteq_{\mathcal{J}} \bigsqcap V^{II\triangle\mathcal{L}} \equiv_{\emptyset} (\bigsqcap V)^{\mathcal{I}\mathcal{I}_d\triangle\mathcal{T}_d}.$$

Summing up, it holds true that $\mathcal{J}$ is a model of the complete TBox $\mathcal{B}_2$ from Proposition 39, which implies completeness of $\mathcal{B}_3$. □

As an immediate consequence of our previous results we find that there exists always a *canonical* finite concept inclusion base for $\phi_{\mathcal{I},d} \triangle \phi_{\mathcal{T},d}$.

Corollary 42. *For each finitely representable interpretation $\mathcal{I}$, for each TBox $\mathcal{T}$, and for any role-depth bound $d \in \mathbb{N}$, the following TBox, called* canonical base*, is sound and complete for the concept inclusions that are both valid in $\mathcal{I}$ as well as entailed by $\mathcal{T}$ and further have a role depth not exceeding d.*

$$\mathsf{Can}(\mathcal{I}, \mathcal{T}, d) := \{ \bigsqcap P \sqsubseteq \bigsqcap P^{II\triangle\mathcal{L}} \mid P \textit{ is a pseudo-closure of } \phi_{\mathbb{K}} \triangle \phi_{\mathcal{L}} \}$$

8.5. Error-Tolerant Axiomatization of Concept Inclusions from Interpretations

Assume that an interpretation $\mathcal{I}$ as well as a TBox $\mathcal{T}$ are given such that $\mathcal{I}$ contains observations that could possibly be faulty due to inaccurate generation methods, and that $\mathcal{T}$ is certainly valid in the domain of interest, e.g., as it has been hand-crafted by experts. In particular, we assume that $\mathcal{I}$ is not a model of $\mathcal{T}$, i.e., that at least one domain element in $\mathcal{I}$ exists which serves as a counterexample against at least one concept inclusion from $\mathcal{T}$. However, we are expected to axiomatize terminological knowledge from $\mathcal{I}$

which is valid in the domain of interest. As a solution, we suggest to construct the concept inclusion base of the supremum of the closure operators that are induced by $\mathcal{I}$, and by $\mathcal{T}$, respectively. It is then ensured that only those concept inclusions are axiomatized which are valid for all those domain elements of $\mathcal{I}$ that respect the concept inclusions in $\mathcal{T}$, i.e., that we axiomatize concept inclusions from $\mathcal{I}$ that are compatible with the axioms contained in $\mathcal{T}$. In a certain sense this yields a method for an error correction in $\mathcal{I}$ when learning concept inclusions. We will define a short motivating example as follows.

$$\begin{aligned}\Sigma_{\mathsf{C}} &\coloneqq \{\mathsf{Person}, \mathsf{Car}, \mathsf{Wheel}\}\\ \Sigma_{\mathsf{R}} &\coloneqq \{\mathsf{child}\}\\ \mathcal{T} &\coloneqq \{\exists\,\mathsf{child}.\top \sqsubseteq \mathsf{Person},\ \mathsf{Person} \sqcap \mathsf{Car} \sqsubseteq \bot\}\end{aligned}$$

$\mathcal{I}$:

Consider the concept inclusion $\mathsf{Car} \sqsubseteq \exists\,\mathsf{child}.\mathsf{Wheel}$. Of course, it is valid in $\mathcal{I}$ and, thus, it is entailed by the canonical base for $\mathcal{I}$ when applying the construction from Distel (2011) or from Borchmann, Distel and Kriegel (2016). We can show that this concept inclusion is also valid for the supremum $\phi_{\mathcal{I},d} \triangledown \phi_{\mathcal{T},d}$ for any $d \geq 1$. The closure of Car with respect to $\phi_{\mathcal{I},d} \triangledown \phi_{\mathcal{T},d}$ is the least common subsumer of all those concept descriptions that are closures of both $\phi_{\mathcal{I},d}$ and $\phi_{\mathcal{T},d}$, and that are subsumed by Car. It is easy to see that this closure can be computed by an exhaustive repeated application of both closure operators until a fixed point is reached. As we shall see below, the fixed point $\bot$ is reached after the first iteration, and hence $\bot$ is the closure.

