The Unification Type of an Equational Theory May Depend on the Instantiation Preorder

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Abstract

The unification type of an equational theory is defined using a preorder on substitutions, called the instantiation preorder, whose scope is either restricted to the variables occurring in the unification problem, or unrestricted such that all variables are considered. It has been known for more than three decades that the unification type of an equational theory may vary, depending on which instantiation preorder is used. More precisely, it was shown in 1991 that the theory ACUI of an associative, commutative, and idempotent binary function symbol with a unit is unitary w.r.t. the restricted instantiation preorder, but not unitary w.r.t. the unrestricted one. In 2016 this result was strengthened by showing that the unrestricted type of this theory also cannot be finitary. Here, we considerably improve on this result by proving that ACUI is infinitary w.r.t. the unrestricted instantiation preorder, thus precluding type zero. We also show that, w.r.t. this preorder, the unification type of ACU (where idempotency is removed from the axioms) and of AC (where additionally the unit is removed) is infinitary, though it is respectively unitary and finitary in the restricted case. In the other direction, we prove (using the example of unification in the description logic $\mathcal{EL}$) that the unification type may actually improve from type zero to infinitary when switching from the restricted instantiation preorder to the unrestricted one. In addition, we establish some general results on the relationship between the two instantiation preorders.

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1 Introduction

Syntactic unification of terms was independently introduced by Robinson [47] and Knuth and Bendix [38] as a tool for computing resolvents in resolution-based theorem proving and critical pairs in the completion of term rewriting systems. Both showed the important result that any solvable unification problem has a most general unifier (mgu), i.e., a unifier that has all other unifiers as instances. In these papers, a substitution θ is defined to be an instance of a substitution σ if there is a substitution λ such that $\lambda\sigma = \theta$, i.e., $\lambda(\sigma(x)) = \theta(x)$ holds for all variables x in the countably infinite set of variables V available for building terms. In this paper, we call the preorder on substitutions obtained this way the unrestricted instantiation preorder and write it as $\sigma \leq_\emptyset^V \theta$, where the index $\emptyset$ indicates that terms are to be made syntactically equal. In addition to many papers on how to compute the mgu efficiently (e.g., [43, 42, 25]), properties of the preorder on substitutions defined this way have, for instance, been investigated in [28, 41].

In his seminal paper [45], Plotkin proposed to build certain equational theories (such as associativity or commutativity) into the unification algorithm rather than treating their axiomatization within the general theorem proving process. Similar proposals were also made in the setting of Knuth-Bendix completion in term rewriting [44, 37]. As already pointed out by Plotkin, in the equational setting most general unifiers need not exist and their rôle is instead taken on by minimal complete sets of unifiers, i.e., sets of unifiers such that every unifier is an instance of a unifier in this set, and no distinct elements in the set are comparable w.r.t. the instantiation preorder.[1] As instantiation preorder he uses what we call the restricted instantiation preorder, i.e., $\sigma \leq_E^X \theta$ where E is the theory modulo which unification is considered and X is the set of variables occurring in the unification problem. This preorder requires the existence of a substitution λ such that $\lambda(\sigma(x)) \approx_E \theta(x)$ holds for all variables $x \in X$. He explains the use of equality modulo E ($\approx_E$) in this definition, but does not comment on the restriction to the variables of the unification problem. Plotkin also gives an example of an equational theory (associativity A) where minimal complete sets of unifiers may become infinite, and conjectures that there may exist theories for which such sets do not exist.

Siekmann proposed to characterize equational theories according to the cardinality and existence of minimal complete sets of unifiers into the types unitary, finitary, infinitary, and zero. However, in the first overview paper on results in this direction [46], he uses the unrestricted instantiation preorder, and the same is true for his work on unification modulo commutativity [50]. In later overview papers [51, 52, 53] he describes the unrestricted instantiation preorder in the introduction, but employs the restricted one in the formal definition of unification types, again without explanation. Due to potential applications of equational unification in resolution-based theorem proving and term rewriting, unification properties (among them the unification type) of frequently encountered equational axioms such as associativity, commutative, idempotency, distributivity and their combinations were extensively studied in the automated deduction community in the 1980s and 1990s (see [36, 18, 19] for overviews). More recently, unification in certain logics such as modal and description logics has drawn considerable interest [31, 13, 8], where the goal is to make a formula valid or two formulas equivalent by applying a substitution. In particular, the unification types of various modal logics have been determined (see, e.g., [35, 34, 26, 22, 1, 21, 27]). In both areas, the authors usually employ the restricted instantiation preorder.

In the present paper, we investigate the impact that using the unrestricted rather than the restricted instantiation preorder has on the unification type. Until now, there were only two partial results in this direction. Already in [4] it was shown that the theory ACUI of an associative, commutative, and idempotent binary function symbol f with a unit 0, which is unitary w.r.t. the restricted instantiation preorder [7] for elementary[2] unification, is not unitary w.r.t. the unrestricted one, and thus must be finitary, infinitary or of type zero. In [10], this result was strengthened by demonstrating that also type finitary is not possible. In the present paper, we prove that the unification type of ACUI is actually infinitary w.r.t. the unrestricted instantiation preorder. We show the same result for the theory AC of an

[1] Plotkin actually calls these sets “maximally general set of unifiers” and requires two additional technical conditions.

[2] This means that unification problems may only contain terms built using variables, 0, and f.

associative and commutative binary function symbol and for ACU, which extends AC with a unit. Note that ACU is unitary and AC is finitary for elementary unification w.r.t. the restricted instantiation preorder [56] (see also Section 10.3 in [16]). Quite surprisingly, we are also able to show that the unification type of the description logic $\mathcal{EL}$ actually improves from type zero [13] to infinitary when switching from the restricted instantiation preorder to the unrestricted one. In addition to these results for specific theories/logics, we establish some general results on the relationship between the two instantiation preorders, which among other things imply that for associativity A and for commutativity C the unification type does not depend on which of the two instantiation preorders is employed.

2 Basic definitions and general results

Given a *signature* Σ consisting of a finite set of function symbols (with associated arities) and a countably infinite set of *variables* V, the set $T(\Sigma, V)$ of *terms* over Σ with variables in V is defined in the usual way [16]. An equational theory E is given by a finite set of identities $s \approx t$ between terms, which are (implicitly) assumed to be universally quantified. Such a set of identities E induces the congruence relation $\approx_E$ on terms, which can either be defined syntactically through rewriting or semantically through first-order interpretations of Σ, with $\approx$ as identity relation [16].

A *substitution* σ is a mapping from V to $T(\Sigma, V)$ that has a finite *domain* $\mathrm{Dom}(\sigma) := \{x \in V \mid \sigma(x) \neq x\}$. It can be homomorphically extended to a mapping from $T(\Sigma, V)$ to $T(\Sigma, V)$ by defining $\sigma(f(t_1, \ldots, t_n)) := f(\sigma(t_1), \ldots, \sigma(t_n))$. The *variable range* $\mathrm{VRan}(\sigma)$ of σ consists of the set of variables occurring in the terms $\sigma(x)$ for $x \in \mathrm{Dom}(\sigma)$. Substitutions can be compared using the instantiation preorder: given an equational theory E, a set of variables $X \subseteq V$, and two substitutions σ, τ, we say that σ is *more general* than τ (or τ is an *instance* of σ) w.r.t. E and X (written $\sigma \leq_E^X \tau$) if there is a substitution λ such that $\lambda\sigma \approx_E^X \tau$, i.e., $\lambda(\sigma(x)) \approx_E \tau(x)$ holds for all $x \in X$. In case $X = V$ we also write $\lambda\sigma \approx_E \tau$ in place of $\lambda\sigma \approx_E^V \tau$. It is easy to see that $\leq_E^X$ is indeed a preorder, i.e., reflexive and transitive, but in general not antisymmetric. We write $\sim_E^X$ for the equivalence relation induced by $\leq_E^X$, i.e., $\sigma \sim_E^X \tau$ iff $\sigma \leq_E^X \tau$ and $\tau \leq_E^X \sigma$. We say that σ is *strictly more general* than τ (or τ is a *strict instance* of σ) w.r.t. E and X (written $\sigma <_E^X \tau$) if $\sigma \leq_E^X \tau$ and $\sigma \not\sim_E^X \tau$.

An *E-unification problem* is a finite set of equations of the form $\Gamma = \{s_1 \approx_E^? t_1, \ldots, s_n \approx_E^? t_n\}$ such that $s_1, t_1, \ldots, s_n, t_n$ are terms in $T(\Sigma, V)$. An *E-unifier* of Γ is a substitution σ that solves all the equations in Γ, i.e., satisfies $\sigma(s_i) \approx_E \sigma(t_i)$ for all $i, 1 \leq i \leq n$. The unification problem Γ is *solvable* if it has an E-unifier. The set of all E-unifiers of Γ is denoted as $\mathcal{U}_E(\Gamma)$. For *elementary E-unification* it is assumed that Σ (and thus also Γ) contains only function symbols occurring in E. For *E-unification with constants* Σ and Γ may contain additional constant symbols, and for *general E-unification* Σ and Γ may contain additional function symbols of arbitrary arity.

Unification types for non-empty sets of identities E are usually defined w.r.t. the *restricted instantiation preorder*, which is $\leq_E^X$ where X is the finite set $\mathrm{Var}(\Gamma)$ of all variables occurring in the given unification problem Γ, but some authors also use the *unrestricted instantiation preorder* $\leq_E^V$. In this section, we will additionally consider settings where X is between these two extremes. Note that $\mathrm{Var}(\Gamma) \subseteq X$ is required for the set of E-unifiers to be closed under instantiation.

▶ **Lemma 1.** *If Γ is an E-unification problem and $X \subseteq V$ a set of variables satisfying $\mathrm{Var}(\Gamma) \subseteq X$, then $\sigma \in \mathcal{U}_E(\Gamma)$ implies $\theta \in \mathcal{U}_E(\Gamma)$ for all substitutions θ such that $\sigma \leq_E^X \theta$.*

Given an E-unification problem Γ and some set of variables X with $\mathrm{Var}(\Gamma) \subseteq X \subseteq V$, we say that a set $\mathcal{S}$ of substitutions is a *complete set* of E-unifiers of Γ w.r.t. $\leq_E^X$ if it consists of E-unifiers of Γ, and every E-unifier of Γ is an instance of an element of the complete set, i.e., for every $\theta \in \mathcal{U}_E(\Gamma)$ there exists $\sigma \in \mathcal{S}$ such that $\sigma \leq_E^X \theta$. Such a set is called *minimal* if it does not contain two distinct elements that are comparable w.r.t. $\leq_E^X$. It is easy to see that minimal complete sets of E-unifiers of a given unification problem Γ are unique up to the equivalence relation $\sim_E^X$ induced by the preorder $\leq_E^X$ (see, e.g., Corollary 3.13 in [19] and Theorem 3 below), and thus all have the same cardinality.

▶ **Definition 2.** *Let Γ be a solvable E-unification problem and X a set of variables such that* $\mathrm{Var}(\Gamma) \subseteq X \subseteq V$*. Then the* unification type *of Γ w.r.t. $\leq_E^X$ is*

- unitary *if Γ has a minimal complete set of E-unifiers of cardinality one w.r.t. $\leq_E^X$, whose single element is then called* most general E-unifier (mgu)*,*
- finitary *if Γ has a finite minimal complete set of E-unifiers of cardinality greater than one w.r.t. $\leq_E^X$,*
- infinitary *if Γ has an infinite minimal complete set of E-unifiers w.r.t. $\leq_E^X$,*
- zero *if Γ does not have a minimal complete set of E-unifiers w.r.t. $\leq_E^X$, i.e., every complete set is redundant in the sense that it must contain two distinct elements that are comparable w.r.t. $\leq_E^X$.*

Minimal complete sets of unifiers can alternatively be characterized using the following order-theoretic point of view [3, 19]. Let Γ be an E-unification problem and $X \subseteq V$ a set of variables satisfying $\mathrm{Var}(\Gamma) \subseteq X$. We denote the $\sim_E^X$-equivalence class of a unifier σ as $[\sigma]_E^X$ and the set of all equivalence classes of unifiers as $[\mathcal{U}_E(\Gamma)]_E^X$. The partial order $\preceq_E^X$ on $[\mathcal{U}_E(\Gamma)]_E^X$ induced by the instantiation preorder $\leq_E^X$ on unifiers is defined as $[\sigma]_E^X \preceq_E^X [\tau]_E^X$ if $\sigma \leq_E^X \tau$. We say that $S \subseteq [\mathcal{U}_E(\Gamma)]_E^X$ is complete w.r.t. $\preceq_E^X$ if every element of $[\mathcal{U}_E(\Gamma)]_E^X$ is above (w.r.t. $\preceq_E^X$) some element of S.

