Maximum Entropy Reasoning via Model Counting in (Description) Logics that Count

Franz Baader and Anton Claußnitzer
TU Dresden and SCADS.AI Dresden/Leipzig, Dresden, Germany

Abstract

In previous work it was shown that the logic $\mathcal{ALC}^{\mathsf{ME}}$, which extends the description logic (DL) $\mathcal{ALC}$ with probabilistic conditionals, has domain-lifted inference. Here, we extend this result from the base logic $\mathcal{ALC}$ to two logics that can count, the two-variable fragment C^2 of first-order logic (FOL) with counting quantifiers, and the DL $\mathcal{ALCSCC}$, which is not a fragment of FOL. As an auxiliary result, we prove that model counting in $\mathcal{ALCSCC}$ can be realized in a domain-liftable way.

1 Introduction

Description logics (DLs) (Baader et al. 2003; 2017) are a well-investigated family of logic-based knowledge representation formalisms, which can be used to formalize the terminological knowledge of an application domain in a machine-processable way. For instance, large medical ontologies such as SNOMED CT[1] and Galen[2] have been developed using an appropriate DL. While classical DLs are often sufficient for formalizing certain knowledge like the definition of medical terminology, they cannot adequately express uncertain knowledge, which may, e.g., be needed for medical diagnosis. Using a non-medical example, the concept of a father can be formalized by the concept inclusion (CI) $\mathit{Father} \sqsubseteq \mathit{Human} \sqcap \mathit{Male} \sqcap \exists \mathit{child}.\mathit{Human}$, which says that fathers are male humans that have a human child. However, a statement like "Rich persons usually have rich children" should not be expressed with a CI since it does not hold for all rich persons. It is more appropriate to use a probabilistic conditional (PC) of the form $(\forall \mathit{child}.\mathit{Rich} \mid \mathit{Person} \sqcap \mathit{Rich})[p]$, where the probability p may be based on statistical knowledge or express the degree of a subjective belief. The CI and PC of our example can be phrased in the probabilistic DL $\mathcal{ALC}^{\mathsf{ME}}$ (Wilhelm et al. 2019; Baader et al. 2019; Wilhelm and Kern-Isberner 2019), which extends the well-known DL $\mathcal{ALC}$ with probabilistic conditionals that are interpreted based on the aggregating semantics and the maximum entropy principle. Compared to other probabilistic extensions of DL (such as (Lukasiewicz 2008; Peñaloza and Potyka 2017; Gutiérrez-Basulto et al. 2017)), $\mathcal{ALC}^{\mathsf{ME}}$ has the advantage that the aggregating semantics smoothly combines the statistical and the subjective view on probabilities and that the maximum entropy approach fulfills a number of reasonable commonsense principles (Paris 1999; Kern-Isberner and Thimm 2010; Beierle, Finthammer, and Kern-Isberner 2015). Like other approaches for probabilistic reasoning in a first-order setting, the aggregating semantics assumes that interpretations are built over a fixed finite domain Δ. To be able to deal with large domain sizes, one needs reasoning to be domain-lifted (Van den Broeck et al. 2011), which means that inferences can be drawn in time polynomial in the size of Δ. The main results of (Baader et al. 2019; Wilhelm et al. 2019; Wilhelm and Kern-Isberner 2019) are that $\mathcal{ALC}^{\mathsf{ME}}$ allows for domain-lifted inference.

In the present paper we extend these results from the base logic $\mathcal{ALC}$ to logics that can count. Number restrictions (Hollunder, Nutt, and Schmidt-Schauß 1990; Hollunder and Baader 1991) are DL concept constructors that can express simple numerical constraints on the number of role successors of an individual, such as that it has three children that are rich, and only two that are not rich, whereas cardinality restrictions on concepts (Baader, Buchheit, and Hollunder 1996; Tobies 2000) can constrain the overall number of elements of a concept, e.g., expressing that there are more than 500,000 rich people living in Florida. Description logics offering such counting features are contained in C^2, the two-variable fragment of FOL with counting quantifiers, and are thus decidable (Grädel, Otto, and Rosen 1997; Pacholski, Szwast, and Tendera 1997). In (Kuželka 2021; Tóth and Kuželka 2024) it was recently shown that (extended versions of) model counting in C^2 can be realized in a domain-liftable way. We will use this result to prove that $C^{2\,\mathsf{ME}}$ allows for domain-lifted inference. The DL $\mathcal{ALCSCC}$ (Baader 2017) offers more expressive counting constraints on role successors, which in general cannot be expressed in C^2 or even full FOL (Baader and Bortoli 2019). For example, in $\mathcal{ALCSCC}$ we can describe persons that have more rich than non-rich children without specifying how many children of each type the person actually has, and in $\mathcal{ALCSCC}^{\mathsf{ME}}$ we can say that, with a high probability (say .8), rich persons have more rich than non-rich children. We will show that (an extended version of) model counting in $\mathcal{ALCSCC}$ can be realized in a domain-liftable way, and

[1] https://www.snomed.org/

[2] https://bioportal.bioontology.org/ontologies/GALEN

use this result to prove that $\mathcal{ALCSCC}^{\mathsf{ME}}$ allows for domain-lifted inference.

2 The Logics C^2 and $\mathcal{ALCSCC}$

We briefly sketch syntax and semantics of these logics. More detailed formal definitions can, e.g., be found in (Tóth and Kuželka 2024; Baader and Bortoli 2019).

Two-Variable Logic with Counting To define C^2, we consider the two-variable fragment L^2 of FOL, where only two variables x, y are available for building formulas, but in addition to quantifiers $\exists$ and $\forall$, we also allow the use of counting-quantifiers $\exists^{\geq k}$ and $\exists^{\leq k}$, where k is a non-negative integer. The formula $\exists^{\geq k}.\phi(x)$ ($\exists^{\leq k}.\phi(x)$) holds in an interpretation I if its domain Δ^I contains at least (at most) k elements d such that $\phi(d)$ is true in I. Note that counting quantifiers can be expressed in FOL, but not in L^2 since this would require more than two variables.

In the following, we want to view C^2 as a DL. For this purpose, we define C^2 *concepts* to be C^2 formulas with free variable x. For such a concept $C = \phi(x)$, its extension C^I in an interpretation I is the set $C^I := \{d \in \Delta^I \mid \phi(d) \text{ is true in } I\}$. To be consistent with DL notation, we sometimes write the conjunction of C^2 concepts as $C \sqcap D$ rather than $C \wedge D$. A C^2 *terminology (TBox)* is a C^2 sentence, i.e., a C^2 formula without free variables.

