Gärdenfors's Supplementary Postulates for Partial Product Contractions

Franz Baader[1] [0000−0002−4049−221X] and Renata Wassermann[2] [0000−0001−8065−1433]

[1] TU Dresden and Center for Scalable Data Analytics and AI (ScaDS.AI), Germany
[2] University of São Paulo and Center for AI (C4AI), Brazil

Abstract. In the area of belief change, contraction operations are used to modify a given belief set or belief base such that certain unwanted consequences no longer follow. In previous work we have introduced a framework for constructing contraction operations that generalizes the well-known partial meet contraction approach, called partial product contractions (PPCs). The main idea was to replace the remainders employed by partial meet contractions with optimal repairs, which were first considered in ontology engineering. We were able to characterize PPCs with variants of well-known rationality postulates, and provided a large number of concrete instances of the general framework. In the present work, we start to investigate whether the rather weak conditions imposed by our framework are sufficient to generalize further classical results from belief change to this setting. To this purpose, we consider Gärdenfors's supplementary postulates for belief contractions. We are able to show that, under two reasonable additional conditions, PPCs induced by maximizingly and transitively relational selection functions indeed satisfy these postulates, similarly to the classical case.

1 Introduction

Getting rid of knowledge that has undesired consequences has been investigated in the area of belief change under the name of *contraction* [1,13] and in ontology engineering under the name of *repair* [17,28,8,29]. In their seminal paper [1], Alchourrón, Gärdenfors, and Makinson introduce *partial meet contractions* and characterize them using certain *rationality postulates*, i.e., properties for which they argue that a reasonable contraction operation should satisfy them. Basically, this approach works on *belief sets*, i.e., deductively closed sets of formulas, considers all maximal subsets of a given belief set that do not contain the undesired consequence, called *remainders*, selects a subset of the set of remainders using a *selection function*, and then returns the *meet* (i.e., intersection) of the selected remainders as contraction. From a practical point of view, a disadvantage of working on belief sets is that the belief set returned by such a contraction operation may not be representable as the deductive closure of a finite *belief base*, even if the original belief set was finitely representable. To overcome this problem, Nebel [23] and Hansson [13] apply the partial meet approach directly to belief bases, i.e., remainders are now maximal subset of the base that do not imply the undesired consequence, and the contraction is again the intersection of a non-empty collection of these remainders. In ontology engineering, such remainders are called *optimal classical repairs* [8]. Both partial meet base contractions and optimal classical repairs have have been criticized for the fact that they are syntax-dependent and may remove too many consequences [15,8,27,22,2]. In the context of knowledge bases (KBs) expressed using certain Description Logics (DLs), *optimal repairs* yield a syntax-independent repair approach that does not lose consequences unnecessarily [5,6,7,19,4]

The original motivation for the new contraction approach presented in [9] was to leverage these advances on the side of ontology engineering to obtain a contraction approach that combines the advantages of belief set and belief base contractions without sharing their disadvantages. The main idea underlying this approach was to use, in the partial meet contraction approach, optimal repairs as remainders. However, instead of introducing this new approach in the specific instance of certain DL KBs, we consider in [9] a very general setup, where knowledge bases are not necessarily sets of formulas, the meet operation is replaced by an abstract product operation on KBs, and the repair goal may be different from non-entailment of an undesired consequence (such as forgetting [20,21,10,18] of certain parts of the signature). We were able to characterize the *partial product contractions* (PPCs) obtained this way using certain well-known rationality postulates and to describe a large variety of different instances of the general framework (see [9] and Section 2 for more details).

In the present work, we start to investigate whether the rather weak conditions imposed by our framework are sufficient to show that further classical results from belief change can be generalized to this setting. To this purpose, we consider Gärdenfors's *supplementary postulates* for belief contractions (*conjunctive overlap* and *conjunctive inclusion*), which were already investigated in the original AGM paper [1], but also considered in other settings (e.g., [15,16,11,26]). Although the supplementary postulates are typically studied in connection to belief sets, Ribeiro [24] presented a first characterization for belief bases. Basically, these postulates characterize a restriction of partial meet contractions where the selection functions are defined using a certain transitive relation on remainders. Under two additional assumptions (conditions (1) and (2)) we are able to show that Gärdenfors's supplementary postulates indeed hold for partial product contractions if we require the transitive relation defining the selection function to be also *maximizing*, a condition that has been introduced before [14,16]. We also give DL-based examples that show that the additional conditions are necessary.

2 The general framework and $\mathcal{EL}$ concepts as instance

We briefly introduce the general framework and illustrate it using the instance of $\mathcal{EL}$ concepts since this instance (in different variants) will also be employed in the (counter)examples in the rest of the paper (see [9] for more details).

Partial product contractions We assume that we are given a set of *knowledge bases* (KBs) and a reflexive and transitive *entailment relation* between knowledge bases. We write KBs as $\mathcal{K}$, possibly primed ($\mathcal{K}'$) or with an index ($\mathcal{K}_i$), and entailment as $\models$, i.e., $\mathcal{K} \models \mathcal{K}'$ means that $\mathcal{K}$ *entails* $\mathcal{K}'$, or equivalently that $\mathcal{K}'$ is *entailed by* $\mathcal{K}$. We call two KBs $\mathcal{K}, \mathcal{K}'$ *equivalent* (write $\mathcal{K} \equiv \mathcal{K}'$) if they entail each other. We say that $\mathcal{K}$ *strictly entails* $\mathcal{K}'$ ($\mathcal{K} \models_s \mathcal{K}'$) if $\mathcal{K} \models \mathcal{K}'$, but $\mathcal{K}' \not\models \mathcal{K}$.

We make no assumptions on the inner structure of knowledge bases, but we assume that we have operations *sum* ($\oplus$) and *product* ($\otimes$) available that are akin to conjunction and disjunction. For each finite, non-empty set of KBs $\mathfrak{K}$:

- $\oplus\mathfrak{K} \models \mathcal{K}$ for all $\mathcal{K} \in \mathfrak{K}$ and $\oplus\mathfrak{K}$ is the least KB satisfying this property, i.e., if $\mathcal{K}'$ is a KB satisfying $\mathcal{K}' \models \mathcal{K}$ for all $\mathcal{K} \in \mathfrak{K}$, then $\mathcal{K}' \models \oplus\mathfrak{K}$;
- $\mathcal{K} \models \otimes\mathfrak{K}$ for all $\mathcal{K} \in \mathfrak{K}$ and $\otimes\mathfrak{K}$ is the greatest KB satisfying this property, i.e., if $\mathcal{K}'$ is a KB satisfying $\mathcal{K} \models \mathcal{K}'$ for all $\mathcal{K} \in \mathfrak{K}$, then $\otimes\mathfrak{K} \models \mathcal{K}'$.

