An Algebraic View on p-Admissible Concrete Domains for Lightweight Description Logics

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Abstract. Concrete domains have been introduced in Description Logics (DLs) to enable reference to concrete objects (such as numbers) and predefined predicates on these objects (such as numerical comparisons) when defining concepts. To retain decidability when integrating a concrete domain into a decidable DL, the domain must satisfy quite strong restrictions. In previous work, we have analyzed the most prominent such condition, called ω-admissibility, from an algebraic point of view. This provided us with useful algebraic tools for proving ω-admissibility, which allowed us to find new examples for concrete domains whose integration leaves the prototypical expressive DL $\mathcal{ALC}$ decidable.

When integrating concrete domains into lightweight DLs of the $\mathcal{EL}$ family, achieving decidability is not enough. One wants reasoning in the resulting DL to be tractable. This can be achieved by using so-called p-admissible concrete domains and restricting the interaction between the DL and the concrete domain. In the present paper, we investigate p-admissibility from an algebraic point of view. Again, this yields strong algebraic tools for demonstrating p-admissibility. In particular, we obtain an expressive numerical p-admissible concrete domain based on the rational numbers. Although ω-admissibility and p-admissibility are orthogonal conditions that are almost exclusive, our algebraic characterizations of these two properties allow us to locate an infinite class of p-admissible concrete domains whose integration into $\mathcal{ALC}$ yields decidable DLs.

Keywords: Description logic · Concrete domains · p-admissibility · Convexity · ω-admissibility · Finite boundedness · Tractability · Decidability · Constraint satisfaction

1 Introduction

Description Logics (DLs) [3,5] are a well-investigated family of logic-based knowledge representation languages, which are frequently used to formalize ontologies for application domains such as the Semantic Web [25] or biology

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and medicine [24]. A DL-based ontology consists of inclusion statements (so-called GCIs) between concepts defined using the DL at hand. For example, the GCI $\mathit{Human} \sqsubseteq \exists \mathit{parent}.\mathit{Human}$, which says that every human being has a human parent, uses concepts expressible in $\mathcal{EL}$. This GCI clearly implies the inclusion $\mathit{Human} \sqsubseteq \exists \mathit{parent}.\exists \mathit{parent}.\mathit{Human}$, i.e., *Human* is subsumed by $\exists \mathit{parent}.\exists \mathit{parent}.\mathit{Human}$ w.r.t. any ontology containing the above GCI. Keeping the subsumption problem decidable, and preferably of a low complexity, is an important design goal for DLs. While subsumption in the lightweight DL $\mathcal{EL}$ is tractable (i.e., decidable in polynomial time), it is ExpTime-complete in $\mathcal{ALC}$, which is obtain from $\mathcal{EL}$ by adding negation [5].

If information about the age of human beings is relevant in the application at hand, then one would like to associate humans with their ages and formulate constraints on these numbers. This becomes possible by integrating concrete domains into DLs [4]. Using the concrete domain $(\mathbb{Q}, >)$, we can express that children cannot be older than their parents with the GCI $>(\mathit{age}, \mathit{parent\,age}) \sqsubseteq \bot$, where $\bot$ is the bottom concept (always interpreted as the empty set) and *age* is a concrete feature that maps from the abstract domain populating concepts into the concrete domain $(\mathbb{Q}, >)$. While integrating $(\mathbb{Q}, >)$ leaves $\mathcal{ALC}$ decidable [30], this is no longer the case if we integrate $(\mathbb{Q}, +_1)$, where $+_1$ is a binary predicate that is interpreted as incrementation [5,7]. In [31], ω-admissibility was introduced as a condition on concrete domains that ensures decidability. It was shown in that paper that Allen's interval logic [1] as well as the region connection calculus RCC8 [33] can be represented as ω-admissible concrete domains. Since ω-admissibility is a collection of rather complex technical conditions, it is not easy to show that a given concrete domain satisfies this property. In [7], we relate ω-admissibility to well-known notions from model theory, which allows us to prove ω-admissibility of certain concrete domains (among them Allen and RCC8) using known results from model theory. A different algebraic condition (called *EHD*) that ensures decidability was introduced in [18], and used in [29] to show decidability and complexity results for a concrete domains based on the integers.

When integrating a concrete domain into a lightweight DL like $\mathcal{EL}$, one wants to preserve tractability rather than just decidability. To achieve this, the notion of p-admissible concrete domains was introduced in [2] and paths of length > 1 were disallowed in concrete domain constraints. Regarding the latter restriction, note that, in the above example, we have used the path *parent age*, which has length 2. The restriction to paths of length 1 means (in our example) that we can no longer compare the ages of different humans, but we can still define concepts like teenager, using the GCI

$$\mathit{Teenager} \sqsubseteq \mathit{Human} \sqcap \geq_{10}(\mathit{age}) \sqcap \leq_{19}(\mathit{age}),$$

where $\geq_{10}$ and $\leq_{19}$ are unary predicates respectively interpreted as the rational numbers greater equal 10 and smaller equal 19. In a p-admissible concrete domain, satisfiability of conjunctions of atomic formulae and validity of implications between such conjunctions must be tractable. In addition, the concrete domain must be *convex*, which roughly speaking means that a conjunction cannot imply a true disjunction. For example, the concrete domain $(\mathbb{Q}, >, =, <)$ is ω-admissible [7], but it is not convex since $x < y \wedge x < z$ implies $y < z \vee y = z \vee y > z$, but none of the disjuncts. In [2], two p-admissible concrete domains were exhibited, where one of them is based on $\mathbb{Q}$ with unary predicates $=_p, >_p$ and binary predicates $+_p, =$. To the best of our knowledge, since then no other p-admissible concrete domains have been described in the literature.