The DL $\mathcal{ALCSCC}$ Our introduction of the DL $\mathcal{ALCSCC}$ is based on the presentation in (Baader and Bortoli 2019) since it is more streamlined than the one in (Baader 2017). $\mathcal{ALCSCC}$ *concepts* are built from concept names and role names using the concept constructors conjunction ($C \sqcap D$), disjunction ($C \sqcup D$), negation ($\neg C$), and successor constraint (succ(Con)), where Con is a cardinality constraint. *Cardinality constraints* are of the form

$$N_0 + N_1|s_1| + \cdots + N_k|s_k| \leq M_0 + M_1|t_1| + \cdots + M_\ell|t_\ell|,$$

where the s_i, t_j are set terms and the N_i, M_j are non-negative integers. Finally, *set terms* are built from set variables using intersection ($s \cap t$), union ($s \cup t$), and complement ($\overline{s}$), where role names and $\mathcal{ALCSCC}$ concepts can be used as set variables. An interpretation I consists of a non-empty domain Δ^I and assigns sets $A^I \subseteq \Delta^I$ to concept names A and binary relations r^I on Δ^I to role names r. The extension C^I of compound concepts is defined by induction, where conjunction, disjunction, and negation are interpreted as intersection, union, and complement. To define whether or not an element d of Δ^I belongs to $\text{succ}(\text{Con})^I$, one considers the set of all role successors[3] of d (i.e., elements e such that there is a role name r with $(d, e) \in r^I$) as set universe $\mathcal{U}_d$ and interprets role names r as the set of r-successors of d and concepts C as $C^I \cap \mathcal{U}_d$. The interpretation of set terms as subsets of $\mathcal{U}_d$ and the validity of Con, which is required for d to belong to $\text{succ}(\text{Con})^I$, are then defined in the obvious way. For example, the $\mathcal{ALCSCC}$ concept $\mathit{Person} \sqcap \text{succ}(|\mathit{child} \cap \neg \mathit{Rich}| + 1 \leq |\mathit{child} \cap \mathit{Rich}|)$ describes persons that have more rich than non-rich children.

An $\mathcal{ALCSCC}$ *terminology (TBox)* is a finite set of CIs of the form $C \sqsubseteq D$, where C, D are $\mathcal{ALCSCC}$ concepts. The interpretation I is a *model* of such a TBox $\mathcal{T}$ ($I \models \mathcal{T}$) if $C^I \subseteq D^I$ holds for all elements $C \sqsubseteq D$ of $\mathcal{T}$.

[3]In (Baader 2017), interpretations are restricted such that this set is finite, whereas Baader and Bortoli (2019) also allow for infinite sets. Since we will only consider finite interpretations, this difference is irrelevant here.

Model Counting Model counting usually asks how many models over a given finite domain Δ a given sentence has. Here, we consider a slightly extended version of this task, called in the following *concept-constrained model counting*, where the underlying logic is either C^2 or $\mathcal{ALCSCC}$. Let $\mathcal{T}$ be a TBox, $C_1, \dots, C_n$ concepts, $c_1, \dots, c_\ell$ non-negative integers, and Δ a finite set. Then

$$\text{ccmc}(\mathcal{T}, C_1, \dots, C_\ell, c_1, \dots, c_\ell, \Delta)$$

is defined to be the number of models I of $\mathcal{T}$ with domain Δ that satisfy $|C_i^I| = c_i$ ($1 \leq i \leq \ell$). We say that concept-constrained model counting is *domain-liftable* if this number can be computed in polynomial time in the size of the input Δ (i.e., where the other inputs of the function ccmc are assumed to be of constant size). We will show later that this is the case for both C^2 and $\mathcal{ALCSCC}$, but first define our probabilistic extensions of these two logics.

3 The Logics $\mathcal{ALCSCC}^{\mathsf{ME}}$ and $C^{2^{\mathsf{ME}}}$

In the following, let $\mathcal{L}$ be either $\mathcal{ALCSCC}$ or C^2. In the logic $\mathcal{L}^{\mathsf{ME}}$, we consider *probabilistic conditionals (PCs)* of the form $(D \mid C)[p]$, where C, D are $\mathcal{L}$ concepts and p is a rational number. An $\mathcal{L}$ *knowledge base* $\mathcal{K} = (\mathcal{T}, \mathcal{C})$ consists of an $\mathcal{L}$ TBox $\mathcal{T}$ together with a finite set $\mathcal{C}$ of PCs. To define the semantics of such a knowledge base $\mathcal{K}$, we follow (Wilhelm et al. 2019; Baader et al. 2019) and consider interpretations over a fixed, finite domain Δ of the signature of $\mathcal{K}$. We denote the (finite) set of all these interpretations with $\mathcal{I}^\Delta$ and the set of probability distributions $P\colon \mathcal{I}^\Delta \to [0,1]$ over $\mathcal{I}^\Delta$ with $\mathfrak{P}^\Delta$. The distribution $P \in \mathfrak{P}^\Delta$ is a *model* of $\mathcal{K} = (\mathcal{T}, \mathcal{C})$ if all interpretations I that are not models of $\mathcal{T}$ satisfy $P(I) = 0$ and the following holds for all PCs $(F_i \mid E_i)[p_i]$ in $\mathcal{C}$: $\sum_{I \in \mathcal{I}^\Delta} P(I) \cdot |E_i^I| > 0$ and

$$\sum_{I \in \mathcal{I}^\Delta} P(I) \cdot |E_i^I \cap F_i^I| = p_i \cdot \sum_{I \in \mathcal{I}^\Delta} P(I) \cdot |E_i^I|. \quad (1)$$

This semantics for PCs is called *aggregating semantics* (Kern-Isberner and Thimm 2010). A knowledge base with at least one model is *consistent*. Under the assumption that concept-constrained model counting for $\mathcal{L}$ is domain-lifted (which will be shown for C^2 and $\mathcal{ALCSCC}$ in the next section), consistency checking for $\mathcal{L}^{\mathsf{ME}}$ is also domain-lifted.