The goal of the contraction operation to be defined is to "repair" a certain defect of the given knowledge base, but this defect is not restricted to entailment of some unwanted consequence. Formally, we assume that we have additional syntactic entities called repair requests: given a KB $\mathcal{K}$, a *repair request* α determines a set of KBs $\mathrm{Rep}(\mathcal{K}, \alpha)$ such that $\mathcal{K} \models \mathcal{K}'$ holds for every element $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$, and $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ and $\mathcal{K}' \models \mathcal{K}''$ imply $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$. We call the elements of $\mathrm{Rep}(\mathcal{K}, \alpha)$ *repairs* of $\mathcal{K}$ for α. Two repair requests α and α' are *equivalent w.r.t.* $\mathcal{K}$ ($\alpha \equiv_{\mathcal{K}} \alpha'$) if $\mathrm{Rep}(\mathcal{K}, \alpha) = \mathrm{Rep}(\mathcal{K}, \alpha')$.

Note that this notion of repairs generalizes the classical no-entailment condition for contractions, where a repair request α is also a KB and contractions are required (by the *success* postulate) not to entail α. In this restricted setting, the set of repairs is defined as $\mathrm{Rep}(\mathcal{K}, \alpha) = \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}' \not\models \alpha\}$.

Finally, we assume the *optimal repair property*, which says that, for every pair $\mathcal{K}, \alpha$ consisting of a KB and a repair request (called a *repair problem*), there exists a finite set of KBs $\mathrm{Orep}(\mathcal{K}, \alpha)$ satisfying

- $\mathrm{Orep}(\mathcal{K}, \alpha) \subseteq \mathrm{Rep}(\mathcal{K}, \alpha)$ (repair property),
- every element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K}, \alpha)$ is *optimal*, i.e., there is no $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ such that $\mathcal{K}'' \models_s \mathcal{K}'$ (optimality),
- $\mathrm{Orep}(\mathcal{K}, \alpha)$ *covers* all repairs, i.e., for every $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ there is $\mathcal{K}' \in \mathrm{Orep}(\mathcal{K}, \alpha)$ such that $\mathcal{K}' \models \mathcal{K}''$ (coverage).

Note that $\mathrm{Orep}(\mathcal{K}, \alpha)$ is unique up to equivalence and that coverage of $\mathrm{Orep}(\mathcal{K}, \alpha)$ implies that this set is empty iff $\mathrm{Rep}(\mathcal{K}, \alpha) = \emptyset$. As mentioned before, optimal repairs will play the role of remainders, and thus this equivalence says that there are remainders iff the repair goal can be achieved.

Given a set of KBs $\mathcal{K}$, a set of repair requests α inducing repair sets $\mathrm{Rep}(\mathcal{K}, \alpha)$, and a reflexive and transitive binary relation $\models$ between KBs, we call $\models$ *partial product contraction (PPC) enabling* if all the properties introduced above are satisfied.

Let $\mathcal{K}$ be a KB and $\mathrm{Orep}(\mathcal{K}, \alpha)$ for each repair request α the corresponding set of optimal repairs, which covers all repairs of $\mathcal{K}$ for α. A *selection function* γ

for $\mathcal{K}$ takes such sets of optimal repairs as input and must satisfy the following properties, for each repair request α:

- If $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$, then $\emptyset \neq \gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \subseteq \mathrm{Orep}(\mathcal{K},\alpha)$.
- If $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$, then $\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) = \{\mathcal{K}\}$.
- If $\mathrm{Orep}(\mathcal{K},\alpha)$ and $\mathrm{Orep}(\mathcal{K},\alpha')$ are equal up to equivalence, then so are $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ and $\gamma(\mathrm{Orep}(\mathcal{K},\alpha'))$.

Each selection function γ induces a *PPC operation* ctr_γ as follows:

$$\mathrm{ctr}_\gamma(\mathcal{K},\alpha) := \otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha)).$$

It was shown in [9] that, among all operations receiving as input a KB and a repair request and returning as output a KB, the PPC operations are exactly those operations that satisfy the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance*.

$\mathcal{EL}$ concepts as KBs $\mathcal{EL}$ concepts are built inductively, starting with concept names A from a set N_C of such names, and using the concept constructors $\top$ (top concept), $C \sqcap D$ (conjunction), and $\exists r.C$ (existential restriction), where C, D are $\mathcal{EL}$ concepts and r belongs to a set N_R of role names. A general concept inclusion (GCI) of $\mathcal{EL}$ is of the form $C \sqsubseteq D$ for $\mathcal{EL}$ concepts C, D, and an $\mathcal{EL}$ TBox is a finite set of such GCIs. Given an $\mathcal{EL}$ concept C, its signature $\mathrm{Sig}(C)$ consists of the concept and role names occurring in C.

The semantics of $\mathcal{EL}$ is defined in a model-theoretic way, using the notion of an *interpretation* $\mathcal{I}$, which is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the domain $\Delta^\mathcal{I}$ is a non-empty set and the interpretation function $\cdot^\mathcal{I}$ maps each concept name $A \in N_C$ to $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and each role name $r \in N_R$ to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The interpretation of an $\mathcal{EL}$ concept is defined inductively as follows: $\top^\mathcal{I} := \Delta^\mathcal{I}$, $(C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}$, and $(\exists r.C)^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid \exists e \in \Delta^\mathcal{I}$ such that $(d,e) \in r^\mathcal{I}$ and $e \in C^\mathcal{I}\}$. A model $\mathcal{I}$ of the $\mathcal{EL}$ TBox $\mathcal{T}$ is an interpretation that satisfies all its GCIs, i.e., $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for all $C \sqsubseteq D \in \mathcal{T}$. Given $\mathcal{EL}$ concepts C, D and an $\mathcal{EL}$ TBox $\mathcal{T}$, we say that C is *subsumed by D w.r.t. $\mathcal{T}$* ($C \sqsubseteq^\mathcal{T} D$) if $C^\mathcal{I} \subseteq D^\mathcal{I}$ in all models $\mathcal{I}$ of $\mathcal{T}$. The $\mathcal{EL}$ concepts C, D are *equivalent w.r.t. $\mathcal{T}$* (written $C \equiv^\mathcal{T} D$) if $C \sqsubseteq^\mathcal{T} D$ and $D \sqsubseteq^\mathcal{T} C$. The $\mathcal{EL}$ TBox $\mathcal{T}$ is *cycle-restricted* if there is no $\mathcal{EL}$ concept C and $m \geq 1$ (not necessarily distinct) role names $r_1, \ldots, r_m$ such that $C \sqsubseteq^\mathcal{T} \exists r_1. \cdots \exists r_m.C$.