One of the main contributions of the present paper is to devise algebraic characterizations of convexity in different settings. We start by noting that the definition of convexity given in [2] is ambiguous, and that what was really meant is what we call *guarded* convexity. However, in the presence of the equality predicate (which is available in the two p-admissible concrete domains introduced in [2]), the two notions of convexity coincide. Then we devise a general characterization of convexity based on the notion of *square embeddings*, which are embeddings of the product $\mathfrak{B}^2$ of a relational structure $\mathfrak{B}$ into $\mathfrak{B}$. We investigate the implications of this characterization further for so-called ω-categorical structures, finitely bounded structures, and numerical concrete domains. For *ω-categorical structures*, the square embedding criterion for convexity can be simplified, and we use this result to obtain new p-admissible concrete domains: countably infinite vector spaces over finite fields. *Finitely bounded structures* can be defined by specifying finitely many forbidden patterns, and are of great interest in the constraint satisfaction (CSP) community [15]. We show that, for such structures, convexity is a necessary *and sufficient* condition for p-admissibility. This result provides use with many examples of p-admissible concrete domains, but their usefulness in practice still needs to be investigated. Regarding *numerical concrete domains*, we exhibit a new and quite expressive p-admissible concrete domain based on the rational numbers, whose predicates are defined by linear equations over $\mathbb{Q}$.

Next, the paper investigates the connection between p-admissibility and ω-admissibility. We show that only trivial concrete domains can satisfy both properties. However, by combining the results on finitely bounded structures of the present paper with results in [7], we can show that convex finitely bounded homogeneous structures, which are p-admissible, can be integrated into $\mathcal{ALC}$ (even without the length 1 restriction on role paths) without losing decidability. Whereas these structures are not ω-admissible, they can be expressed in an ω-admissible concrete domain [7]. Finally, we show that, in general, the restriction to paths of length 1 is needed when integrating a p-admissible concrete domain into $\mathcal{EL}$, not only to stay tractable, but even to retain decidability.

2 Preliminaries

In this section, we introduce the algebraic and logical notions that will be used in the rest of the paper. The set $\{1, \ldots, n\}$ is denoted by $[n]$. We use the bar notation for tuples; for a tuple $\bar{t}$ indexed by a set I, the value of $\bar{t}$ at the position $i \in I$ is denoted by $\bar{t}[i]$. For a function $f\colon A^k \to B$ and n-tuples $\bar{t}_1, \ldots, \bar{t}_k \in A^n$, we use $f(\bar{t}_1, \ldots, \bar{t}_k)$ as a shortcut for the tuple $\big(f(\bar{t}_1[1], \ldots \bar{t}_k[1]), \ldots, f(\bar{t}_1[n], \ldots, \bar{t}_k[n])\big)$.

From a mathematical point of view, concrete domains are relational structures. A *relational signature* τ is a set of *relation symbols*, each with an associated natural number called *arity*. For a relational signature τ, a *relational τ-structure* $\mathfrak{A}$ (or simply τ-structure or structure) consists of a set A (the *domain*) together with the relations $R^{\mathfrak{A}} \subseteq A^k$ for each relation symbol $R \in \tau$ of arity k. Such a structure $\mathfrak{A}$ is *finite* if its domain A is finite. We often describe structures by listing their domain and relations, i.e., we write $(A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \dots)$.

An *expansion* of a τ-structure $\mathfrak{A}$ is a σ-structure $\mathfrak{B}$ with $A = B$ such that $\tau \subseteq \sigma$ and $R^{\mathfrak{B}} = R^{\mathfrak{A}}$ for each relation symbol $R \in \tau$. Conversely, we call $\mathfrak{A}$ a *reduct* of $\mathfrak{B}$. The *product* of a family $(\mathfrak{A}_i)_{i\in I}$ of τ-structures is the τ-structure $\prod_{i\in I} \mathfrak{A}_i$ over $\prod_{i\in I} A_i$ such that, for each $R \in \tau$ of arity k, we have $(\bar{a}_1, \dots, \bar{a}_k) \in R^{\Pi_{i\in I}\mathfrak{A}_i}$ iff $(\bar{a}_1[i], \dots, \bar{a}_k[i]) \in R^{\mathfrak{A}_i}$ for every $i \in I$. We denote the binary product of a structure $\mathfrak{A}$ with itself as $\mathfrak{A}^2$.

A *homomorphism* $h\colon \mathfrak{A} \to \mathfrak{B}$ for τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a mapping $h\colon A \to B$ that *preserves* each relation of $\mathfrak{A}$, i.e., if $\bar{t} \in R^{\mathfrak{A}}$ for some k-ary relation symbol $R \in \tau$, then $h(\bar{t}) \in R^{\mathfrak{B}}$. A homomorphism $h\colon \mathfrak{A} \to \mathfrak{B}$ is *strong* if it additionally satisfies the inverse condition: for every k-ary relation symbol $R \in \tau$ and $\bar{t} \in A^k$ we have $h(\bar{t}) \in R^{\mathfrak{B}}$ only if $\bar{t} \in R^{\mathfrak{A}}$. An *embedding* is an injective strong homomorphism. We write $\mathfrak{A} \hookrightarrow \mathfrak{B}$ if $\mathfrak{A}$ embeds into $\mathfrak{B}$. The class of all finite τ-structures that embed into $\mathfrak{B}$ is denoted by $\mathrm{Age}(\mathfrak{B})$. A *substructure* of $\mathfrak{B}$ is a structure $\mathfrak{A}$ over the domain $A \subseteq B$ such that the inclusion map $i\colon A \to B$ is an embedding. Conversely, we call $\mathfrak{B}$ an extension of $\mathfrak{A}$. An *isomorphism* is a surjective embedding. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* (written $\mathfrak{A} \cong \mathfrak{B}$) if there exists an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. An *automorphism* of $\mathfrak{A}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$.