$$\begin{aligned}\mathsf{Car}^{\mathcal{I}\mathcal{I}_d} &\equiv \mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel}\\ (\mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel})^{\mathcal{T}_d} &\equiv \mathsf{Car} \sqcap \exists\,\mathsf{child}.\mathsf{Wheel} \sqcap \mathsf{Person} \sqcap \bot \equiv \bot\end{aligned}$$

However, the considered concept inclusion $\mathsf{Car} \sqsubseteq \exists\,\mathsf{child}.\mathsf{Wheel}$ is also a consequence of the stronger concept inclusion $\mathsf{Car} \sqsubseteq \bot$, and hence it would not have been axiomatized in a construction of the canonical base. In particular, it is readily verified that the object δ is not compatible with $\mathcal{T}$—in contrast to the other objects ϵ, ζ, and η. Eventually, Car is a pseudo-closure of the supremum, and hence the canonical base contains the axiom expressing the non-existence of cars.

So far, no effective procedure for axiomatizing concept inclusions from such a supremum $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$ for the unrestricted case or from $\phi_{\mathcal{I},d} \triangledown \phi_{\mathcal{T},d}$ for the role-depth bounded case has been developed. It is straight-forward to claim that one could suitably generalize the techniques of Distel (2011) or of Borchmann, Distel and Kriegel (2016), much like this has been achieved in Section 8.4 for the infimum. This will be subject of a future publication of the author. Some first results in that direction have already been found, and shall be presented in the following. The next lemma shows how closures in such a supremum can be computed.

Lemma 43. *Let $\mathcal{I}$ be an interpretation and consider an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox $\mathcal{T}$. Then, for each $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C, the following statement holds true.*

$$C^{\mathcal{I}\mathcal{I}\triangledown\mathcal{T}} \equiv_{\emptyset} \left(\bigcup\{\, X \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\mathcal{T}\mathcal{I}} \,\}\right)^{\mathcal{I}}$$

Proof. We have the following which shall be justified below.

$$
\begin{aligned}
C^{\mathcal{I}\mathcal{I}\triangledown\mathcal{T}} &\equiv_\emptyset \bigvee\{\, D \mid D \in \mathcal{EL}^\bot(\Sigma) \text{ and } D \equiv_\emptyset D^{\mathcal{I}\mathcal{I}} \equiv_\emptyset D^{\mathcal{T}} \sqsubseteq_\emptyset C \,\} && (1)\\
&\equiv_\emptyset \bigvee\{\, X^{\mathcal{I}} \mid X \subseteq \Delta^{\mathcal{I}} \text{ and } X^{\mathcal{I}} \equiv_\emptyset X^{\mathcal{I}\mathcal{T}} \sqsubseteq_\emptyset C \,\} && (2)\\
&\equiv_\emptyset \bigvee\{\, X^{\mathcal{I}} \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\mathcal{T}\mathcal{I}} \,\} && (3)\\
&\equiv_\emptyset (\bigcup\{\, X \mid X \subseteq C^{\mathcal{I}} \text{ and } X \subseteq X^{\mathcal{I}\mathcal{T}\mathcal{I}} \,\})^{\mathcal{I}} && (4)
\end{aligned}
$$