For a given cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$, the following *instance* of the general framework was introduced in [2,9]: *KBs* are $\mathcal{EL}$ concepts, *entailment* is subsumption (i.e., $C \models D$ iff $C \sqsubseteq^\mathcal{T} D$), and *repair requests* are $\mathcal{EL}$ concepts with associated repair sets $\mathrm{Rep}^\mathcal{T}_{\mathrm{ent}}(C,D) := \{C' \mid C \sqsubseteq^\mathcal{T} C', C' \not\sqsubseteq^\mathcal{T} D\}$. As shown in [9], the entailment relation $\sqsubseteq^\mathcal{T}$ is PPC enabling for this repair setting. In fact, the *sum* is conjunction of concepts, and the *product* is the least common subsumer (lcs) w.r.t. the TBox $\mathcal{T}$, which exists according to [30] since the TBox is cycle-restricted. Cycle-restrictedness also ensures that the optimal repair property is satisfied. In [2], this was shown using results for optimal ABox repairs from [6]. A more generic argument, which can also be used for repair goals other than non-entailment follows from the following lemma (see Proposition 3 in [3]).

Lemma 1 ([3]). *If $\mathcal{T}$ is a cycle-restricted $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept, then* $\mathrm{Subs}^{\mathcal{T}}(C) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C'\}$ *is finite up to equivalence.*

Since repairs of C are subsumers of C, the set of repairs is finite up to equivalence, and the optimal repairs are the ones that are the minimal elements w.r.t. subsumption of this finite set.

$\mathcal{EL}$ concept contraction was considered in [25] for the empty TBox and in [2] for cycle-restricted TBoxes. For the sake of simplicity, we use (variants of) $\mathcal{EL}$ concept contraction in our examples, but note that it is a special case of the practically more relevant ABox contractions considered as an instance of the general framework in [9].

3 The supplementary postulates

As mentioned above, in [9] it was shown that, for PPC enabling entailments, a contraction operation can be obtained by the partial product contraction approach iff it satisfies the rationality postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance*. We now investigate under what additional conditions Gärdenfors's supplementary postulates *conjunctive overlap* and *conjunctive inclusion* for belief contractions [12,1] also hold. In the setting considered in [12,1], KBs are set of formulas from a logic with conjunction, repair requests are formulas, and the repair goal is non-entailment of the repair request. The supplementary postulates put contraction w.r.t. $\alpha \wedge \beta$ in relation to contraction w.r.t. α and contraction w.r.t. β.

To generalize this setting to our framework, we first assume that repair requests are KBs and that the repair goal is non-entailment of the repair request. If we then use sum as stand-in for conjunction, we obtain the following identity, due to the fact that a KB entails a sum iff it entails all of it summands (see Lemma 3 in [9]):

$$\begin{aligned}\mathrm{Rep}(\mathcal{K}, \alpha \oplus \beta) &= \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}' \wedge \mathcal{K}' \not\models \alpha \oplus \beta\} \\ &= \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}' \wedge (\mathcal{K}' \not\models \alpha \vee \mathcal{K}' \not\models \beta)\} = \mathrm{Rep}(\mathcal{K}, \alpha) \cup \mathrm{Rep}(\mathcal{K}, \beta).\end{aligned}$$

In our general framework, where repair requests need not be KBs and the repair goals may be different from non-entailment, we now make the additional assumption that there is an operation $\boxplus$ on repair requests such that the following identity holds:

$$\mathrm{Rep}(\mathcal{K}, \alpha \boxplus \beta) = \mathrm{Rep}(\mathcal{K}, \alpha) \cup \mathrm{Rep}(\mathcal{K}, \beta). \tag{1}$$

Since union of sets is associative, commutative, and idempotent, the operation $\boxplus$ also satisfies these properties, up to equivalence $\equiv_{\mathcal{K}}$ of repair requests.

The following is an example of a repair setting where repair requests are not KBs, but (1) nevertheless holds.

Example 1. We consider forgetting for $\mathcal{EL}$ concepts (for simplicity in the case $\mathcal{T} = \emptyset$), where repair requests are finite sets of concept and role names, and repairs are defined as follows:

$$\mathrm{Rep}_{\mathrm{for}}(C,\alpha) = \{C' \mid C \sqsubseteq^{\emptyset} C' \text{ and } \alpha \not\subseteq \mathrm{Sig}(C')\}.^{3}$$

Since it is well-known that, for $\mathcal{EL}$ concepts, $C' \sqsubseteq^{\emptyset} C''$ implies $\mathrm{Sig}(C') \supseteq \mathrm{Sig}(C'')$, such repair sets are closed under entailment. Since the empty TBox is cycle-restricted, the entailment relation $\sqsubseteq^{\emptyset}$ is known to have product (least common subsumer) and sum (conjunction). The optimal repair property is an easy consequence of Lemma 1.

If we define $\alpha \boxplus \beta := \alpha \cup \beta$, then (1) holds: for all concepts C' satisfying $C \sqsubseteq^{\emptyset} C'$ we have $C' \in \mathrm{Rep}_{\mathrm{for}}(C, \alpha \boxplus \beta)$ iff $\alpha \cup \beta \not\subseteq \mathrm{Sig}(C')$ iff $\alpha \not\subseteq \mathrm{Sig}(C')$ or $\beta \not\subseteq \mathrm{Sig}(C')$ iff $C' \in \mathrm{Rep}_{\mathrm{for}}(C,\alpha) \cup \mathrm{Rep}_{\mathrm{for}}(C,\beta)$.

In general, however, (1) need not hold.

Example 2. Consider $\mathcal{EL}$ concepts as KBs, subsumption w.r.t. the empty TBox as entailment, and finite sets of concept and role names as repair requests, and define $\mathrm{Rep}(C,\alpha) = \{C' \mid C \sqsubseteq^{\emptyset} C' \text{ and } \mathrm{Sig}(C') \subseteq \alpha\}$. Closure under entailment holds for the same reason as in the previous example.

We show that the repair sets defined here cannot satisfy (1) for any operation $\boxplus$ on repair requests. In fact, to satisfy the inclusion from right to left of (1) the sum operation on repair requests must satisfy $\alpha \cup \beta \subseteq \alpha \boxplus \beta$. But then one can easily generate an example where the other inclusion is not satisfied: $A \sqcap B \in \mathrm{Rep}(A \sqcap B, \{A\} \boxplus \{B\})$, but it belongs neither to $\mathrm{Rep}(A \sqcap B, \{A\})$ nor to $\mathrm{Rep}(A \sqcap B, \{B\})$.

