Treating Role Assertions as First-class Citizens in Repair and Error-tolerant Reasoning

Franz Baader
Technische Universität Dresden and
Center for Scalable Data Analytics
and Artificial Intelligence (ScaDS.AI)
Dresden, Germany
franz.baader@tu-dresden.de

Francesco Kriegel
Technische Universität Dresden
Dresden, Germany
francesco.kriegel@tu-dresden.de

Adrian Nuradiansyah
Technische Universität Dresden
Dresden, Germany
adrian.nuradiansyah@tu-dresden.de

ABSTRACT

Errors in Description Logic (DL) ontologies are often detected when reasoning yields unintuitive consequences. The question is then how to repair the ontology in an optimal way, i.e., such that the unwanted consequences are removed, but a maximal set of the unobjected consequences is kept. Error-tolerant reasoning does not commit to a single optimal repair: brave reasoning asks whether the consequence is entailed by some repair and cautious reasoning whether it is entailed by all repairs. Previous research on repairing ABoxes w.r.t. TBoxes formulated in the DL $\mathcal{EL}$ has developed methods for computing optimal repairs, and has recently also determined the complexity of error-tolerant reasoning: brave reasoning is in P and cautious reasoning is in coNP. However, in this work the unwanted consequences were restricted to being $\mathcal{EL}$ instance assertions. In the present paper, we show that the mentioned results can be extended to a setting where also role assertions can be required to be removed. Our solution is based on a two-stage approach where first the unwanted role assertions and then the unwanted concept assertions are removed. We also investigate the complexity of error-tolerant reasoning w.r.t. classical repairs, which are maximal subsets of the ABox that do not have the unwanted consequences, and show that, in this setting, brave reasoning is NP-complete and cautious reasoning is coNP-complete.

CCS CONCEPTS

• **Computing methodologies** → **Ontology engineering**; **Description logics**; • **Theory of computation** → **Description logics**;

KEYWORDS

Description logic, repairs, error-tolerant reasoning

ACM Reference Format:
Franz Baader, Francesco Kriegel, and Adrian Nuradiansyah. 2023. Treating Role Assertions as First-class Citizens in Repair and Error-tolerant Reasoning. In *The 37th ACM/SIGAPP Symposium on Applied Computing (SAC '23), March 27-31, 2023, Tallinn, Estonia.* ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/3555776.3577630

SAC'23, March 27 –31, 2023, Tallinn, Estonia

ACM ISBN 978-1-4503-9517-5/23/03…$15.00
https://doi.org/10.1145/3555776.3577630

1 INTRODUCTION

Description logics (DLs) [2] are a well-investigated class of knowledge representation languages, which can be used to describe the data and the background knowledge of an application domain in a structured and formally well-understood way. The data are represented in an ABox using concept and role assertions, such as *has_parent*(*SOUTH*, *KIM*) and *Famous*(*KIM*), which say that South has the famous parent Kim. Background information is represented in a TBox using concept inclusions (CIs), such as $\exists\, has\_parent. Famous \sqsubseteq Rich$, which says that children of famous parents are rich. Queries can then be used to derive consequences from the TBox and the ABox, such as that *SOUTH* belongs to the concepts $\exists\, has\_parent. Famous$ and *Rich*. For the DL $\mathcal{EL}$, which we have used in this example and on which we will concentrate in this paper, such instance queries can be answered in polynomial time [1].

Errors in ontologies are often detected when unintuitive consequences are derived. In our example, the user might decide that Kim actually is not a parent of South, i.e., that the consequence *has_parent*(*SOUTH*, *KIM*) is incorrect, and request that it is removed. In the setting of this paper, such a repair request may contain both concept and role assertions. The classical approach for satisfying a repair request is to construct a maximal subset of the ABox that has none of the unwanted consequences. In our example, this means that the role assertion *has_parent*(*SOUTH*, *KIM*) must be removed from the ABox. However, this also removes the consequences $\exists\, has\_parent. Famous(SOUTH)$ and *Rich*(*SOUTH*), which the user did not complain about. In fact, it might be that the user knows that South has a famous parent, but made an error when naming this parent. Thus, they want to get rid of the erroneous role assertion, but not of the concept instance relationships for South that it entails.

Optimal repairs try to keep a maximal set of consequences, rather than a maximal subset of the input ABox. The rational behind retaining a maximal set of consequences, instead of just some more consequences than the classical approach, is that this is the best we can do when the only information that we have is that certain consequences (as specified by the user) need to be removed. In a series of papers, the authors of [3, 6, 7, 9] have investigated settings in which such optimal repairs exist and can effectively by computed. In particular, in [3] they consider (among others) the case where the background knowledge is static (i.e., cannot be changed) and formulated as an $\mathcal{EL}$ TBox, the data are given by a generalization of ABoxes, called quantified ABoxes, which may contain anonymous individuals, the repair request consists of $\mathcal{EL}$ concept assertions, and one is only interested in answers to instance queries. The last restriction means that, when judging optimality of a repair, only implied instance relations need to be taken into account. Formally, this is reflected by the fact that qABoxes are compared using IQ-entailment rather than classical first-order entailment. It is shown in [3] how a set of repairs, called canonical repairs, can be constructed that covers the set of all repairs w.r.t. IQ-entailment. The optimal ones can then be obtained from this set by removing redundant elements, i.e., ones that are strictly IQ-entailed by another one.

In this paper, we extend the results of [3] to a setting where the repair request may also contain role assertions. First, we consider role repair requests, which contain only role assertions, and show that in this case there exists a single optimal repair covering all repairs that can be computed in polynomial time. In our example, $\exists\{x\}.\{\mathit{has\_parent}(\mathit{SOUTH}, x), \mathit{Famous}(x), \mathit{Famous}(\mathit{KIM})\}$ is this optimal repair. This example also illustrates why it is useful to have anonymous individuals (represented as existentially quantified variables). It allows us to retain the consequences $\mathit{Rich}(\mathit{SOUTH})$ and $\exists\,\mathit{has\_parent}.\mathit{Famous}(\mathit{SOUTH})$, although there is no longer a named individual in the ABox that is a parent of *SOUTH*. If the repair request contains also concept assertions, then one first applies this approach to the subset consisting of its role assertions, and then the approach of [3] to the resulting qABox and the concept assertions in the repair request. However, when removing canonical repairs that are redundant, one needs to use IRQ-entailment (which takes both concept and role assertions into account) rather than IQ-entailment (see Example 4.10 for an explanation why IRQ-entailment is needed). Role assertions in the repair request can also be dealt with by the repair approach in [6], however not as first-class citizens, but by expressing them as concept assertions involving nominals. They are then treated in one go together with the unwanted concept assertions, rather than a two-stage approach as in the present paper.

In error-tolerant reasoning, one does not commit to a single repair, but rather reasons w.r.t. all repairs. Brave entailment asks whether a given query is entailed by some repair whereas cautious entailment asks whether the query is entailed by all repairs. This was originally considered for classical TBox repairs [16, 17], and only recently investigated in [8] for the optimal repairs of [3]. It is shown there that brave entailment is in P and cautious entailment is in coNP, but without a matching lower bound. We show that the same holds also in the setting considered in the present paper, where repair requests may also contain role assertions. This extensions is non-trivial, mainly due to the fact that IRQ-entailment must be used to compare repairs. We complement these complexity results for optimal repairs with hardness results for classical repairs, where brave reasoning is NP-complete and cautious reasoning is coNP-complete. Similar results are shown in [16, 17], but for a setting where TBoxes rather than ABoxes are repaired.

2 PRELIMINARIES

First, we introduce the syntax of the DL $\mathcal{EL}$ and of quantified ABoxes, to fix the notation, but refer the reader to standard texts on DLs [2] for the semantics of the former and to [3, 9] for more information on the latter. Then, we introduce the relevant entailment relations and recall some useful results regarding them from [3, 4].

Starting with disjoint sets Σ_C of *concept names* and Σ_R of *role names*, $\mathcal{EL}$ *concept descriptions* are built using the constructors top concept ($\top$), conjunction ($C \sqcap D$), and existential restriction ($\exists r.C$). An $\mathcal{EL}$ *concept inclusion (CI)* is of the form $C \sqsubseteq D$ for concept descriptions C, D, and an $\mathcal{EL}$ *TBox* $\mathcal{T}$ is a finite set of CIs. Given an additional set Σ_I of *individual names*, disjoint with Σ_C and Σ_R, an $\mathcal{EL}$ *concept assertion* is of the form $C(a)$, where C is an $\mathcal{EL}$ concept description and $a \in \Sigma_\mathsf{I}$, and a *role assertion* is of the form $r(a, b)$, where $r \in \Sigma_\mathsf{R}$ and $a, b \in \Sigma_\mathsf{I}$. An $\mathcal{EL}$ *ABox* $\mathcal{A}$ is a finite set of concept assertions and role assertions.

A *quantified ABox (qABox)* $\exists X.\mathcal{A}$ consists of a finite set X of *variables*, which is disjoint with $\Sigma = \Sigma_\mathsf{I} \cup \Sigma_\mathsf{C} \cup \Sigma_\mathsf{R}$, and a *matrix* $\mathcal{A}$, which is a finite set of concept assertions $A(u)$ and role assertions $r(u, v)$, where $A \in \Sigma_\mathsf{C}$, $r \in \Sigma_\mathsf{R}$ and $u, v \in \Sigma_\mathsf{I} \cup X$. Thus, the matrix is an ABox built over the extended signature $\Sigma \cup X$, but cannot contain complex concept descriptions. An *object* of $\exists X.\mathcal{A}$ is either an individual name in Σ_I or a variable in X.