We begin with observing that the equivalence in Equation (1) is satisfied; it follows from the characterization of suprema of closure operators in Section 3. As all concept descriptions D over which the least common subsumer is constructed are model-based most specific concept descriptions for $\mathcal{I}$, we infer that Equation (2) holds true. Now we find that $X^{\mathcal{I}} \sqsubseteq_\emptyset C$ is equivalent to $X \subseteq C^{\mathcal{I}}$, cf. the Galois properties as described by Distel (2011, Lemma 4.1) and by Borchmann, Distel and Kriegel (2016, Lemmas 4.3 and 4.4), and further that $X^{\mathcal{I}} \equiv_\emptyset X^{\mathcal{I}\mathcal{T}}$ is satisfied if, and only if, $X^{\mathcal{I}} \sqsubseteq_\emptyset X^{\mathcal{I}\mathcal{T}}$ as well as $X^{\mathcal{I}} \sqsupseteq_\emptyset X^{\mathcal{I}\mathcal{T}}$, where the former is equivalent to $X \subseteq X^{\mathcal{I}\mathcal{T}\mathcal{I}}$ and the latter is trivially true, cf. Lemma 18. We conclude the validity of Equation (3). Eventually, Equation (4) is well known. □

As an immediate corollary, we obtain that the supremum $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$ has only finitely many closures if $\mathcal{I}$ is finitely representable—a fact that does not analogously hold true for the infimum $\phi_{\mathcal{I}} \vartriangle \phi_{\mathcal{T}}$.

In the following lemma, we show that lifting a concept inclusion by existentially quantifying both premise and conclusion preserves validity in the supremum $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}}$.

Lemma 44. *Fix some interpretation $\mathcal{I}$ as well as an $\mathcal{EL}^\bot_{\mathsf{st}}$ terminological box $\mathcal{T}$. Then, for each $\mathcal{EL}^\bot_{\mathsf{st}}$ concept inclusion $C \sqsubseteq D$ and for each role name $r \in \Sigma_{\mathsf{R}}$, the following statement is satisfied.*

$$\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}} \models C \sqsubseteq D \textit{ implies } \phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}} \models \exists\, r.\, C \sqsubseteq \exists\, r.\, D$$

Proof. Assume that $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}} \models C \sqsubseteq D$, that is, for every concept description E such that $E \sqsubseteq_\emptyset C$ and $E \equiv_\emptyset E^{\mathcal{I}\mathcal{I}} \equiv_\emptyset E^{\mathcal{T}}$, it holds true that $E \sqsubseteq_\emptyset D$. Now consider some concept description F satisfying $F \sqsubseteq_\emptyset \exists\, r.\, C$ as well as $F \equiv_\emptyset F^{\mathcal{I}\mathcal{I}} \equiv_\emptyset F^{\mathcal{T}}$; we shall show that $F \sqsubseteq_\emptyset \exists\, r.\, D$ holds true. Without loss of generality, we assume that F is reduced, which implies that, in particular, all top-level conjuncts are incomparable with respect to $\sqsubseteq_\emptyset$, and further we assume that each filler of an existential restriction on the top-level of $F^{\mathcal{I}\mathcal{I}}$ and of $F^{\mathcal{T}}$, respectively, is itself closed, that is, $X \equiv_\emptyset X^{\mathcal{I}\mathcal{I}}$ for each $\exists\, r.\, X \in \mathsf{Conj}(F^{\mathcal{I}\mathcal{I}})$ and $Y \equiv_\emptyset Y^{\mathcal{T}}$ for each $\exists\, s.\, Y \in \mathsf{Conj}(F^{\mathcal{T}})$.

From $F \sqsubseteq_\emptyset \exists\, r.\, C$ we infer that there is some $\exists\, r.\, E \in \mathsf{Conj}(F)$ such that $E \sqsubseteq_\emptyset C$. Furthermore, $F \equiv_\emptyset F^{\mathcal{I}\mathcal{I}} \equiv_\emptyset F^{\mathcal{T}}$ implies the existence of concept descriptions $\exists\, r.\, E_1^{\mathcal{I}\mathcal{I}} \in \mathsf{Conj}(F^{\mathcal{I}\mathcal{I}})$, $\exists\, r.\, E_2^{\mathcal{T}} \in \mathsf{Conj}(F^{\mathcal{T}})$, $\exists\, r.\, E_3^{\mathcal{I}\mathcal{I}} \in \mathsf{Conj}(F^{\mathcal{I}\mathcal{I}})$, and $\exists\, r.\, E_4 \in \mathsf{Conj}(F)$ such that