In the following, we assume that the entailment relation is PPC enabling w.r.t. the repair sets at hand, and that contractions are built using the PPC approach. In addition, we assume that the identity (1) holds. The following lemma is an easy consequence of this identity.

Lemma 2. *Up to equivalence, the following inclusion holds:* $\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta) \subseteq \mathrm{Orep}(\mathcal{K},\alpha) \cup \mathrm{Orep}(\mathcal{K},\beta)$.

Proof. Assume that $\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta)$. Then $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K},\alpha) \cup \mathrm{Rep}(\mathcal{K},\beta)$ and there is no $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha) \cup \mathrm{Rep}(\mathcal{K},\beta)$ such that $\mathcal{K}' \models_s \mathcal{K}''$. If $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K},\alpha)$, then $\mathcal{K}''$ is an optimal repair of $\mathcal{K}$ for α since there is no $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$ such that $\mathcal{K}' \models_s \mathcal{K}''$. Consequently, coverage of $\mathrm{Orep}(\mathcal{K},\alpha)$ implies that $\mathcal{K}''$ is equivalent to an element of $\mathrm{Orep}(\mathcal{K},\alpha)$. If $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\beta)$, then the fact that $\mathcal{K}''$ is equivalent to an element of $\mathrm{Orep}(\mathcal{K},\beta)$ can by shown in the same way ☐

The proof of this lemma actually shows the following stronger result.

Lemma 3. *Up to equivalence, the following holds:* $\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta) \cap \mathrm{Rep}(\mathcal{K},\alpha) \subseteq \mathrm{Orep}(\mathcal{K},\alpha)$ *and* $\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta) \cap \mathrm{Rep}(\mathcal{K},\beta) \subseteq \mathrm{Orep}(\mathcal{K},\beta)$.

[3] In [9], we used the condition $\alpha \cap \mathrm{Sig}(C') = \emptyset$, which corresponds to a package repair setting, whereas the condition introduced here corresponds to a choice repair setting.

The inclusion in the other direction in Lemma 2 does not hold in general, even if we use non-entailment as repair goal.

Example 3. As entailment we consider subsumption $\sqsubseteq^{\mathcal{T}}$ between $\mathcal{EL}$ concepts w.r.t. the following $\mathcal{EL}$ TBox $\mathcal{T} := \{\exists r.A \sqsubseteq P, \exists r.B \sqsubseteq P\}$, and repairs are defined by non-entailment of the repair request, which is an $\mathcal{EL}$ concept, i.e. $\mathrm{Rep}(C,\alpha) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C' \wedge C' \not\sqsubseteq^{\mathcal{T}} \alpha\}$. Since the TBox $\mathcal{T}$ is cycle-restricted, the entailment relation $\models := \sqsubseteq^{\mathcal{T}}$ is known to be PPC-enabling w.r.t. this definition of repair sets [9]. As operation $\boxplus$ on repair requests, we use $\oplus$, which here is conjunction.

Let $C := \exists r.(A \sqcap B)$, $\alpha := P$, and $\beta := \exists r.A \sqcap \exists r.B$. Then, we obtain the following optimal repair sets (up to equivalence):

$$\begin{aligned}&\mathrm{Orep}(C,\alpha) = \{\exists r.\top\}, \quad \mathrm{Orep}(C,\beta) = \{\exists r.A, \exists r.B\},\\ &\mathrm{Orep}(C,\alpha \boxplus \beta) = \mathrm{Orep}(C,\alpha \sqcap \beta) = \{\exists r.A, \exists r.B\}.\end{aligned}$$

Thus, the union of $\mathrm{Orep}(C,\alpha)$ and $\mathrm{Orep}(C,\beta)$ contains $\exists r.\top$, whereas $\exists r.\top$ does not belong to $\mathrm{Orep}(C,\alpha \boxplus \beta)$ since it is not optimal in the presence of the other two existential restrictions.

3.1 Conjunctive overlap

The supplementary postulate *conjunctive overlap* can be formulated in our general setting as follows:

– $\mathrm{ctr}(\mathcal{K},\alpha \boxplus \beta) \models \mathrm{ctr}(\mathcal{K},\alpha) \otimes \mathrm{ctr}(\mathcal{K},\beta)$ (conjunctive overlap)

In the classical setting of [1], KBs are deductively closed sets of formulas (and thus entailment is the superset relation and product is intersection), repair requests are formulas, and $\boxplus$ is conjunction. Thus, *conjunctive overlap* is formulated as follows: $\mathrm{ctr}(\mathcal{K},\alpha \wedge \beta) \supseteq \mathrm{ctr}(\mathcal{K},\alpha) \cap \mathrm{ctr}(\mathcal{K},\beta)$.

As in this classical case, to ensure that *conjunctive overlap* is satisfied, we must make additional assumptions on the selection function (see, e.g., [16]). We say that the selection function γ is *transitively relational* if there is a transitive relation $\trianglelefteq$ on $\mathrm{Con}(\mathcal{K}) := \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}'\}$ such that equivalent KBs are in this relation and

$$\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) = \{\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K},\alpha) \mid \mathcal{K}' \trianglelefteq \mathcal{K}'' \text{ for all } \mathcal{K}' \in \mathrm{Orep}(\mathcal{K},\alpha)\},$$

whenever $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$. Note that the conditions imposed on selection functions in [9] require that non-emptiness of the optimal repair set implies that $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ is non-empty as well. This is an additional condition that a transitive relation must satisfy to be able to induce a selection function. Invariance under equivalence is taken care of by our requirement that equivalent KBs are in the relation $\trianglelefteq$. For the sake of simplicity, we subsume this property under transitivity, i.e., whenever we say in the following that $\trianglelefteq$ is transitive, this also means that equivalent KBs are in the relation $\trianglelefteq$.

In addition, we assume that $\trianglelefteq$ is *weakly maximizing*, i.e., $\mathcal{K}' \models \mathcal{K}''$ implies $\mathcal{K}' \trianglerighteq \mathcal{K}''$. In the literature [16], the stronger property of being *maximizing* has been considered in this context. However, for showing the postulate *conjunctive overlap*, the weak version turns out to be sufficient. The relation $\trianglelefteq$ is *maximizing* if $\mathcal{K}' \models_s \mathcal{K}''$ implies $\mathcal{K}' \rhd \mathcal{K}''$ (i.e., $\mathcal{K}' \trianglerighteq \mathcal{K}''$, but $\mathcal{K}'' \ntrianglerighteq \mathcal{K}'$).