The semantics of $\mathcal{EL}$ can be defined in a model-theoretic way or by translating concepts C into first-order formulas $\phi_C(x)$ with one free variable x, and CIs, assertions, TBoxes, and ABoxes into first-order sentences [2]. For example, the concept assertion $C(a)$ is translated into $\phi_C(a)$, where $a \in \Sigma_\mathsf{I}$ is now viewed as a constant symbol. Like $\mathcal{EL}$ ABoxes, quantified ABoxes $\exists X.\mathcal{A}$ can be equipped with a model-theoretic semantics [9] or translated into FO sentences [4], but where the elements of X are viewed as existentially quantified first-order variables rather than constants.

Let α, β be (q)ABoxes, concept inclusions, or assertions, $\mathcal{T}$ an $\mathcal{EL}$ TBox, and $\phi_\alpha, \phi_\beta, \phi_\mathcal{T}$ their first-order translations. Then, α *entails* β *w.r.t.* $\mathcal{T}$ ($\alpha \models^\mathcal{T} \beta$) if the implication $(\phi_\alpha \wedge \phi_\mathcal{T}) \to \phi_\beta$ is valid according to the semantics of first-order logic. If $\exists X.\mathcal{A} \models^\mathcal{T} C(a)$, then a is called an *instance* of C w.r.t. $\exists X.\mathcal{A}$ and $\mathcal{T}$. In case $\mathcal{T} = \emptyset$, we will sometimes write $\models$ instead of $\models^\emptyset$. If $\emptyset \models^\mathcal{T} C \sqsubseteq D$, then we also write $C \sqsubseteq^\mathcal{T} D$ and say that C *is subsumed by* D *w.r.t.* $\mathcal{T}$; in case $\mathcal{T} = \emptyset$ we simply say that C is subsumed by D. The subsumption problem in $\mathcal{EL}$ is decidable in polynomial time [1], and the same is true for entailment between $\mathcal{EL}$ ABoxes and entailment of a concept assertion by a qABox. A role assertion between individuals is entailed by a qABox iff it is contained in its matrix. The entailment problem between qABoxes is NP-complete [3, 9].

If one is only interested in instance queries, then using IQ-*entailment* rather than first-order entailment between qABoxes is more appropriate. The qABox $\exists X.\mathcal{A}$ IQ-*entails* $\exists Y.\mathcal{B}$ *w.r.t.* $\mathcal{T}$ (written $\exists X.\mathcal{A} \models^\mathcal{T}_\mathsf{IQ} \exists Y.\mathcal{B}$) if $\exists Y.\mathcal{B} \models^\mathcal{T} C(a)$ implies $\exists X.\mathcal{A} \models^\mathcal{T} C(a)$ for each $\mathcal{EL}$ concept assertion $C(a)$. Given a qABoxes $\exists X.\mathcal{A}$ and a TBox $\mathcal{T}$, one can compute the IQ-saturation $\mathrm{sat}^\mathcal{T}_\mathsf{IQ}(\exists X.\mathcal{A})$ of $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ in polynomial time, and this saturation satisfies $\exists X.\mathcal{A} \models^\mathcal{T}_\mathsf{IQ} \exists Y.\mathcal{B}$ iff $\mathrm{sat}^\mathcal{T}_\mathsf{IQ}(\exists X.\mathcal{A}) \models_\mathsf{IQ} \exists Y.\mathcal{B}$ for each qABox $\exists Y.\mathcal{B}$ [3]. IQ-entailment w.r.t. the empty TBox in turn corresponds to the existence of a simulation in the other direction [9]. Since existence of a simulation is in P [13], IQ-entailment between qABoxes (with our without TBox) is also in P.

In this paper, we are interested in role relationships in addition to instance relationships. For this reason, we use IRQ-*entailment*, which was first introduced in [4]. The qABox $\exists X.\mathcal{A}$ IRQ-*entails* $\exists Y.\mathcal{B}$ *w.r.t.* $\mathcal{T}$ (written $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Y.\mathcal{B}$) if every concept and role assertion entailed by the latter w.r.t. $\mathcal{T}$ is also entailed by the former w.r.t. $\mathcal{T}$. As shown in [4], $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Y.\mathcal{B}$ iff $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IQ}} \exists Y.\mathcal{B}$ and $r(a,b) \in \mathcal{B}$ implies $r(a,b) \in \mathcal{A}$ for all $r \in \Sigma_{\mathsf{R}}$ and $a,b \in \Sigma_{\mathsf{I}}$. Since deciding IQ-entailment and checking the inclusion relation for the role assertions can be done in polynomial time, IRQ-entailment is also tractable. Since ABoxes $\mathcal{B}$ consist of concept and role assertions only, $\exists X.\mathcal{A} \models^{\mathcal{T}} \mathcal{B}$ iff $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \mathcal{B}$, which shows that entailment of an ABox by a qABox w.r.t. $\mathcal{T}$ can be decided in polynomial time [4].

3 IRQ-REPAIRS FOR ABOX REPAIR REQUESTS

Let $\exists X.\mathcal{A}$ be a qABox and $\mathcal{T}$ an $\mathcal{EL}$ TBox. When defining a repair framework, one needs to decide which entailments from $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ one is interested in, and how the unwanted consequences that should be removed may look like. The former determines which entailment relation between qABoxes needs to be considered, and the later is expressed in the repair request. In our previous work on repairs in [3], we considered two settings: in both, the repair request consisted of instance assertions, but the relevant consequences were either also instance assertions or (Boolean) conjunctive queries. In the present paper, both the relevant consequences and the repair requests are ABoxes. Note, however, that an ABox as repair request is viewed to be a disjunction of its assertions, i.e., to satisfy the repair request, *none* of the assertions contained in it are allowed to follow. In this setting, where not only entailed concept assertions, but also role assertions are of interest, qABoxes need to be compared using IRQ-entailment.

Definition 3.1. Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $\exists X.\mathcal{A}$ be a qABox.

- An *ABox repair request* $\mathcal{P}$ is an $\mathcal{EL}$ ABox.
- Given an ABox repair request $\mathcal{P}$, an IRQ-*repair of* $\exists X.\mathcal{A}$ *for* $\mathcal{P}$ *w.r.t.* $\mathcal{T}$ is a qABox $\exists Y.\mathcal{B}$ such that $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Y.\mathcal{B}$ and no assertion in $\mathcal{P}$ is entailed by $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$.
- Such an IRQ-repair is *optimal* if there is no IRQ-repair $\exists Z.\mathcal{C}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ such that $\exists Z.\mathcal{C} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Y.\mathcal{B}$ and $\exists Y.\mathcal{B} \not\models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Z.\mathcal{C}$.

Given an ABox repair request $\mathcal{P}$, we denote the set of its concept assertions as $\mathcal{P}_C$ and the set of its role assertions as $\mathcal{P}_R$. The request $\mathcal{P}$ is an *instance repair request* if $\mathcal{P} = \mathcal{P}_C$ and a *role repair request* if $\mathcal{P} = \mathcal{P}_R$. We say that a set $\mathfrak{R}$ of IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ IRQ-*covers* all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ if for every IRQ-repair $\exists Y.\mathcal{B}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ there exists an element $\exists Z.\mathcal{C}$ of $\mathfrak{R}$ such that $\exists Z.\mathcal{C} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists Y.\mathcal{B}$.

In [4], the problem of computing optimal IRQ-repairs has been considered for the case of instance repair requests. To be more precise, the following is shown there.

Proposition 3.2 (Proposition 5 in [4]). *If* $\mathcal{P}$ *is an instance repair request, then up to* IRQ*-equivalence, the set of optimal* IRQ*-repairs of* $\exists X.\mathcal{A}$ *for* $\mathcal{P}$ *w.r.t.* $\mathcal{T}$ *can be computed in exponential time, and it* IRQ*-covers all* IRQ*-repairs of* $\exists X.\mathcal{A}$ *for* $\mathcal{P}$ *w.r.t.* $\mathcal{T}$.

We now show how this result can be extended from instance repair requests to ABox repair requests $\mathcal{P}$. The basic idea is to develop a repair approach that can deal with role repair requests, and then first apply this approach to the input qABox and afterwards apply the approach from [4] to the result.

3.1 Dealing with role repair requests

Since role assertions are implied by a qABox iff they occur in its matrix, one might think that the only repair option for a role repair request is to remove the assertions in the request. Whereas the qABox obtained this way clearly is a repair, it need not cover all such repairs.

Example 3.3. Consider the qABox $\exists\emptyset.\{r(a,b), A(b)\}$, where a, b are individual names, the empty TBox, and the role repair request $\mathcal{P} = \{r(a,b)\}$. Removing the role assertion $r(a,b)$ yields the qABox $\exists\emptyset.\{A(b)\}$, which is a repair for $\mathcal{P}$, but this qABox does not entail all repairs for $\mathcal{P}$: for example, $\exists\{x\}.\{r(a,x), A(x)\}$ is a repair for $\mathcal{P}$, but it is not entailed by $\exists\emptyset.\{A(b)\}$.

For the naïve idea of just removing the role assertions in the repair request to work, one must first modify the input ABox appropriately, using the construction sketched in the proof of the following lemma.

Lemma 3.4. *Let* $\exists X.\mathcal{A}$ *be a qABox. Then one can construct in polynomial time a qABox* $\exists Y.\mathcal{B}$ *such that*

(1) $\exists Y.\mathcal{B}$ *is* IRQ*-equivalent to* $\exists X.\mathcal{A}$ *w.r.t.* $\mathcal{T}$*, and*
(2) if $\exists Z.\mathcal{C}$ *is obtained from* $\exists Y.\mathcal{B}$ *by removing some role assertions between individuals, then* $\exists Z.\mathcal{C}$ *is* IQ*-equivalent to* $\exists X.\mathcal{A}$ *w.r.t.* $\mathcal{T}$.