Given a relational signature τ, we can build first-order formulae using the relation symbols of τ in the usual way. Relational τ-structures then coincide with first-order interpretations. In the context of p-admissibility, we are interested in quite simple formulae. A τ-atom is of the form $R(x_1, \dots, x_n)$, where $R \in \tau$ is an n-ary relation symbol and $x_1, \dots, x_n$ are variables. For a fixed τ-structure $\mathfrak{A}$, the *constraint satisfaction problem (CSP)* for $\mathfrak{A}$ [11] asks whether a given finite conjunction of atoms is satisfiable in $\mathfrak{A}$. An *implication* is of the form $\forall \bar{x}.\, (\phi \Rightarrow \psi)$ where ϕ is a conjunction of atoms, ψ is a disjunction of atoms, and the tuple $\bar{x}$ consists of the variables occurring in ϕ or ψ. Such an implication is a *Horn-implication* if ψ is the empty disjunction (corresponding to falsity $\bot$) or a single atom. The CSP for $\mathfrak{A}$ can be reduced in polynomial time to the validity problem for Horn-implications since ϕ is satisfiable in $\mathfrak{A}$ iff $\forall \bar{x}.\, (\phi \Rightarrow \bot)$ is not valid in $\mathfrak{A}$. Conversely, validity of Horn implications in a structure $\mathfrak{A}$ can be reduced in polynomial time to the CSP in the expansion $\mathfrak{A}^{\neg}$ of $\mathfrak{A}$ by the complements of all relations. In fact, the Horn implication $\forall \bar{x}.\, (\phi \Rightarrow \psi)$ is valid in $\mathfrak{A}$ iff $\phi \wedge \neg\psi$ is not satisfiable in $\mathfrak{A}^{\neg}$. In the signature of $\mathfrak{A}^{\neg}$, $\neg\psi$ can then be expressed by an atom.

3 Integrating p-Admissible Concrete Domains into $\mathcal{EL}$

Given countably infinite sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of concept and role names, $\mathcal{EL}$ concepts are built using the concept constructors top concept ($\top$), conjunction ($C \sqcap D$), and existential restriction ($\exists r.C$). The semantics of the constructors is defined in the usual way (see, e.g., [3,5]). It assigns to every $\mathcal{EL}$ concept C a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, where $\Delta^{\mathcal{I}}$ is the interpretation domain of the given interpretation $\mathcal{I}$.

As mentioned before, a concrete domain is a τ-structure $\mathfrak{D}$ with a relational signature τ. To integrate such a structure into $\mathcal{EL}$, we complement concept and role names with a set of *feature names* $\mathsf{N_F}$, which provide the connection between the abstract domain $\Delta^{\mathcal{I}}$ and the concrete domain D. A *path* is of the form $r\,f$ or f where $r \in \mathsf{N_R}$ and $f \in \mathsf{N_F}$. In our example in the introduction, *age* is both a feature name and a path of length 1, and *parent age* is a path of length 2. The DL $\mathcal{EL}(\mathfrak{D})$ extends $\mathcal{EL}$ with the new concept constructor

$$R(p_1, \ldots, p_k) \quad \text{(concrete domain restriction)},$$

where $p_1, \ldots, p_k$ are paths, and $R \in \tau$ is a k-ary relation symbol. We use $\mathcal{EL}[\mathfrak{D}]$ to denote the sublanguage of $\mathcal{EL}(\mathfrak{D})$ where paths in concrete domain restrictions are required to have length 1. Note that $\mathcal{EL}(\mathfrak{D})$ is the restriction to $\mathcal{EL}$ of the way concrete domains were integrated into $\mathcal{ALC}$ in [31], whereas our definition of $\mathcal{EL}[\mathfrak{D}]$ describes how concrete domains were integrated into $\mathcal{EL}$ in [2].

To define the semantics of concrete domain restrictions, we assume that an interpretation $\mathcal{I}$ assigns functional binary relations $f^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times D$ to feature names $f \in \mathsf{N_F}$, where *functional* means that $(a, d) \in f^{\mathcal{I}}$ and $(a, d') \in f^{\mathcal{I}}$ imply $d = d'$. We extend the interpretation function to paths of the form $p = r\,f$ by setting $(r\,f)^{\mathcal{I}} = \{(a, d) \in \Delta^{\mathcal{I}} \times D \mid \text{there is } b \in \Delta^{\mathcal{I}} \text{ such that } (a, b) \in r^{\mathcal{I}} \text{ and } (b, d) \in f^{\mathcal{I}}\}$. The semantics of concrete domain restrictions is now defined as follows:

$$R(p_1, \ldots, p_k)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \text{there are } d_1, \ldots, d_k \in D \text{ such that}$$
$$(a, d_i) \in p_i^{\mathcal{I}} \text{ for all } i \in [k] \text{ and } (d_1, \ldots, d_k) \in R^{\mathfrak{D}}\}.$$

As usual, an $\mathcal{EL}(\mathfrak{D})$ TBox is defined to be a finite set of GCIs $C \sqsubseteq D$, where C, D are $\mathcal{EL}(\mathfrak{D})$ concepts. The interpretation $\mathcal{I}$ is a *model* of such a TBox if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all GCIs $C \sqsubseteq D$ occurring in it. Given $\mathcal{EL}(\mathfrak{D})$ concept descriptions C, D and an $\mathcal{EL}(\mathfrak{D})$ TBox $\mathcal{T}$, we say that C is *subsumed by* D w.r.t. $\mathcal{T}$ (written $C \sqsubseteq_{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all models of $\mathcal{T}$. For the subsumption problem in $\mathcal{EL}[\mathfrak{D}]$, to which we restrict our attention for the moment, only $\mathcal{EL}[\mathfrak{D}]$ concepts may occur in $\mathcal{T}$, and C, D must also be $\mathcal{EL}[\mathfrak{D}]$ concepts.