▶ **Theorem 3** ([19])**.** *Let M be the set of $\preceq_E^X$-minimal elements of $[\mathcal{U}_E(\Gamma)]_E^X$. If $\mathcal{S}$ is a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^X$, then $M = \{[\sigma]_E^X \mid \sigma \in \mathcal{S}\}$. Conversely, if M is complete in $[\mathcal{U}_E(\Gamma)]_E^X$, then any set of substitutions obtained by picking one representative for each element of M is a minimal complete set of E-unifiers of Γ.*

Consequently, unification type zero corresponds to the case where the set M of minimal elements is not complete, whereas the other types are determined by the cardinality of the set M in case it is complete. Theorem 3.1 in [3] establishes conditions that are necessary, sufficient, or both for proving unification type zero.[3] Here, we present one of these sufficient conditions since we will use it in Section 3 to show that the unification type of the description logic $\mathcal{EL}$ is zero w.r.t. the restricted instantiation preorder. It was also employed in the first paper showing unification type zero for an equational theory [29, 30].

▶ **Lemma 4** ([3])**.** *If there exists a strictly decreasing chain $\sigma_1 >_E^X \sigma_2 >_E^X \sigma_3 >_E^X \cdots$ such that the set $\mathcal{S} = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$ is a complete set of E-unifiers of Γ w.r.t. $\leq_E^X$, then Γ has type zero w.r.t. $\leq_E^X$.*

[3] Note that, in [3], the instantiation preorder is written the other way round, i.e., more general substitutions are larger.

Proof. Suppose there exists a chain $\sigma_1 >_E^X \sigma_2 >_E^X \sigma_3 >_E^X \cdots$ satisfying the conditions stated in the lemma. This chain satisfies the following properties:

1. It has no lower bound in $\mathcal{U}_E(\Gamma)$, i.e., there is no E-unifier τ of Γ such that $\sigma_i \geq_E^X \tau$ for all $i \geq 1$. To see why this is true, suppose such a unifier τ does exist. Since $\mathcal{S}$ is complete, there is $j \geq 1$ such that $\tau \geq_E^X \sigma_j >_E^X \sigma_{j+1} \geq_E^X \tau$. Transitivity of $\geq_E^X$ yields that $\sigma_{j+1} \geq_E^X \sigma_j$, but this contradicts $\sigma_j >_E^X \sigma_{j+1}$.
2. For all $i \geq 1$, if there is $\tau \in \mathcal{U}_E(\Gamma)$ such that $\sigma_i \geq_E^X \tau$, then there exists $\tau' \in \mathcal{U}_E(\Gamma)$ such that $\tau \geq_E^X \tau'$ and $\sigma_{i+1} \geq_E^X \tau'$. To show this, assume that $\sigma_i \geq_E^X \tau$. The completeness of $\mathcal{S}$ yields $j \geq 1$ such that $\sigma_i \geq_E^X \tau \geq_E^X \sigma_j$. This implies that $\sigma_i \geq_E^X \sigma_j$ because $\geq_E^X$ is a transitive relation. Hence, since the chain is strictly decreasing, it follows that $i \leq j$, and thus $\sigma_{i+1} \geq_E^X \sigma_{j+1}$ and $\tau \geq_E^X \sigma_{j+1}$. Consequently, we can set $\tau' := \sigma_{j+1}$.

Now, let M be the set of $\preceq_E^X$-minimal elements of $[\mathcal{U}_E(\Gamma)]_E^X$. To prove that Γ has type zero, it suffices to show that M is not complete. Assume to the contrary that M is complete. Then there exists $[\tau]_E^X \in M$ such that $[\tau]_E^X \preceq_E^X [\sigma_1]_E^X$, and thus $\sigma_1 \geq_E^X \tau$. Since our chain does not have a lower bound, there must be an $i \geq 1$ such that $\sigma_i \geq_E^X \tau$ and $\sigma_{i+1} \not\geq_E^X \tau$. Using the second property shown for our chain, there exists a unifier τ' such that $\tau \geq_E^X \tau'$ and $\sigma_{i+1} \geq_E^X \tau'$. Minimality of $[\tau]_E^X$ implies that $\tau \sim_E^X \tau'$, and thus $\sigma_{i+1} \geq_E^X \tau' \geq_E^X \tau$, which yields a contradiction. Thus, we have shown that M cannot be complete, which implies that Γ has type zero. ◀

As usual, we order unification types w.r.t. how bad they are (larger is worse) by setting

$$\text{zero} > \text{infinitary} > \text{finitary} > \text{unitary}.$$

The unification type w.r.t. $\leq_E^X$ of an equational theory E is the worst type of any solvable E-unification problem w.r.t. $\leq_E^X$. The *unrestricted unification type* of Γ (E) is the one w.r.t. $\leq_E^V$ and the *restricted unification type* of Γ (E) is the one w.r.t. $\leq_E^X$ for $X = \mathrm{Var}(\Gamma)$.

The results on the unification type of equational theories in the literature are usually shown for the restricted case. As we will demonstrate in this paper, it may indeed make a considerable difference for the unification type which instantiation preorder is employed in its definition. This is however not the case in the following situation.

▶ **Lemma 5.** *Let E be an equational theory, Γ an E-unification problem, $X_0 := \mathrm{Var}(\Gamma)$, and $X \subseteq V$ a set of variables such that $X_0 \subseteq X$ and $V \setminus X$ is infinite. If Γ has a minimal complete set of E-unifiers w.r.t. $\leq_E^X$, then it has a minimal complete set of E-unifiers w.r.t. $\leq_E^{X_0}$ of the same cardinality, and vice versa.*

Proof. Let $\mathcal{S}$ be a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^X$. For every unifier θ in $\mathcal{S}$ we can rename the variables in $\mathrm{VRan}(\theta)$ such that they do not belong to X by applying an appropriate permutation π that maps the variables of $\mathrm{VRan}(\theta)$ bijectively to a set of variables Y with $|Y| = |\mathrm{VRan}(\theta)|$ and $Y \cap (X \cup \mathrm{VRan}(\theta)) = \emptyset$. Such a finite set Y of variables exists since $\mathrm{VRan}(\theta)$ is finite and $V \setminus X$ is infinite. For a given bijection $p : \mathrm{VRan}(\theta) \to Y$, we can define π as follows: $\pi(z) := p(z)$ for all $z \in \mathrm{VRan}(\theta)$, $\pi(y) := p^{-1}(y)$ for all $y \in Y$, and $\pi(x) = x$ for all other variables. Note that π is a substitution since its domain is $\mathrm{VRan}(\theta) \cup Y$, which is finite. To show that π really is a permutation, i.e., a bijective mapping from V to V, it is sufficient to show that it is an injective mapping from V to V (see Lemma 2.6 in [28]). This is an immediate consequence of the facts that p and p^{-1} are bijections and Y and $\mathrm{VRan}(\theta)$ are disjoint.

The substitution $\pi\theta$ is equivalent to θ w.r.t. the equivalence relation $\sim_\emptyset^V$ induced by $\leq_\emptyset^V$, and thus also w.r.t. $\sim_E^X$. In fact, $\pi^{-1}\pi\theta = \theta$. If we restrict the domain of this substitution to X, then the resulting substitution $\pi\theta|_X$ is still equivalent to θ w.r.t. $\sim_E^X$ and also satisfies

$\mathrm{VRan}(\pi\theta|_X) \cap X = \emptyset$. To see the latter, consider a variable $x \in \mathrm{Dom}(\pi\theta|_X)$. First assume that $x \in \mathrm{Dom}(\theta) \cap X$. Then all the variables in $\theta(x)$ belong to $\mathrm{VRan}(\theta)$, and thus all the variables in $\pi\theta|_X(x)$ belong to Y, which is disjoint with X. If $x \in X \setminus \mathrm{Dom}(\theta)$, then $\pi\theta|_X(x) = \pi(x) \neq x$. Since $\mathrm{Dom}(\pi) = \mathrm{VRan}(\theta) \cup Y$ and Y is disjoint with X, this implies $x \in \mathrm{VRan}(\theta)$, and thus $\pi(x) \in Y$.

Let $\mathcal{S}'$ be the set of substitutions obtained from $\mathcal{S}$ by applying this renaming and domain restriction process to every element of $\mathcal{S}$. Due to the $\sim_E^X$-equivalence of the elements of $\mathcal{S}$ with their modified variants in $\mathcal{S}'$ and the fact that X contains all the variables occurring in Γ, it is easy to see that $\mathcal{S}'$ is also a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^X$. In a second step, we restrict the domains of the elements of $\mathcal{S}'$ to X_0. Given an element θ of $\mathcal{S}'$, we define $\theta|_{X_0}$ to be the substitution that coincides with θ on X_0 and maps each variable $x \notin X_0$ to x. Since θ is an E-unifier of Γ and this unification problem contains only variables from X_0, the substitution $\theta|_{X_0}$ is clearly also an E-unifier of Γ. We claim that $\theta|_{X_0} \leq_E^X \theta$. To prove this, we define the substitution λ by setting $\lambda(x) := \theta(x)$ if $x \in X \setminus X_0$ and $\lambda(x) := x$ for all other variables. We now show that $\lambda\theta|_{X_0} \approx_E^X \theta$. If $x \in X_0 \cap \mathrm{Dom}(\theta)$, then $\lambda(\theta|_{X_0}(x)) = \lambda(\theta(x)) = \theta(x)$. The first identity holds by the definition of $\theta|_{X_0}$ and the second since the variables occurring in $\theta(x)$ are elements of $\mathrm{VRan}(\theta)$, and thus do not belong to X. If $x \in X_0 \setminus \mathrm{Dom}(\theta)$, then $\lambda(\theta|_{X_0}(x)) = \lambda(x) = x = \theta(x)$. Finally, if $x \in X \setminus X_0$, then $\lambda(\theta|_{X_0}(x)) = \lambda(x) = \theta(x)$. Again, the first identity holds by the definition of $\theta|_{X_0}$ and the second by the definition of λ. Summing up, we have shown that the following holds for every element θ of $\mathcal{S}'$: $\theta|_{X_0}$ is an E-unifier of Γ and $\theta|_{X_0} \leq_E^X \theta$. Since $\mathcal{S}'$ is a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^X$, its elements are minimal w.r.t. $\leq_E^X$, which yields $\theta|_{X_0} \sim_E^X \theta$. Consequently, we know that the set $\mathcal{S}'|_{X_0} := \{\theta|_{X_0} \mid \theta \in \mathcal{S}'\}$ is also a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^X$.