Proof. For each individual a occurring in $\exists X.\mathcal{A}$, we introduce a fresh variable x_a as a copy of a, which we add to the quantifier-prefix. In addition, we add the following assertions to the matrix:

$$\{A(x_a) \mid A(a) \in \mathcal{A}\} \cup$$
$$\{r(x_a,u) \mid r(a,u) \in \mathcal{A}\} \cup \{r(u,x_a) \mid r(u,a) \in \mathcal{A}\}.$$

Let $\exists Y.\mathcal{B}$ be the qABox obtained this way. To see that $\exists Y.\mathcal{B}$ is IRQ-equivalent to $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$, first note that both qABoxes contain exactly the same role assertions between individuals. Thus, it is sufficient to show that they are IQ-equivalent w.r.t. $\mathcal{T}$. Since $\exists Y.\mathcal{B}$ extends $\exists X.\mathcal{A}$, the former obviously entails the latter, and thus also IQ-entails it w.r.t. $\mathcal{T}$. To prove that $\exists X.\mathcal{A}$ IQ-entails $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$, it is sufficient to show that there is a simulation $\mathfrak{S}$ from $\exists Y.\mathcal{B}$ to $\exists X.\mathcal{A}$. We can define $\mathfrak{S}$ as the extension of the identity on the objects of $\mathcal{A}$ that additionally relates the copies x_a with the respective individuals a. It is easy to see that $\mathfrak{S}$ really is a simulation.

Regarding Statement 2 of the lemma, it is again clear that $\exists Y.\mathcal{B}$ IQ-entails $\exists Z.\mathcal{C}$ w.r.t. $\mathcal{T}$ since the latter is obtained from the former by removing assertions. To prove that the entailment in the other direction holds as well, it is again enough to show that there is a simulation from $\exists Y.\mathcal{B}$ to $\exists Z.\mathcal{C}$. On the variables, this simulation

$\mathfrak{S}$ is the identity (note that $Z = Y$ since only role assertions were removed) and every individual a is related to itself as well as to its copy x_a.

This shows that $\exists Z.\mathcal{C}$ is IQ-equivalent to $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$. □

Example 3.5. Consider the qABox $\exists\emptyset.\{r(a,b), A(b)\}$ from Example 3.3. The qABox $\exists Y.\mathcal{B}$ that satisfies the properties stated in Lemma 3.4, as constructed in the proof of this lemma, is

$$\exists Y.\mathcal{B} := \exists\{x_a, x_b\}.\{r(a,b), r(a,x_b), A(b), A(x_b), r(x_a,b)\}.$$

Theorem 3.6. *Let $\exists X.\mathcal{A}$ be a qABox, $\mathcal{T}$ an $\mathcal{EL}$ TBox, and $\mathcal{P}$ a role repair request. Consider the qABox $\exists Y.\mathcal{B}$ constructed from $\exists X.\mathcal{A}$ according to Lemma 3.4, and let $\exists Z.\mathcal{C}$ be the qABox obtained from $\exists Y.\mathcal{B}$ by removing the role assertions between individuals that belong to $\mathcal{P}$. Then, $\exists Z.\mathcal{C}$ is an optimal IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$, and the singleton set $\{\exists Z.\mathcal{C}\}$ IRQ-covers all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. This implies that the set of all optimal IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ (up to IRQ-equivalence) can be computed in polynomial time.*

Proof. First, we show that $\exists Z.\mathcal{C}$ is indeed an IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. By Lemma 3.4, $\exists Y.\mathcal{B}$ is IRQ-equivalent to $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ and IQ-equivalent to $\exists Z.\mathcal{C}$. Since $\exists Z.\mathcal{C}$ is obtained from $\exists Y.\mathcal{B}$ by removing role assertions between individuals, it is IRQ-entailed by $\exists Y.\mathcal{B}$, and thus IRQ-entailed by $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$. By construction, $\mathcal{C}$ does not contain, and thus $\exists Z.\mathcal{C}$ does not entail, any of the role assertions in $\mathcal{P}$.

Second, assume that $\exists V.\mathcal{D}$ is an IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. Since $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists V.\mathcal{D}$ and $\exists Z.\mathcal{C}$ is IQ-equivalent to $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ and IQ-saturated, we know that $\exists Z.\mathcal{C} \models_{\mathsf{IQ}} \exists V.\mathcal{D}$. In addition, $\exists X.\mathcal{A} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists V.\mathcal{D}$ also implies that $\mathcal{D}$ only contains role assertions between individuals that also occur in $\mathcal{A}$. Since $\exists V.\mathcal{D}$ is an IRQ-repair for $\mathcal{P}$, $\mathcal{D}$ does not contain role assertions from $\mathcal{P}$. Thus, by the construction of $\exists Z.\mathcal{C}$, every role assertion between individuals in $\mathcal{D}$ also belongs to $\mathcal{C}$, which together with $\exists Z.\mathcal{C} \models_{\mathsf{IQ}} \exists V.\mathcal{D}$ yields $\exists Z.\mathcal{C} \models^{\mathcal{T}}_{\mathsf{IRQ}} \exists V.\mathcal{D}$. This shows that $\{\exists Z.\mathcal{C}\}$ IRQ-covers all IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.

Finally, optimality of $\exists Z.\mathcal{C}$ is an immediate consequence of the covering property we have just shown. In fact, since each IRQ-repair $\exists V.\mathcal{D}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ is IRQ-covered by $\{\exists Z.\mathcal{C}\}$, no such repair can strictly entail $\exists Z.\mathcal{C}$. □

Example 3.7. Consider the TBox, qABox, and role repair request from Example 3.3. If we remove $r(a,b)$ from the IRQ-equivalent qABox $\exists Y.\mathcal{B}$ constructed in Example 3.5, we obtain the optimal IRQ-repair $\exists Z.\mathcal{C} := \exists\{x_a, x_b\}.\{r(a,x_b), A(b), A(x_b), r(x_a,b)\}$.

3.2 Iterated repair

If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then one can first repair for $\mathcal{P}_1$ and then the resulting qABoxes for $\mathcal{P}_2$. The following theorem can easily be shown using transitivity of entailment.

Theorem 3.8. *Let $\exists X.\mathcal{A}$ be a qABox, $\mathcal{T}$ an $\mathcal{EL}$ TBox, and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ an ABox repair request, and let $\mathfrak{R}$ be a set of IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}_1$ w.r.t. $\mathcal{T}$ that IRQ-covers all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}_1$ w.r.t. $\mathcal{T}$. Consider the set $\mathfrak{R}'$ of qABoxes that is obtained from $\mathfrak{R}$ by replacing each element $\exists Y.\mathcal{B}$ of $\mathfrak{R}$ by the elements of a set of IRQ-repairs of $\exists Y.\mathcal{B}$ for $\mathcal{P}_2$ w.r.t. $\mathcal{T}$ that IRQ-covers all IRQ-repairs of $\exists Y.\mathcal{B}$ for $\mathcal{P}_2$ w.r.t. $\mathcal{T}$. Then, $\mathfrak{R}'$ is a set of IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ that IRQ-covers all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.*

If the covering sets used in the theorem consist of optimal repairs, then the set $\mathfrak{R}'$ also consists of optimal repairs if additionally the first covering set $\mathfrak{R}$ is a singleton set.

Corollary 3.9. *Assume that, in Theorem 3.8, the set $\mathfrak{R}$ consists of a single qABox $\exists Y.\mathcal{B}$ (which is then necessarily an optimal repair), and the set of repairs for $\exists Y.\mathcal{B}$ consists of optimal repairs. Then, $\mathfrak{R}'$ consists (up to IRQ-equivalence) of the optimal IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.*

Proof. We know from Theorem 3.8 that $\mathfrak{R}'$ IRQ-covers all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. Thus, it must contain (up to IRQ-equivalence) all optimal IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. Assume that one of its elements is not optimal. Then it is strictly entailed by another element. This contradicts the assumption that this set only contains optimal repairs of $\exists Y.\mathcal{B}$. □

Given Theorem 3.6, this corollary actually applies in the setting where $\mathcal{P}_1 = \mathcal{P}_R$ and $\mathcal{P}_2 = \mathcal{P}_C$. Since we know from Proposition 3.2 that the set of optimal IRQ-repairs for an instance repair request can be computed in exponential time, and it IRQ-covers all IRQ-repairs, this yields the following result for ABox repair requests.

Corollary 3.10. *Let $\exists X.\mathcal{A}$ be a qABox, $\mathcal{T}$ an $\mathcal{EL}$ TBox, and $\mathcal{P}$ an ABox repair request. The set of all optimal IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ (up to IRQ-equivalence) can be computed in exponential time, and it IRQ-covers all IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.*

4 ERROR-TOLERANT REASONING

In error-tolerant reasoning, one does not commit to a single (optimal or classical) repairs, but instead reasons with respect to all repairs. Brave entailment asks whether a given query is entailed by some repair whereas cautious entailment asks whether the query is entailed by all repairs. In this section, we investigate the complexity of error-tolerant reasoning problems both for optimal IRQ-repairs and for classical repairs.

4.1 Based on Optimal IRQ-Repairs

We assume in the following that the given repair problem has a repair, which is the case if the repair request $\mathcal{P}$ is *solvable w.r.t. $\mathcal{T}$*, i.e., if $\top \not\sqsubseteq^{\mathcal{T}} C$ for all $C(a) \in \mathcal{P}$ [3].

Definition 4.1. Let $\exists X.\mathcal{A}$ be a qABox, $\mathcal{T}$ an $\mathcal{EL}$ TBox, $\mathcal{P}$ an ABox repair request that is solvable w.r.t. $\mathcal{T}$, and $\mathcal{Q}$ an $\mathcal{EL}$ ABox. Then, $\mathcal{Q}$ is *bravely* IRQ-*entailed* by $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ and $\mathcal{P}$ if there is an optimal IRQ-repair $\exists Z.\mathcal{C}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ such that $\exists Z.\mathcal{C} \models^{\mathcal{T}} \mathcal{Q}$. It is *cautiously* IRQ-*entailed* by $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ and $\mathcal{P}$ if $\exists Z.\mathcal{C} \models^{\mathcal{T}} \mathcal{Q}$ holds for all optimal IRQ-repairs $\exists Z.\mathcal{C}$ of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.