Subsumption in $\mathcal{EL}$ is known to be decidable in polynomial time [16]. For $\mathcal{EL}[\mathfrak{D}]$, this is the case if the concrete domain is p-admissible [2]. According to [2], a concrete domain $\mathfrak{D}$ is *p-admissible* if it satisfies the following conditions: (i) satisfiability of conjunctions of atoms and validity of Horn implications in $\mathfrak{D}$ are tractable; and (ii) $\mathfrak{D}$ is convex. Unfortunately, the definition of convexity in [2] (below formulated using our notation) is ambiguous:

($*$) If a conjunction of atoms of the form $R(x_1, \ldots, x_k)$ implies a disjunction of such atoms, then it also implies one of its disjuncts.

The problem is that this definition does not say anything about which variables may occur in the left- and right-hand sides of such implications. To illustrate this, let us consider the structure $\mathfrak{N} = (\mathbb{N}, E, O)$ in which the unary predicates E and O are respectively interpreted as the even and odd natural numbers. If the right-hand side of an implication considered in the definition of convexity may contain variables not occurring on the left-hand side, then $\mathfrak{N}$ is not convex: $\forall x, y. (E(x) \Rightarrow E(y) \vee O(y))$ holds in $\mathfrak{N}$, but neither $\forall x, y. (E(x) \Rightarrow E(y))$ nor $\forall x, y. (E(x) \Rightarrow O(y))$ does. However, for *guarded implications*, where all variables occurring on the right-hand side must also occur on the left-hand side, the structure $\mathfrak{N}$ satisfies the convexity condition $(*)$. We say that a structure is *convex* if $(*)$ is satisfied without any restrictions on the occurrence of variables, and *guarded convex* if $(*)$ is satisfied for guarded implications. Clearly, any convex structure is guarded convex, but the converse implication does not hold, as exemplified by $\mathfrak{N}$.

We claim that, what was actually meant in [2], was guarded convexity rather than convexity. In fact, it is argued in that paper that non-convexity of $\mathfrak{D}$ allows one to express disjunctions in $\mathcal{EL}[\mathfrak{D}]$, which makes subsumption in $\mathcal{EL}[\mathfrak{D}]$ ExpTime-hard. However, this argument works only if the counterexample to convexity is given by a guarded implication. Let us illustrate this again on our example $\mathfrak{N}$. Whereas $\forall x, y.\ (E(x) \Rightarrow E(y) \vee O(y))$ holds in $\mathfrak{N}$, the subsumption $E(f) \sqsubseteq_{\emptyset} E(g) \sqcup O(g)$ does not hold in the extension of $\mathcal{EL}[\mathfrak{D}]$ with disjunction since the feature g need not have a value. For this reason, we use guarded convexity rather than convexity in our definition of p-admissibility. For the same reason, we also restrict the tractability requirement in this definition to validity of guarded Horn implications.

Definition 1. *A relational structure $\mathfrak{D}$ is* p-admissible *if it is guarded convex and validity of guarded Horn implications in $\mathfrak{D}$ is tractable*

Using this notion, the main results of [2] concerning concrete domains can now be summarized as follows.

Theorem 1 (Baader, Brandt, and Lutz [2]). *Let $\mathfrak{D}$ be a relational structure. Then subsumption in $\mathcal{EL}[\mathfrak{D}]$ is*

1. *decidable in polynomial time if $\mathfrak{D}$ is p-admissible;*
2. *ExpTime-hard if $\mathfrak{D}$ is not guarded convex.*

The two p-admissible concrete domains introduced in [2] have equality as one of their relations. For such structures, convexity and guarded convexity obviously coincide since one can use $x = x$ as a trivially true guard. For example, the extension $\mathfrak{N}_=$ of $\mathfrak{N}$ with equality is no longer guarded convex since the implication $\forall x. (x = x \Rightarrow E(x) \vee O(x))$ holds in $\mathfrak{N}_=$, but neither $\forall x. (x = x \Rightarrow E(x))$ nor $\forall x. (x = x \Rightarrow O(x))$.

In the next section, we will show algebraic characterizations of (guarded) convexity. Regarding the tractability condition in the definition of p-admissibility, we have seen that it is closely related to the constraint satisfaction problem for $\mathfrak{D}$ and $\mathfrak{D}^{\neg}$. Characterizing tractability of the CSP in a given structure is a very hard problem. Whereas the Feder-Vardi conjecture [20] has recently been confirmed after 25 years of intensive research in the field by giving an algebraic criterion that can distinguish between *finite* structures with tractable and with NP-complete CSPs [17,34], finding comprehensive criteria that ensure tractability for the case of infinite structures is a wide open problem, though first results for special cases have been found (see, e.g., [13,14]).

4 Algebraic Characterizations of Convexity

Before we can formulate our characterization of (guarded) convexity, we need to introduce a semantic notion of guardedness. We say that the relational τ-structure $\mathfrak{A}$ is *guarded* if for every $a \in A$ there is a relation $R \in \tau$ such that a appears in a tuple in $R^{\mathfrak{A}}$.