Theorem 3.1. *Consistency of an $\mathcal{L}^{\mathsf{ME}}$ knowledge base $\mathcal{K}$ for a finite domain Δ can be checked in time polynomial in $|\Delta|$.*

Proof. Let $\mathcal{K} = (\mathcal{T}, \mathcal{C})$ and $\mathcal{C} = \{(D_i \mid C_i)[p_i] \mid 1 \leq i \leq n\}$. We must check whether there is a distribution P such that $P(I) = 0$ if $I \not\models \mathcal{T}$, and the (in)equations defining the semantics of PCs are satisfied. Note that we cannot directly build and solve this system of linear (in)equations since $|\mathcal{I}^\Delta|$ is exponential in $|\Delta|$. However, interpretations that behave the same for $E_1, \ldots, E_n, E_1 \sqcap F_1, \ldots, E_n \sqcap F_n$ in the sense that the cardinalities of the extensions of these concepts coincide in these interpretations need not be treated separately. Instead, we only need to count how many such interpretations that are models of $\mathcal{T}$ there are. Thus, let $\boldsymbol{C}$ be the tuple of concepts $E_1, \ldots, E_n, E_1 \sqcap F_1, \ldots, E_n \sqcap F_n$. For $i = 1, \ldots, n$, we consider the linear equations

$$\sum_{\mathrm{ccmc}(\mathcal{T},\boldsymbol{C},\boldsymbol{c},\Delta)\neq 0} x_{\boldsymbol{c}} \cdot c_{n+i} = p_i \cdot \sum_{\mathrm{ccmc}(\mathcal{T},\boldsymbol{C},\boldsymbol{c},\Delta)\neq 0} x_{\boldsymbol{c}} \cdot c_i,$$

where $\boldsymbol{c} = (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n})$ ranges over all tuples of numbers in $\{0, \ldots, |\Delta|\}^{2n}$ and the expressions $x_{\boldsymbol{c}}$ are the variables. In addition, we require $x_{\boldsymbol{c}} \geq 0$ for these variables and add the (in)equations

$$\sum_{\mathrm{ccmc}(\mathcal{T},\boldsymbol{C},\boldsymbol{c},\Delta)\neq 0} x_{\boldsymbol{c}} = 1 \;\text{ and }\; \sum_{\mathrm{ccmc}(\mathcal{T},\boldsymbol{C},\boldsymbol{c},\Delta)\neq 0} x_{\boldsymbol{c}} \cdot c_i > 0.$$

A solution $x_{\boldsymbol{c}} = q_{\boldsymbol{c}}$ for the variables in this system of linear (in)equations yields a model P of $\mathcal{K}$ as follows. For $I \in \mathcal{I}^\Delta$, let $\boldsymbol{c}$ be defined by setting $c_i := |E_i^I|$ and $c_{n+i} := |E_i^I \cap F_i^I|$ $(1 \leq i \leq n)$. Then we set $P(I) := 0$ if $\mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta) = 0$ and $P(I) := q_{\boldsymbol{c}}/\mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta)$ otherwise. It is easy to see that P is indeed a model of $\mathcal{K}$. Conversely, it is also easy to see that a model of $\mathcal{K}$ can be used to construct a solution of the above system of linear (in)equations.

Finally, note that this system can be constructed and solved in time polynomial in $|\Delta|$ since the cardinality of $\{0, \ldots, |\Delta|\}^{2n}$ is polynomial in $|\Delta|$ and (as we will show in the next section) $\mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta)$ can be computed in time polynomial in $|\Delta|$. □

Instead of reasoning w.r.t. all models of a consistent knowledge base, we use the maximum entropy distribution as preferred model. In fact, as pointed out by Wilhelm et al. (2019), according to Paris (1999), this distribution is the most appropriate choice of model in this setting. The entropy of a probability distribution P is $-\sum_{I\in\mathcal{I}^\Delta} P(I) \cdot \log_2 P(I)$, where we use the convention $0 \cdot \infty = 0$. For every consistent knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{C})$, there is exactly one model of $\mathcal{K}$ with maximal entropy, i.e., the optimization problem

$$-\sum_{I\in\mathcal{I}^\Delta} P(I) \cdot \log_2 P(I) \stackrel{!}{=} \max \;\text{ with the conditions}$$

$$\sum_{I\in\mathcal{I}^\Delta} P(I) = 1, \quad \sum_{I\in\mathcal{I}^\Delta} P(I)|E^I| > 0 \text{ for } (F|E)[p] \in \mathcal{C},$$

$$\sum_{I\in\mathcal{I}^\Delta} P(I)|E^I \cap F^I| = p \sum_{I\in\mathcal{I}^\Delta} P(I)|E^I| \text{ for } (F|E)[p] \in \mathcal{C},$$

$$\forall I \in \mathcal{I}^\Delta\colon P(I) \geq 0 \text{ and } P(I) = 0 \text{ if } I \not\models \mathcal{T},$$

has exactly one solution $P_\mathcal{K}^{\mathsf{ME}}$ (Kern-Isberner and Thimm 2010). Instead of solving this optimization problem directly, one usually considers the dual optimization problem, whose solutions represent $P_\mathcal{K}^{\mathsf{ME}}$ in a compact way.

Assume that $\mathcal{C} = \{(F_i \,|\, E_i)[p_i] \mid i = 1, \ldots, n\}$ and define the functions f_i $(1 \leq i \leq n)$ as $f_i(I) := |E_i^I \cap F_i^I| - p_i|E_i^I|$. An application of the Lagrange multiplier method to the above optimization problem then yields $P_\mathcal{K}^{\mathsf{ME}}(I) = 0$ if $I \not\models \mathcal{T}$ and $P_\mathcal{K}^{\mathsf{ME}}(I) = \alpha_0 \alpha_1^{f_1(I)} \cdots \alpha_n^{f_n(I)}$ if $I \models \mathcal{T}$, where the values $\alpha_i > 0$ are solutions to the equations $\sum_{I\in\mathcal{I}^\Delta, I\models\mathcal{T}} f_i(I) \alpha_1^{f_1(I)} \cdots \alpha_n^{f_n(I)} = 0$, $i = 1, \ldots, n$, and $\alpha_0 = \left(\sum_{I\in\mathcal{I}^\Delta, I\models\mathcal{T}} \alpha_1^{f_1(I)} \cdots \alpha_n^{f_n(I)}\right)^{-1}$ is a normalization constant.

Since the numbers α_i are solutions of a non-linear optimization problem, they can in general only be approximated (e.g., using Newton's method). Following (Wilhelm et al. 2019), we do not investigate this approximation process here, but assume that a rational approximation $\boldsymbol{\beta} \in \mathbb{Q}^n_{>0}$ of the exact solution $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{>0}$ is given. For such an approximation $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, the induced probability distribution $P_\mathcal{K}^{\boldsymbol{\beta}}$ on $\mathcal{I}^\Delta$ is defined by Wilhelm et al. (2019) as

$$P_\mathcal{K}^{\boldsymbol{\beta}}(I) = \begin{cases} \beta_0 \beta_1^{f_1(I)} \cdots \beta_n^{f_n(I)} & \text{if } I \models \mathcal{T}, \\ 0 & \text{else,} \end{cases}$$

where the normalization constant β_0 is defined analogously to α_0. Our goal is to show that domain-lifted inference w.r.t. $P_\mathcal{K}^{\boldsymbol{\beta}}$ is possible.