Brave entailment. Brave IRQ-entailment can be decided in polynomial time since it can be reduced to the ABox entailment and the instance problems in $\mathcal{EL}$, which are both in P. In fact, it is easy to see that $\mathcal{Q}$ is bravely IRQ-entailed by $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ iff $\exists X.\mathcal{A} \models^{\mathcal{T}} \mathcal{Q}$ and no assertion in $\mathcal{P}$ is entailed by $\mathcal{Q}$ w.r.t. $\mathcal{T}$. Intuitively, this equivalence holds since the latter statement means that $\mathcal{Q}$ is itself a repair, and requiring entailment from some optimal repair is the same as requiring entailment from an arbitrary repair (see the argument used to show Lemma 16 in [8] for more details).

Theorem 4.2. *Brave entailment w.r.t. optimal* IRQ*-repairs for ABox repair requests is in* P.

Basically, such a reduction is possible whenever the considered consequence can itself be viewed as a repair. For the setting of classical repairs (see Section 4.2 below), this is not the case since the considered consequences need not be subsets of the given qABox. We will show that, in this setting, brave reasoning is NP-complete.

As argued in [8], brave entailment can also be used to support computing a specific repair that not only removes unwanted consequences (specified by $\mathcal{P}$), but also retains wanted ones (specified by $\mathcal{Q}$), if such a repair exists at all. We refer the reader to [8] for details, and just point out that the results shown in Section 4.1 of that paper for instance repair requests also hold in our more general setting of ABox repair requests.

Cautious entailment. This type of entailment cannot be solved by a simple reduction to classical reasoning in $\mathcal{EL}$, and the naïve approach of computing all optimal repairs and then checking entailment from them would be too complex since there may be exponentially many optimal repairs, each of which can have exponential size [7]. To avoid this problem, we need to look more closely at how optimal repairs for instance repair requests are constructed.

Recall that, given an ABox repair request $\mathcal{P}$, we can compute the optimal IRQ-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ by first computing the optimal IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}_R$ w.r.t. $\mathcal{T}$. By Theorem 3.6, this can be done in polynomial time, and the resulting qABox $\exists Z.\mathcal{C}$ is of polynomial size. The next step is then to compute the optimal IRQ-repairs of $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$, which is where the exponential blow-up could happen. As shown in [4], the canonical IQ-repairs introduced in [3] are also IRQ-repairs, and they IRQ-cover all IRQ-repairs of $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$. The optimal IRQ-repairs are obtained from this covering set by removing redundant elements, i.e., elements that are strictly IRQ-entailed by another one.

To obtain the canonical IQ-repairs of $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$, the approach in [3] first saturates this qABox w.r.t. $\mathcal{T}$. Thus, we consider $\exists Y.\mathcal{B} := \mathsf{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.\mathcal{C})$, which can again be computed in polynomial time. We denote the set of all subconcepts of concepts occurring in $\mathcal{P}_C$ or $\mathcal{T}$ with $\mathsf{Sub}(\mathcal{P}_C, \mathcal{T})$. An *atom* is a concept name or an existential restriction. We denote the set of atoms in $\mathsf{Sub}(\mathcal{P}_C, \mathcal{T})$ with $\mathsf{Atoms}(\mathcal{P}_C, \mathcal{T})$. Important ingredients in the definition of canonical repairs are repair types and seed functions.

Definition 4.3. A *repair type* $\mathcal{K}$ for an object u of $\exists Y.\mathcal{B}$ is a subset of $\mathsf{Atoms}(\mathcal{P}_C, \mathcal{T})$ that satisfies the following three conditions:

1. $K \not\sqsubseteq_\emptyset K'$ for all distinct atoms $K, K' \in \mathcal{K}$.
2. $\mathcal{B} \models K(u)$ for every atom $K \in \mathcal{K}$.
3. $\mathcal{K}$ is *premise-saturated*, i.e., for each atom $K \in \mathcal{K}$ and each subconcept $C \in \mathsf{Sub}(\mathcal{P}_C, \mathcal{T})$, if $\mathcal{B} \models C(u)$ and $C \sqsubseteq^{\mathcal{T}} K$, then there is an atom $K' \in \mathcal{K}$ with $C \sqsubseteq_\emptyset K'$. [1]

A *repair type assignment (rta)* on $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$ assigns to each individual a a repair type for a. Such a function is called a *repair seed function (rsf)* if for each $P(a) \in \mathcal{P}_C$, there is $K \in s(a)$ such that $P \sqsubseteq^{\emptyset} K$.

[1]A *repair pre-type* need only satisfy the first two conditions.

It is shown in [3] that each repair seed function s induces a *canonical* IQ*-repair*, denoted by $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.\mathcal{C}, s)$. Since these canonical repairs IRQ-cover all repairs of $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$ [4], Theorem 3.8 and Corollary 3.9 yield the following result.

Proposition 4.4. *Let $\mathcal{P}$ be an ABox repair request and $\exists Z.\mathcal{C}$ be the optimal* IRQ*-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}_R$ w.r.t. $\mathcal{T}$. Then the set of all canonical* IQ*-repairs* $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.\mathcal{C}, s)$ *(where s ranges over all repair seed functions on $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$)* IRQ*-covers all* IRQ*-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$. Removing all elements from this set that are strictly* IRQ*-entailed by another element yields the set of all optimal* IRQ*-repairs of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.*

Following [8], we do not compute the (exponentially large) canonical repairs, but instead work directly with the (polynomially small) seed functions. To make this feasible, we need to show that entailment from the induced canonical repairs as well as optimality of these repairs can be decided in time polynomial in the size of the seed function, the qABox $\exists Z.\mathcal{C}$, and the TBox $\mathcal{T}$.

Tractability of the first problem is an immediate consequence of the following lemma, for whose formulation we need to introduce some notation. Given sets $\mathcal{K}$ and $\mathcal{L}$ of concept descriptions, we say that $\mathcal{K}$ is *covered* by $\mathcal{L}$ (written $\mathcal{K} \leq \mathcal{L}$) if, for each $K \in \mathcal{K}$, there is $L \in \mathcal{L}$ such that $K \sqsubseteq_\emptyset L$. If $\mathcal{K}$ is a repair type and v an object in the saturated qABox $\exists Y.\mathcal{B}$, then we define $\mathsf{Succ}(\mathcal{K}, r, v) := \{C \mid \exists r.C \in \mathcal{K} \text{ and } \mathcal{B} \models^{\mathcal{T}} C(v)\}$.

Lemma 4.5. *Let $\mathcal{Q}$ be an $\mathcal{EL}$ ABox such that $\exists Z.\mathcal{C} \models^{\mathcal{T}} \mathcal{Q}$ and s be a repair seed function on $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$. Then, $\mathcal{Q}$ is entailed by the canonical* IQ*-repair induced by s iff for each $C(a) \in \mathcal{Q}$, C is not subsumed by any atom in $s(a)$ w.r.t. $\mathcal{T}$, and $\mathsf{Succ}(s(a), r, b) \leq s(b)$ holds for each $r(a, b) \in \mathcal{Q}$.*

Restricted to concept assertions, this result was already stated in [8]. The part on role assertions is an easy consequence of the way role assertions in the canonical repair are defined in [3]. Since the conditions on the right-hand sides of the equivalences can obviously be checked in polynomial time, we obtain the following tractability result.

Proposition 4.6. *Given a qABox $\exists Z.\mathcal{C}$, a TBox $\mathcal{T}$, an instance repair request $\mathcal{P}_C$, a repair seed function s, and an ABox $\mathcal{Q}$, we can decide in polynomial time whether $\mathcal{Q}$ is entailed by the canonical* IQ*-repair induced by s w.r.t. $\mathcal{T}$.*

To address the problem of checking optimality, we use a pre-order on seed functions that reflects IRQ-entailment between the induced canonical repairs. For IQ-entailment, such a pre-order was already introduced in [5] and used in [8]. Its extension to IRQ-entailment was also sketched in [5].

Definition 4.7. Let s, t be rtas on $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$. We say that s is $\leq_{\mathsf{IQ}}$*-covered* by t (denoted $s \leq_{\mathsf{IQ}} t$) if $s(a) \leq t(a)$ holds for each $a \in \Sigma_{\mathsf{I}}$. Additionally, s is $\leq_{\mathsf{IRQ}}$*-covered* by t (denoted $s \leq_{\mathsf{IRQ}} t$) if $s \leq_{\mathsf{IQ}} t$ and if, for all $r(a, b) \in \mathcal{C}$ with $a, b \in \Sigma_{\mathsf{I}}$, it holds that $\mathsf{Succ}(t(a), r, b) \leq t(b)$ implies $\mathsf{Succ}(s(a), r, b) \leq s(b)$.

The following lemma is an easy consequence of Lemma 4.5.

Lemma 4.8 ([5]). *Let s, t be rsfs on $\exists Z.C$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$. Then $s \leq_{\mathsf{IRQ}} t$ iff $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.C, s) \models^{\mathcal{T}}_{\mathsf{IRQ}} \mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.C, t)$.*

Given any pre-order $\leq$, we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\beta \not\leq \alpha$, and say that α is $\leq$-minimal if there is no β such that $\beta < \alpha$. The previous lemma together with Proposition 4.4 implies that minimal seed functions correspond to optimal repairs in the following sense.

Proposition 4.9. *If s is a $\leq_{\mathsf{IRQ}}$-minimal rsf, then $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.C, s)$ is an optimal IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$, and every optimal IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ is IRQ-equivalent to a canonical repair $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Z.C, s)$ for a $\leq_{\mathsf{IRQ}}$-minimal rsf s.*

The following example illustrates why we must use $\leq_{\mathsf{IRQ}}$-minimality rather than $\leq_{\mathsf{IQ}}$-minimality in this proposition.