We claim that the same is true w.r.t. the smaller set of variables X_0, i.e., that $\mathcal{S}'|_{X_0}$ is also minimal and complete w.r.t. $\leq_E^{X_0}$. Completeness trivially follows from the fact that $\leq_E^X \subseteq \leq_E^{X_0}$. To prove minimality, assume that θ and τ are two distinct elements of $\mathcal{S}'|_{X_0}$. Then these two substitutions are not comparable w.r.t. $\leq_E^X$. Assume that they are comparable w.r.t. $\leq_E^{X_0}$, i.e., there is a substitution λ such that $\lambda(\theta(x)) \approx_E \tau(x)$ holds for all $x \in X_0$. By our construction of $\mathcal{S}'|_{X_0}$, we know that $\mathrm{Dom}(\theta) \subseteq X_0$, $\mathrm{Dom}(\tau) \subseteq X_0$, and $\mathrm{VRan}(\theta) \cap X = \emptyset$. If x is a variable in $X \setminus X_0$, then $\lambda(\theta(x)) = \lambda(x)$ and $\tau(x) = x$. Thus, if we modify λ to λ' such that $\lambda'(x) = x$ holds for all $x \in X \setminus X_0$, then $\lambda'(\theta(x)) = \tau(x)$ holds for all $x \in X \setminus X_0$. This modification has no effect on the variables $x \in X_0$. In fact, let x be such a variable. If $x \in \mathrm{Dom}(\theta)$, then $\theta(x)$ does not contain any variable from X, and thus $\lambda'(\theta(x)) = \lambda(\theta(x)) \approx_E \tau(x)$ holds. If $x \notin \mathrm{Dom}(\theta)$, then $\theta(x) = x$, and thus again $\lambda'(\theta(x)) = \lambda'(x) = \lambda(x) = \lambda(\theta(x)) \approx_E \tau(x)$ since λ' coincides with λ on the variables in X_0. Summing up, we have shown that the assumption $\theta \leq_E^{X_0} \tau$ implies $\theta \leq_E^X \tau$, which contradicts the fact that $\mathcal{S}'|_{X_0}$ is minimal w.r.t. $\leq_E^X$. Consequently, $\mathcal{S}'|_{X_0}$ is a minimal complete set w.r.t. $\leq_E^{X_0}$, and this set has the same cardinality as the minimal complete set $\mathcal{S}$ of E-unifiers of Γ w.r.t. $\leq_E^X$ we have started with.

Conversely, let $\mathcal{S}$ be a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^{X_0}$. By applying the same construction as in the proof of the other direction, we can assume without loss of generality that every unifier $\theta \in \mathcal{S}$ satisfies $\mathrm{Dom}(\theta) \subseteq X_0$ and $\mathrm{VRan}(\theta) \cap X = \emptyset$. In fact, by applying this construction to a minimal complete set $\mathcal{S}'$ of E-unifiers of Γ w.r.t. $\leq_E^{X_0}$, we obtain a new set $\mathcal{S}$ where every element θ' of the original set $\mathcal{S}'$ is replaced with an element θ that satisfies the above conditions and is $\sim_E^X$-equivalent to θ', and thus also $\sim_E^{X_0}$-equivalent. Consequently, this new set $\mathcal{S}$ is also a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^{X_0}$.

We claim that $\mathcal{S}$ is also minimal and complete w.r.t. the larger set of variables X. Minimality trivially follows from the fact that $\leq_E^X \subseteq \leq_E^{X_0}$. To prove completeness, assumed that τ is an E-unifier of Γ. Completeness of $\mathcal{S}$ w.r.t. $\leq_E^{X_0}$ yields an element θ of $\mathcal{S}$ such that $\theta \leq_E^{X_0} \tau$, i.e., there is a substitution λ such that $\lambda\theta(x) \approx_E \tau(x)$ holds for all $x \in X_0$. Let x be a variable in $X \setminus X_0$. Then $\lambda\theta(x) = \lambda(x)$. Thus, if we modify λ to λ' such that $\lambda'(x) = \tau(x)$ holds for all $x \in X \setminus X_0$, then $\lambda'(\theta(x)) = \tau(x)$ holds for all $x \in X \setminus X_0$. The claim that this modification has no effect for the variables $x \in X_0$ can be shown as in the proof of minimality of $\mathcal{S}'|_{X_0}$ w.r.t. $\leq_E^{X_0}$ above. This proves that $\lambda'\theta(x) \approx_E \tau(x)$ holds for all $x \in X$, and thus $\mathcal{S}$ is also complete w.r.t. $\leq_E^X$. ◀

The following theorem is an immediate consequence of this lemma.

▶ **Theorem 6.** *Let E be an equational theory, Γ an E-unification problem, and $X \subseteq V$ a set of variables such that $\mathrm{Var}(\Gamma) \subseteq X$ and $V \setminus X$ is infinite. Then the restricted unification type of Γ coincides with the unification type of Γ w.r.t. $\leq_E^X$.*

Note that the condition of the theorem is in particular satisfied if X is a finite superset of $\mathrm{Var}(\Gamma)$. However, it is clearly not satisfied for $X = V$, which corresponds to the unrestricted instantiation preorder setting. We will see below that in this case the unification type can indeed depend on whether the restricted or the unrestricted instantiation preorder is used. However, there is a special case where this cannot happen.

▶ **Theorem 7.** *Let E be an equational theory, Γ an E-unification problem, and $\mathcal{S}$ a set of E-unifiers of Γ such that $\mathrm{VRan}(\sigma) \cup \mathrm{Dom}(\sigma) \subseteq \mathrm{Var}(\Gamma)$ holds for all $\sigma \in \mathcal{S}$. Then $\mathcal{S}$ is a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^V$ iff it is a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$.*

Proof. Let $\mathcal{S}$ be a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^V$. Since $\leq_E^V \subseteq \leq_E^{\mathrm{Var}(\Gamma)}$, this set is also complete w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$. To show minimality w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$, assume to the contrary that σ, θ are two distinct elements of $\mathcal{S}$ such that $\sigma \leq_E^{\mathrm{Var}(\Gamma)} \theta$, i.e., there is a substitution λ such that $\lambda\sigma(x) \approx_E \theta(x)$ holds for all $x \in \mathrm{Var}(\Gamma)$. We modify λ to λ' by setting $\lambda'(x) = x$ for all variables $x \in V \setminus \mathrm{Var}(\Gamma)$. For $x \in \mathrm{Var}(\Gamma)$, we know that $\sigma(x)$ contains only variables from $\mathrm{VRan}(\sigma) \subseteq \mathrm{Var}(\Gamma)$ if $x \in \mathrm{Dom}(\sigma)$ or $\sigma(x) = x \in \mathrm{Var}(\Gamma)$. Since λ and λ' coincide on $\mathrm{Var}(\Gamma)$, this yields $\lambda'\sigma(x) = \lambda\sigma(x) \approx_E \theta(x)$. For $x \in V \setminus \mathrm{Var}(\Gamma)$, this variable does not belong to any of the sets $\mathrm{Dom}(\sigma)$, $\mathrm{Dom}(\theta)$, and $\mathrm{Dom}(\lambda')$, and thus $\lambda'\sigma(x) = \lambda'(x) = x = \theta(x)$. Summing up, we have shown that $\sigma \leq_E^V \theta$, which contradicts our assumption that $\mathcal{S}$ is minimal w.r.t. $\leq_E^V$.

Conversely, assume that $\mathcal{S}$ is a minimal complete set of E-unifiers of Γ w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$. Since $\leq_E^V \subseteq \leq_E^{\mathrm{Var}(\Gamma)}$, minimality also holds w.r.t. $\leq_E^V$. To show completeness w.r.t. $\leq_E^V$, assume that θ is an E-unifier of Γ. Completeness of $\mathcal{S}$ w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$ yields a substitution $\sigma \in \mathcal{S}$ such that $\sigma \leq_E^{\mathrm{Var}(\Gamma)} \theta$. Similarly to the first part of the proof, we can show that this also implies $\sigma \leq_E^V \theta$. The difference is that now θ is an arbitrary unifier, and thus $\mathrm{VRan}(\theta) \cup \mathrm{Dom}(\theta) \subseteq \mathrm{Var}(\Gamma)$ need not hold. Let λ be such that $\lambda\sigma(x) \approx_E \theta(x)$ holds for all $x \in \mathrm{Var}(\Gamma)$. We modify λ to λ' by setting $\lambda'(x) = \theta(x)$ for all variables $x \in V \setminus \mathrm{Var}(\Gamma)$. For $x \in \mathrm{Var}(\Gamma)$, we obtain $\lambda'\sigma(x) = \lambda\sigma(x) \approx_E \theta(x)$ as in the first part of the proof. For $x \in V \setminus \mathrm{Var}(\Gamma)$, this variable does not belong to $\mathrm{Dom}(\sigma)$, and thus $\lambda'\sigma(x) = \lambda'(x) = \theta(x)$. ◀

Examples of equational theories where the conditions of this theorem are always satisfied are the empty theory (unitary), the theory C axiomatizing commutativity of a binary function symbol (finitary), and the theory A axiomatizing associativity of a binary function symbol (infinitary). This is an easy consequence of the known algorithms [42, 50, 45] computing (or enumerating, in the case of A) minimal complete sets of unifiers for these theories.

The following result is an easy consequence of the fact that $\leq_E^V \subseteq \leq_E^{\mathrm{Var}(\Gamma)}$.

▶ **Theorem 8.** *Let E be an equational theory. If E has unrestricted unification type unitary (finitary), then it has restricted unification type unitary (finitary or unitary).*

Proof. A given finite minimal complete set $\mathcal{S}$ of E-unifiers of Γ w.r.t. $\leq_E^V$ is also complete w.r.t. $\leq_E^{\mathrm{Var}(\Gamma)}$. If it is not minimal (which can only happen in the finitary case), then one can make it minimal by removing redundant element (i.e., elements θ such that $\mathcal{S}$ contains an element $\sigma <_E^{\mathrm{Var}(\Gamma)} \theta$) without destroying completeness. ◀

3 The unification type of $\mathcal{EL}$

Unification was introduced in description logics as a tool for detecting redundancies in large knowledge bases [14]. The description logic $\mathcal{EL}$ [6] has drawn considerable attention since its standard reasoning problems can be solved in polynomial time while the logic is still expressive enough for formalizing bio-medical ontologies [11]. In [12, 13] it was shown that, in the setting of unification with constants, $\mathcal{EL}$ has unification type zero w.r.t. the restricted instantiation preorder. In the following, we will analyze the example used in [13] to prove this result in more detail, and explain why it does not work for the unrestricted instantiation preorder. Then we show that the unification type of $\mathcal{EL}$ in the unrestricted setting is actually infinitary, not just for this example, but in general.

But first, we briefly introduce $\mathcal{EL}$ and recall why unification in $\mathcal{EL}$ can be seen as unification modulo an equational theory (see [6, 13, 15] for more detailed descriptions). Given sets of concept names (unary predicates) and role names (binary predicates), $\mathcal{EL}$ *concept descriptions* (or simply concepts) are built from concept names using the concept constructors top concept ($\top$), conjunction ($C \sqcap D$), and existential restriction ($\exists r.C$). In the model-theoretic semantics of $\mathcal{EL}$, a given interpretation $\mathcal{I}$ assigns sets $C^{\mathcal{I}}$ to concept descriptions C according to the semantics of the constructors. To be more precise, an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty interpretation domain and an extension function that assigns subsets of this domain to concept names and binary relations on the domain to role names. This interpretation function is extended to concept descriptions as follows:

$$\top^{\mathcal{I}} := \Delta^{\mathcal{I}}, \quad (C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}, \quad (\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I}} \text{ such that } (d,e) \in r^{\mathcal{I}}\}.$$

The concept description C is *subsumed* by the concept description D (written $C \sqsubseteq D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all interpretations $\mathcal{I}$, and C and D are *equivalent* (written $C \equiv D$) if they subsume each other.

In [13], the notion of a reduced $\mathcal{EL}$ concept is employed to derive characterizations of equivalence and subsumption of $\mathcal{EL}$ concepts. Here, we recall this notion and the characterizations since they are used later on to explain why $\mathcal{EL}$ has unification type zero w.r.t. the restricted instantiation preorder. Küsters [39] introduces the following rules for reducing $\mathcal{EL}$ concept descriptions:

$$\begin{array}{ll} C \sqcap \top \to C & \text{for all } \mathcal{EL} \text{ concept descriptions } C, \\ A \sqcap A \to A & \text{for all concept names } A, \\ \exists r.C \sqcap \exists r.D \to \exists r.C & \text{for all } \mathcal{EL} \text{ concept descriptions } C, D \text{ with } C \sqsubseteq D. \end{array}$$

A *reduced form* of a given $\mathcal{EL}$ concept description C is then obtained from C by applying these rules exhaustively modulo associativity and commutativity of $\sqcap$. The following theorem is stated in [13] as an easy consequence of Corollary 6.3.1 on page 181 of [39].

▶ **Theorem 9** (Theorem 3.1 of [13]). *Let C, D be $\mathcal{EL}$ concept descriptions, and $\widehat{C}, \widehat{D}$ reduced forms of C, D, respectively. Then $C \equiv D$ iff $\widehat{C}$ is identical to $\widehat{D}$ up to associativity and commutativity of $\sqcap$.*

This theorem is used in [13] to derive a recursive characterization of subsumption in $\mathcal{EL}$.