Lemma 4. *Let $\trianglelefteq$ be a transitive relation on KBs such that equivalent KBs are in this relation. If $\trianglelefteq$ is maximizing, then it is also weakly maximizing.*

Proof. Assume that $\mathcal{K}' \models \mathcal{K}''$. If $\mathcal{K}' \equiv \mathcal{K}''$, then $\mathcal{K}' \trianglerighteq \mathcal{K}''$ due to the assumption that equivalent KBs are in the relation $\trianglelefteq$. Otherwise, $\mathcal{K}' \models_s \mathcal{K}''$, and thus maximizing yields $\mathcal{K}' \rhd \mathcal{K}''$, which implies $\mathcal{K}' \trianglerighteq \mathcal{K}''$. □

We call a selection function *(weakly) maximizingly and transitively relational* if it is transitively relational w.r.t. a (weakly) maximizing relation $\trianglelefteq$. Due to the properties of the product, the postulate *conjunctive overlap* is an easy consequence of the following lemma.

Lemma 5. *Let γ by a weakly maximizingly and transitively relational selection function. Then, up to equivalence and under the assumption that* $\mathrm{Rep}(\mathcal{K},\alpha) \neq \emptyset \neq \mathrm{Rep}(\mathcal{K},\beta)$*, the following inclusion holds:*

$$\gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)) \subseteq \gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \cup \gamma(\mathrm{Orep}(\mathcal{K},\beta)).$$

Proof. Note that non-emptiness of the repair sets implies that all optimal repair sets under consideration are also non-empty. Assume that $\mathcal{K}'' \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$. Then (up to equivalence) $\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K},\alpha) \cup \mathrm{Orep}(\mathcal{K},\beta)$ and $\mathcal{L} \trianglelefteq \mathcal{K}''$ holds for all element $\mathcal{L}$ of $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$. Assume without loss of generality that $\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K},\alpha)$. To prove that $\mathcal{K}'' \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha))$, we consider an arbitrary element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K},\alpha)$ and show that $\mathcal{K}' \trianglelefteq \mathcal{K}''$. If $\mathcal{K}'$ is also an element of $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$, then we have $\mathcal{K}' \trianglelefteq \mathcal{K}''$. Otherwise, $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$ implies that $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha \boxplus \beta)$, and thus there is $\mathcal{L} \in \mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$ such that $\mathcal{L} \models \mathcal{K}'$. But then $\mathcal{L} \trianglelefteq \mathcal{K}''$ and the weakly maximizing property yields $\mathcal{L} \trianglerighteq \mathcal{K}'$. Thus, transitivity of $\trianglelefteq$ implies $\mathcal{K}' \trianglelefteq \mathcal{K}''$. □

Theorem 1. *Assume that $\models$ is PPC enabling and that the identity (1) holds. If the selection function γ is weakly maximizingly and transitively relational, then the PPC operation* ctr_γ *satisfies the postulate* conjunctive overlap*.*

Proof. Note that the coverage property of optimal repairs implies that $\mathrm{Rep}(\mathcal{K},\alpha) = \emptyset$ iff $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$. Thus, if $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) = \emptyset$, then $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset = \mathrm{Orep}(\mathcal{K},\beta)$. In this case, $\mathrm{ctr}_\gamma(\mathcal{K},\alpha \boxplus \beta) = \mathcal{K} = \mathrm{ctr}_\gamma(\mathcal{K},\alpha) = \mathrm{ctr}_\gamma(\mathcal{K},\beta)$, and thus *conjunctive overlap* clearly holds.

If all three sets of optimal repairs are non-empty, then *conjunctive overlap* is an immediate consequence of Lemma 5.

Now assume that $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) \neq \emptyset \neq \mathrm{Orep}(\mathcal{K},\alpha)$ and $\mathrm{Orep}(\mathcal{K},\beta) = \emptyset$. In this case, $\mathrm{Rep}(\mathcal{K},\alpha \boxplus \beta) = \mathrm{Rep}(\mathcal{K},\alpha)$, and thus $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) = \mathrm{Orep}(\mathcal{K},\alpha)$ up to equivalence, which implies that $\mathrm{ctr}_\gamma(\mathcal{K},\alpha \boxplus \beta) \equiv \mathrm{ctr}_\gamma(\mathcal{K},\alpha)$. In addition, $\mathrm{Orep}(\mathcal{K},\beta) = \emptyset$ implies that $\mathrm{ctr}_\gamma(\mathcal{K},\beta) = \mathcal{K}$. Since $\mathcal{K} \models \mathrm{ctr}_\gamma(\mathcal{K},\alpha)$, we thus have $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \otimes \mathrm{ctr}_\gamma(\mathcal{K},\beta) \equiv \mathrm{ctr}_\gamma(\mathcal{K},\alpha)$, which shows that *conjunctive overlap* also holds in this case. The symmetric case where $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$ can be treated analogously. □

In a setting where KBs are $\mathcal{EL}$ concepts and entailment is subsumption, we can define transitive and weakly maximizing relations $\trianglelefteq$ as follows.

Example 4. Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox. Given a KB C together with concepts $D_1, \ldots, D_n$ such that $C \sqsubseteq^{\mathcal{T}} D_i$ for $i = 1, \ldots, n$, we define (for all C' with $C \sqsubseteq^{\mathcal{T}} C'$) the number $\kappa(C') := |\{D_i \mid C' \sqsubseteq^{\mathcal{T}} D_i \text{ for } i = 1, \ldots, n\}|$, which counts how many of the concepts D_i are still subsumers of C'. These numbers yield the following relation $\trianglelefteq$: $C'' \trianglelefteq C'$ iff $\kappa(C'') \leq \kappa(C')$.

We claim that the relation $\trianglelefteq$ is transitive and weakly maximizing. In fact, transitivity of $\trianglelefteq$ is obvious since the relation $\leq$ on natural numbers is transitive. To show the weakly maximizing property, assume that $C' \sqsubseteq^{\mathcal{T}} C''$. Then $C'' \sqsubseteq^{\mathcal{T}} D_i$ implies $C' \sqsubseteq^{\mathcal{T}} D_i$, and thus $\kappa(C') \geq \kappa(C'')$.

To be able to use this relation for defining a selection function γ, we must check whether $\mathrm{Orep}(C, \alpha) \neq \emptyset$ implies that the set

$$\{C'' \in \mathrm{Orep}(C, \alpha) \mid C' \trianglelefteq C'' \text{ for all } C' \in \mathrm{Orep}(C, \alpha)\}$$

is non-empty. This is clearly the case independently of what kind of repair requests are employed. In fact, a non-empty set $\mathrm{Orep}(C, \alpha)$ clearly contains an element with maximal κ-value since this set is finite.