$$E \sqsupseteq_\emptyset E_1^{\mathcal{I}\mathcal{I}} \sqsupseteq_\emptyset E_2^{\mathcal{T}} \sqsupseteq_\emptyset E_3^{\mathcal{I}\mathcal{I}} \sqsupseteq_\emptyset E_4,$$

and so it follows that $E = E_4$, since F is reduced. Furthermore, we conclude that $E \equiv_\emptyset E^{\mathcal{I}\mathcal{I}} \equiv_\emptyset E^{\mathcal{T}}$ and, thus, that $E \sqsubseteq_\emptyset D$. Eventually, we infer that $\exists\, r.\, E \sqsubseteq_\emptyset \exists\, r.\, D$, and as $\exists\, r.\, E$ is a top-level conjunct in F it follows that $F \sqsubseteq_\emptyset \exists\, r.\, D$. □

Corollary 45. *Let $\mathcal{I}$ be some interpretation and $\mathcal{T}$ an $\mathcal{EL}^{\bot}_{\mathsf{st}}$ TBox. Then, for each $\mathcal{EL}^{\bot}_{\mathsf{st}}$ concept description C and for each role name r, the following holds true.*

$$(\exists\, r.\, C)^{\mathcal{I}\mathcal{I}\triangledown\mathcal{T}} \sqsubseteq_{\emptyset} \exists\, r.\, C^{\mathcal{I}\mathcal{I}\triangledown\mathcal{T}}$$

Proof. The statement immediately follows from $\phi_{\mathcal{I}} \triangledown \phi_{\mathcal{T}} \models C \sqsubseteq C^{\mathcal{I}\mathcal{I}\triangledown\mathcal{T}}$ by an application of Lemma 44. □

8.6. Axiomatization of Concept Inclusions from ABoxes

Assume that $\mathcal{A}$ is a *simple* ABox which may not only contain positive assertions, but also negative assertions, i.e., $\mathcal{A}$ may consist of axioms of the forms

$$a \mathrel{\varepsilon} A,\ a \mathrel{\not\varepsilon} A,\ (a, b) \mathrel{\varepsilon} r,\ (a, b) \mathrel{\not\varepsilon} r,$$

where $a, b \in \Sigma_{\mathsf{I}}$ are individual names, $A \in \Sigma_{\mathsf{C}}$ is a concept name, and $r \in \Sigma_{\mathsf{R}}$ is a role name. We further require $\mathcal{A}$ to be consistent, that is, it has a model. Apparently, $\mathcal{A}$ is consistent if, and only if, it does not contain $a \mathrel{\varepsilon} A$ and $a \mathrel{\not\varepsilon} A$ at the same time, and similarly it does not simultaneously contain $(a, b) \mathrel{\varepsilon} r$ and $(a, b) \mathrel{\not\varepsilon} r$.

It is readily verified that ABoxes cannot be axiomatized with respect to default semantics, i.e., when we only adopt the *Unique Name Assumption* (abbrv. *UNA*), i.e., different individual names address different individuals, and the *Open World Assumption* (abbrv. *OWA*), i.e., an axiom may be true in the domain of interest irrespective of it being entailed by the ABox $\mathcal{A}$, or alternatively, there may be axioms the validity of which cannot be decided with only the information contained in the ABox $\mathcal{A}$. This is due to the fact that the size of the domain of a model of the considered ABox is not bounded, and hence for each concept inclusion $C \sqsubseteq D$, we can construct a model of the ABox but which also contains a counterexample against $C \sqsubseteq D$. Consequently, when we aim at learning terminological boxes from assertional boxes as above, we have to impose further restrictions on the allowed models. An idea which would probably perform well in practice would be to further require the *Domain Closure Assumption* (abbrv. *DCA*), i.e., all individuals/objects of the domain of interest are known. The DCA is also utilized in *Database Theory*, where it is assumed that every individual/object which occurs in the domain of interest also occurs in the data set. Applying the DCA to the *Description Logic* setting, we would enforce that there are no individuals except explicitly described in the signature or used in the ABox, or when applying it to interpretations $\mathcal{I}$, the restriction of the extension function to Σ_{I} is surjective. Analogously, requiring the UNA to hold true for interpretations $\mathcal{I}$ implies that the restriction of the extension function to Σ_{I} is injective. It is readily verified that, for interpretations $\mathcal{I}$ satisfying both the UNA and DCA, the mapping $\cdot^{\mathcal{I}}\restriction_{\Sigma_{\mathsf{I}}}$ is bijective, and w.l.o.g. we shall hence simply assume that $\Delta^{\mathcal{I}} = \Sigma_{\mathsf{I}}$.