▶ **Corollary 10** (Corollary 3.2 of [13]). *Let $C = A_1 \sqcap \ldots \sqcap A_k \sqcap \exists r_1.C_1 \sqcap \ldots \sqcap \exists r_m.C_m$ and $D = B_1 \sqcap \ldots \sqcap B_\ell \sqcap \exists s_1.D_1 \sqcap \ldots \sqcap \exists s_n.D_n$, where $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ are concept names. Then $C \sqsubseteq D$ iff $\{B_1, \ldots, B_\ell\} \subseteq \{A_1, \ldots, A_k\}$ and for every $j, 1 \leq j \leq n$, there exists an $i, 1 \leq i \leq m$, such that $r_i = s_j$ and $C_i \sqsubseteq D_j$.*

From this corollary, the following lemma is then derived in [13].

▶ **Lemma 11** (Lemma 3.3 of [13]). *If C, D are reduced $\mathcal{EL}$ concept descriptions such that $\exists r.D \sqsubseteq C$, then C is either $\top$, or of the form $C = \exists r.C_1 \sqcap \ldots \sqcap \exists r.C_n$ where $n \geq 1$; $C_1, \ldots, C_n$ are reduced and pairwise incomparable w.r.t. subsumption; and $D \sqsubseteq C_1, \ldots, D \sqsubseteq C_n$. Conversely, if C, D are $\mathcal{EL}$ concept descriptions such that $C = \exists r.C_1 \sqcap \ldots \sqcap \exists r.C_n$ and $D \sqsubseteq C_1, \ldots, D \sqsubseteq C_n$, then $\exists r.D \sqsubseteq C$.*

Equivalence of $\mathcal{EL}$ concept descriptions can be axiomatized by the equational theory bSLmO of *bounded semilattices with monotone operators* [54, 15, 55]. For this purpose, we view the conjunction operator $\sqcap$ as a binary function symbol, $\top$ as a constant symbol, and $\exists r.$ for a role name r as a unary function symbol. The theory bSLmO over this signature then consists of the identities stating that $\sqcap$ is associative, commutative, and idempotent, has $\top$ as unit, and existential restrictions as monotone operators:

$$x \sqcap y \approx y \sqcap x,\ (x \sqcap y) \sqcap z \approx x \sqcap (y \sqcap z),\ x \sqcap x \approx x,\ x \sqcap \top \approx x, \exists r.x \sqcap \exists r.(x \sqcap y) \approx \exists r.(x \sqcap y).$$

For *unification in $\mathcal{EL}$* we partition the set of concept names into concept variables and concept constants. Substitutions can then replace concept variables in $\mathcal{EL}$ concept descriptions with $\mathcal{EL}$ concept descriptions. Unifiers are supposed to solve equations between concept descriptions with variables by making them equivalent. This corresponds to unification with constants modulo the equational theory bSLmO. In the following, we use $\approx_{\mathcal{EL}}$ (rather than $\approx_{\mathsf{bSLmO}}$) to denote equivalence between $\mathcal{EL}$ concept descriptions and also employ the subscript $\mathcal{EL}$ when writing the respective instantiation preorders.

▶ **Example 12.** Consider the $\mathcal{EL}$-unification problem $\Gamma := \{x \sqcap \exists r.y \approx^?_{\mathcal{EL}} \exists r.y\}$. Then, for every $n \geq 0$, the substitution

$$\sigma_n := \{x \mapsto \exists r.z_1 \sqcap \ldots \sqcap \exists r.z_n,\quad y \mapsto z_1 \sqcap \ldots \sqcap z_n \sqcap z\}$$

for distinct variables $x, y, z_1, \ldots, z_n, z$ is an $\mathcal{EL}$-unifier of Γ, where the empty conjunction is $\top$, i.e., $\sigma_0 = \{x \mapsto \top, y \mapsto z\}$. We will show below that, w.r.t. the restricted instantiation preorder, the set $\{\sigma_n \mid n \geq 0\}$ is a complete set of $\mathcal{EL}$-unifiers that constitutes a strictly decreasing chain $\sigma_0 >^{\{x,y\}}_{\mathcal{EL}} \sigma_1 >^{\{x,y\}}_{\mathcal{EL}} \sigma_2 >^{\{x,y\}}_{\mathcal{EL}} \cdots$ of more and more general unifiers. According to Lemma 4 this implies that Γ has unification type zero, and thus $\mathcal{EL}$ has unification type zero.

▶ **Lemma 13.** *Let Γ and σ_n for $n \geq 0$ be defined as in Example 12. Then the set $\{\sigma_n \mid n \geq 0\}$ is a complete set of $\mathcal{EL}$-unifiers of Γ w.r.t. $\leq^{\{x,y\}}_{\mathcal{EL}}$.*

Proof. Let σ be an $\mathcal{EL}$-unifier of Γ and $C := \sigma(y)$ and $D := \sigma(x)$. Then $\exists r.C \sqsubseteq D$. In addition, we can assume without loss of generality that C and D are reduced. The characterization of subsumption in $\mathcal{EL}$ (see Corollary 10) yields $D = \exists r.D_1 \sqcap \ldots \sqcap \exists r.D_n$ for $n \geq 0$ $\mathcal{EL}$ concept descriptions $D_1, \ldots, D_n$ satisfying $C \sqsubseteq D_1, \ldots, C \sqsubseteq D_n$ (see Lemma 11). Thus, if we define $\lambda := \{z_1 \mapsto D_1, \ldots, z_n \mapsto D_n, z \mapsto C\}$, then $\lambda\sigma_n \approx_{\mathcal{EL}}^{\{x,y\}} \sigma$, which shows $\sigma_n \leq_{\mathcal{EL}}^{\{x,y\}} \sigma$. ◀

▶ **Lemma 14.** *If σ_n for $n \geq 0$ are defined as in Example 12, then $\sigma_0 >_{\mathcal{EL}}^{\{x,y\}} \sigma_1 >_{\mathcal{EL}}^{\{x,y\}} \sigma_2 >_{\mathcal{EL}}^{\{x,y\}} \cdots$.*

Proof. First note that $\sigma_{n+1} \leq_{\mathcal{EL}}^{\{x,y\}} \sigma_n$ since $\lambda\sigma_{n+1} \approx_{\mathcal{EL}}^{\{x,y\}} \sigma_n$ for $\lambda := \{z_{n+1} \mapsto \top\}$. Second, assume that $\sigma_n \leq_{\mathcal{EL}}^{\{x,y\}} \sigma_{n+1}$, i.e., there is a substitution τ such that $\tau\sigma_n \approx_{\mathcal{EL}}^{\{x,y\}} \sigma_{n+1}$. Then $\exists r.\tau(z_1) \sqcap \ldots \sqcap \exists r.\tau(z_n) \approx_{\mathcal{EL}} \exists r.z_1 \sqcap \ldots \sqcap \exists r.z_{n+1}$. Consequently, by Theorem 9, the reduced forms of these two concept descriptions would need to be equal up to associativity and commutativity of $\sqcap$. However, the reduced form of the concept description on the left-hand side has at most n existential restrictions, whereas the reduced form of the concept description on the right-hand side has $n+1$ existential restrictions, which yields a contradiction. Thus, we have shown $\sigma_{n+1} <_{\mathcal{EL}}^{\{x,y\}} \sigma_n$. ◀

The following theorem is now an immediate consequence of Lemma 4.

▶ **Theorem 15.** *The restricted unification type of $\mathcal{EL}$ is zero.*

With respect to the unrestricted instantiation preorder, the instance relationship between the substitutions σ_n and σ_{n+1} no longer holds. More generally, we can show the following result.

▶ **Lemma 16.** *If σ_n for $n \geq 0$ are defined as in Example 12, then $\sigma_n \not\leq_{\mathcal{EL}}^{V} \sigma_m$ for all distinct $n, m \geq 0$.*

Proof. Let $n < m$. Then we know from Lemma 14 that $\sigma_n \not\leq_{\mathcal{EL}}^{\{x,y\}} \sigma_m$, which implies $\sigma_n \not\leq_{\mathcal{EL}}^{V} \sigma_m$ since $\leq_{\mathcal{EL}}^{V} \subseteq \leq_{\mathcal{EL}}^{\{x,y\}}$.

Assume that $n > m$ and that there is a substitution λ such that $\lambda\sigma_n \approx_{\mathcal{EL}} \sigma_m$. Then $\lambda\sigma_n(x) \approx_{\mathcal{EL}} \sigma_m(x)$, and since $\sigma_m(x)$ does not contain z_n, but $\sigma_n(x)$ does, λ must replace z_n by a concept description not containing z_n, and thus in particular $\lambda(z_n) \not\approx_{\mathcal{EL}} z_n$. But then $\lambda\sigma_n(z_n) = \lambda(z_n) \not\approx_{\mathcal{EL}} z_n$ and $\sigma_m(z_n) = z_n$ yield a contradiction to $\lambda\sigma_n \approx_{\mathcal{EL}} \sigma_m$. ◀

Thus, w.r.t. the unrestricted instantiation preorder, the set $\{\sigma_n \mid n \geq 0\}$ consists of unifiers that are incomparable. However, this set is also no longer complete.

▶ **Lemma 17.** *Let Γ and σ_n for $n \geq 0$ be defined as in Example 12. The $\mathcal{EL}$-unifier $\sigma := \{x \mapsto \exists r.A, y \mapsto A\}$ of Γ is not an instance of any of the substitutions σ_n for $n \geq 0$ if we use the unrestricted instantiation preorder.*

Proof. First, note that $\sigma_0 \leq_{\mathcal{EL}}^{V} \sigma$ is not possible since $\sigma_0(x) = \top$, and thus $\lambda\sigma_0(x) = \top$ holds for all substitutions λ, whereas $\sigma(x) = \exists r.A$. If $n \geq 1$ and $\lambda\sigma_n \approx_{\mathcal{EL}} \sigma$, then λ must replace all variables z_i occurring in $\sigma_n(x)$ with A, and thus $\lambda\sigma_n(z_i) = \lambda(z_i) \approx_{\mathcal{EL}} A$. However, $\sigma(z_i) = z_i$, which yields a contradiction to $\lambda\sigma_n \approx_{\mathcal{EL}} \sigma$. ◀

Summing up, we have seen that the $\mathcal{EL}$-unification problem Γ of Example 12 has unification type zero for the restricted instantiation preorder, but the proof that we have used to show this result does not work if we employ the unrestricted instantiation preorder instead.

In the following, we prove the general result that the unrestricted unification type of $\mathcal{EL}$ is not zero. Note that, by Theorem 8, this type cannot be unitary or finitary, and thus must be infinitary. The main idea underlying our proof of this result is to show that, up to the equivalence relation $\sim^V_{\mathcal{EL}}$, every substitution has only finitely many more general substitutions w.r.t. $\leq^V_{\mathcal{EL}}$. The most challenging technical result needed to prove this is the following lemma.

▶ **Lemma 18.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathcal{EL}} \theta$. Then there is a substitution σ' such that $\sigma \sim^V_{\mathcal{EL}} \sigma'$ and $\mathrm{Dom}(\sigma') \subseteq \mathrm{Dom}(\theta)$.*

Proof. First note that we can assume without loss of generality that $\mathrm{VRan}(\sigma) \cap \mathrm{Dom}(\theta) = \emptyset$. Otherwise, we can apply a permutation to σ that renames the variables in $\mathrm{VRan}(\sigma)$ appropriately, which yields a substitution that is $\sim^V_{\mathcal{EL}}$-equivalent to σ and satisfies the required disjointness condition (see the proof of Lemma 5 for how such a permutation can be obtained).