3.2 Conjunctive inclusion

The supplementary postulate *conjunctive inclusion* can be formulated in our general setting as follows:

- if $\mathrm{ctr}(\mathcal{K}, \alpha \boxplus \beta) \in \mathrm{Rep}(\mathcal{K}, \alpha)$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \models \mathrm{ctr}(\mathcal{K}, \alpha \boxplus \beta)$ (conjunctive inclusion)

In the classical case, repairs are deductively closed sets of formulas not containing the repair request, and thus *conjunctive inclusion* can be formulated as follows: if $\alpha \notin \mathrm{ctr}(\mathcal{K}, \alpha \wedge \beta)$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \supseteq \mathrm{ctr}(\mathcal{K}, \alpha \wedge \beta)$.

In our general setting, to draw conclusions from the left-hand side of this implication (which for a PPC contraction states that a product belongs to a repair set), it might be useful to have the following property connecting products with repair sets, where $\mathfrak{K}$ is a finite set of KBs that are entailed by $\mathcal{K}$:

$$\otimes\mathfrak{K} \in \mathrm{Rep}(\mathcal{K}, \alpha) \text{ iff } \mathcal{K}' \in \mathit{Rep}(\mathcal{K}, \alpha) \text{ for some } \mathcal{K}' \in \mathfrak{K}. \tag{2}$$

The implication from right to left always holds in our framework since repair sets are assumed to be closed under entailment. The other directions does not hold in general, but it holds if the repair goal is non-entailment.

Lemma 6. *Let* $\mathrm{Rep}(\mathcal{K}, \alpha)$ *be closed under entailment. Then* $\mathcal{K}' \in \mathit{Rep}(\mathcal{K}, \alpha)$ *for some* $\mathcal{K}' \in \mathfrak{K}$ *implies* $\otimes\mathfrak{K} \in \mathrm{Rep}(\mathcal{K}, \alpha)$.

Proof. This is an immediate consequence of the fact that $\mathcal{K}' \models \otimes\mathfrak{K}$ for all $\mathcal{K}' \in \mathfrak{K}$. □

Lemma 7. *Let* $\text{Rep}(\mathcal{K}, \alpha) = \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}' \wedge \mathcal{K}' \not\models \alpha\}$ *and let* $\mathfrak{K}$ *be a finite set of KBs that are entailed by* $\mathcal{K}$. *Then* $\otimes\mathfrak{K} \in \text{Rep}(\mathcal{K}, \alpha)$ *implies that there is a KB* $\mathcal{K}' \in \mathfrak{K}$ *such that* $\mathcal{K}' \in Rep(\mathcal{K}, \alpha)$.

Proof. We show the contraposition. If $\mathcal{K}' \notin Rep(\mathcal{K}, \alpha)$ for all $\mathcal{K}' \in \mathfrak{K}$, then $\mathcal{K}' \models \alpha$ for all $\mathcal{K}' \in \mathfrak{K}$. Due to the definition of the product, this implies $\otimes\mathfrak{K} \models \alpha$, and thus $\otimes\mathfrak{K} \notin \text{Rep}(\mathcal{K}, \alpha)$. □

The following examples demonstrate that *conjunctive inclusion* need not hold if (2) is not satisfied, both in the weakly maximizing and in the maximizing case.

Example 5. We use $\mathcal{EL}$ concepts as knowledge bases and subsumption $\sqsubseteq^\emptyset$ w.r.t. the empty TBox as entailment between knowledge bases. Then, conjunction is the sum operation $\oplus$ and the least common subsumer is the product operation $\otimes$. As repair requests we also consider $\mathcal{EL}$ concepts and the repair goal is non-subsumption, but now w.r.t. the following TBox $\mathcal{T}$:

$$\{\exists r.A_1 \sqsubseteq P_1 \sqcap P_3, \exists r.A_2 \sqsubseteq P_1 \sqcap P_2, \exists r.A_3 \sqsubseteq P_2 \sqcap P_3\},$$

i.e., $\text{Rep}(C, \alpha) := \{C' \mid C \sqsubseteq^\emptyset C' \wedge C' \not\sqsubseteq^\mathcal{T} \alpha\}$. It is easy to see that these repair sets are closed under entailment $\sqsubseteq^\emptyset$ since $\sqsubseteq^\emptyset \subseteq \sqsubseteq^\mathcal{T}$. The optimal repair property is satisfied since (up to equivalence $\equiv^\emptyset$) a given $\mathcal{EL}$ concept has only finitely many subsumers w.r.t. $\sqsubseteq^\emptyset$. As operation $\boxplus$ on repair requests we also use conjunction. It is easy to see that we are in a PPC enabling setting and that (1) is satisfied. If we set $C := \exists r.(A_1 \sqcap A_2 \sqcap A_3)$, $\alpha := P_1 \sqcap P_2 \sqcap P_3$, $\beta := \exists r.A_2 \sqcap \exists r.A_3$, then

$$\text{Orep}(C, \alpha) = \{\exists r.A_1, \exists r.A_2, \exists r.A_3\},\ \text{Orep}(C, \beta) = \{\exists r.(A_1 \sqcap A_2), \exists r.(A_1 \sqcap A_3)\},$$
$$\text{Orep}(C, \alpha \boxplus \beta) = \{\exists r.(A_1 \sqcap A_2), \exists r.(A_1 \sqcap A_3)\}.$$

In fact, it is easy to see that the $\mathcal{EL}$ concepts C' satisfying $C \sqsubseteq^\emptyset C'$ are (up to equivalence) conjunctions of concepts of the form C, $\top$, $\exists r.\top$, $\exists r.A_i$ for $i \in \{1, 2, 3\}$, and $\exists r.(A_i \sqcap A_j)$ for $i, j \subseteq \{1, 2, 3\}$ with $i \neq j$. For such a concept C' not to entail $\alpha = P_1 \sqcap P_2 \sqcap P_3$ w.r.t. $\mathcal{T}$, there must be an $i \in \{1, 2, 3\}$ such C' does not entail P_i. Assume prototypically that $i = 1$ (the other cases are symmetric). Then C' entails neither $\exists r.A_1$ nor $\exists r.A_2$. The most specific concept entailed by C and satisfying this is $\exists r.A_3$. Regarding non-entailment of β by such a concept C', this is the case if C' does not entail $\exists r.A_2$ or it does not entail $\exists r.A_3$. Assume prototypically that $\exists r.A_2$ is not entailed by C'. The most specific concept entailed by C and satisfying this is $\exists r.(A_1 \sqcap A_3)$