Example 4.10. Consider the qABox $\exists\emptyset.\mathcal{A} = \exists\emptyset.\{A(b), r(a,b)\}$, the empty TBox, and the repair request $\mathcal{P} = \{\exists r.A(a)\}$. Since there is no role assertion in $\mathcal{P}$, the optimal role repair is $\exists\emptyset.\mathcal{A}$ itself. There is only one $\leq_{\mathsf{IQ}}$-minimal rsf s with $s(a) = \{\exists r.A\}$ and $s(b) = \emptyset$, which yields the canonical IQ-repair $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists\emptyset.\mathcal{A}, s)$ in which $r(a,b)$ does not occur, but the individual b still belongs to A.

The rsf t with $t(a) = \{\exists r.A\}$ and $t(b) = \{A\}$ is not $\leq_{\mathsf{IQ}}$-minimal since $s <_{\mathsf{IQ}} t$. It is, however, $\leq_{\mathsf{IRQ}}$-minimal. In fact, $s \not\leq_{\mathsf{IRQ}} t$ since $\mathsf{Succ}(t(a), r, b) = \{A\} \leq t(b) = \{A\}$, but $\mathsf{Succ}(s(a), r, b) = \{A\} \not\leq s(b) = \emptyset$. The function t induces the canonical IQ-repair $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists\emptyset.\mathcal{A}, t)$, in which $r(a,b)$ occurs, but the individual b is no longer an instance of A. If we are only interested in concept assertions, this repair is not optimal, but in the context of this paper, where we are also interested in role assertion, it has the consequences $r(a,b)$, which $\mathsf{rep}^{\mathcal{T}}_{\mathsf{IQ}}(\exists\emptyset.\mathcal{A}, s)$ does not have.

It remains to show that checking $\leq_{\mathsf{IRQ}}$-minimality is tractable. Since the proof of this result is rather technical, we defer it to Section 5 and only state the main result here.

Proposition 4.11. *$\leq_{\mathsf{IRQ}}$-minimality of seed functions can be decided in polynomial time.*

With this result in place, we can now show that cautious entailment is in coNP by describing an NP-procedure for non-entailment. As sketched above, we first compute in polynomial time the optimal repair $\exists Z.C$ of $\exists X.\mathcal{A}$ for $\mathcal{P}_R$ w.r.t. $\mathcal{T}$ based on the construction described in the proof of Theorem 3.6. Then, we execute a guess-and-check NP-procedure as follows: we guess a function $s : \Sigma_{\mathsf{I}} \to \wp(\mathsf{Atoms}(\mathcal{R}, \mathcal{T}))$ and check whether (i) s is a repair seed function on $\exists Z.C$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$; (ii) s is $\leq_{\mathsf{IRQ}}$-minimal; and (iii) s does not satisfy one of the conditions described in Lemma 4.5, and thus the canonical IQ-repair induced by s does not entail Q. Note that these three tasks can be performed in polynomial time. This guess-and-check procedure is successful iff there is an optimal IRQ-repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ that does not entail Q.

Theorem 4.12. *Cautious entailment w.r.t. optimal IRQ-repairs for ABox repair requests is in coNP.*

4.2 Based on Classical Repairs

If we consider classical repairs [14, 20, 21] rather than optimal ones, then we obtain a matching coNP lower bound for cautious reasoning, and even brave reasoning becomes NP-hard. First, we recall the relevant definitions for classical repairs, adapted to qABoxes. Note that, for classical repairs, the optimality criterion of being subset maximal is already built into the definition of the repair.

Definition 4.13. Let $\exists X.\mathcal{A}$ be a qABox, $\mathcal{T}$ an $\mathcal{EL}$ TBox, and $\mathcal{P}$ an ABox repair request. The qABox $\exists X.\mathcal{B}$ is a *classical repair* of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ if $\mathcal{B} \subseteq \mathcal{A}$, no assertion in $\mathcal{P}$ is entailed by $\exists X.\mathcal{B}$ w.r.t. $\mathcal{T}$, and for each qABox $\exists X.\mathcal{C}$ such that $\mathcal{B} \subset \mathcal{C} \subseteq \mathcal{A}$, there is an assertion in $\mathcal{P}$ that is entailed by $\exists X.\mathcal{C}$ w.r.t. $\mathcal{T}$.

For a given query Q, the *brave entailment* problem for classical repairs asks whether there is a classical repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ such that the repair entails Q, whereas the *cautious entailment* problem asks if Q is entailed by each classical repair of $\exists X.\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$.

Proposition 4.14. *Brave entailment for classical repairs of qABoxes w.r.t. $\mathcal{EL}$ TBoxes is NP-complete.*

Proof. An NP procedure for deciding brave entailment can be obtained as follows. Given $\exists X.\mathcal{A}, \mathcal{T}, \mathcal{P}$, and Q, we guess a subset of $\mathcal{A}$ and check whether it entails Q w.r.t. $\mathcal{T}$, but does not entail any of the assertions in $\mathcal{P}$ w.r.t. $\mathcal{T}$. To test whether $\mathcal{B}$ is maximal, we check whether there is no element $\alpha \in \mathcal{A} \setminus \mathcal{B}$ such that $\exists X.\mathcal{B} \cup \{\alpha\}$ does not entail any of the assertions in $\mathcal{P}$ w.r.t. $\mathcal{T}$. It is clear that all these tests can be performed in polynomial time.

We show NP-hardness by a reduction from the NP-complete problem *monotone 1-in-3-SAT* [19]. Given a set $\mathcal{V} = \{p_1, \ldots, p_m\}$ of propositional variables and n 3-clauses $c_1, \ldots, c_n$ such that each c_i is of the form $p^i_1 \vee p^i_2 \vee p^i_3$, where $p^i_1, p^i_2, p^i_3 \in \mathcal{V}$, a solution of the problem is an assignment of truth values to the variables that makes exactly one variable in each clause true. We assume w.l.o.g. that every variable is contained in at least one clause.

Given such an instance of the monotone 1-in-3-SAT problem, we construct an instance of the brave entailment problem for classical repairs as follows. We introduce a concept name P_j for each $p_j \in \mathcal{V}$ and concept names B_i, C_i for each clause c_i, and consider the ABox $\mathcal{A} := \{P_1(a), \ldots, P_m(a)\}$. The TBox $\mathcal{T}$ contains the following CIs for each clause $c_i = p^i_1 \vee p^i_2 \vee p^i_3$:

(∗) $P^i_1 \sqcap P^i_2 \sqsubseteq B_i, P^i_1 \sqcap P^i_3 \sqsubseteq B_i, P^i_2 \sqcap P^i_3 \sqsubseteq B_i$ and

(∗∗) $P^i_1 \sqsubseteq C_i, P^i_2 \sqsubseteq C_i, P^i_3 \sqsubseteq C_i$.

Then, we define the repair request $\mathcal{P} = \{B_1(a), \ldots, B_n(a)\}$ and the query $Q = \{C_1(a), \ldots, C_n(a)\}$.

Intuitively, making the variable p_j true corresponds to keeping $P_j(a)$ in the repair. Thus, the CIs of the form (∗) together with the repair request express the condition that at most one of the variables in c_i can be true. The CIs of the form (∗∗) together with the query express that at least one of the variables in c_i must be true. Before we can conclude that the given monotone 1-in-3-SAT problem has a solution iff Q is entailed by some classical repair of $\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$, we need to explain why the maximality condition in the definition of classical repairs is satisfied. The reason is that, for every $P_j(a)$ that is removed from the repair, the corresponding variable p_j belongs to some clause c_i. This clause is then satisfied by another of its variables, and thus adding back $P_j(a)$ would then trigger a CI of the form (∗∗), thus entailing $B_i(a)$. ◻

A standard way for obtaining classical repairs is to compute all justifications and then use Reiter's hitting set duality [18] to generate the repairs from them. The qABox $\exists X.\mathcal{C}$ is a *justification* of $\mathcal{P}$ for $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$ if $\mathcal{C}$ is a minimal subset of $\mathcal{A}$ such that $\mathcal{P}$ is entailed by $\exists X.\mathcal{C}$ w.r.t. $\mathcal{T}$. The classical repairs are obtained by computing all minimal hitting sets $\mathcal{D}$ of the collection of justifications, and then considering all ABoxes $\mathcal{B}$ generated from $\mathcal{A}$ by removing one of these sets $\mathcal{D}$. Given a collection of sets, a *hitting set* is a set that has a non-empty intersection with each element of the collection. An easy consequence of this duality, which will be used in the proof of the next proposition, is that an assertion not belonging to any justification is contained in every classical repair. The hardness proof given below is inspired by the proof of a similar hardness result for cautious reasoning in the setting of repairing $\mathcal{EL}$ TBoxes [17].

Proposition 4.15. *Cautious entailment for classical repairs of qABoxes w.r.t. $\mathcal{EL}$ TBoxes is* coNP*-complete.*

Proof. An NP procedure for cautious non-entailment can be obtained by guessing a subset of the ABox and then checking whether it is a classical repair and whether $\mathcal{Q}$ is not entailed, similarly to what we have described in the previous proof.

We show NP-hardness of cautious non-entailment by a reduction of the NP-complete *"path via a node" problem* [11, 15]: given a directed graph (V, E) and vertices $s, t, m \in V$, this problem asks whether there is a simple path[2] from s to t via m?

To define the reduction, we consider a directed graph $\mathbf{G} := (V, E)$ and vertices $s, t, m \in V$. We first split the vertex m into a new edge from m_1 to m_2, which yields the directed graph $\mathbf{G}' := (V', E')$ with vertex set $V' := (V \setminus \{m\}) \cup \{m_1, m_2\}$ and edge set

$$\begin{aligned} E' := {} & \{\, (v,w) \mid (v,w) \in E,\ v \neq m,\ w \neq m \,\} \cup \{(m_1, m_2)\} \\ & \cup \{\, (v, m_1) \mid (v,m) \in E,\ v \neq m \,\} \\ & \cup \{\, (m_2, w) \mid (m,w) \in E,\ w \neq m \,\}. \end{aligned}$$

Each path in $\mathbf{G}$ via m corresponds to a path in $\mathbf{G}'$ containing the new edge (m_1, m_2) in the obvious way.