Since $\sigma \leq^V_{\mathcal{EL}} \theta$, we know that there is a substitution λ such that $\lambda\sigma \approx_{\mathcal{EL}} \theta$. Consider the set $\mathcal{X}$ of all variables x such that $x \in \mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$. Then $\lambda\sigma(x) \approx_{\mathcal{EL}} \theta(x) = x$ holds for all $x \in \mathcal{X}$. Consequently, the top-level conjunction of $\sigma(x)$ cannot contain concept constants (i.e., constants) or existential restrictions (i.e., terms starting with a function application). This means that $\sigma(x)$ is a conjunction of variables:

$$\sigma(x) = z^x_1 \sqcap \ldots \sqcap z^x_{n_x},$$

where we assume (without loss of generality) that all the variables z^x_i for a fixed x are pairwise distinct. To obtain $\lambda\sigma(x) \approx_{\mathcal{EL}} x$, the substitution λ must thus assign (modulo $\approx_{\mathcal{EL}}$) x or $\top$ to the variables z^x_i for all $i, 1 \leq i \leq n_x$, and x to at least one of these variables. Let us assume without loss of generality that λ assigns x to z^x_1 for all $x \in \mathcal{X}$. This implies that $z^x_1 \neq z^y_1$ for different elements x, y of $\mathcal{X}$.

Since $\mathrm{VRan}(\sigma) \cap \mathrm{Dom}(\theta) = \emptyset$, we know that $z^x_i \notin \mathrm{Dom}(\theta)$ for all $x \in \mathcal{X}$ and $1 \leq i \leq n_x$. We claim that $z^x_i \in \mathrm{Dom}(\sigma)$ also holds, and thus $z^x_i \in \mathcal{X}$. Otherwise, $\lambda\sigma(z^x_i) = \lambda(z^x_i) \in \{x, \top\}$. However, $\lambda\sigma(z^x_i) \approx_{\mathcal{EL}} \theta(z^x_i) = z^x_i$. This yields a contradiction unless $z^x_i = x$. But then we also have $z^x_i \in \mathrm{Dom}(\sigma)$ since $x \in \mathrm{Dom}(\sigma)$. We have thus shown that all the variables z^x_i for $x \in \mathcal{X}$ also belong to $\mathcal{X} = \mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$. Since $z^x_1 \neq z^y_1$ for different elements x, y of $\mathcal{X}$, this implies that the z-variables with index 1 already "use up" all of $\mathcal{X}$, and thus $n_x = 1$ holds for all $x \in \mathcal{X}$.

Consequently, σ is a W-renaming for $W = \mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$, where according to Definition 2.11 in [28] a substitution τ is a W-renaming if $\tau(x)$ is a variable for all $x \in W$ and τ is injective on W. Since σ is the identity on $V \setminus \mathrm{Dom}(\sigma)$ and elements of $\mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$ are mapped to $\mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$ by σ, the substitution σ is also a W-renaming for $W = V \setminus \mathrm{Dom}(\theta)$. By Lemma 2.12 in [28], there is a permutation π that coincides with σ on $W = V \setminus \mathrm{Dom}(\theta)$. Let $\sigma' := \pi^{-1}\sigma$. Then $\sigma' \sim^V_{\emptyset} \sigma$, and thus also $\sigma' \sim^V_{\mathcal{EL}} \sigma$. In addition, for all $x \notin \mathrm{Dom}(\theta)$ we have $\sigma'(x) = \pi^{-1}\sigma(x) = x$, which yields $x \notin \mathrm{Dom}(\sigma')$. This shows that $\mathrm{Dom}(\sigma') \subseteq \mathrm{Dom}(\theta)$. ◀

This lemma together with the next result shows that, up to equivalence, we can bound the variables occurring in more general substitutions by the substitutions they have as instance.

▶ **Lemma 19.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathcal{EL}} \theta$ and $\mathrm{Dom}(\sigma) \subseteq \mathrm{Dom}(\theta)$. Then $\mathrm{VRan}(\sigma) \subseteq \mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$.*

Proof. Let λ be such that $\lambda\sigma \approx_{\mathcal{EL}} \theta$. Assume that $z \in \mathrm{VRan}(\sigma)$, but $z \notin \mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$. Then $z \notin \mathrm{Dom}(\sigma)$, and thus $\lambda(z) = \lambda\sigma(z) \approx_{\mathcal{EL}} \theta(z) = z$. However, since $z \in \mathrm{VRan}(\sigma)$, there is a variable $x \in \mathrm{Dom}(\sigma)$ such that $\sigma(x)$ contains z. Then $x \in \mathrm{Dom}(\theta)$, and thus all variables occurring in $\theta(x)$ belong to $\mathrm{VRan}(\theta)$. Since $\lambda\sigma(x) \approx_{\mathcal{EL}} \theta(x)$, this means that $z \in \mathrm{VRan}(\theta)$, which contradicts our assumptions on z. ◀

The following lemma bounds the role depth (i.e., the maximal nesting of existential restrictions) as well as the concept constants and role names occurring in more general substitutions. Formally, the role depth $\mathrm{rd}(C)$ of an $\mathcal{EL}$ concept description C is defined inductively as follows:

- $\mathrm{rd}(A) := \mathrm{rd}(\top) := 0$ for all concept names A,
- $\mathrm{rd}(C \sqcap D) := \max\{\mathrm{rd}(C), \mathrm{rd}(D)\}$ and $\mathrm{rd}(\exists r.C) := 1 + \mathrm{rd}(C)$.

▶ **Lemma 20.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathcal{EL}} \theta$. Then the following holds for all $x \in V$: the role depth of $\sigma(x)$ is bounded by the role depth of $\theta(x)$, and the concept constants and role names occurring in $\sigma(x)$ also occur in $\theta(x)$.*

Proof. This is an easy consequence of the fact that $\approx_{\mathcal{EL}}$ preserves the role depth as well as the set of concept constants and role names occurring in a concept description, and applying a substitution to a concept description can only increase the role depth and add concept constants or role names, but not decrease the role depth or remove concept or role names. ◀

As a consequence of the sequence of lemmas we have just shown we know that, for a given substitution θ, the set of more general substitutions is finite up to the equivalence relation $\sim^V_{\mathcal{EL}}$. In fact, Lemmas 18 and 19 show that one can restrict the attention to substitutions σ satisfying $\mathrm{Dom}(\sigma) \cup \mathrm{VRan}(\sigma) \subseteq \mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$. In addition, for all $x \in \mathrm{Dom}(\sigma)$, the concept descriptions $\sigma(x)$ are built using role names and concept constants occurring in $\theta(x)$ as well as variables from $\mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$, and have a role depth that is bounded by the one of $\theta(x)$. It is well-known [13] that there are up to equivalence $\approx_{\mathcal{EL}}$ only finitely many $\mathcal{EL}$-concept descriptions satisfying these properties.

▶ **Lemma 21.** *For a given substitution θ, the set $\{\sigma \mid \sigma \leq^V_{\mathcal{EL}} \theta\}$ is finite up to $\sim^V_{\mathcal{EL}}$-equivalence.*

As an immediate consequence we obtain that the set of substitutions more general than θ contains elements that are minimal w.r.t. $\leq^V_{\mathcal{EL}}$. This obviously shows that the unrestricted unification type of $\mathcal{EL}$ cannot be zero. Since the restricted unification type is zero, the unrestricted type cannot be unitary or finitary by Theorem 8.

▶ **Theorem 22.** *The unrestricted unification type of $\mathcal{EL}$ is infinitary.*

4 The unrestricted unification types of ACUI, ACU, and AC

We consider a signature consisting of a binary function symbol f and a constant symbol 0. The theory ACU consists of identities stating that f is associative and commutative and that 0 is a unit for f. The theory ACUI additionally states that f is idempotent. The theory AC is obtained from ACU by removing the unit 0 from the signature and the identity involving it from the axiomatization. For elementary unification (where unification problems may only contain f, 0, and variables), ACU and ACUI are unitary w.r.t. the restricted instantiation preorder, whereas AC is finitary [19]. In the following, we show that all three theories are infinitary w.r.t. the unrestricted instantiation preorder. These results extend the one from [10] in two directions. First, [10] considers only ACUI, whereas here we also investigate ACU and AC. Second, [10] provides only the lower bound "at least infinitary" for ACUI, whereas here we determine the exact unification type (infinitary) for the three theories.

We start with proving that the unrestricted unification type of AC, ACU, and ACUI is at least infinitary. In contrast to $\mathcal{EL}$, where we were able to use Theorem 8 to deduce that it is not unitary or finitary w.r.t. the unrestricted instantiation preorder from the fact that it is of type zero for the restricted instantiation preorder, we must prove this directly for ACUI, ACU, and AC. Actually, following [10], we show a more general result that holds for regular theories satisfying certain *additional restrictions*.

Recall that a set of identities E is *regular* if $\mathrm{Var}(s) = \mathrm{Var}(t)$ holds for all identities $s \approx t$ in E. Regularity of the defining set of identities of an equational theory implies regularity of the whole theory, i.e., if E is a regular set of identities, then $\mathrm{Var}(s) = \mathrm{Var}(t)$ holds for all terms s, t satisfying $s \approx_E t$ [58]. The identities of AC, ACU, and ACUI are regular, and thus the following result from [10] applies to them.

▶ **Lemma 23** ([10])**.** *Let E be a regular theory and $\Gamma = \{s \approx_E^? t\}$ an E-unification problem s.t. $\mathrm{Var}(s) \cap \mathrm{Var}(t) = \emptyset$. Then the set $\mathcal{C}_E(\Gamma)$ consisting of all E-unifiers σ of Γ satisfying*

$$\forall y \in \mathrm{VRan}(\sigma).\exists x, x' \in V \text{ s.t. } x \neq x' \text{ and } y \in \mathrm{Var}(\sigma(x)) \cap \mathrm{Var}(\sigma(x'))$$

is complete w.r.t. $\leq_E^V$.

Together with Theorem 3, this lemma yields the following result.

▶ **Lemma 24.** *Let E be a regular theory and $\Gamma = \{s \approx_E^? t\}$ an E-unification problem s.t. $\mathrm{Var}(s) \cap \mathrm{Var}(t) = \emptyset$. If Γ has a minimal complete set of E-unifiers w.r.t. $\leq_E^V$, then it has one that is contained in $\mathcal{C}_E(\Gamma)$.*

Proof. Since Γ has a minimal complete set of E-unifiers, the set M of minimal elements of $[\mathcal{U}_E(\Gamma)]_E^V$ is complete. For $\mathcal{C}_E(\Gamma)$ to be complete, it must contain for every equivalence class in M at least one representative. Thus, by selecting for each class in M one of its representatives in $\mathcal{C}_E(\Gamma)$, we obtain a minimal complete set that is contained in $\mathcal{C}_E(\Gamma)$. ◀

We are now ready to formulate the "additional restrictions" mentioned above.

▶ **Definition 25.** *Given a regular equational theory E, we say that the E-unification problem Γ is* NUOF *if the following conditions are satisfied:*

- $\Gamma = \{s \approx_E^? t\}$ *for terms s, t satisfying* $\mathrm{Var}(s) \cap \mathrm{Var}(t) = \emptyset$*,*
- *there is a $\leq_E^V$-minimal unifier σ of Γ that uses fresh variables, i.e.,* $\mathrm{VRan}(\sigma) \setminus X \neq \emptyset$ *where* $X = \mathrm{Var}(s) \cup \mathrm{Var}(t)$*, and*
- *this unifier σ belongs to the set $\mathcal{C}_E(\Gamma)$ defined in the formulation of Lemma 23.*

Intuitively, NUOF stands for "not unitary or finitary," but we still need to show that this name is justified. Given a NUOF E-unification problem Γ, let $x_0 \in \mathrm{VRan}(\sigma) \setminus X$ and consider the following construction of substitutions:

$$\sigma_z := \sigma\tau_{x_0,z} \quad \text{where } z \in V \text{ and } \tau_{x_0,z} := \{x_0 \mapsto z, z \mapsto x_0\}.$$

One can show that, under certain conditions on z, such substitutions σ_z are $\leq_E^V$-minimal unifiers that are incomparable to each other w.r.t. $\leq_E^V$. By Theorem 3, this implies that Γ cannot have a finite minimal complete set of unifiers w.r.t. $\leq_E^V$ since there are infinitely many variables z satisfying these conditions.