If we take as relation $\trianglelefteq$ the universal relation on $\mathcal{EL}$ concepts, then $\trianglelefteq$ is clearly transitive and weakly maximizing. For this relation, the induced selection function γ always selects the whole set of optimal repairs. Consequently $\text{ctr}(C, \alpha) = \text{lcs}(\exists r.A_1, \exists r.A_2, \exists r.A_3) = \exists r.\top$ and $\text{ctr}(C, \alpha \boxplus \beta) = \text{lcs}(\exists r.(A_1 \sqcap A_2), \exists r.(A_1 \sqcap A_3)) = \exists r.A_1$. Thus, we have $\text{ctr}(C, \alpha \boxplus \beta) \in \text{Rep}(C, \alpha)$, but $\text{ctr}(C, \alpha) = \exists r.\top \not\sqsubseteq^\mathcal{T} \exists r.A_1$, which shows that *conjunctive inclusion* is not satisfied. Note that (2) is not satisfied in this example. In fact, neither $\exists r.(A_1 \sqcap A_2)$ nor $\exists r.(A_1 \sqcap A_3)$ belongs to $\text{Rep}(C, \alpha)$, but their product $\exists r.A_1$ does.

In this example, we could not have employed $\sqsubseteq^{\mathcal{T}}$ as entailment relation for comparing repairs. While (up to equivalence w.r.t. $\mathcal{T}$) the optimal repair sets would have been the same, the lcs of $\exists r.(A_1 \sqcap A_2)$ and $\exists r.(A_1 \sqcap A_3)$ w.r.t. $\mathcal{T}$ would be $P_1 \sqcap P_2 \sqcap P_3 \sqcap \exists r.A_1$ and thus not a repair for α. A modified version of Example 5 can be used to show that such a counterexample also exists if a maximizing (rather than just weakly maximizing) transitive relation is used.

Example 6. In this example, we basically employ the same setup as in the previous example, but restrict the set of KBs to consist of all existential restrictions subsuming $C := \exists r.(A_1 \sqcap A_2 \sqcap A_3)$ w.r.t. the empty TBox. Between these KBs we again use $\sqsubseteq^{\emptyset}$ as entailment relation. Up to equivalence, the set of KBs considered here consist of the concepts $\exists r.\sqcap M$ for $M \subseteq \{A_1, A_2, A_3\}$, where $\sqcap\emptyset = \top$ and otherwise $\sqcap M$ is the conjunction of the elements of M. For such concepts, we have $\exists r.\sqcap M \sqsubseteq^{\emptyset} \exists r.\sqcap N$ iff $M \supseteq N$, and thus sum corresponds to set union and product to set intersection, i.e., $\exists r.\sqcap M \oplus \exists r.\sqcap N = \exists r.\sqcap(M \cup N)$ and $\exists r.\sqcap M \otimes \exists r.\sqcap N = \exists r.\sqcap(M \cap N)$.

As repair requests we consider arbitrary $\mathcal{EL}$ concepts α, which define repair sets $\mathrm{Rep}(C, \alpha) := \{C' \mid C \sqsubseteq^{\emptyset} C' \wedge C' \not\sqsubseteq^{\mathcal{T}} \alpha\}$, where $\mathcal{T}$ is the TBox of Example 5. As in Example 5 it is easy to see that repair sets are closed under entailment and satisfy the optimal repair property. As operation $\boxplus$ on repair requests we again use conjunction, which ensures that (1) is satisfied. If we set $C := \exists r.(A_1 \sqcap A_2 \sqcap A_3)$, $\alpha := P_1 \sqcap P_2 \sqcap P_3$, and $\beta := \exists r.A_2 \sqcap \exists r.A_3$, then we obtain the same sets of optimal repairs as in Example 5.

Now, we define a maximizing and transitive relation $\trianglelefteq$ on knowledge bases: $\exists r.\sqcap M \trianglelefteq \exists r.\sqcap N$ iff $|M| \leq |N|$. This relation is clearly transitive since $\leq$ on natural numbers is transitive. If $\exists r.\sqcap N \sqsubset^{\emptyset} \exists r.\sqcap M$, then $N \supseteq M$, and thus $|N| \geq |M|$. Since $|N| = |M|$ would imply $N = M$, and thus $\exists r.\sqcap N \equiv^{\emptyset} \exists r.\sqcap M$, we actually have $|N| > |M|$, which implies $\exists r.\sqcap N \rhd \exists r.\sqcap M$.

For our optimal repair sets, the selection function induced by this relation $\trianglelefteq$ again selects all elements. We can now proceed as in Example 5 to show that *conjunctive inclusion* and (2) are not satisfied.

Now, we investigate whether the postulate *conjunctive inclusion* holds under the assumption that (1) and (2) are satisfied. We start with a lemma that looks similar to Lemma 5. Its proof is inspired by the proof of *condition* T in the proof of Observation 2.76 in [16]. For the case of belief set contraction, this lemma is crucial for showing *conjunctive inclusion*. Unfortunately, due to the fact that the inclusion $\mathrm{Orep}(\mathcal{K}, \alpha) \cup \mathrm{Orep}(\mathcal{K}, \beta) \subseteq \mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta)$ need not hold, we do not obtain a direct inclusion relation between the sets selected by γ.

Lemma 8. *Let γ be a weakly maximizingly and transitively relational selection function and assume that (1) and (2) hold. If* $\mathrm{Rep}(\mathcal{K}, \alpha) \neq \emptyset \neq \mathrm{Rep}(\mathcal{K}, \beta)$ *and* $\mathrm{ctr}(\mathcal{K}, \alpha \boxplus \beta) \in \mathrm{Rep}(\mathcal{K}, \alpha)$*, then for every element $\mathcal{Z}$ of* $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$ *there exists an element $\mathcal{Z}'$ of* $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta))$ *such that $\mathcal{Z}' \models \mathcal{Z}$.*

Proof. Since (2) holds, $\mathrm{ctr}(\mathcal{K}, \alpha \boxplus \beta) \in \mathrm{Rep}(\mathcal{K}, \alpha)$ implies that there is an $\mathcal{X} \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta))$ such that $\mathcal{X} \in \mathrm{Rep}(\mathcal{K}, \alpha)$. Lemma 3 yields $\mathcal{X} \in \mathrm{Orep}(\mathcal{K}, \alpha)$.

Now, assume that $\mathcal{Z} \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha))$. Then $\mathcal{Z} \in \mathrm{Rep}(\mathcal{K},\alpha) \subseteq \mathrm{Rep}(\mathcal{K},\alpha \boxplus \beta)$, and thus there is $\mathcal{Z}' \in \mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$ such that $\mathcal{Z}' \models \mathcal{Z}$. This implies $\mathcal{Z}' \trianglerighteq \mathcal{Z}$.