In particular, we then restrict the semantics as follows. A Σ_{I}-*interpretation* is an interpretation $\mathcal{I}$ where $\Delta^{\mathcal{I}} := \Sigma_{\mathsf{I}}$ and where $a^{\mathcal{I}} := a$ for all individual names $a \in \Sigma_{\mathsf{I}}$.[2] Furthermore, we call a Σ_{I}-interpretation $\mathcal{I}$ a Σ_{I}-*model* of $\mathcal{A}$ if $\mathcal{I}$ is a model of $\mathcal{A}$, and we shall then write $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$. The ABox $\mathcal{A}$ Σ_{I}-*entails* a concept inclusion $C \sqsubseteq D$ if, for each Σ_{I}-interpretation $\mathcal{I}$, it holds true that $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$ implies $\mathcal{I} \models C \sqsubseteq D$, and we denote this as $\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D$. If Σ_{I} is finite, then reasoning with respect to Σ_{I}-semantics can be reduced to reasoning with respect to default semantics when we

[2]Note that this somehow corresponds to the Herbrand universe of a FO-theory.

further allow for the use of nominals. A *nominal* is a concept description of the form $\{a_1, \ldots, a_n\}$ where $a_1, \ldots, a_n \in \Sigma_{\mathsf{I}}$ are individuals, and its extension is defined by

$$\{a_1, \ldots, a_n\}^{\mathcal{I}} \coloneqq \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$$

for each interpretation $\mathcal{I}$. It is easy to verify that, for any concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D$ if, and only if, $\mathcal{A} \cup \mathcal{T}_{\Sigma_{\mathsf{I}}} \models C \sqsubseteq D$ where the TBox $\mathcal{T}_{\Sigma_{\mathsf{I}}}$ encodes the Σ_{I}-semantics, i.e., that objects do not have multiple names (UNA) and that all objects are known, i.e., named (DCA). In particular, $\mathcal{T}_{\Sigma_{\mathsf{I}}}$ is defined as follows.

$$\mathcal{T}_{\Sigma_{\mathsf{I}}} \coloneqq \{\, \{a\} \sqcap \{b\} \sqsubseteq \bot \mid a, b \in \Sigma_{\mathsf{I}} \text{ and } a \neq b \,\} \cup \{\top \sqsubseteq \{\, a \mid a \in \Sigma_{\mathsf{I}} \,\}\}$$

Our goal now is to find a technique for the axiomatization of assertional boxes with respect to Σ_{I}-semantics, that is, to compute a concept inclusion base for a given ABox $\mathcal{A}$ that is sound and complete for all $C \sqsubseteq D$ satisfying $\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D$. Before we investigate the technical details, we first demonstrate that using an ABox (with UNA, DCA, OWA) as input yields indeed different results than using an interpretation (with UNA, DCA, CWA). Both the ABox and each of its Σ_{I}-model have in common that the set of individuals/objects is fully known. However, an ABox allows for the presence of unknown facts, i.e., by leaving out both assertions $a \sqsubseteq A$ and $a \not\sqsubseteq A$ we leave it open whether a is an instance of A, simply because we do not know it. This degree of freedom is not possible in an interpretation $\mathcal{I}$: either an object $\delta \in \Delta^{\mathcal{I}}$ belongs to an extension $A^{\mathcal{I}}$ or not; there is no means to express that it is not known. Consequently, utilizing ABoxes as input data to learn from allows for more practical use cases.