Theorem 2. *For a relational τ structure $\mathfrak{B}$, the following are equivalent:*

1. *$\mathfrak{B}$ is* guarded *convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$ whose σ-reduct is* guarded, *there exists a strong homomorphism from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{B}$.*

We concentrate here on proving "2 ⇒ 1" since this is the direction that will be used later on. Alternatively, we could obtain "2 ⇒ 1" by adapting the proof of McKinsey's lemma [22]. A proof of the other direction can be found in [6].

Proof of "2 ⇒ 1" of Theorem 2. Suppose to the contrary that the implication $\forall x_1, \ldots, x_n.\, (\phi \Rightarrow \psi)$ is valid in $\mathfrak{B}$, where ϕ is a conjunction of atoms such that each variable x_i is present in some atom of ϕ, and ψ is a disjunction of atoms $\psi_1, \ldots, \psi_k$, but we also have $\mathfrak{B} \not\models \forall x_1, \ldots, x_n.\, (\phi \Rightarrow \psi_i)$ for every $i \in [k]$. Without loss of generality, we assume that $\phi, \psi_1, \ldots, \psi_k$ all have the same free variables $x_1, \ldots, x_n$, some of which might not influence their truth value. For every $i \in [k]$, there exists a tuple $\bar{t}_i \in B^n$ such that

$$\mathfrak{B} \models \phi(\bar{t}_i) \wedge \neg\psi_i(\bar{t}_i). \qquad (*)$$

We show by induction on i that, for every $i \in [k]$, there exists a tuple $\bar{s}_i \in B^n$ that satisfies the *induction hypothesis*

$$\mathfrak{B} \models \phi(\bar{s}_i) \wedge \neg \bigvee_{\ell \in [i]} \psi_\ell(\bar{s}_i). \qquad (\dagger)$$

In the *base case* ($i = 1$), it follows from ($*$) that $\bar{s}_1 := \bar{t}_1$ satisfies (†).

In the *induction step* ($i \to i + 1$), let $\bar{s}_i \in B^n$ be any tuple that satisfies (†). Let $\sigma \subseteq \tau$ be the finite set of relation symbols occurring in the implication $\forall x_1, \ldots, x_n.\, (\phi \Rightarrow \psi)$, and let $\mathfrak{A}_i$ be the substructure of $\mathfrak{B}^2$ on the set $\{(\bar{s}_i[1], \bar{t}_{i+1}[1]), \ldots, (\bar{s}_i[n], \bar{t}_{i+1}[n])\}$. Since $\mathfrak{B} \models \phi(\bar{s}_i)$ by (†), $\mathfrak{B} \models \phi(\bar{t}_{i+1})$ by ($*$), and ϕ contains an atom for each variable x_i, we conclude that the σ-reduct of $\mathfrak{A}_i$ is guarded. By 2., there exists a strong homomorphism f_i from the σ-reduct of $\mathfrak{A}_i$ to the σ-reduct of $\mathfrak{B}$. Since ϕ is a conjunction of atoms and f_i is a homomorphism, we have that $\mathfrak{B} \models \phi\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$. Suppose that $\mathfrak{B} \models \psi_{i+1}\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_{i+1}(\bar{t}_{i+1})$, a contradiction to ($*$). Now suppose that $\mathfrak{B} \models \psi_j\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$ for some $j \leq i$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_j(\bar{s}_i)$, a contradiction to (†). We conclude that $\bar{s}_{i+1} := f_i(\bar{s}_i, \bar{t}_{i+1})$ satisfies (†).

Since $\mathfrak{B} \models \forall x_1, \ldots, x_n. (\phi \Rightarrow \psi)$, the existence of a tuple $\bar{s}_i \in B^n$ that satisfies (†) for $i = k$ leads to a contradiction. □

As an easy consequence of Theorem 2, we also obtain a characterization of (unguarded) convexity. This is due to the fact that the structure $\mathfrak{B}$ is convex iff its expansion with the full unary predicate (interpreted as B) is guarded convex. In addition, in the presence of this predicate, any structure is guarded.

Corollary 1. *For a relational τ-structure $\mathfrak{B}$, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$, there exists a strong homomorphism from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{B}$.*

As an example, the structure $\mathfrak{N} = (\mathbb{N}, E, O)$ introduced in the previous section is guarded convex, but not convex. According to the corollary, the latter should imply that there is a finite substructure $\mathfrak{A}$ of $\mathfrak{N}^2$ that has no strong homomorphism to $\mathfrak{N}$. In fact, if we take as $\mathfrak{A}$ the substructure of $\mathfrak{N}^2$ induced by the tuple $(1, 2)$, then this tuple belongs neither to E nor to O in the product. However, a strong homomorphism to $\mathfrak{N}$ would need to map this tuple either to an odd or an even number. But then the tuple would need to belong to either E or O since the homomorphism is strong. This example does not work for the case of guarded convexity, because the considered substructure is not guarded. In fact, a guarded substructure of $\mathfrak{N}^2$ can only contain tuples where both components are even or both components are odd. In the former case, the tuple can be mapped to an even number, and in the latter to an odd number.

In the presence of the equality predicate, strong homomorphisms are embeddings and guarded convexity is the same as convexity.

Corollary 2. *For a structure $\mathfrak{B}$ with a relational signature τ* with equality*, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$, the σ-reduct of $\mathfrak{A}$ embeds into the σ-reduct of $\mathfrak{B}$.*

5 Examples of Convex and p-Admissible Structures

We consider three different kinds of structures (ω-categorical, finitely bounded, numerical) and show under which conditions such structures are convex. This provides us with new examples for p-admissible concrete domains.