For a vector $\boldsymbol{\beta} \in \mathbb{Q}^n_{>0}$ and given $\mathcal{L}$ concepts E, F, we consider the problem of computing the probability p such that $P_\mathcal{K}^{\boldsymbol{\beta}} \models (F \,|\, E)[p]$, i.e., $(F \,|\, E)[p]$ holds in $P_\mathcal{K}^{\boldsymbol{\beta}}$ w.r.t. the aggregating semantics. For the base logic $\mathcal{ALC}$, it is claimed in (Wilhelm et al. 2019) that p is always a rational number and can be computed in time polynomial in $|\Delta|$. Unfortunately, the former claim is not true (see (Baader and Claußnitzer 2025) for an example): due to a faulty calculation, certain roots are canceled out by Wilhelm et al., which actually cannot be removed. For this reason, our correction and extension of this result has a more involved formulation.

Theorem 3.2. *Let E, F be $\mathcal{L}$ concepts, $\mathcal{K} = (\mathcal{T}, \mathcal{C})$ with $\mathcal{C} = \{(D_i \,|\, C_i)[p_i] \mid 1 \leq i \leq n\}$ a consistent $\mathcal{L}$ knowledge base where $p_i = s_i/t_i$ for natural numbers s_i, t_i, and let $P_\mathcal{K}^{\boldsymbol{\beta}}$ be an approximation of the maximum entropy distribution, as defined above. Then we can compute (in time polynomial in $|\Delta|$) a polynomial $P(X_1, \ldots, X_n)$ in n indeterminates and with rational coefficients such that $p := P(\sqrt[t_1]{\beta_1}, \ldots, \sqrt[t_n]{\beta_n})$ satisfies $P_\mathcal{K}^{\boldsymbol{\beta}} \models (F \,|\, E)[p]$.*

Proof. Let $\boldsymbol{C}$ be the tuple of concepts $E_1, \ldots, E_n, E_1 \sqcap F_1, \ldots, E_n \sqcap F_n, E, E \sqcap F$. By the aggregating semantic, the desired probability p can be obtained as follows:

$$p = \frac{\sum_{I\in\mathcal{I}^\Delta} |E^I \cap F^I| P_\mathcal{K}^{\boldsymbol{\beta}}(I)}{\sum_{I\in\mathcal{I}^\Delta} |E^I| P_\mathcal{K}^{\beta}(I)} =$$

$$\frac{\sum_{I\in\mathcal{I}^\Delta} |E^I \cap F^I| \beta_0 \beta_1^{f_1(I)} \cdots \beta_n^{f_n(I)}}{\sum_{I\in\mathcal{I}^\Delta} |E^I| \beta_0 \beta_1^{f_1(I)} \cdots \beta_n^{f_n(I)}} =$$

$$\frac{\sum_{I\in\mathcal{I}^\Delta} |E^I \cap F^I| \cdot \prod_{i=1}^n \sqrt[t_i]{\beta_i}^{\,t_i|E_i^I \cap F_i^I| - s_i|E_i^I|}}{\sum_{I\in\mathcal{I}^\Delta} |E^I| \cdot \prod_{i=1}^n \sqrt[t_i]{\beta_i}^{\,t_i|E_i^I \cap F_i^I| - s_i|E_i^I|}} =$$

$$\frac{\sum_{I\in\mathcal{I}^\Delta} |E^I \cap F^I| \cdot \prod_{i=1}^n \sqrt[t_i]{\beta_i}^{\,t_i|E_i^I \cap F_i^I| - s_i|E_i^I| + s_i|\Delta|}}{\sum_{I\in\mathcal{I}^\Delta} |E^I| \cdot \prod_{i=1}^n \sqrt[t_i]{\beta_i}^{\,t_i|E_i^I \cap F_i^I| - s_i|E_i^I| + s_i|\Delta|}} =$$

$$\frac{\sum_{\boldsymbol{c}} \mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta) \cdot c_{2n+2} \cdot \prod_{i=1}^{n} \sqrt[t_i]{\beta_i}^{t_i c_{n+i} - s_i c_i + s_i |\Delta|}}{\sum_{\boldsymbol{c}} \mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta) \cdot c_{2n+1} \cdot \prod_{i=1}^{n} \sqrt[t_i]{\beta_i}^{t_i c_{n+i} - s_i c_i + s_i |\Delta|}},$$

where $\boldsymbol{c} = (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}, c_{2n+1}, c_{2n+2})$ in the last quotient ranges of all tuples in $\{0, \ldots, |\Delta|\}^{2n+2}$. Note that this yields a quotient of polynomials (with the roots as indeterminates), which can be constructed in time polynomial in $|\Delta|$ since the numbers $\mathrm{ccmc}(\mathcal{T}, \boldsymbol{C}, \boldsymbol{c}, \Delta)$ can be computed time polynomial in $|\Delta|$ and the cardinality of $\{0, \ldots, |\Delta|\}^{2n+2}$ is also polynomial in $|\Delta|$. The time needed for checking whether the denominator of this quotient is 0 is also polynomial in $|\Delta|$ (see Lemma 3.3 below). If this is the case, then there is no $p \in [0, 1]$ such that $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models (D \,|\, C)[p]$ holds. Otherwise, it is a well-known fact from the theory of algebraic field extensions that the above quotient of polynomials in $\sqrt[t_i]{\beta_i}$ can be rewritten as a polynomial in $\sqrt[t_i]{\beta_i}$, $i = 1, \ldots, n$, with rational coefficients (see, e.g., (Jacobson 1974), Theorem 4.1). For this purpose, one needs to solve a system of linear equations whose size does not depend on $|\Delta|$. In addition, all numbers encountered in the necessary calculations are exponentially bounded in $|\Delta|$, and thus the sizes of their binary representations are bounded by a polynomial in $|\Delta|$. □

As a consequence of this theorem, we can prove that certain inferences from $P^{\boldsymbol{\beta}}_{\mathcal{K}}$ are domain-lifted. To show this, we need the following lemma (see (Baader and Claußnitzer 2025) for a proof).

Lemma 3.3. *Let $P(X_1, \ldots, X_n)$ be a polynomial in n indeterminates and with rational coefficients, $t_1, \ldots, t_n$ be natural numbers ≥ 1, and $q, \beta_1, \ldots, \beta_n$ be rational numbers. Then we can decide whether $q = P(\sqrt[t_1]{\beta_1}, \ldots, \sqrt[t_n]{\beta_n})$ in time that is polynomial in the degree and the size of the binary representation of the coefficients of $P(X_1, \ldots, X_n)$.*

Note that, for the polynomials constructed in the proof of Theorem 3.2, the degree and the size of the binary representation of the coefficients is polynomially bounded by $|\Delta|$. Thus, the above lemma yields a time bound that is polynomial in $|\Delta|$.