▶ **Lemma 26** ([10])**.** *Let E be a regular equational theory E, Γ a NUOF E-unification problem, and X and σ_z for $z \in V$ be defined as above.*

- *For each $z \in V \setminus X$, σ_z is a minimal E-unifier of Γ w.r.t. $\leq_E^V$.*
- *For any two distinct variables $z, z' \in V \setminus (\mathrm{Dom}(\sigma) \cup \mathrm{VRan}(\sigma))$, σ_z and $\sigma_{z'}$ are incomparable w.r.t. $\leq_E^V$.*

Since $V \setminus (X \cup \mathrm{Dom}(\sigma) \cup \mathrm{VRan}(\sigma))$ is infinite, this lemma together with Theorem 3 implies that Γ cannot have a finite minimal complete set w.r.t. $\leq_E^V$.

▶ **Theorem 27.** *If E is a regular equational theory and Γ a NUOF E-unification problem, then Γ does not have a finite minimal complete set of E-unifiers w.r.t. the unrestricted instantiation preorder $\leq_E^V$.*

We are now ready to apply this result to ACUI.

▶ **Corollary 28** ([10]). *The unrestricted unification type of* ACUI *for elementary unification is at least infinitary.*

Proof. Since ACUI is regular, it is sufficient to show that there is an ACUI-unification problem Γ that is NUOF. According to Corollary 3.6 in [7], any most general unifier (w.r.t. restricted instantiation) of the ACUI-unification problem $\Gamma = \{f(x, f(y, z)) \approx^?_{\mathsf{ACUI}} f(u, v)\}$ must use a fresh variable. Let θ be such an mgu.

If Γ does not have a minimal complete set of ACUI-unifiers w.r.t. unrestricted instantiation, then we are done. Thus, assume that Γ has a minimal complete set $\mathcal{S}$ w.r.t. unrestricted instantiation. By Lemma 24, we can assume without loss of generality that $\mathcal{S} \subseteq \mathcal{C}_E(\Gamma)$, and by Theorem 3 we know that the elements of $\mathcal{S}$ are $\leq^V_{\mathsf{ACUI}}$-minimal. Since θ is an ACUI-unifier of Γ, there is a $\sigma \in \mathcal{S}$ such that $\sigma \leq^V_{\mathsf{ACUI}} \theta$. Since $\leq^V_{\mathsf{ACUI}} \subseteq \leq^{\mathrm{Var}(\Gamma)}_{\mathsf{ACUI}}$, this implies that σ is also an mgu of Γ w.r.t. the restricted instantiation preorder, and thus it introduces a fresh variable.

Consequently, we have shown that Γ is NUOF, and thus Theorem 27 is applicable, which proves the corollary. ◀

It remains to show that type zero is not possible for ACUI. This is actually an easy consequence of the results for $\mathcal{EL}$ we have shown in the previous section. In fact, for elementary unification in ACUI, we consider a term set that is a subset of the one for $\mathcal{EL}$ if we use the conjunction operator of $\mathcal{EL}$ as f and the top concept of $\mathcal{EL}$ as unit 0. On such terms (which we will call ACUI-terms in the following), the equational theories $\approx_{\mathcal{EL}}$ and $\approx_{\mathsf{ACUI}}$ coincide. If we consider substitutions σ, θ using only ACUI-terms, then $\sigma \leq^V_{\mathsf{ACUI}} \theta$ clearly implies $\sigma \leq^V_{\mathcal{EL}} \theta$. To show the other direction, assume that $\lambda\sigma \approx_{\mathcal{EL}} \theta$, but there is a variable z such that $\lambda(z)$ is not an ACUI term, i.e., contains a concept constant or an existential restriction. If $z \notin \mathrm{Dom}(\sigma)$ or $z \in \mathrm{VRan}(\sigma)$, then this leads to a contradiction. Otherwise, we can modify λ to λ' by setting $\lambda'(z) = z$ and still have $\lambda'\sigma \approx_{\mathcal{EL}} \theta$. This shows that we can assume without loss of generality that λ uses only ACUI-terms, which completes the proof that $\sigma \leq^V_{\mathcal{EL}} \theta$ implies $\sigma \leq^V_{\mathsf{ACUI}} \theta$. Thus, Lemma 21 entails that the set $\{\sigma \mid \sigma \leq^V_{\mathsf{ACUI}} \theta\}$ is finite up to $\sim^V_{\mathcal{EL}}$-equivalence, and thus also up to $\sim^V_{\mathsf{ACUI}}$-equivalence.

▶ **Theorem 29.** *The unrestricted unification type of* ACUI *for elementary unification is infinitary.*

Theorem 27 also applies to elementary unification in ACU. It is well-known (see, e.g., Section 10.3 in [16]) that a given elementary ACU-unification problem Γ can be translated into a system of homogeneous linear diophantine equations. W.r.t. the restricted instantiation preorder, the mgu of the problem Γ can then be obtained from the minimal generating set of the solutions of this system, also called its Hilbert base, where the number of variables used in the range of this mgu corresponds to the cardinality of the Hilbert base of the system. As pointed out in Example 2 of [40], the cardinality of the Hilbert base for equations of the form $ny = x_1 + 2x_2 + \ldots nx_n$ grows at least exponentially in n, and thus there are clearly instances where the mgu of the corresponding ACU-unification problem needs more than $n+1$ variables. Given this, one can now proceed as in the case of ACUI to show that Theorem 27 applies.

▶ **Corollary 30.** *The unrestricted unification type of* ACU *for elementary unification is at least infinitary.*

Proving that type infinitary is also the upper bound in these cases turns out to be more involved than for ACUI. In fact, our proof of the finiteness result used for $\mathcal{EL}$ and then adapted to ACUI depends on idempotency, and thus does not apply to ACU. Instead, we show that a set of the form $\{\sigma \mid \sigma \leq^{V}_{\mathsf{ACU}} \theta\}$ for a given unifier θ cannot contain an infinite decreasing chain w.r.t. $>^{V}_{\mathsf{ACU}}$, and thus θ must be above a minimal unifier. Our proof of this result in Section A uses the fact that analogues of Lemmas 18 and 19 also hold for ACU, but requires quite some additional effort to prove the non-existence of infinite decreasing chains.

▶ **Theorem 31.** *The unrestricted unification type of* ACU *for elementary unification is infinitary.*

The theory AC, which is obtained from ACU by removing the unit 0 from the signature and the identity containing it from the axiomatization, is finitary w.r.t. the restricted instantiation preorder. Proving that the unrestricted unification type of AC is infinitary turns out to be easier than for ACU.

Showing that the unrestricted unification type of AC cannot be unitary or finitary is very similar to our proofs for ACUI and ACU. In contrast to ACU, the theory AC is finitary rather than unitary in the restricted setting. A minimal complete set of unifiers is obtained by taking appropriate subsets of the Hilbert base and turning them into unifiers that have a variable in the range for each element of the subset (see, e.g., [16, 32]). Since the full Hilbert base is an appropriate subset, the minimal complete set of AC-unifiers for the unification problem Γ_n corresponding to the linear diophantine equation $ny = x_1 + 2x_2 + \ldots nx_n$ for a large enough natural number n must contain a unifier θ that introduces a fresh variable. As in the case of ACUI we can now show that, under the assumption that Γ has a minimal complete set $\mathcal{S}$ of AC-unifiers w.r.t. the unrestricted instantiation preorder, this set $\mathcal{S}$ contains a $\leq^{V}_{\mathsf{AC}}$-minimal AC-unifier σ satisfying $\sigma \leq^{V}_{\mathsf{AC}} \theta$, and thus also $\sigma \leq^{\mathrm{Var}(\Gamma_n)}_{\mathsf{AC}} \theta$. Since θ contains a variable in the range for each element of the Hilbert base and elements of the Hilbert base cannot be generated by a sum of other vectors, this implies that σ must also contain at least as many variables in its range as θ. Consequently, we have shown that Γ_n is NUOF, and thus Theorem 27 is applicable.

▶ **Corollary 32.** *The unrestricted unification type of* AC *for elementary unification is at least infinitary.*

To prove that the unrestricted unification type of AC cannot be zero, we employ the same approach as for unification in $\mathcal{EL}$. First note that Lemmas 18 and 19 also hold if we replace $\mathcal{EL}$ with AC. The main difference is that the crucial argument in the proof of Lemma 18 becomes simpler. There, we consider a setting where $\lambda\sigma(x) \approx_{\mathcal{EL}} \theta(x) = x$ holds for all variables $x \in \mathcal{X}$, and conclude that then $\sigma(x) = z^x_1 \sqcap \ldots \sqcap z^x_{n_x}$ for some variables $z^x_1, \ldots, z^x_{n_x}$. Thus is trivially the case for AC if we replace $\approx_{\mathcal{EL}}$ with $\approx_{\mathsf{AC}}$ and the binary conjunction operator $\sqcap$ of $\mathcal{EL}$ with the AC function symbol f. Then it is proved (by a somewhat involved counting argument) that $n_x = 1$ for all $x \in \mathcal{X}$ and $z^x_1 \neq z^y_1$ for distinct elements $x, y \in \mathcal{X}$. For AC, this is again trivially the case since there is no idempotency and the unit 0 (corresponding to $\top$ in $\mathcal{EL}$) is not available. The rest of the proof of Lemma 18 and the proof of Lemma 19 then work as in the case of $\mathcal{EL}$. Lemma 20 can be replaced by the following result.

▶ **Lemma 33.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathsf{AC}} \theta$. Then the following holds for all $x \in V$: if $\sigma(x)$ and $\theta(x)$ respectively contain m and n occurrences of (not necessarily distinct) variables, then $m \leq n$.*

Proof. Since we consider elementary unification in AC, all terms are built using the AC symbol f and variables. If $\lambda\sigma(x) \approx_{\mathsf{AC}} \theta(x)$, then every occurrence of a variable in $\sigma(x)$ is replaced by a term containing one or more variables. Consequently, $\lambda\sigma(x)$ is a term containing occurrences of at least as many variables as $\sigma(x)$. Since the theory cannot change the number of variable occurrences, but only rearrange parentheses and reorder variables, the same is true for $\theta(x)$. ◀

Together with the AC analogues of Lemmas 18 and 19, this lemma yields an AC analogue of Lemma 21, and thus the following theorem.

▶ **Theorem 34.** *The unrestricted unification type of* AC *for elementary unification is infinitary.*

Note that, due to the availability of the unit 0, an ACU-analogue of Lemma 33 does not hold. This is why we need the more involved argument in Section A.

5 Conclusion

In this paper, we have investigated the effect that the employed instantiation preorder has on the unification type of an equational theory. As a rule of thumb, one can extract from this investigation that nothing changes if unifiers in a minimal complete set w.r.t. the restricted instantiation preorder do not need fresh variables (Theorem 7), whereas the unification type switches from unitary or finitary to at least infinitary otherwise (Theorem 27), though the latter result was only shown for regular theories. We have employed Theorem 27 to prove that the unification type of the frequently used theories ACUI, ACU, and AC is at least infinitary w.r.t. the unrestricted instantiation preorder (Corollaries 28, 30, and 32), and were also able to show the matching upper bound, i.e., that these three theories are indeed infinitary rather than of type zero (Theorems 29, 31, and 34). We have clarified that the reason for such changes is not that the unrestricted instantiation preorder considers infinitely many variables, but that it does not leave infinitely many variables unobserved (Theorem 6). In particular, this shows that nothing changes compared to the restricted setting if one compares unifiers on finite supersets of the set of variables occurring in the unification problem instead. Rather surprisingly, we have found an example (unification in the description logic $\mathcal{EL}$) where the unification type improves from type zero to infinitary when going from the restricted to the unrestricted instantiation preorder (Theorem 22).

While the contributions of this work are primarily foundational, one can nevertheless ask whether the obtained results also have a practical impact. The answer to the question which instantiation preorder should be used in practice mainly depends on the application that employs unification, i.e., which variables are relevant in the overall procedure and for which of them need the instantiation relation between the unifiers hold. In case the restricted unification type is unitary/finitary, the restricted instantiation preorder should be used unless the application enforces comparison on all variables. For example, in Knuth-Bendix completion modulo equational theories [44, 37, 20], where unification is employed to test confluence by computing critical pairs, the restricted instantiation preorder is clearly sufficient, and thus should be used for theories that are unitary or finitary w.r.t. it. In case the restricted unification type is zero and the unrestricted one is infinitary, it may be useful to employ unrestricted unification if one can find a unification algorithm that enumerates a minimal complete set of unifiers w.r.t. the unrestricted preorder. From the restricted preorder point of view, such an algorithm would enumerate a complete set of unifiers that is non-minimal, but usually still much smaller than the set of all unifiers. With few exceptions, where the restricted and the unrestricted types coincide according to our Theorem 7 (e.g., the empty theory, commutativity C, or associativity A) there are no unification algorithms known for the unrestricted case since research on equational unification concentrated on the restricted case. For the cases where the unrestricted type is better (i.e., infinitary), it is thus a new challenge to find an algorithm enumerating a minimal complete set of unifiers. Our proof of type infinitary for $\mathcal{EL}$ does not directly yield a practical algorithm.