5.1 Convex ω-Categorical Structures

A structure is called *ω-categorical* if its first-order theory has a unique countable model up to isomorphism. A well-known example of such a structure is $(\mathbb{Q}, <)$, whose first-order theory is the theory of linear orders without first and last element. Such structures have drawn considerable attention in the CSP community since their CSPs can, to some extent, be investigated using the algebraic tools originally developed for finite structures. Countably infinite ω-categorical structures can be characterized using automorphisms and orbits. For every structure $\mathfrak{A}$, the set of all automorphisms of $\mathfrak{A}$, denoted by $\mathrm{Aut}(\mathfrak{A})$, forms a permutation group with composition as group operation [23]. The *orbit* of a tuple $\bar{t} \in A^k$ under $\mathrm{Aut}(\mathfrak{A})$ is the set $\{(g(\bar{t}[1]), ..., g(\bar{t}[k])) \mid g \in \mathrm{Aut}(\mathfrak{A})\}$. The following result is due to Engeler, Ryll-Nardzewski, and Svenonius (see Theorem 6.3.1 in [23]).

Theorem 3. *For a countably infinite structure $\mathfrak{D}$ with a countable signature, the following are equivalent:*

1. *$\mathfrak{D}$ is ω-categorical.*
2. *Every relation preserved by $\mathrm{Aut}(\mathfrak{D})$ has a first-order definition in $\mathfrak{D}$.*
3. *For every $k \geq 1$, there are only finitely many orbits of k-tuples under $\mathrm{Aut}(\mathfrak{D})$.*

For countably infinite ω-categorical structures the characterization of convexity of Corollary 2 can be improved to the following simpler statement.

Theorem 4. *For a countably infinite ω-categorical relational structure $\mathfrak{B}$ with a countable signature τ with equality, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *$\mathfrak{B}^2$ embeds into $\mathfrak{B}$.*

The proof of this theorem combines the proof of Corollary 2 with the following two facts, which are implied by ω-categoricity of $\mathfrak{B}$. First, there exists a strong homomorphism from $\mathfrak{B}^2$ to $\mathfrak{B}$ iff there exists a strong homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ for every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$ (see, e.g., Lemma 3.1.5 in [11]). Second, to deal with the fact that τ may be infinite (which is problematic for the proof of "1 $\Rightarrow$ 2"), we can use Theorem 3, which ensures that, for every $k \geq 1$, there are only finitely many inequivalent k-ary formulae over $\mathfrak{B}$ consisting of a single τ-atom.

In the CSP literature, one can find two examples of countably infinite ω-categorical structure that satisfy the square embedding condition of the above theorem: atomless Boolean algebras and countably infinite vector spaces over finite fields. Since the CSP for atomless Boolean algebras is NP-complete [9], this example does not provide us with a p-admissible concrete domain. Things are more rosy for the vector space example. As shown in [12], the relational representation $\mathfrak{V}_q = (V_q, R^+, R^{s_0}, \ldots, R^{s_{q-1}})$ of the countably infinite vector space over a finite field $\mathrm{GF}(q)$ is ω-categorical, satisfies $\mathfrak{V}_q^2 \cong \mathfrak{V}_q$, and its CSP is decidable in polynomial time, even if the complements of all predicates are added. Here R^+ is a ternary predicate corresponding to addition of vectors, and the R^{s_i} are binary predicates corresponding to scalar multiplication of a vector with the element s_i of GF(q). We can show that these properties are preserved if we add finitely many unary predicates R^{e_i} that correspond to unit vectors $e_1, \ldots, e_k$ [6].

Corollary 3. *The structure* $\mathfrak{V}_q$ *expanded with predicates* $R^{e_1}, \ldots, R^{e_k}$ *for unit vectors* $e_1, \ldots, e_k$ *is p-admissible.*

For the case $q = 2$, the vectors in V_q are one-sided infinite tuples of zeros and ones containing only finitely many ones, which can be viewed as representing finite subsets of $\mathbb{N}$. For example, $(0, 1, 1, 0, 1, 0, 0, \ldots)$ represents the set $\{1, 2, 4\}$. Thus, if we use $\mathfrak{V}_2$ as concrete domain, the features assign finite sets of natural numbers to individuals. For example, assume that the feature *daughters-ages* assigns the set of ages of female children to a person, and *sons-ages* the set of ages of male children. Then $R^+(\mathit{daughters\text{-}ages}, \mathit{sons\text{-}ages}, \mathit{zero})$ describes persons that, for every age, have either both a son and a daughter of this age, or no child at all of this age. The feature *zero* is supposed to point to the zero vector, which can, e.g., be enforced using the GCI $\top \sqsubseteq R^+(\mathit{zero}, \mathit{zero}, \mathit{zero})$.

5.2 Convex Structures with Forbidden Patterns

For a class $\mathcal{F}$ of τ-structures, $\mathrm{Forb}_e(\mathcal{F})$ stands for the class of all finite τ-structures that do not embed any member of $\mathcal{F}$. A structure $\mathfrak{B}$ is *finitely bounded* if its signature is finite and $\mathrm{Age}(\mathfrak{B}) = \mathrm{Forb}_e(\mathcal{F})$ for some finite set $\mathcal{F}$ of *bounds*. Alternatively, one can say that $\mathfrak{B}$ is finitely bounded if its signature is finite and there is a universal first-order sentence Φ with equality such that $\mathrm{Age}(\mathfrak{B})$ consists precisely of the finite models of Φ [8]. A well-known example of a finitely bounded structure is $(\mathbb{Q}, >, =)$, for which the self loop, the 2-cycle, the 3-cycle, and two isolated vertices can be used as bounds (see Fig. 1 in [7]). As universal sentence defining $\mathrm{Age}(\mathbb{Q}, >, =)$ we can take the conjunction of the usual axioms defining linear orders. For finitely bounded structures, p-admissibility turns out to be equivalent to convexity.