Corollary 3.4. *Let E, F be $\mathcal{L}$ concepts, $q \in [0, 1]$ a rational number, $\mathcal{K} = (\mathcal{T}, \mathcal{C})$ a consistent $\mathcal{L}$ knowledge base, and $P^{\boldsymbol{\beta}}_{\mathcal{K}}$ a rational approximation of the maximum entropy distribution. Then $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models (F \,|\, E)[q]$ and $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models E \sqsubseteq F$ can be decided in time polynomial in $|\Delta|$.*

Proof. Consider the quotient constructed in the proof of Theorem 3.2. We can use Lemma 3.3 to decide whether the denominator is equal to zero or not. If it is zero, then $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models (F \,|\, E)[q]$ does not hold according to the aggregation semantics. However, $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models E \sqsubseteq F$ does hold since then $E^I = \emptyset$ for all worlds I with $P^{\boldsymbol{\beta}}_{\mathcal{K}}(I) \neq 0$. If the denominator is not zero, then we can turn the quotient into a polynomial in the indeterminates $\sqrt[t_i]{\beta_i}$, $i = 1, \ldots, n$, as pointed out in the proof of Theorem 3.2. We can then decide whether $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models (F \,|\, E)[q]$ holds by checking whether the polynomial evaluates to q. To decide whether $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models E \sqsubseteq F$ holds it is enough to check whether the polynomial evaluates to 1. □

Note that it would also be interesting to know, for a given rational number q, whether q is larger or smaller than the probability p for which $P^{\boldsymbol{\beta}}_{\mathcal{K}} \models (F \,|\, E)[p]$ holds. However, while we know how to decide this problem (see (Baader and Claußnitzer 2025)), it is not clear to us whether deciding the problem can be done in time polynomial in $|\Delta|$.

4 Concept-Constrained Model Counting

We prove here that concept-constrained model counting for C^2 and $\mathcal{ALCSCC}$ is domain-lifted. For C^2, this is an easy consequence of the results by Kuželka (2021), whereas for $\mathcal{ALCSCC}$ we show this from scratch.

Concept-Constrained Model Counting in C^2 In Definition 4 of his 2021 paper, Kuželka introduces the following model-counting task: given a sentence Γ, a finite domain Δ, a list $\Psi = (R_1, \ldots, R_m)$ of m predicates, and an m-tuple $\boldsymbol{n} = (n_1, \ldots, n_m)$ of non-negative integers, $\mathrm{MC}_{\Psi,\Gamma,\Delta}(\boldsymbol{n})$ counts the number of models I of Γ over the domain Δ such that $|R_i^I| = n_i$ for $i = 1, \ldots, m$. He shows that a weighted version of this task, which has the unweighted one as a special case, is domain-liftable for C^2 (Proposition 4 together with Theorem 4 of (Kuželka 2021)).

Our concept-constrained model counting task $\mathrm{ccmc}(\mathcal{T}, C_1, \ldots, C_\ell, c_1, \ldots, c_\ell, \Delta)$ can be seen as special case. In fact, let $A_1, \ldots, A_\ell$ be fresh unary predicates (i.e., ones not occurring in $\mathcal{T}$ or $C_1, \ldots, C_\ell$), and let Γ be obtained from the sentence $\mathcal{T}$ by conjoining the sentences $\forall x.(A_i(x) \leftrightarrow C_i)$ for $i = 1, \ldots, \ell$. Then $\mathrm{ccmc}(\mathcal{T}, C_1, \ldots, C_\ell, c_1, \ldots, c_\ell, \Delta) = \mathrm{MC}_{\Psi,\Gamma,\Delta}(\boldsymbol{n})$, where $\Psi = (A_1, \ldots, A_\ell)$ and $\boldsymbol{n} = (c_1, \ldots, c_\ell)$.

Theorem 4.1. *Concept-constrained model counting in C^2 is domain-liftable.*

This result covers concept-constrained model counting for quite a number of expressive DLs since the constructors of $\mathcal{ALC}$ as well as qualified number restrictions, nominals, inverse roles, role hierarchies, concept and role assertions, and even cardinality constraints on concepts are expressible in C^2. Note, however, that DLs with transitive roles are not covered since expressing transitivity requires three variables. In fact, it is even open whether counting the number of transitive relations on a finite domain Δ is domain-liftable (see, e.g., (Pfeiffer 2004; Mala 2022) for research in this direction).

Concept-Constrained Model Counting in $\mathcal{ALCSCC}$ We use the type-based approach employed by Wilhelm et al. (2019) for $\mathcal{ALC}$, but need to extend it considerably due to the more expressive constraints on role successors.

For a given concept-constrained model counting task $\mathrm{ccmc}(\mathcal{T}, C_1, \ldots, C_\ell, c_1, \ldots, c_\ell, \Delta)$, let $S = \{A_1, \ldots, A_L\}$ be the set of all concept names and $T = \{D_1, \ldots, D_M\}$ the set of all concepts of the form $\mathrm{succ}(\ldots)$ occurring in $\mathcal{T}$ or $C_1, \ldots, C_\ell$. A *type for S and T* (simply called *type* in the following) is of the form

$$\tau^{p_1, \ldots, p_L, p'_1, \ldots, p'_M} := A_1^{p_1} \sqcap \ldots \sqcap A_L^{p_L} \sqcap D_1^{p'_1} \sqcap \ldots \sqcap D_M^{p'_M},$$

where $p_i \in \{0, 1\}$ for all $i \in \{1, \ldots, L\}$ and $A_i^1 := A_i$ and $A_i^0 := \neg A_i$ (and analogously for p'_i and D_i). It is easy to see that $(\tau^{\boldsymbol{p}})^I \cap (\tau^{\boldsymbol{q}})^I = \emptyset$ holds for all distinct tuples $\boldsymbol{p}, \boldsymbol{q} \in \{0,1\}^{L+M}$ and interpretations I. In addition, for any concept F containing only concept names from S and successor constraints in T, there is a set $W_F \subseteq \{0,1\}^{L+M}$ such that $F^I = \bigcup_{\boldsymbol{p} \in W_F} (\tau^{\boldsymbol{p}})^I$ holds for all interpretations I. This set can be computed from F in time not depending on $|\Delta|$. By a slight abuse of notation we write $\tau^{\boldsymbol{p}} \in F$ to indicate that $\boldsymbol{p} \in W_F$. The following lemma is easy to see for all such concepts F and interpretations I.