To show that $\mathcal{Z}'$ is selected by γ, we consider an arbitrary element $\mathcal{V}$ of $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$ and prove that $\mathcal{Z} \trianglerighteq \mathcal{V}$. Since $\mathcal{X} \in \mathrm{Orep}(\mathcal{K},\alpha)$, we know that $\mathcal{Z} \trianglerighteq \mathcal{X}$. In addition, $\mathcal{V} \in \mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$ and $\mathcal{X} \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$ yield $\mathcal{X} \trianglerighteq \mathcal{V}$. Putting all inequalities together yields $\mathcal{Z}' \trianglerighteq \mathcal{Z} \trianglerighteq \mathcal{X} \trianglerighteq \mathcal{V}$, and thus $\mathcal{Z}' \trianglerighteq \mathcal{V}$ by transitivity. Since $\mathcal{V}$ was assumed to be an arbitrary element of $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$, this proves $\mathcal{Z}' \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$. □

In the proof of this lemma it is sufficient to know that $\trianglelefteq$ is weakly maximizing. If we make the stronger assumption that $\trianglelefteq$ is maximizing, then we obtain the following stronger lemma, which will allow us to prove *conjunctive inclusion*.

Lemma 9. *Assume in addition to the assumptions of the previous lemma that* $\trianglelefteq$ *is maximizing. Then for every element* $\mathcal{Z}$ *of* $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ *there exists an element* $\mathcal{Z}'$ *of* $\gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$ *such that* $\mathcal{Z}' \equiv \mathcal{Z}$.

Proof. If none of the entailments $\mathcal{Z}' \models \mathcal{Z}$ in the previous lemma is strict, then we are done. We now show that assuming that one of these entailments is strict leads to a contradiction. Thus, assume that $\mathcal{Z}$ is an element of $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ such that there exists an element $\mathcal{Z}'$ of $\gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$ satisfying $\mathcal{Z}' \models_s \mathcal{Z}$. Then the maximizing property yields $\mathcal{Z}' \triangleright \mathcal{Z}$. Since $\mathcal{Z}$ is selected, we also know that $\mathcal{Z} \trianglerighteq \mathcal{Z}_0$ holds for all $\mathcal{Z}_0 \in \mathrm{Orep}(\mathcal{K},\alpha)$. In addition, $\mathcal{Z}_1 \trianglerighteq \mathcal{Z}'$ holds for all $\mathcal{Z}_1 \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$. Transitivity thus yields $\mathcal{Z}_1 \triangleright \mathcal{Z}_0$ for all $\mathcal{Z}_1 \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$ and all $\mathcal{Z}_0 \in \mathrm{Orep}(\mathcal{K},\alpha)$.

The assumption $\mathrm{ctr}(\mathcal{K},\alpha \boxplus \beta) \in \mathrm{Rep}(\mathcal{K},\alpha)$ implies that there is an $\mathcal{X} \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta))$ such that $\mathcal{X} \in \mathrm{Rep}(\mathcal{K},\alpha)$. Coverage yields an element $\mathcal{X}_0 \in \mathrm{Orep}(\mathcal{K},\alpha)$ such that $\mathcal{X}_0 \models \mathcal{X}$, and thus $\mathcal{X}_0 \trianglerighteq \mathcal{X}$. However, we have shown in the previous paragraph that actually $\mathcal{X} \triangleright \mathcal{X}_0$ must hold. □

Theorem 2. *Assume that* $\models$ *is PPC enabling and that the identities (1) and (2) hold. If the selection function* γ *is maximizingly and transitively relational, then the partial product contraction operation* ctr_γ *satisfies the postulate* conjunctive inclusion*.*

Proof. As already pointed out in the proof of Theorem 1, the coverage property of optimal repairs implies that $\mathrm{Rep}(\mathcal{K},\alpha) = \emptyset$ iff $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$. Thus, if $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) = \emptyset$, then $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset = \mathrm{Orep}(\mathcal{K},\beta)$. In this case, $\mathrm{ctr}_\gamma(\mathcal{K},\alpha \boxplus \beta) = \mathcal{K} = \mathrm{ctr}_\gamma(\mathcal{K},\alpha)$, and thus *conjunctive overlap* clearly holds. If all three sets of optimal repairs are non-empty, then *conjunctive inclusion* is an immediate consequence of Lemma 9.

Now assume that $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) \neq \emptyset \neq \mathrm{Orep}(\mathcal{K},\alpha)$ and $\mathrm{Orep}(\mathcal{K},\beta) = \emptyset$. In this case, $\mathrm{Rep}(\mathcal{K},\alpha \boxplus \beta) = \mathrm{Rep}(\mathcal{K},\alpha)$, and thus $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta)$ and $\mathrm{Orep}(\mathcal{K},\alpha)$ are equal up to equivalence. This implies that $\mathrm{ctr}(\mathcal{K},\alpha) \equiv \mathrm{ctr}(\mathcal{K},\alpha \boxplus \beta)$ due to the properties required for selection functions, which shows that *conjunctive inclusion* also holds in this case. Finally, assume $\mathrm{Orep}(\mathcal{K},\alpha \boxplus \beta) \neq \emptyset \neq \mathrm{Orep}(\mathcal{K},\beta)$ and $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$. In this case, the precondition $\mathrm{ctr}(\mathcal{K},\alpha \boxplus \beta) \in \mathrm{Rep}(\mathcal{K},\alpha)$ of *conjunctive inclusion* is false. Thus the postulate trivially holds. □

4 Conclusion

We have seen that, under reasonable additional assumptions, Gärdenfors's supplementary postulates can be shown to hold for the partial product contractions produced by the general framework of [9]. This clarifies what conditions are really needed for these postulates to hold. At the moment, it remains open whether the postulates in [9] together with *conjunctive overlap* and *conjunctive inclusion* characterize the partial product contractions obtained from all maximizingly and transitively relational selection functions. If one considers the proofs of such characterization theorems involving the supplementary postulates in the literature (see, e.g., [16], where quite a number of such theorems are shown), then one sees that they strongly make use of the "remainder variant" of the equation $\mathrm{Orep}(\mathcal{K}, \alpha \boxplus \beta) = \mathrm{Orep}(\mathcal{K}, \alpha) \cup \mathrm{Orep}(\mathcal{K}, \beta)$, which holds for the remainders considered there, but not in our general setting (see Example 3). Also note that such proofs usually depend on the fact that KBs are sets of formulas of a logic in which certain Boolean operators are available. It needs to be seen whether the proof of an appropriate characterization theorem requires additional conditions on our framework, or whether the development of new proof approaches is sufficient.

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