This modified graph is represented in the ABox by using concept names $\mathsf{Edge}\langle v \to w\rangle$ for all $v, w \in V'$:

$$\mathcal{A} := \{\, \mathsf{Edge}\langle v \to w\rangle(a) \mid (v,w) \in E' \,\},$$

where a is some fixed individual name. To represent paths from any node v to the terminal node t, we employ concept names $\mathsf{Path}\langle v \to t\rangle$ for all $v \in V'$. The connection between such paths and the edges is axiomatized in the TBox:

$$\begin{aligned} \mathcal{T} := {} & \{\top \sqsubseteq \mathsf{Path}\langle t \to t\rangle\} \cup \\ & \{\, \mathsf{Edge}\langle v \to w\rangle \sqcap \mathsf{Path}\langle w \to t\rangle \sqsubseteq \mathsf{Path}\langle v \to t\rangle \mid (v,w) \in E' \,\}. \end{aligned}$$

Obviously, $\mathsf{Path}\langle s \to t\rangle(a)$ is entailed by $\mathcal{A}$ w.r.t. $\mathcal{T}$ iff there is a path from s to t in $\mathbf{G}'$. Moreover, it is easy to see that the simple paths from s to t in $\mathbf{G}'$ correspond to the justifications for $\mathsf{Path}\langle s \to t\rangle(a)$ in the obvious way. Consequently, there is a simple path from s to t via m in the original graph $\mathbf{G}$ iff there is a simple path from s to t using the edge (m_1, m_2) in $\mathbf{G}'$ iff there is a justification for $\mathsf{Path}\langle s \to t\rangle(a)$ containing $\mathsf{Edge}\langle m_1 \to m_2\rangle(a)$. Thus, if we define $\mathcal{P} := \{\mathsf{Path}\langle s \to t\rangle(a)\}$ and $\mathcal{Q} := \{\mathsf{Edge}\langle m_1 \to m_2\rangle(a)\}$, then there is a simple path from s to t via m in $\mathbf{G}$ iff there is a classical repair of $\mathcal{A}$ for $\mathcal{P}$ w.r.t. $\mathcal{T}$ that does not contain $\mathsf{Edge}\langle m_1 \to m_2\rangle(a)$ iff $\mathsf{Edge}\langle m_1 \to m_2\rangle(a)$ is not a cautious consequence of $\mathcal{A}$ w.r.t. $\mathcal{T}$ and $\mathcal{P}$ for classical repairs. □

[2] A path is *simple* if it does not contain any vertex more than once.

5 IRQ-MINIMALITY OF SEED FUNCTIONS

In this section, we assume that the qABoxes $\exists Z.\mathcal{C}$ and $\exists Y.\mathcal{B}$ are defined as in Section 4.1 and that repair types, rtas and rsfs are defined for these qABoxes w.r.t. the instance repair request $\mathcal{P}_C$. We prove that $\leq_{\mathsf{IRQ}}$-minimality of seed functions can be decided in polynomial time. Basically, we show that, if s is not $\leq_{\mathsf{IRQ}}$-minimal, then we can effectively find an rsf t with $t <_{\mathsf{IRQ}} s$ among a polynomial number of candidates. We start with characterizing the relation $<_{\mathsf{IRQ}}$.

Lemma 5.1. *Let s, t be repair type assignments on $\exists Z.\mathcal{C}$ for $\mathcal{P}_C$ w.r.t. $\mathcal{T}$. Then, $t <_{\mathsf{IRQ}} s$ iff the following two conditions hold:*

(1) $t(b) \leq s(b)$ for all $b \in \Sigma_\mathsf{I}$ and there exists $a \in \Sigma_\mathsf{I}$ such that $t(a) < s(a)$;
(2) $\mathsf{Succ}(s(c), r, d) \leq s(d)$ implies $\mathsf{Succ}(t(c), r, d) \leq t(d)$ for all $r(c,d) \in \mathcal{A}$ with $c, d \in \Sigma_\mathsf{I}$.

Proof. The "if" direction is trivial. To show the "only if" direction, assume that $t <_{\mathsf{IRQ}} s$. Then $t \leq_{\mathsf{IRQ}} s$ and thus $t(b) \leq s(b)$ for all $b \in \Sigma_\mathsf{I}$ and Statement 2 of the lemma hold. Assume that Statement 1 does not hold. Then $s(b) \leq t(b)$ holds for all $b \in \Sigma_\mathsf{I}$. We claim that this implies $s \leq_{\mathsf{IRQ}} t$. For this, it is enough to show, for all $r(c,d) \in \mathcal{A}$ with $c, d \in \Sigma_\mathsf{I}$, that $\mathsf{Succ}(t(c), r, d) \leq t(d)$ implies $\mathsf{Succ}(s(c), r, d) \leq s(d)$. This is an immediate consequence of the fact that $t(d) \leq s(d)$ and transitivity of $\leq$. □

On the way towards satisfying Statement 1, we recall from [8] how, for a given repair type $\mathcal{K}$, a non-empty set $\mathcal{M}$ of atoms covered by it can be employed to construct a repair pre-type that is strictly covered by $\mathcal{K}$.

Definition 5.2 ([8]). Let $\mathcal{K}$ be a repair type for u and $\mathcal{M}$ be a non-empty subset of $\mathsf{Atoms}(\mathcal{P}_C, \mathcal{T})$ such that $\mathcal{M} \leq \mathcal{K}$. We define the *lowering of $\mathcal{K}$ w.r.t. $\mathcal{M}$* by

$$\mathsf{low}(\mathcal{K}, \mathcal{M}) := \mathsf{Max}\left\{ E \;\middle|\; \begin{array}{l} E \in \mathsf{Atoms}(\mathcal{P}_C, \mathcal{T}),\ \mathcal{B} \models E(u), \\ E \sqsubseteq_\emptyset K \text{ for some } K \in \mathcal{K}, \\ M \not\sqsubseteq_\emptyset E \text{ for each } M \in \mathcal{M} \end{array} \right\}.$$

It was shown in [8] that $\mathsf{low}(\mathcal{K}, \mathcal{M})$ is a repair pre-type that satisfies $\mathsf{low}(\mathcal{K}, \mathcal{M}) < \mathcal{K}$. The idea is now to use this lowering function to construct, for a given rta s and an atom $D \in s(a)$ for some individual a, a strictly smaller rta t for which $D \notin t(a)$. There are, however, two problems to overcome. First, we must ensure that the sets $t(d)$ for $d \in \Sigma_\mathsf{I}$ are repair types. Second, we must guarantee that Statement 2 of Lemma 5.1 also holds. This is taken care of by the construction introduced in the next definition.

Definition 5.3. Let s be an rta and $D \in s(a)$ for some $a \in \Sigma_\mathsf{I}$. We inductively define the following sets:

- $\mathcal{M}_a^0 := \{D\}$ and $\mathcal{M}_b^0 := \emptyset$ for each $b \in \Sigma_\mathsf{I} \setminus \{a\}$,
- for each $c \in \Sigma_\mathsf{I}$, the set $\mathcal{M}_c^{i+1}$ consists of
  (1) all atoms in $\mathcal{M}_c^i$,
  (2) all atoms $F \in \mathsf{low}(s(c), \mathcal{M}_c^i)$ for which there exists $C \in \mathsf{Sub}(\mathcal{P}_C, \mathcal{T})$ such that $\mathcal{B} \models C(u)$, $C \sqsubseteq^{\mathcal{T}} F$, and $\{C\} \not\leq \mathsf{low}(s(c), \mathcal{M}_c^i)$, and

(3) all atoms $\exists r.E \in \mathsf{low}(s(c), \mathcal{M}_c^i)$ for which there is $r(c,d) \in C$ with $d \in \Sigma_\mathsf{I}$ such that $\mathsf{Succ}(s(c), r, d) \leq s(d)$, $E \in \mathsf{Succ}(\mathsf{low}(s(c), \mathcal{M}_c^i), r, d)$, and $\{E\} \not\leq \mathsf{low}(s(d), \mathcal{M}_d^i)$.

For $c \in \Sigma_\mathsf{I}$, we set $\mathcal{M}_c := \mathcal{M}_c^j$, where j is minimal such that $\mathcal{M}_d^{j+1} = \mathcal{M}_d^j$ for all $d \in \Sigma_\mathsf{I}$. We define the *lowering function* $\mathsf{low}(s, D(a))$ of s w.r.t. $D(a)$ as $\mathsf{low}(s, D(a)) : c \mapsto \mathsf{low}(s(c), \mathcal{M}_c)$ for all $c \in \Sigma_\mathsf{I}$.[3]

Using an argument similar to the one employed in the proof for Lemma 10 in [8], we can show that $\mathsf{low}(s(c), \mathcal{M}_c)$ is a repair type for c, for each $c \in \Sigma_\mathsf{I}$, and thus $\mathsf{low}(s, D(a))$ is a repair type assignment. Next, we show that it is strictly smaller than s.

Lemma 5.4. *If s is an rta and $D \in s(a)$, then $\mathsf{low}(s, D(a)) <_\mathsf{IRQ} s$.*

Proof. Let $s' := \mathsf{low}(s, D(a))$. We first show that (1) $s'(a) < s(a)$ and $s'(b) \leq s(b)$ for each $b \in \Sigma_\mathsf{I} \setminus \{a\}$. We can show by induction that $\mathcal{M}_a^i$ is not empty and is covered by $s(a)$ for each $i \geq 0$. As shown in Lemma 8 of [8], $s'(a) = \mathsf{low}(s(a), \mathcal{M}_a) < s(a)$. Likewise, for each individual $b \in \Sigma_\mathsf{I} \setminus \{a\}$, we can also show by induction that $\mathcal{M}_b^i$ is covered by $s(b)$ for each i. If $\mathcal{M}_b$ is not empty, then, by Lemma 8 of [8], we have $s'(b) = \mathsf{low}(s(b), \mathcal{M}_b) < s(b)$. Otherwise, $s'(b) = \mathsf{low}(s(b), \mathcal{M}_b) = \mathsf{low}(s(b), \emptyset) = s(b)$. Consequently, we have $s'(b) \leq s(b)$ for each $b \in \Sigma_\mathsf{I} \setminus \{a\}$.