Lemma 4.2. $|F^I| = \sum_{\tau^{\boldsymbol{p}} \in F} |(\tau^{\boldsymbol{p}})^I|$.

The CIs in $\mathcal{T}$ force certain types to be empty in all models. Let $F_{\mathcal{T}} := \bigsqcup_{C \sqsubseteq D \in \mathcal{T}} C \sqcap \neg D$. We call a type $\tau^{\boldsymbol{p}}$ *forbidden by* $\mathcal{T}$ (or simply *forbidden*) if $\tau^{\boldsymbol{p}} \in F_{\mathcal{T}}$. The forbidden types can again be computed in time not depending on $|\Delta|$.

Lemma 4.3. *The interpretation I is a model of $\mathcal{T}$ iff $|(\tau^{\boldsymbol{p}})^I| = 0$ for all types $\tau^{\boldsymbol{p}}$ forbidden by $\mathcal{T}$.*

Instead of solving the concept-constrained model counting task for $C_1, \ldots, C_\ell$, we consider this task for the sequence of all types, i.e., show how to compute $\text{ccmc}(\mathcal{T}, \Theta, \boldsymbol{k}, \Delta)$, where Θ ranges over all types $\tau^{\boldsymbol{p}}$ for $\boldsymbol{p} \in \{0,1\}^{L+M}$ and $\boldsymbol{k}$ is an 2^{L+M}-tuple of numbers $k_{\boldsymbol{p}}$ whose sum is $|\Delta|$ satisfying $k_{\boldsymbol{p}} = 0$ if the corresponding type $\tau^{\boldsymbol{p}}$ is forbidden by $\mathcal{T}$. We call this the *type-constrained model counting* task. Domain-liftability of the latter task implies domain-liftability of the former.

Lemma 4.4. *$ccmc(\mathcal{T}, C_1, \ldots, C_\ell, c_1, \ldots, c_\ell, \Delta)$ can be obtained as the sum of all numbers $ccmc(\mathcal{T}, \Theta, \boldsymbol{k}, \Delta)$, where in addition to the conditions on the inputs for the type-constrained model counting task we require that $\boldsymbol{k}$ satisfies $c_i = \sum_{\tau^{\boldsymbol{p}} \in C_i} k_{\boldsymbol{p}}$ for $i = 1, \ldots, \ell$.*

In the following, let $\boldsymbol{k}$ be an admissible input tuple for the type-constrained model counting task $\text{ccmc}(\mathcal{T}, \Theta, \boldsymbol{k}, \Delta)$. To construct a model of $\mathcal{T}$ with domain Δ, we must assign the elements of Δ to the types such that the cardinality constraints given by $\boldsymbol{k}$ are satisfied. However, note that we do not really construct all such assignments (since there are exponentially many of them in $|\Delta|$), but rather count their number. Obviously, there are

$$\frac{|\Delta|!}{\prod_{\boldsymbol{p} \in \{0,1\}^{L+M}} k_{\boldsymbol{p}}!} \tag{2}$$

many possible ways of making such an assignment. A given assignment determines to what concept names the elements $d \in \Delta$ belong in the interpretation I to be constructed: $d \in A_i^I$ iff A_i occurs positively in the type to which d was assigned.

It remains to construct the role successors in a way such that the successor constraints D_i or $\neg D_i$ occurring in the type $\tau^{\boldsymbol{p}}$ assigned to d are satisfied. The set terms occurring in such constraints are Boolean combinations of concept names in S, successor constraints in T, and role names $r_1, \ldots, r_K$ occurring in $\mathcal{T}$ or $C_1, \ldots, C_\ell$. Analogously to types, we define *Venn regions* (or simply *regions*) as in (Baader 2017): given a tuple $(\tilde{p}_1, \ldots, \tilde{p}_L, \tilde{p}'_1, \ldots, \tilde{p}'_M, \tilde{q}_1, \ldots, \tilde{q}_K) \in \{0,1\}^{L+M+K}$, the associated Venn region is

$$A_1^{\tilde{p}_1} \cap \ldots \cap A_L^{\tilde{p}_L} \cap D_1^{\tilde{p}'_1} \cap \ldots \cap D_M^{\tilde{p}'_M} \cap r_1^{\tilde{q}_1} \cap \ldots \cap r_K^{\tilde{q}_K},$$

where $A_i^{\tilde{p}_i}, D_i^{\tilde{p}'_i}$ are defined as before, and $r_i^1 = r_i$ and $r_i^0 = \overline{r_i}$. We denote such a region as $\rho^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$. Again, regions associated with different tuples are disjoint and any set term can be written as a union of certain regions. For this reason, the cardinality $|s|$ of a set term s occurring in a successor constraint can be written as the sum of cardinalities of certain regions. Thus, we can assume without loss of generality that the cardinality constraints inside the successor constraints D_i are linear inequations between cardinalities of regions.

We now introduce symbolic names (later used as variables) for these cardinalities. For a prototypical element d of Δ with assigned type $\tau^{\boldsymbol{p}}$, let $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$ (with $\tilde{p}_1, \ldots, \tilde{p}'_1, \ldots, \tilde{q}_1, \ldots \in \{0,1\}$) stand for the number of role successors of d with assigned type $\tau^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}$ that are r_j-successors of d iff $\tilde{q}_j = 1$ (for $j = 1, \ldots, K$). Note that, to have a role successor, at least one $\tilde{q}_j$ must be 1, which means that $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{0,\ldots,0} = 0$. Let us now investigate what additional conditions the cardinalities $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$ must satisfy. These are given by the (possibly negated) successor constraints in $\tau^{\boldsymbol{p}}$, where we replace each cardinality $|\rho^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}|$ of a region with the corresponding μ-variable. For each successor constraint $D_i = \text{succ}(\text{Con}_i)$, let $\kappa_i \leq \lambda_i$ be the linear inequation obtained from Con_i by replacing all cardinalities $|\rho^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}|$ with the corresponding variable $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$. Correspondingly, for $\neg D_i$ we obtain $\kappa_i > \lambda_i$. In order to satisfy the successor constraints in $\tau^{\boldsymbol{p}}$, the values chosen for the variables $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$ must thus satisfy the following conditions $(*)$:

- $\sum_{\tilde{q}_1,\ldots,\tilde{q}_K \in \{0,1\}} \mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K} \leq k_{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}$,
- $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{0,\ldots,0} = 0$ for all $\tilde{p}_1, \ldots, \tilde{p}_L, \tilde{p}'_1, \ldots, \tilde{p}'_M \in \{0,1\}$,
- $\kappa_i \leq \lambda_i$ if D_i is a conjunct in $\tau^{\boldsymbol{p}}$,
- $\kappa_i > \lambda_i$ if $\neg D_i$ is a conjunct in $\tau^{\boldsymbol{p}}$.