Theorem 5. *Let* $\mathfrak{B}$ *be a finitely bounded* τ*-structure with equality. Then the following statements are equivalent:*

1. $\mathfrak{B}$ *is convex,*
2. $\mathrm{Age}(\mathfrak{B})$ *is defined by a conjunction of Horn implications,*
3. $\mathfrak{B}$ *is p-admissible.*

The structure $(\mathbb{Q}, >, =)$ is not convex. In fact, since it is also ω-categorical, convexity would imply that its square $(\mathbb{Q}, >, =) \times (\mathbb{Q}, >, =)$ embeds into $(\mathbb{Q}, >, =)$, by Theorem 4. This cannot be the case since the product contains incomparable elements, whereas $(\mathbb{Q}, >, =)$ does not. In the universal sentence defining $\mathrm{Age}(\mathbb{Q}, >, =)$, the totality axiom $\forall x, y. (x < y \vee x = y \vee x > y)$ is the culprit since it is not Horn. If we remove this axiom, we obtain the theory of strict partial orders.

Example 1. It is well-known that there exists a unique countable homogeneous[1] strict partial order $\mathfrak{O}$ [32], whose age is defined by the universal sentence $\forall x, y, z. (x < y \wedge y < z \Rightarrow x < z) \wedge \forall x. (x < x \Rightarrow \bot)$, which is a Horn implication. Thus, $\mathfrak{O}$ extended with equality is finitely bounded and convex. Using $\mathfrak{O}$ as a concrete domain means that the feature values satisfy the theory of strict partial orders, but not more. One can, for instance, use this concrete domain to model preferences of people; e.g., the concept $\mathit{Italian} \sqcap {>}(\mathit{pizzapref}, \mathit{pastapref})$ describes Italians that like pizza more than pasta. Using $\mathfrak{O}$ here means that preferences may be incomparable. As we have seen above, adding totality would break convexity and thus p-admissibility.

Beside finitely bounded structures, the literature also considers structures whose age can be described by a finite set of forbidden homomorphic images [19, 26]. For a class $\mathcal{F}$ of τ-structures, $\mathrm{Forb}_h(\mathcal{F})$ stands for the class of all finite τ-structures that do not contain a homomorphic image of any member of $\mathcal{F}$. A structure is *connected* if its so-called Gaifman graph is connected.

Theorem 6 (Cherlin, Shelah, and Shi [19]). *Let $\mathcal{F}$ be a finite family of connected relational structures with a finite signature τ. Then there exists an ω-categorical τ-structure $\mathrm{CSS}(\mathcal{F})$ that is a reduct of a finitely bounded homogeneous structure and such that $\mathrm{Age}(\mathrm{CSS}(\mathcal{F})) = \mathrm{Forb}_h(\mathcal{F})$.*

We can show [6] that the structures of the form $\mathrm{CSS}(\mathcal{F})$ provided by this theorem are always p-admissible.

Proposition 1. *Let $\mathcal{F}$ be a finite family of connected relational structures with a finite signature τ. Then the expansion of $\mathrm{CSS}(\mathcal{F})$ by the equality predicate is p-admissible.*

This proposition actually provides us with infinitely many examples of countable p-admissible concrete domains, which all yield a different extension of $\mathcal{EL}$: the so-called Henson digraphs [21] (see [6] for details). The usefulness of these concrete domains for defining interesting concepts is, however, unclear.

5.3 Convex Numerical Structures

We exhibit two new p-admissible concrete domain that are respectively based on the real and the rational numbers, and whose predicates are defined by linear equations. Let $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ be the relational structure over $\mathbb{R}$ that has, for every linear equation system $A\bar{x} = \bar{b}$ over $\mathbb{Q}$, a relation consisting of all its solutions in $\mathbb{R}$. We define $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ as the substructure of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ on $\mathbb{Q}$. For example, using the matrix $A = (2\ 1\ {-1})$ and the vector $\bar{b} = (0)$ one obtains the ternary relation $\{(p, q, r) \in \mathbb{Q}^3 \mid 2p + q = r\}$ in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$.

Theorem 7. *The relational structures $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ and $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ are p-admissible.*

[1] A structure is *homogeneous* if every isomorphism between its finite substructures extends to an automorphism of the whole structure.

To prove this theorem for $\mathbb{R}$, we start with the well-known fact that $(\mathbb{R}, +, 0)^2$ and $(\mathbb{R}, +, 0)$ are isomorphic [28], and show that it can be extended to $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$. This yields convexity of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$. For $\mathbb{Q}$, we cannot employ the same argument since $(\mathbb{Q}, +, 0)^2$ is not isomorphic to $(\mathbb{Q}, +, 0)$. Instead, we use the well-known fact that the structures $(\mathbb{Q}, +, 0)$ and $(\mathbb{R}, +, 0)$ satisfy the same first-order-sentences [28] to show that convexity of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ implies convexity of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. Tractability can be shown for both structures using a variant of the Gaussian elimination procedure. A detailed proof can be found in [6].

It is tempting to claim that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ is considerably more expressive than the p-admissible concrete domain $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ with domain $\mathbb{Q}$, unary predicates $=_p, >_p$, and binary predicates $+_p, =$ exhibited in [2]. However, formally speaking, this is not true since the relations $>_p$ cannot be expressed in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. In fact, adding such a relation to $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ would destroy convexity. Conversely, adding the ternary addition predicate, which is available in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$, to $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ also destroys convexity. Using these observations, we can actually show that the expressive powers of $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ and $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ are incomparable [6]. We expect, however, that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ will turn out to be more useful in practice than $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$.