Next, we show that (2) for each $r(c,d) \in \mathcal{A}$ with $c, d \in \Sigma_\mathsf{I}$, if $\mathsf{Succ}(s(c), r, d) \leq s(d)$, then $\mathsf{Succ}(s'(c), r, d) \leq s'(d)$. Let j be the index that is defined in Definition 5.3 and $\mathcal{M}_c = \mathcal{M}_c^j$. Assume to the contrary that $\mathsf{Succ}(s(c), r, d) \leq s(d)$, but $\mathsf{Succ}(s'(c), r, d) \not\leq s'(d)$. This would imply that there is $\exists r.E \in s'(c) = \mathsf{low}(s(c), \mathcal{M}_c^j)$ such that $E \in \mathsf{Succ}(s'(c), r, d)$, but $\{E\} \not\leq s'(d)$. It would follow that $\exists r.E \in \mathcal{M}_c^{j+1} = \mathcal{M}_c^j$. However, this is a contradiction to $\exists r.E \in \mathsf{low}(s(c), \mathcal{M}_c^j)$ since the lowering set requires $\exists r.E$ to not be subsumed by any element in $\mathcal{M}_c^j$.

Given (1) and (2), we can infer $\mathsf{low}(s, D(a)) <_\mathsf{IRQ} s$ by Lemma 5.1. □

The next lemma connects rtas smaller than s with a certain lowering of s.

Lemma 5.5. *If $t <_\mathsf{IRQ} s$ for rtas s, t, then there exist $a \in \Sigma_\mathsf{I}$ and $D \in s(a)$ such that $t \leq_\mathsf{IQ} \mathsf{low}(s, D(a))$.*

Proof. Since $t <_\mathsf{IRQ} s$, we know that the two conditions in Lemma 5.1 are fulfilled. Specifically, Condition (1) yields an individual a such that $t(a) < s(a)$. Since $s(a)$ is not covered by $t(a)$, there must be an atom D in $s(a)$ that is not subsumed by any atom in $t(a)$. We consider the lowering $\mathsf{low}(s, D(a))$, and show that t is $\leq_\mathsf{IQ}$-covered by it. To do so, we prove by induction on i that, for each individual name c, the repair type $t(c)$ is covered by $\mathsf{low}(s(c), \mathcal{M}_c^i)$.

Base case ($i = 0$). If $c = a$, then $\mathcal{M}_c^0 = \{D\}$ and the claim follows as in the base case of the proof of Lemma 11 in [8] (treating $t(a)$ as $\mathcal{L}$ and $s(a)$ as $\mathcal{K}$). If $c \neq a$, then $\mathcal{M}_c^0 = \emptyset$ and the claim follows since $\mathsf{low}(s(c), \emptyset) = s(c)$.

Induction step ($i \longrightarrow i+1$). Consider an individual name $c \in \Sigma_\mathsf{I}$ and an atom $L \in t(c)$. We must show that there is an atom in $\mathsf{low}(s(c), \mathcal{M}_c^{i+1})$ that subsumes L. We prove this by verifying that L satisfies the conditions stated for E in Definition 5.2. The proof is similar to that of Lemma 11 in [8] (treating $t(c)$ as $\mathcal{L}$ and $s(c)$ as $\mathcal{K}$), with the exception of Case (3) below, which corresponds to the new case (3) in Definition 5.3.

- Since $t(c)$ is a repair type for c, we know that $\mathcal{B} \models L(c)$.
- Due to $t(c) \leq s(c)$, there is $K \in s(c)$ with $L \sqsubseteq^\emptyset K$.
- It remains to show that $M \not\sqsubseteq^\emptyset L$ for each $M \in \mathcal{M}_c^{i+1}$. Assume the contrary. According to Definition 5.3, there are three possible reasons for M to belong to $\mathcal{M}_c^{i+1}$. That the cases (1) and (2) lead us to a contradiction can be shown as in the inductive case of the proof of Lemma 11 in [8]. Thus, we concentrate on the new case (3). In this case, M is of the form $\exists r.E$ for an existential restriction $\exists r.E \in \mathsf{low}(s(c), \mathcal{M}_c^i)$ that satisfies the conditions formulated in case (3) of Definition 5.3.
  By assumption we have that $\exists r.E \sqsubseteq^\emptyset L$, and thus the recursive characterization of subsumption in [10] implies that $L = \exists r.F$ for a concept F satisfying $E \sqsubseteq^\emptyset F$. Since $E \in \mathsf{Succ}(\mathsf{low}(s(c), \mathcal{M}_c^i), r, d)$, we have $\mathcal{B} \models E(d)$, and thus $E \sqsubseteq^\emptyset F$ implies $\mathcal{B} \models F(d)$. Now, $\{E\} \not\leq \mathsf{low}(s(d), \mathcal{M}_d^i)$ together with $E \sqsubseteq^\emptyset F$ implies that $\{F\} \not\leq \mathsf{low}(s(d), \mathcal{M}_d^i)$.
  Now recall that $L = \exists r.F$ is in $t(c)$, and so $\mathcal{B} \models F(d)$ implies that $F \in \mathsf{Succ}(t(c), r, d)$. Furthermore, it follows from Condition (2) of Lemma 5.1 that $\mathsf{Succ}(t(c), r, d) \leq t(d)$, and so we obtain that $\{F\} \leq t(d)$. Since $t(d) \leq \mathsf{low}(s(d), \mathcal{M}_d^i)$ by the induction hypothesis, it follows that $\{F\} \leq \mathsf{low}(s(d), \mathcal{M}_d^i)$, a contradiction! □

In general, $t \leq_\mathsf{IRQ} \mathsf{low}(s, D(a))$ need not hold. The following example shows that $t <_\mathsf{IRQ} s$ does not imply the existence of $D \in s(a)$ such that $t \leq_\mathsf{IRQ} \mathsf{low}(s, D(a))$.

Example 5.6. Consider the TBox $\mathcal{T} = \{\exists r_2.A \sqsubseteq \exists r_1.\exists r_2.\top\}$, the qABox $\exists\emptyset.\mathcal{A} = \exists\emptyset.\{r_1(a,a), r_2(a,b), A(b), B(b), C(b)\}$, and the repair request $\mathcal{P}_C = \{\exists r_2.(A \sqcap B \sqcap C)(a), \exists r_1.\exists r_2.(A \sqcap B)(b)\}$.

Let s, t be the rtas defined as $s(a) := \{\exists r_2.A, \exists r_1.\exists r_2.\top\}$, $t(a) := \{\exists r_2.(A \sqcap B \sqcap C), \exists r_1.\exists r_2.(A \sqcap B)\}$ and $s(b) = t(b) = \emptyset$. Clearly, $t <_\mathsf{IRQ} s$. Since a is the only individual that is mapped by s to a non-empty set, s can only be lowered either w.r.t. $(\exists r_2.A)(a)$ or w.r.t. $(\exists r_1.\exists r_2.\top)(a)$.

Using the former assertion, we obtain $s_1' = \mathsf{low}(s, (\exists r_2.A)(a))$ such that $s_1'(a) = \{\exists r_2.(A \sqcap B), \exists r_1.\exists r_2.(A \sqcap B)\}$ and $s_1'(b) = \emptyset$, whereas using the latter assertion yields $s_2' = \mathsf{low}(s, (\exists r_1.\exists r_2.\top)(a))$ such that $s_2'(a) = \{\exists r_2.A, \exists r_1.\exists r_2.(A \sqcap B)\}$ and $s_2'(b) = \emptyset$. It is easy to see that t is neither IRQ-covered by s_1' nor by s_2' since both lowering functions ensure that $r_1(a,a)$ is contained in the induced canonical repairs, while t does not ensure this condition.

Nevertheless, Lemma 5.5 is strong enough to yield a characterization of $\leq_\mathsf{IRQ}$-minimality of seed functions.

Lemma 5.7. *The repair seed function s is not $\leq_\mathsf{IRQ}$-minimal iff there exists $D \in s(a)$ such that $\mathsf{low}(s, D(a))$ is an rsf.*

Proof. If $s' = \mathsf{low}(s, D(a))$ is an rsf, then Lemma 5.4 yield $s' <_\mathsf{IRQ} s$, which shows that s is not $\leq_\mathsf{IRQ}$-minimal. If conversely s is not $\leq_\mathsf{IRQ}$-minimal, then there is an rsf t such that $t <_\mathsf{IRQ} s$. By Lemma 5.5, there is $a \in \Sigma_\mathsf{I}$ and $D \in s(a)$ such that $t \leq_\mathsf{IQ} \mathsf{low}(s, D(a))$. Since $t(c) \leq \mathsf{low}(s, D(a))(c)$ for each $c \in \Sigma_\mathsf{I}$, transitivity and the fact that t is an rsf yield, for each $P(c) \in \mathcal{P}_C$ with $\mathcal{B} \models P(c)$, $\{P\} \leq s'(c)$. Thus, $\mathsf{low}(s, D(a))$ is an rsf. □

[3] Strictly speaking, $\mathsf{low}(s(c), \mathcal{M}_c)$ is not defined if $\mathcal{M}_c = \emptyset$. In this case, we set $\mathsf{low}(s(c), \emptyset) := s(c)$.