Regarding future foundational work, it is probably not very interesting to find further equational theories where the phenomena already exhibited in this paper also occur, unless one can prove a meta-theorem that exactly characterizes under what conditions these changes in the unification type happen for a given class of equational theories. A candidate class for such a meta-result are commutative/monoidal theories [17]: ACUI and ACU belong to this class and AC is obtained from an element of this class by removing the unit, but the equational theory corresponding to $\mathcal{EL}$ does not belong to it. A description logic of restricted unification type zero whose equational theory is commutative/monoidal is the DL $\mathcal{FL}_0$ [14, 5], but its unrestricted unification type is unknown. On the level of single equational theories, some interesting questions still remain. Are there actually equational theories of unification type zero w.r.t. the unrestricted instantiation preorder? If the answer is yes, which restricted unification types can such theories have? For instance, the theory AI of an associative and idempotent binary function symbol has restricted unification type zero [2, 48], but its unrestricted type is still unclear, and thus might be also zero. From the order-theoretic point of view, which only takes into account that the unrestricted preorder is a subset of the restricted one, it is also conceivable that there might be theories whose restricted type is infinitary whereas the unrestricted type is zero. But finding an example of an equational theory where this happens may be hindered by the fact that only very few natural theories of restricted unification type infinitary are known. The overview article [19] mentions only two: associativity (A) and both-sided distributivity (D). Now, A is not a candidate since our Theorem 7 implies that its unrestricted type is also infinitary. The theory D might be a candidate, but unification in it is rather complicated, though it is decidable [49].

Regarding related work, let us point out that recently there has also been other work on the impact that changing the preorder on substitutions has on the unification type [23, 24, 9, 33, 57], but the preorders investigated there differ from the (un)restricted instantiation preorder.

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A The unification type of ACU

In this section, we complete the proof of Theorem 31 by showing that type infinitary is also the upper bound for ACU. In contrast to the other equational theories considered in Sections 3 and 4, an analog of Lemma 21 does not hold for ACU. Instead, we show that a set of the form $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ for a given unifier θ cannot contain an infinite decreasing chain w.r.t. $>^V_{\mathsf{ACU}}$. We do this in two steps:

1. We recall a well-known characterization of the congruence classes of $\approx_{\mathsf{ACU}}$, and then employ this characterization to show an auxiliary result about $\leq^V_{\mathsf{ACU}}$.
2. Finally, we use this auxiliary result to provide the final argument that shows that type zero is not possible for ACU.

Let $\Sigma = \{f, 0\}$ be the signature introduced in Section 4 to define ACU. Given a term $t \in T(\Sigma, V)$, we denote by c^t_x the number of occurrences of the variable x in t. The congruence classes of $\approx_{\mathsf{ACU}}$ can be characterized in the following way (Lemma 10.3.1 of [16]).

▶ **Lemma 35** ([16]). *Let $s, t \in T(\Sigma, V)$. Then, $s \approx_{\mathsf{ACU}} t$ iff $c^t_x = c^s_x$ for all $x \in V$.*

This lemma tells us that, given a finite set $X_n = \{x_1, \dots, x_n\} \subseteq V$, the $\approx_{\mathsf{ACU}}$-equivalence class of a term $t \in T(\Sigma, X_n)$ can be uniquely represented by the vector $\vec{v}_n(t) = (c^t_{x_1}, \dots, c^t_{x_n})$. If $t \approx_{\mathsf{ACU}} 0$, then each component $c^t_{x_j}$ $(1 \leq j \leq n)$ of $\vec{v}_n(t)$ has value zero.

Based on this representation, a substitution σ with $\mathrm{Dom}(\sigma) \subseteq X_n$ and $\mathrm{VRan}(\sigma) \subseteq X_m$ $(n, m \geq 0)$ can be represented as an $n \times m$ matrix with rows $\vec{v}_m(\sigma(x_1)), \dots, \vec{v}_m(\sigma(x_n))$. Then, applying a substitution σ to a term t corresponds to multiplying the vector for t with the matrix for σ, i.e., the representation $\vec{v}_m(\sigma(t))$ of $\sigma(t)$ can be obtained by computing the following sums (for all $j, 1 \leq j \leq m$):

$$c^{\sigma(t)}_{x_j} \;=\; \sum_{k=1}^{n} c^t_{x_k} \cdot c^{\sigma(x_k)}_{x_j}.$$

For example, if $t = f(x_1, f(x_2, x_2))$ and $\sigma = \{x_1 \mapsto x_2, x_2 \mapsto f(x_1, x_1)\}$, then $\vec{v}_2(t) = (1, 2)$ and the matrix representing σ has the rows $(0, 1)$ and $(2, 0)$. The vector representing $\sigma(t) = f(x_2, f(f(x_1, x_1), f(x_1, x_1)))$ is $(4, 1)$. Note that $4 = 1 \cdot 0 + 2 \cdot 2$ and $1 = 1 \cdot 1 + 2 \cdot 0$.

Based on these representations, we next introduce some notions, and show a property about $\leq^V_{\mathsf{ACU}}$ that will be useful later on. For simplicity, given a substitution σ, we will write $(c^\sigma_{i1}, \dots, c^\sigma_{im})$ instead of $(c^{\sigma(x_i)}_{x_1}, \dots, c^{\sigma(x_i)}_{x_m})$ to refer to the vector $\vec{v}_m(\sigma(x_i))$. In addition, we denote by P_n the set of pairs $\{1, \dots, n\} \times \{1, \dots, n\}$.

▶ **Definition 36.** *Let $n \geq 0$ and $P \subseteq P_n$. Further, let σ_1 and σ_2 be substitutions whose domains and variable ranges are contained in X_n. We say that σ_1* is greater than σ_2 w.r.t. P *(denoted as $>_P$), if*

- *for all $(i, j) \in P$ and $k \in \{1, \dots, n\}$: $c^{\sigma_1}_{ij} \;>\; c^{\sigma_2}_{ik}$.*

Further, we say that σ_1 and σ_2 coincide on P, *if*

- *for all $(i, j) \in P$: $c^{\sigma_1}_{ij} \;= c^{\sigma_2}_{ij}$.*

It is not hard to show that $>_P$ is a transitive relation.

▶ **Lemma 37.** *Let $n \geq 0$ and $P \subseteq P_n$. The relation $>_P$ is transitive.*

Proof. Let σ_1, σ_2 and σ_3 be substitutions such that $\sigma_1 >_P \sigma_2$ and $\sigma_2 >_P \sigma_3$. To prove transitivity of $>_P$, we need to show that $\sigma_1 >_P \sigma_3$.

Let us take any pair $(i, j) \in P$. Since $\sigma_1 >_P \sigma_2$, we have that $c^{\sigma_1}_{ij} \;>\; c^{\sigma_2}_{ik}$ for all $k \in \{1, \dots, n\}$). In particular, this yields $c^{\sigma_1}_{ij} \;>\; c^{\sigma_2}_{ij}$. Furthermore, from $\sigma_2 >_P \sigma_3$, we know that $c^{\sigma_2}_{ij} \;>\; c^{\sigma_3}_{ik}$ for all $k \in \{1, \dots, n\}$). Consequently, it follows that $c^{\sigma_1}_{ij} \;>\; c^{\sigma_3}_{ik}$ for all $k \in \{1, \dots, n\}$. Thus, since $(i, j) \in P$ was arbitrarily chosen, we can conclude that $\sigma_1 >_P \sigma_3$. ◀

We continue by proving the following property about $\leq^V_{\mathsf{ACU}}$.

▶ **Lemma 38.** *Let $n \geq 0$ and $P \subseteq P_n$. In addition, let σ_1, σ_2, σ_3 be substitutions such that:*
- $\mathrm{Dom}(\theta_\ell) \cup \mathrm{VRan}(\theta_\ell) \subseteq X_n$ *for* $\ell \in \{1, 2, 3\}$,
- $\sigma_1 \leq^V_{\mathsf{ACU}} \sigma_2$,
- σ_1 *is greater than* σ_2 *w.r.t.* P, *and*
- σ_1 *and* σ_3 *coincide on* $P_n \setminus P$.

Then, $\sigma_3 \leq^V_{\mathsf{ACU}} \sigma_2$.

Proof. Since $\sigma_1 \leq^V_{\mathsf{ACU}} \sigma_2$, there is a substitution λ such that $\lambda\sigma_1 \approx_{\mathsf{ACU}} \sigma_2$. We can assume that the domain and variable range of λ satisfy the following properties:
- $\mathrm{Dom}(\lambda) \subseteq X_n$. For otherwise, if there is a variable z such that $z \in \mathrm{Dom}(\lambda)$ but $z \notin X_n$, then $\mathrm{Dom}(\sigma_1) \subseteq X_n$ implies that $\lambda\sigma_1(z) \neq z$. However, this would contradict $\lambda\sigma_1 \approx_{\mathsf{ACU}} \sigma_2$, since $\mathrm{Dom}(\sigma_2) \subseteq X_n$.
- $\mathrm{VRan}(\lambda) \subseteq X_n$. Note that if $\mathrm{VRan}(\lambda)$ contains a variable z not in X_n, then $\lambda(x_i)$ contains z for some $i \in \{1, \ldots, n\}$. But, since $z \notin \mathrm{VRan}(\sigma_2)$, the variable x_i cannot be in $\mathrm{VRan}(\sigma_1)$. Thus, we can safely remove the occurrences of z from $\lambda(x_i)$, i.e., $\lambda\sigma_1 \approx_{\mathsf{ACU}} \sigma_2$ remains true.

Hence, since $\mathrm{Dom}(\sigma_1) \cup \mathrm{Dom}(\sigma_3) \cup \mathrm{Dom}(\lambda) \subseteq X_n$, we have that $\mathrm{Dom}(\lambda\sigma_1) \subseteq X_n$ and $\mathrm{Dom}(\lambda\sigma_3) \subseteq X_n$. Therefore, to prove that $\sigma_3 \leq^V_{\mathsf{ACU}} \sigma_2$, it suffices to show that

$$\lambda\sigma_1(x_i) \;\approx_{\mathsf{ACU}}\; \lambda\sigma_3(x_i) \quad \text{for all } i \in \{1, \ldots, n\}. \tag{1}$$

Let $i \in \{1, \ldots, n\}$. Since $\mathrm{VRan}(\sigma_1) \cup \mathrm{VRan}(\lambda) \subseteq X_n$, the $\approx_{\mathsf{ACU}}$-equivalence class of $\lambda\sigma_1(x_i)$ can be represented with a vector of the form $(c^{\lambda\sigma_1}_{i1}, \ldots, c^{\lambda\sigma_1}_{in})$, where each value $c^{\lambda\sigma_1}_{ij}$ $(1 \leq j \leq n)$ is determined by the following expression:

$$c^{\lambda\sigma_1}_{ij} \;=\; \sum_{k=1}^{n} c^{\sigma_1}_{ik} \cdot c^{\lambda}_{kj}.$$

As $\lambda\sigma_1(x_i) \approx_{\mathsf{ACU}} \sigma_2(x_i)$, an application of Lemma 35 yields:

$$c^{\lambda\sigma_1}_{ij} \;=\; \sum_{k=1}^{n} c^{\sigma_1}_{ik} \cdot c^{\lambda}_{kj} \;=\; c^{\sigma_2}_{ij} \quad \text{for all } j \in \{1, \ldots, n\}. \tag{2}$$