6 ω-Admissibility versus p-Admissibility

The notion of ω-admissibility was introduced in [31] as a condition on concrete domains $\mathfrak{D}$ that ensures that the subsumption problem in $\mathcal{ALC}(\mathfrak{D})$ w.r.t. TBoxes remains decidable. This is a rather complicated condition, but for our purposes it is sufficient to know that, according to [31], an ω-admissible concrete domain $\mathfrak{D}$ has finitely many binary relations, which are jointly exhaustive (i.e., their union yields $D \times D$) and pairwise disjoint (i.e., for two different relation symbols R_i, R_j we have $R_i^{\mathfrak{D}} \cap R_j^{\mathfrak{D}} = \emptyset$). In the presence of equality, these two conditions do not go well together with convexity.

Proposition 2. *Let $\mathfrak{D}$ be a structure with a finite binary relational signature that includes equality. If $\mathfrak{D}$ is convex, jointly exhaustive, and pairwise disjoint, then its domain D satisfies $|D| \leq 1$.*

This proposition shows that there are no non-trivial concrete domains with equality that are at the same time p-admissible and ω-admissible. Without equality, there are some, but they are still not very interesting [6]. Nevertheless, by combining the results of Sect. 5.2 with Corollary 2 in [7], we obtain non-trivial p-admissible concrete domains with equality for which subsumption in $\mathcal{ALC}(\mathfrak{D})$ is decidable.

Corollary 4. *Let $\mathfrak{D}$ be a finitely bounded convex structure with equality that is a reduct of a finitely bounded homogeneous structure. Then subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}]$ and decidable in $\mathcal{ALC}(\mathfrak{D})$.*

The Henson digraphs already mentioned in Sect. 5.2 provide us with infinitely many examples of structures that satisfy the conditions of this corollary.

In general, however, p-admissibility of $\mathfrak{D}$ does *not* guarantee decidability of subsumption in $\mathcal{ALC}(\mathfrak{D})$. For example, subsumption w.r.t. TBoxes is undecidable in $\mathcal{ALC}(\mathfrak{D}_{\mathbb{Q},\mathrm{dist}})$ and $\mathcal{ALC}(\mathfrak{D}_{\mathbb{Q},\mathrm{lin}})$ since this is already true for their common reduct $(\mathbb{Q}, +_1)$ [7].

Even for $\mathcal{EL}$, integrating a p-admissible concrete domain may cause undecidability if we allow for role paths of length 2. To show this, we consider the relational structure $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ over $\mathbb{Q}^2$, which has, for every affine transformation $\mathbb{Q}^2 \to \mathbb{Q}^2 : \bar{x} \mapsto A\bar{x}+\bar{b}$, the binary relation $R_{A,\bar{b}} \coloneqq \{(\bar{x}, \bar{y}) \in (\mathbb{Q}^2)^2 \mid \bar{y} = A\bar{x}+\bar{b}\}$.

Theorem 8. *The relational structure $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ is p-admissible, which implies that subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}]$. However, subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$.*

In [6], we show p-admissibility of $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ using the fact that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ is p-admissible. Tractability of subsumption in $\mathcal{EL}[\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}]$ is then an immediate consequence of Theorem 1. Undecidability of subsumption w.r.t. TBoxes in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$ can be shown by a reduction from *2-Dimensional Affine Reachability*, which is undecidable by Corollary 4 in [10]. For this problem, one is given vectors $\bar{v}, \bar{w} \in \mathbb{Q}^2$ and a finite set S of affine transformations from $\mathbb{Q}^2$ to $\mathbb{Q}^2$. The question is then whether $\bar{w}$ can be obtained from $\bar{v}$ by repeated application of transformations from S. It is not hard to show that 2-Dimensional Affine Reachability can effectively be reduced to subsumption w.r.t. TBoxes in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$.

7 Conclusion

The notion of p-admissible concrete domains was introduced in [2], where it was shown that integrating such concrete domains into the lightweight DL $\mathcal{EL}$ (and even the more expressive DL $\mathcal{EL}^{++}$) leaves the subsumption problem tractable. The paper [2] contains two examples of p-admissible concrete domains, and since then no new examples have been exhibited in the literature. This appears to be mainly due to the fact that it is not easy to show the convexity condition required by p-admissibility "by hand". The main contribution of the present paper is that it provides us with a useful algebraic tool for showing convexity: the square embedding condition. We have shown that this tool can indeed be used to exhibit new p-admissible concrete domains, such as countably infinite vector spaces over finite field, the countable homogeneous partial order, and numerical concrete domains over $\mathbb{R}$ and $\mathbb{Q}$ whose relations are defined by linear equations. The usefulness of these numerical concrete domains for defining concepts should be evident. For the other two we have indicated their potential usefulness by small examples.

We have also shown that, for finitely bounded structures, convexity is equivalent to p-admissibility, and that this corresponds to the finite substructures being definable by a conjunction of Horn implications. Interestingly, this provides us with infinitely many examples of countable p-admissible concrete domains, which all yield a different extension of $\mathcal{EL}$: the Henson digraphs. From a theoretical point of view, this is quite a feat, given that before only two p-admissible concrete domains were known.

Finitely bounded structures also provide us with examples of structures $\mathfrak{D}$ that can be used both in the context of $\mathcal{EL}$ and $\mathcal{ALC}$, in the sense that subsumption is tractable in $\mathcal{EL}[\mathfrak{D}]$ and decidable in $\mathcal{ALC}(\mathfrak{D})$. Finally, we have shown that, when embedding p-admissible concrete domains into $\mathcal{EL}$, the restriction to paths of length 1 in concrete domain restrictions (indicated by the square brackets) is needed since there is a p-admissible concrete domains $\mathfrak{D}$ such that subsumption in $\mathcal{EL}(\mathfrak{D})$ is undecidable.

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