For instance, define $\mathcal{A} \coloneqq \{a \sqsubseteq A,\ a \sqsubseteq B\}$ over the signature Σ with $\Sigma_{\mathsf{C}} \coloneqq \{A, B\}$, $\Sigma_{\mathsf{R}} \coloneqq \emptyset$, and $\Sigma_{\mathsf{I}} \coloneqq \{a, b\}$. Then, the concept inclusion $A \sqsubseteq B$ is no consequence of $\mathcal{A}$, but it would be if we consider $\mathcal{A}$ as an interpretation—more specifically, $A \sqsubseteq B$ is valid in the *canonical model* $\mathcal{I}_{\mathcal{A}}$. This is due to the definition of such a canonical model:

$$\begin{aligned} &\Delta^{\mathcal{I}_{\mathcal{A}}} \coloneqq \Sigma_{\mathsf{I}} \\ &\cdot^{\mathcal{I}_{\mathcal{A}}} \colon \begin{cases} A \mapsto \{\, a \mid a \sqsubseteq A \in \mathcal{A} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (a,b) \mid (a,b) \sqsubseteq r \in \mathcal{A} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases} \end{aligned}$$

The *dual canonical model* $\mathcal{I}_{\mathcal{A}}^{\partial}$ is given as follows.

$$\begin{aligned} &\Delta^{\mathcal{I}_{\mathcal{A}}^{\partial}} \coloneqq \Sigma_{\mathsf{I}} \\ &\cdot^{\mathcal{I}_{\mathcal{A}}^{\partial}} \colon \begin{cases} A \mapsto \{\, a \mid a \not\sqsubseteq A \notin \mathcal{A} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, (a,b) \mid (a,b) \not\sqsubseteq r \notin \mathcal{A} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases} \end{aligned}$$

Clearly, both $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}_{\mathcal{A}}^{\partial}$ are Σ_{I}-models of $\mathcal{A}$. Furthermore, it holds true that any Σ_{I}-model of $\mathcal{A}$ is *between* these two canonical models and, more specifically, for each Σ_{I}-model $\mathcal{I}$ of $\mathcal{A}$, it holds true that

$$\begin{gathered} A^{\mathcal{I}_{\mathcal{A}}} \subseteq A^{\mathcal{I}} \subseteq A^{\mathcal{I}_{\mathcal{A}}^{\partial}} \text{ for any } A \in \Sigma_{\mathsf{C}} \\ \text{and } r^{\mathcal{I}_{\mathcal{A}}} \subseteq r^{\mathcal{I}} \subseteq r^{\mathcal{I}_{\mathcal{A}}^{\partial}} \text{ for each } r \in \Sigma_{\mathsf{R}}. \end{gathered}$$

As an immediate consequence we obtain that, for each finite signature Σ, there are only finitely many Σ_{I}-models of a simple ABox.

The following lemma states some equivalent characterizations of Σ_{I}-entailment.

Lemma 46. *Let $\mathcal{A}$ be an ABox, and assume that $C \sqsubseteq D$ is a concept inclusion. Then, the following statements are equivalent:*

1. $\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D$
2. $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$ *implies* $\mathcal{I} \models C \sqsubseteq D$ *for each interpretation* $\mathcal{I}$.
3. $\mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A}$ *implies* $\phi_{\mathcal{I}} \models C \sqsubseteq D$ *for any interpretation* $\mathcal{I}$.
4. $\triangle\{ \phi_{\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \} \models C \sqsubseteq D$
5. $\emptyset \models \bigvee\{ C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \} \sqsubseteq D$.