Consider any pair $(i, k) \in P$. Since $\sigma_1 >_P \sigma_2$, this means that $c^{\sigma_1}_{ik} > c^{\sigma_2}_{ij}$ for all $j \in \{1, \ldots, n\}$. Hence, since $\mathrm{VRan}(\lambda) \subseteq X_n$, it must be that $\lambda(x_k) = 0$. For otherwise, $c^{\lambda}_{kj} > 0$ for some $j \in \{1, \ldots, n\}$ and the expression in (2) would not be true for such j. Therefore, (2) can be turned into:

$$c^{\lambda\sigma_1}_{ij} \;=\; \sum_{\substack{k=1\\(i,k)\in P_n\setminus P}}^{n} c^{\sigma_1}_{ik} \cdot c^{\lambda}_{kj} \;=\; c^{\sigma_2}_{ij} \quad \text{for all } j \in \{1, \ldots, n\}.$$

Regarding $\lambda\sigma_3$, we also have $\mathrm{VRan}(\sigma_3) \cup \mathrm{VRan}(\lambda) \subseteq X_n$. Hence, the $\approx_{\mathsf{ACU}}$-equivalence class of $\lambda\sigma_3(x_i)$ has a representation of the form $(c^{\lambda\sigma_3}_{i1}, \ldots, c^{\lambda\sigma_3}_{in})$, where $(1 \leq j \leq n)$:

$$c^{\lambda\sigma_3}_{ij} \;=\; \sum_{\substack{k=1\\(i,k)\in P_n\setminus P}}^{n} c^{\sigma_3}_{ik} \cdot c^{\lambda}_{kj}.$$

But then, since σ_1 and σ_3 coincide on $P_n \setminus P$, we have that:

$$c^{\lambda\sigma_3}_{ij} \;=\; \sum_{\substack{k=1\\(i,k)\in P_n\setminus P}}^{n} c^{\sigma_3}_{ik} \cdot c^{\lambda}_{kj} \;=\; \sum_{\substack{k=1\\(i,k)\in P_n\setminus P}}^{n} c^{\sigma_1}_{ik} \cdot c^{\lambda}_{kj} \;=\; c^{\lambda\sigma_1}_{ij}.$$

Hence, we have shown that $(c_{i1}^{\lambda\sigma_1}, \ldots, c_{in}^{\lambda\sigma_1}) = (c_{i1}^{\lambda\sigma_3}, \ldots, c_{in}^{\lambda\sigma_3})$. Then, an application of Lemma 35 yields that $\lambda\sigma_1(x_i) \approx_{\mathsf{ACU}} \lambda\sigma_3(x_i)$. Thus, since i was arbitrarily selected, we have proved the claim in (1). This concludes the proof of the lemma. ◀

We are now ready to move into the final part of our proof. The main idea is to show the following: if $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ for a given substitution θ contains an infinite decreasing chain w.r.t. $>^V_{\mathsf{ACU}}$, then such a chain contains three substitutions σ_1, σ_2 and σ_3 such that:

- $\sigma_3 >^V_{\mathsf{ACU}} \sigma_2 >^V_{\mathsf{ACU}} \sigma_1$, and
- σ_1, σ_2 and σ_3 satisfy the hypothesis of Lemma 38.

This implies that $\sigma_2 \sim^V_{\mathsf{ACU}} \sigma_3$, which contradicts the existence of such an infinite chain.

In order to apply Lemma 38, we first need to show that there is $n \geq 0$ such that the domain and variable range of each substitution in $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ is contained in X_n. This is a consequence of the analogs of Lemmas 18 and 19 for ACU.

▶ **Lemma 39.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathsf{ACU}} \theta$. Then there is a substitution σ' such that $\sigma \sim^V_{\mathsf{ACU}} \sigma'$ and* $\mathrm{Dom}(\sigma') \subseteq \mathrm{Dom}(\theta)$*.*

Proof. Let $\mathcal{X} = \mathrm{Dom}(\sigma) \setminus \mathrm{Dom}(\theta)$. Since $\sigma \leq^V_{\mathsf{ACU}} \theta$, there exists a substitution λ such that $\lambda\sigma \approx_{\mathsf{ACU}} \theta$. This means that $\lambda\sigma(x) \approx_{\mathsf{ACU}} \theta(x) = x$ for all $x \in \mathcal{X}$. Hence, an application of Lemma 35 (and consequences thereof described above) yields that:

$$c_x^{\lambda\sigma(x)} \;=\; \sum_{z \in V} c_z^{\sigma(x)} \cdot c_x^{\lambda(z)} \;=\; c_x^{\theta(x)} \;=\; 1.$$

This implies that $\sigma(x)$ must contain a single occurrence of some variable z_x such that $\lambda(z_x) = x$, and that $\lambda(y) = 0$ for any other variable y occurring in $\sigma(x)$. In addition, $z_x \neq z_y$ for different elements x, y of $\mathcal{X}$. Once we have this, the same counting argument employed in the proof of Lemma 18 can be applied here to conclude that $\sigma(x) = z_x$ for all $x \in \mathcal{X}$.

The rest of the proof uses the same arguments as our proof of Lemma 18. ◀

The following lemma states that Lemma 19 also holds if we replace $\mathcal{EL}$ with ACU. The proof is the same as for Lemma 19.

▶ **Lemma 40.** *Let σ, θ be substitutions such that $\sigma \leq^V_{\mathsf{ACU}} \theta$ and* $\mathrm{Dom}(\sigma) \subseteq \mathrm{Dom}(\theta)$*. Then* $\mathrm{VRan}(\sigma) \subseteq \mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$*.*

Thus, given a substitution θ and an enumeration $x_1, \ldots, x_n$ of $\mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$, we can assume (modulo $\sim^V_{\mathsf{ACU}}$) that any substitution in $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ is such that $\mathrm{Dom}(\sigma) \cup \mathrm{VRan}(\sigma) \subseteq X_n$. It remains to establish the contradiction mentioned above. We will do this with the help of the following result.

▶ **Lemma 41.** *Let θ be a substitution, $x_1, \ldots, x_n$ an enumeration of* $\mathrm{Dom}(\theta) \cup \mathrm{VRan}(\theta)$*, and $P \subset P_n$. Suppose that $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ contains an infinite decreasing chain $\mathfrak{s} = \sigma_1\sigma_2\ldots$ w.r.t. $>^V_{\mathsf{ACU}}$ such that $\sigma_{\ell+1} >_P \sigma_\ell$ for all $\ell \geq 1$. Then, there exists $(i,j) \in P_n \setminus P$ such that $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ contains an infinite decreasing chain $\tau_1\tau_2\cdots$ w.r.t. $>^V_{\mathsf{ACU}}$ satisfying $\tau_{p+1} >_{P\cup\{(i,j)\}} \tau_p$ for all $p \geq 1$.*

Proof. Suppose such an infinite decreasing chain $\mathfrak{s}$ exists. We claim that there is a pair (i,j) in $P_n \setminus P$ such that the sequence of values $c_{ij}^{\sigma_1}, c_{ij}^{\sigma_2}, \ldots$ is *not bounded*, i.e.,

$$\text{for all } m \in \mathbb{N}, \text{ there exists } \ell \geq 1 \text{ such that: } c_{ij}^{\sigma_\ell} > m. \tag{3}$$

If that were not the case, then there would be infinitely many substitutions in $\mathfrak{s}$ that coincide on $P_n \setminus P$. As a consequence, we could select substitutions $\sigma_{\ell_1}, \sigma_{\ell_2}$ and σ_{ℓ_3} in $\mathfrak{s}$ such that:

- $\sigma_{\ell_3} >^V_{\mathsf{ACU}} \sigma_{\ell_2} >^V_{\mathsf{ACU}} \sigma_{\ell_1}$,
- σ_{ℓ_1} is greater than σ_{ℓ_2} w.r.t. P, and
- σ_{ℓ_1} and σ_{ℓ_3} coincide on $P_n \setminus P$.

Hence, an application of Lemma 38 would yield $\sigma_{\ell_3} \leq^V_{\mathsf{ACU}} \sigma_{\ell_2}$, which contradicts $\sigma_{\ell_3} >^V_{\mathsf{ACU}} \sigma_{\ell_2}$.

Based on such a pair (i, j), we define the infinite chain $\tau_1\tau_2\cdots$ as follows:

1. $\tau_1 = \sigma_1$.
2. By (3), there is $p > 1$ such that $c^{\sigma_p}_{ij} > c^{\sigma_1}_{ik}$ for all $k \in \{1, \ldots, n\}$. We choose τ_2 as σ_p. The following arguments show that τ_1 and τ_2 satisfy the properties required of $\tau_1\tau_2\cdots$ w.r.t. $P \cup \{(i, j)\}$:
   - Since $\sigma_1 >^V_{\mathsf{ACU}} \sigma_p$, we have that $\tau_1 >^V_{\mathsf{ACU}} \tau_2$.
   - By selection of p, we know that $\sigma_p >_{\{(i,j)\}} \sigma_1$. Moreover, since $\sigma_{\ell+1} >_P \sigma_\ell$ for all $\ell \geq 1$, transitivity of $>_P$ yields that $\sigma_p >_P \sigma_1$. Consequently, we can conclude that τ_2 is greater than τ_1 w.r.t. $P \cup \{(i, j)\}$.
3. Once we fix σ_p, (3) also yields $q > p$ such that $c^{\sigma_q}_{ij} > c^{\sigma_p}_{ik}$ for all $k, 1 \leq k \leq n$. We select τ_3 as σ_q. The same arguments yield that τ_2 and τ_3 are as required.
4. By repeating (*ad infinitum*) the described selection process, we can extract from $\mathfrak{s}$ an appropriate remaining sequence of substitutions $\tau_4\tau_5\cdots$.

The described process ensures that each selected substitution τ_ℓ ($\ell \geq 1$) belongs to $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$. Thus, $\tau_1\tau_2\cdots$ is an infinite decreasing chain satisfying the claim of the lemma. ◀

Finally, by using the previous lemma, we can show the main result of this section.

▶ **Lemma 42.** *For a given substitution θ, the set $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ does not contain an infinite decreasing chain w.r.t. $>^V_{\mathsf{ACU}}$.*

Proof. Suppose $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ contains an infinite decreasing chain $\mathfrak{s}$ w.r.t. $>^V_{\mathsf{ACU}}$. By Definition 36, we have $\eta >_\emptyset \eta'$ for all $\eta, \eta' \in \{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$. This means that $\mathfrak{s}$ and $P = \emptyset$ satisfy the hypothesis of Lemma 41. Hence, there is $(i, j) \in P_n$ such that $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ contains an infinite decreasing chain $\tau_1\tau_2\cdots$ satisfying $\tau_{p+1} >_{\{(i,j)\}} \tau_p$ for all $p \geq 1$. This new chain now satisfies the hypothesis of Lemma 41 w.r.t. $P = \{(i, j)\}$. Thus, by a sequence of $n^2 - 1$ further applications of Lemma 41, we conclude that $\{\sigma \mid \sigma \leq^V_{\mathsf{ACU}} \theta\}$ contains an infinite decreasing chain $\sigma_1\sigma_2\cdots$ such that $\sigma_{p+1} >_{P_n} \sigma_p$ for all $p \geq 1$. Finally, it is not hard to verify that this last chain must contain three substitutions $\sigma_{\ell_1}, \sigma_{\ell_2}$ and σ_{ℓ_3} such that

- $\sigma_{\ell_3} >^V_{\mathsf{ACU}} \sigma_{\ell_2} >^V_{\mathsf{ACU}} \sigma_{\ell_1}$,
- σ_{ℓ_1} is greater than σ_{ℓ_2} w.r.t. $P = P_n$, and
- σ_{ℓ_1} and σ_{ℓ_3} coincide on $P_n \setminus P = \emptyset$.

Hence, the application of Lemma 38 yields that $\sigma_{\ell_3} \leq^V_{\mathsf{ACU}} \sigma_{\ell_2}$, which contradicts the existence of $\sigma_1\sigma_2\cdots$ since $\sigma_{\ell_3} >^V_{\mathsf{ACU}} \sigma_{\ell_2}$. Thus, we have derived a contradiction from our initial assumption. This concludes the proof of the lemma. ◀

Lemma 42 shows that the unrestricted unification type of ACU cannot be zero. In fact, this lemma implies that every unifier θ is above a minimal unifier, and thus the set of minimal unifiers is complete. By Theorem 3, we can conclude that the unification type cannot be zero, which completes the proof of Theorem 31.