Since there are only linearly many atoms in $s(a)$ and computing the sets $\mathcal{M}_c$ for $c \in \Sigma_\mathsf{I}$ can be done in polynomial time, this finally proves Proposition 4.11.

6 CONCLUSION

We have shown how results on computing optimal repairs [3] and error-tolerant reasoning w.r.t. optimal repairs [8] can be extended to a setting where the unwanted consequences consist not only of concept assertions, but also of role assertions. Our approach first repairs w.r.t. the role assertions, which yields a single optimal repair that can be computed in polynomial time, and then applies a known approach for computing optimal repairs w.r.t. concept assertions [4] to this repair. For error-tolerant reasoning we prove that, as in the case without role assertions in the repair request, brave entailment is in P and cautious entailment is in coNP. The main technical challenge was here to show that $\leq_{\mathsf{IRQ}}$-minimality of a seed function can be decided in polynomial time, which is more involved than the proof of the corresponding result for $\leq_{\mathsf{IQ}}$-minimality in [8]. We have also shown that brave reasoning is NP-complete and cautious reasoning is coNP-complete if classical repairs are used instead of optimal ones. For cautious reasoning w.r.t. optimal repairs, we were not able to show a matching coNP lower bound, and actually conjecture that the problem is in P. Another topic for future research is to investigate error-tolerant reasoning for optimal repairs in the more expressive DL considered in [6].

ACKNOWLEDGMENTS

This work was partially supported by the AI competence center ScaDS.AI Dresden/Leipzig and the German Research Foundation (DFG) in Project 430150274 and SFB/TRR 248.

REFERENCES

[1] Franz Baader, Sebastian Brandt, and Carsten Lutz. 2005. Pushing the $\mathcal{EL}$ Envelope. In *IJCAI-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, Edinburgh, Scotland, UK, July 30 - August 5, 2005*, Leslie Pack Kaelbling and Alessandro Saffiotti (Eds.). Professional Book Center, 364–369. http://ijcai.org/Proceedings/05/Papers/0372.pdf

[2] Franz Baader, Ian Horrocks, Carsten Lutz, and Ulrike Sattler. 2017. *An Introduction to Description Logic.* Cambridge University Press. https://doi.org/10.1017/9781139025355

[3] Franz Baader, Patrick Koopmann, Francesco Kriegel, and Adrian Nuradiansyah. 2021. Computing Optimal Repairs of Quantified ABoxes w.r.t. Static $\mathcal{EL}$ TBoxes. In *Automated Deduction - CADE 28 - 28th International Conference on Automated Deduction, Virtual Event, July 12-15, 2021, Proceedings (Lecture Notes in Computer Science)*, André Platzer and Geoff Sutcliffe (Eds.), Vol. 12699. Springer, 309–326. https://doi.org/10.1007/978-3-030-79876-5_18

[4] Franz Baader, Patrick Koopmann, Francesco Kriegel, and Adrian Nuradiansyah. 2022. Optimal ABox Repair w.r.t. Static *EL* TBoxes: From Quantified ABoxes Back to ABoxes. In *The Semantic Web - 19th International Conference, ESWC 2022, Proceedings (Lecture Notes in Computer Science)*, Paul Groth, Maria-Esther Vidal, Fabian M. Suchanek, Pedro A. Szekely, Pavan Kapanipathi, Catia Pesquita, Hala Skaf-Molli, and Minna Tamper (Eds.), Vol. 13261. Springer, 130–146. https://doi.org/10.1007/978-3-031-06981-9_8

[5] Franz Baader, Patrick Koopmann, Francesco Kriegel, and Adrian Nuradiansyah. 2022. *Optimal ABox Repair w.r.t. Static $\mathcal{EL}$ TBoxes: from Quantified ABoxes back to ABoxes (Extended Version).* LTCS-Report 22-01. Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany. https://doi.org/10.25368/2022.65

[6] Franz Baader and Francesco Kriegel. 2022. Pushing Optimal ABox Repair from $\mathcal{EL}$ Towards More Expressive Horn-DLs. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning, KR 2022, Haifa, Israel. July 31 - August 5, 2022*, Gabriele Kern-Isberner, Gerhard Lakemeyer, and Thomas Meyer (Eds.). https://doi.org/10.24963/kr.2022/3

[7] Franz Baader, Francesco Kriegel, and Adrian Nuradiansyah. 2019. Privacy-Preserving Ontology Publishing for $\mathcal{EL}$ Instance Stores. In *Logics in Artificial Intelligence - 16th European Conference, JELIA 2019, Rende, Italy, May 7-11, 2019, Proceedings (Lecture Notes in Computer Science)*, Francesco Calimeri, Nicola Leone, and Marco Manna (Eds.), Vol. 11468. Springer, 323–338. https://doi.org/10.1007/978-3-030-19570-0_21

[8] Franz Baader, Francesco Kriegel, and Adrian Nuradiansyah. 2022. Error-Tolerant Reasoning in the Description Logic $\mathcal{EL}$ Based On Optimal Repairs. In *Rules and Reasoning - 6th International Joint Conference, RuleML+RR 2022, Virtual, September 26-28, 2022, Proceedings*, Guido Governatori and Anni-Yasmin Turhan (Eds.). https://doi.org/10.1007/978-3-031-21541-4_15

[9] Franz Baader, Francesco Kriegel, Adrian Nuradiansyah, and Rafael Peñaloza. 2020. Computing Compliant Anonymisations of Quantified ABoxes w.r.t. $\mathcal{EL}$ Policies. In *The Semantic Web - ISWC 2020 - 19th International Semantic Web Conference, Athens, Greece, November 2-6, 2020, Proceedings, Part I (Lecture Notes in Computer Science)*, Jeff Z. Pan, Valentina A. M. Tamma, Claudia d'Amato, Krzysztof Janowicz, Bo Fu, Axel Polleres, Oshani Seneviratne, and Lalana Kagal (Eds.), Vol. 12506. Springer, 3–20. https://doi.org/10.1007/978-3-030-62419-4_1

[10] Franz Baader and Barbara Morawska. 2010. Unification in the Description Logic $\mathcal{EL}$. *Log. Methods Comput. Sci.* 6, 3 (2010). http://arxiv.org/abs/1006.2289

[11] Steven Fortune, John E. Hopcroft, and James Wyllie. 1980. The Directed Subgraph Homeomorphism Problem. *Theor. Comput. Sci.* 10 (1980), 111–121. https://doi.org/10.1016/0304-3975(80)90009-2

[12] Russell Greiner, Barbara A. Smith, and Ralph W. Wilkerson. 1989. A Correction to the Algorithm in Reiter's Theory of Diagnosis. *Artif. Intell.* 41, 1 (1989), 79–88. https://doi.org/10.1016/0004-3702(89)90079-9

[13] Monika Rauch Henzinger, Thomas A. Henzinger, and Peter W. Kopke. 1995. Computing Simulations on Finite and Infinite Graphs. In *36th Annual Symposium on Foundations of Computer Science, Milwaukee, Wisconsin, USA, 23-25 October 1995.* IEEE Computer Society, 453–462. https://doi.org/10.1109/SFCS.1995.492576

[14] Aditya Kalyanpur, Bijan Parsia, Evren Sirin, and Bernardo Cuenca Grau. 2006. Repairing Unsatisfiable Concepts in OWL Ontologies. In *The Semantic Web: Research and Applications, 3rd European Semantic Web Conference, ESWC 2006, Budva, Montenegro, June 11-14, 2006, Proceedings (Lecture Notes in Computer Science)*, York Sure and John Domingue (Eds.), Vol. 4011. Springer, 170–184. https://doi.org/10.1007/11762256_15

[15] Andrea S. LaPaugh and Christos H. Papadimitriou. 1984. The even-path problem for graphs and digraphs. *Networks* 14, 4 (1984), 507–513. https://doi.org/10.1002/net.3230140403

[16] Michel Ludwig and Rafael Peñaloza. 2014. Error-Tolerant Reasoning in the Description Logic $\mathcal{EL}$. In *Logics in Artificial Intelligence - 14th European Conference, JELIA 2014, Funchal, Madeira, Portugal, September 24-26, 2014. Proceedings (Lecture Notes in Computer Science)*, Eduardo Fermé and João Leite (Eds.), Vol. 8761. Springer, 107–121. https://doi.org/10.1007/978-3-319-11558-0_8

[17] Rafael Peñaloza. 2020. Error-Tolerance and Error Management in Lightweight Description Logics. *Künstliche Intell.* 34, 4 (2020), 491–500. https://doi.org/10.1007/s13218-020-00684-5

[18] Raymond Reiter. 1987. A Theory of Diagnosis from First Principles. *Artif. Intell.* 32, 1 (1987), 57–95. https://doi.org/10.1016/0004-3702(87)90062-2 (See the erratum [12]).

[19] Thomas J. Schaefer. 1978. The Complexity of Satisfiability Problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing, May 1-3, 1978, San Diego, California, USA*, Richard J. Lipton, Walter A. Burkhard, Walter J. Savitch, Emily P. Friedman, and Alfred V. Aho (Eds.). ACM, 216–226. https://doi.org/10.1145/800133.804350

[20] Stefan Schlobach. 2005. Debugging and Semantic Clarification by Pinpointing. In *The Semantic Web: Research and Applications, Second European Semantic Web Conference, ESWC 2005, Heraklion, Crete, Greece, May 29 - June 1, 2005, Proceedings (Lecture Notes in Computer Science)*, Asunción Gómez-Pérez and Jérôme Euzenat (Eds.), Vol. 3532. Springer, 226–240. https://doi.org/10.1007/11431053_16

[21] Stefan Schlobach. 2005. Diagnosing Terminologies. In *Proceedings, The Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference, July 9-13, 2005, Pittsburgh, Pennsylvania, USA*, Manuela M. Veloso and Subbarao Kambhampati (Eds.). AAAI Press / The MIT Press, 670–675. http://www.aaai.org/Library/AAAI/2005/aaai05-105.php