The first condition is due to the fact that one cannot have more role successors of a certain type than this type has elements, and we have already explained the second condition. The other two conditions ensure that the assignment of role successors is such that all successor constraints required by the type $\tau^{\boldsymbol{p}}$ are satisfied.

The following formula states how many possible assignments of role successors there are for a fixed element $d \in \Delta$ of type $\tau^{\boldsymbol{p}}$:

$$\sum_{\boldsymbol{\mu} \text{ satisfies } (*)} \prod_{\substack{\tilde{p}_1,\ldots,\tilde{p}_L \in \{0,1\} \\ \tilde{p}'_1,\ldots,\tilde{p}'_M \in \{0,1\}}} h_{\boldsymbol{\mu}}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M} \tag{3}$$

where $\boldsymbol{\mu}$ ranges over all assignments of non-negative integers $0 \leq m^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K} \leq |\Delta|$ to the variables $\mu^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}_{\tilde{q}_1,\ldots,\tilde{q}_K}$ and $h_{\boldsymbol{\mu}}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}$ counts the number

$$h_{\boldsymbol{\mu}}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M} = \frac{k_{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}!}{\prod_{\tilde{q}_1,\ldots,\tilde{q}_K\in\{0,1\}} m_{\tilde{q}_1,\ldots,\tilde{q}_K}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M}! \cdot (k_{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M} - \sum_{\tilde{q}_1,\ldots,\tilde{q}_K\in\{0,1\}} m_{\tilde{q}_1,\ldots,\tilde{q}_K}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M})!}$$

$$\mathrm{ccmc}(\mathcal{T},\Theta,\boldsymbol{k},\Delta) = \frac{|\Delta|!}{\prod_{\boldsymbol{p}\in\{0,1\}^{L+M}} k_{\boldsymbol{p}}!} \prod_{\boldsymbol{p}\in\{0,1\}^{L+M}} \left(\sum_{\boldsymbol{\mu} \text{ satisfies } (*)} \prod_{\substack{\tilde{p}_1,\ldots,\tilde{p}_L\in\{0,1\}\\ \tilde{p}'_1,\ldots,\tilde{p}'_M\in\{0,1\}}} h_{\boldsymbol{\mu}}^{\tilde{p}_1,\ldots,\tilde{p}_L,\tilde{p}'_1,\ldots,\tilde{p}'_M} \right)^{k_{\boldsymbol{p}}}$$

Figure 1: The final formula for type-constrained model counting in $\mathcal{ALCSCC}$

of ways for populating the respective regions with the appropriate number of elements of the respective type (see Fig. 1). The number in (3) is concerned with a single element d of type $\tau^{\boldsymbol{p}}$. Taking into account all elements of this type (of which there are $k_{\boldsymbol{p}}$ many) means that we must take this number to the power of $k_{\boldsymbol{p}}$ since the choices can be made independently for each element. The number obtained this way is with respect to a single type. To take all types into account, we need to multiply the numbers obtained for each type with each other. Finally, this must be considered for all possible type assignments, of which there are as many as described by the formula (2). These considerations yield the overall formula for the type-constrained model counting task shown in Fig. 1.

Proposition 4.5. *Type-constrained model counting in $\mathcal{ALCSCC}$ is domain-liftable.*

Proof. This is an easy consequence of the following facts. First, the number of types and regions does not depend on $|\Delta|$ and all computations related to them (e.g., representing a set term as a union of regions) can be performed in time not depending on $|\Delta|$. Second, the number of assignments $\boldsymbol{\mu}$ in the sum is polynomially bounded by $|\Delta|$. Finally, the values of the numbers considered in calculations are exponentially bounded by $|\Delta|$, and thus their binary representations are of size polynomial in $|\Delta|$. □

Together with Lemma 4.4, this proposition yields the main result of this subsection. Note that the polynomial specifying the time bound in the above proposition and the following theorem does not depend on the specific cardinality vector used as an input.

Theorem 4.6. *Concept-constrained model counting in $\mathcal{ALCSCC}$ is domain-liftable.*

As an easy consequence we obtain that model counting in $\mathcal{ALCSCC}$ is domain-liftable. In fact, we can just use a trivially satisfied concept constraint (such as that the concept $A \sqcap \neg A$ must have cardinality zero) to count all models. The result of the theorem can also be extended to the extension of $\mathcal{ALCSCC}$ with nominals, i.e., concepts that must be interpreted as singleton sets. In fact, we can introduce a new concept name for each nominal, and then use concept constraints to require that the concepts representing nominals must have cardinality 1. In particular, this shows that we can also deal with knowledge bases that, in addition to a TBox, also contain an ABox.

5 Conclusion

In (Wilhelm et al. 2019; Baader et al. 2019), the extension $\mathcal{ALC}^{\mathsf{ME}}$ of the prototypical DL $\mathcal{ALC}$ by probabilistic conditionals has been introduced, and it was shown there that certain inferences in this logic are domain-liftable, i.e., can be solved in time polynomial in the size of the finite domain used to evaluate conditionals. Both papers mention the extension to more expressive DLs as an interesting topic for further research. In the present paper, we show that these results can indeed be extended to two very expressive logics that can formulate counting constraints: the two-variable fragment C^2 of first-order logic with counting quantifiers and the DL $\mathcal{ALCSCC}$, whose counting constraints are so powerful that they can in general not be expressed in FOL.

The domain-liftable approach developed in this paper reduces reasoning in $\mathcal{ALCSCC}^{\mathsf{ME}}$ and $C^{2\,\mathsf{ME}}$ to a certain model counting task, which we call concept-constrained model counting. We show domain-liftability of this task both for C^2 and for $\mathcal{ALCSCC}$. For C^2, this is an easy consequence of known results for weighted model counting (Kuželka 2021), but for $\mathcal{ALCSCC}$ we show this from scratch. For this purpose, we use a type-based approach that is similar to the one employed by Wilhelm et al. (2019) for $\mathcal{ALC}$. But the high expressivity of $\mathcal{ALCSCC}$ necessitates the use of more involved constructions and makes the final counting formula (see Fig. 1) more complicated.

Future work in this direction could, on the one hand, investigate extensions of $\mathcal{ALCSCC}$, for instance, by replacing the CIs in TBoxes with more expressive cardinality constraints on concepts (Baader 2019). On the other hand, as mentioned before, C^2 does not cover DLs with transitive roles, such as $\mathcal{SROIQ}$ (Horrocks, Kutz, and Sattler 2006). It would be interesting to see whether it is possible to develop domain-liftable model counting approaches for such DL. However, this appears to be a very hard problem since it is even open whether counting the number of transitive relations on a finite domain Δ is domain-liftable (see, e.g., (Pfeiffer 2004; Mala 2022) for research in this direction).