Proof. Statements 1 and 2 are equivalent by the very definition of Σ_{I}-semantics. Using a result from Distel (2011, Lemma 4.1) shows that Statements 2 and 3 are equivalent too. The equivalence of Statements 3 to 5 follows immediately from Section 3. □

We define the mapping $\phi_{\mathcal{A}} \colon \mathcal{EL}^{\bot}_{\mathsf{st}}(\Sigma) \to \mathcal{EL}^{\bot}_{\mathsf{st}}(\Sigma)$ induced by some simple ABox $\mathcal{A}$ as above by

$$\phi_{\mathcal{A}} \colon C \mapsto C^{\mathcal{A}} := \bigvee\{ C^{\mathcal{I}\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \}.$$

It then holds true $\phi_{\mathcal{A}} := \triangle\{ \phi_{\mathcal{I}} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \}$, and so $\phi_{\mathcal{A}}$ is a closure operator in the dual of $\boldsymbol{\mathcal{EL}}^{\bot}(\Sigma)$. Furthermore, the interpretation $\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}}$, called *canonical* Σ_{I}*-model* of $\mathcal{A}$, is defined as the disjoint union of all Σ_{I}-models of $\mathcal{A}$, that is, we set

$$\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}} := \biguplus\{ \mathcal{I} \mid \mathcal{I} \models^{\Sigma_{\mathsf{I}}} \mathcal{A} \}.$$

Apparently, it follows that, for any concept inclusion $C \sqsubseteq D$,

$$\mathcal{A} \models^{\Sigma_{\mathsf{I}}} C \sqsubseteq D \text{ if, and only if, } \mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}} \models C \sqsubseteq D,$$

and so the closure operators $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}}}$ coincide. The canonical Σ_{I}-model $\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}}$ is finite if the signature Σ is finite.

Returning back to our initial goal of axiomatizing concept inclusions from some such finite simple ABox $\mathcal{A}$, we can now provide a solution for doing so, namely we suggest to compute some concept inclusion base of this newly introduced closure operator $\phi_{\mathcal{A}}$ or, equivalently, to compute a concept inclusion base of the canonical Σ_{I}-model $\mathcal{I}_{\mathcal{A}}^{\Sigma_{\mathsf{I}}}$. For the latter, we can immediately apply the existing procedures from Distel (2011); from Borchmann, Distel and Kriegel (2016); or from Kriegel (2017).

9. Conclusion

We have defined the notion of most specific consequences with respect to TBoxes in the description logic $\mathcal{EL}^{\bot}$ and some of its extensions with greatest fixed-point semantics, and characterized conditions for the existence of most specific consequences as well as devised means for their computation. Furthermore, we have provided several applications and investigated the interplay of the corresponding closure operator induced by a given TBox with the previously found closure operator induced by an interpretation—more specifically, we have shown how their infimum can be utilized for learning from sequences of interpretations, and have motivated how their supremum can be used for an error-tolerant axiomatization of concept inclusions from interpretations in the presence of a hand-crafted or manually verified TBox that indicates errors in the observed interpretation. Other applications considered a characterization of entailment, a characterization of soundness and completeness, a rather abstract proposal for merging two terminological boxes, and a technique for axiomatizing concept inclusions from simple ABoxes that may also contain negated axioms under Open World Assumption, Domain Closure Assumption, and Unique Name Assumption.

Future research could provide concrete procedures for the proposals in Sections 8.3 and 8.5, could optimize some of the results and procedures, and could investigate how our results can be extended to a more expressive description logic. Please note that the computation of most specific consequences is closely related to the problem of *TBox elimination*. In the description logic $\mathcal{ALC}(\sqcup, *)$, which extends $\mathcal{ALC}$ with *union* of roles and *reflexive-transitive closure* of roles, it is easy to verify that the most specific consequence of C w.r.t. $\mathcal{T}$ is equivalent to the concept description $C \sqcap \forall\, (r_1 \sqcup \ldots \sqcup r_n)^*.\, C_{\mathcal{T}}$ where $r_1, \ldots, r_n$ are the role names occurring in C or in $\mathcal{T}$ and where $C_{\mathcal{T}}$ is defined as $\bigsqcap\{\, \neg C \sqcup D \mid C \sqsubseteq D \in \mathcal{T}\,\}$. However, as we are interested in non-Boolean description logics for inductive learning, this result is not directly helpful for our purposes.

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