Acknowledgment

This work was supported the German Federal Ministry of Education and Research (BMBF, SCADS22B) and the Saxon State Ministry for Science, Culture and Tourism (SMWK) by funding the competence center for Big Data and AI “ScaDS.AI Dresden/Leipzig”.

References

Baader, F., and Bortoli, F. D. 2019. On the expressive power of description logics with cardinality constraints on finite and infinite sets. In Herzig, A., and Popescu, A., eds., *Frontiers of Combining Systems – 12th International Symposium, FroCoS 2019, Proceedings*, volume 11715 of *Lecture Notes in Computer Science*, 203–219. Springer.

Baader, F., and Claußnitzer, A. 2025. Maximum entropy reasoning via model counting in (description) logics that count (extended version). LTCS-Report 25-01, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden. https://doi.org/10.25368/2025.015.

Baader, F.; Calvanese, D.; McGuinness, D. L.; Nardi, D.; and Patel-Schneider, P. F., eds. 2003. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press.

Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.

Baader, F.; Ecke, A.; Kern-Isberner, G.; and Wilhelm, M. 2019. The complexity of the consistency problem in the probabilistic description logic $\mathcal{ALC}^{\mathsf{ME}}$. In Herzig, A., and Popescu, A., eds., *Frontiers of Combining Systems – 12th International Symposium, FroCoS 2019, Proceedings*, volume 11715 of *Lecture Notes in Computer Science*, 167–184. Springer.

Baader, F.; Buchheit, M.; and Hollunder, B. 1996. Cardinality restrictions on concepts. *Artif. Intell.* 88(1-2):195–213.

Baader, F. 2017. A new description logic with set constraints and cardinality constraints on role successors. In Dixon, C., and Finger, M., eds., *Frontiers of Combining Systems – 11th International Symposium, FroCoS 2017, Proceedings*, volume 10483 of *Lecture Notes in Computer Science*, 43–59. Springer.

Baader, F. 2019. Expressive cardinality restrictions on concepts in a description logic with expressive number restrictions. *ACM SIGAPP Applied Computing Review* 19:5–17.

Beierle, C.; Finthammer, M.; and Kern-Isberner, G. 2015. Relational probabilistic conditionals and their instantiations under maximum entropy semantics for first-order knowledge bases. *Entropy* 17(2):852–865.

Grädel, E.; Otto, M.; and Rosen, E. 1997. Two-variable logic with counting is decidable. In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science*, 306–317. IEEE Computer Society.

Gutiérrez-Basulto, V.; Jung, J. C.; Lutz, C.; and Schröder, L. 2017. Probabilistic description logics for subjective uncertainty. *J. Artif. Intell. Res.* 58:1–66.

Hollunder, B., and Baader, F. 1991. Qualifying number restrictions in concept languages. In Allen, J. F.; Fikes, R.; and Sandewall, E., eds., *Proceedings of the 2nd International Conference on Principles of Knowledge Representation and Reasoning (KR'91)*, 335–346. Morgan Kaufmann.

Hollunder, B.; Nutt, W.; and Schmidt-Schauß, M. 1990. Subsumption algorithms for concept description languages. In *9th European Conference on Artificial Intelligence, ECAI 1990*, 348–353.

Horrocks, I.; Kutz, O.; and Sattler, U. 2006. The even more irresistible $\mathcal{SROIQ}$. In Doherty, P.; Mylopoulos, J.; and Welty, C. A., eds., *Proceedings, Tenth International Conference on Principles of Knowledge Representation and Reasoning*, 57–67. AAAI Press.

Jacobson, N. 1974. *Basic Algebra I*. W. H. Freeman and Company.

Kern-Isberner, G., and Thimm, M. 2010. Novel semantical approaches to relational probabilistic conditionals. In Lin, F.; Sattler, U.; and Truszczynski, M., eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the Twelfth International Conference, KR 2010*. AAAI Press.

Kuželka, O. 2021. Weighted first-order model counting in the two-variable fragment with counting quantifiers. *J. Artif. Intell. Res.* 70:1281–1307.

Lukasiewicz, T. 2008. Expressive probabilistic description logics. *Artif. Intell.* 172(6-7):852–883.

Mala, F. A. 2022. On the number of transitive relations on a set. *Indian Journal of Pure and Applied Mathematics* 53(1):228–232.

Pacholski, L.; Szwast, W.; and Tendera, L. 1997. Complexity of two-variable logic with counting. In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science*, 318–327. IEEE Computer Society.

Paris, J. B. 1999. Common sense and maximum entropy. *Synthese* 117(1):75–93.

Peñaloza, R., and Potyka, N. 2017. Towards statistical reasoning in description logics over finite domains. In *Proceedings of the 11th International Conference on Scalable Uncertainty Management (SUM)*, volume 10564 of *Lecture Notes in Computer Science*, 280–294. Springer.

Pfeiffer, G. 2004. Counting transitive relations. *Journal of Integer Sequences* 7:Article 04.3.2.

Tobies, S. 2000. The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. *J. Artif. Intell. Res.* 12:199–217.

Tóth, J., and Kuželka, O. 2024. Complexity of weighted first-order model counting in the two-variable fragment with counting quantifiers: A bound to beat. In Marquis, P.; Ortiz, M.; and Pagnucco, M., eds., *Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning, KR 2024*.

Van den Broeck, G.; Taghipour, N.; Meert, W.; Davis, J.; and Raedt, L. D. 2011. Lifted probabilistic inference by first-order knowledge compilation. In Walsh, T., ed., *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence*, 2178–2185. IJCAI/AAAI.

Wilhelm, M., and Kern-Isberner, G. 2019. Maximum entropy calculations for the probabilistic description logic $\mathcal{ALC}^{\mathsf{ME}}$. In Lutz, C.; Sattler, U.; Tinelli, C.; Turhan, A.; and Wolter, F., eds., *Description Logic, Theory Combination, and All That – Essays Dedicated to Franz Baader on* *the Occasion of His 60th Birthday*, volume 11560 of *Lecture Notes in Computer Science*, 588–609. Springer.

Wilhelm, M.; Kern-Isberner, G.; Ecke, A.; and Baader, F. 2019. Counting strategies for the probabilistic description logic $\mathcal{ALC}^{\mathsf{ME}}$ under the principle of maximum entropy. In Calimeri, F.; Leone, N.; and Manna, M., eds., *Logics in Artificial Intelligence – 16th European Conference, JELIA 2019, Proceedings*, volume 11468 of *Lecture Notes in Computer Science*, 434–449. Springer.