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Combination of Compatible Reduction Orderings that are Total on Ground Terms

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Abstract

Reduction orderings that are compatible with an equational theory E and total on (the E-equivalence classes of) ground terms play an important rôle in automated deduction. We present a general approach for combining such orderings. To be more precise, we show how E_1-compatible reduction orderings total on Σ_1-ground terms and E_2-compatible reduction orderings total on Σ_2-ground terms can be used to construct an $(E_1 \cup E_2)$-compatible reduction ordering total on $(\Sigma_1 \cup \Sigma_2)$-ground terms, provided that the signatures are disjoint and some other (rather weak) restrictions are satisfied. This work was motivated by the observation that it is often easier to construct such orderings for "small" signatures and theories separately, rather than directly for their union.

1 Introduction

Reduction orderings that are *total on ground terms* play an important rôle in many areas of automated deduction. For example, unfailing completion [4]—a variant of Knuth-Bendix completion that avoids failure due to incomparable critical pairs—presupposes such an ordering. In addition, using a reduction ordering that is total on ground terms, one can show that any finite set of ground equations has a decidable word problem [15, 9]. It is, in fact, very easy to obtain such orderings: many of the standard methods for constructing reduction orderings yield orderings that are total on ground terms. For instance, both Knuth-Bendix orderings [14] and lexicographic path orderings [11] are total on ground terms if they are based on a *total precedence ordering* on the set of function symbols.

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Things become more complex if one is interested in reduction orderings that are *compatible with a given equational theory E*. Such orderings, which are, for example, used in rewriting modulo equational theories [18, 10, 2], can be seen as orderings on E-equivalence classes. E-compatible reduction orderings that are total on (E-equivalence classes of) ground terms can be employed for similar purposes as the usual reduction orderings that are total on ground terms. For example, let AC denote a theory that axiomatizes associativity and commutativity of several binary function symbols, where the signature may contain additional free function symbols. An AC-compatible reduction ordering that is total on ground terms can be used to show that for any finite set G of ground equations, the word problem is decidable for $AC \cup G$ [16, 17]. The first AC-compatible reduction ordering total on ground terms was described in [17]. It is based on a relatively complex polynomial interpretation in which the coefficients of the polynomials are again integer polynomials. Surprisingly, it turned out to be rather hard to construct AC-compatible reduction orderings by appropriately modifying standard orderings such as recursive path orderings [7]. The main idea underlying most proposals in this direction (e.g., [5, 3, 12, 6]) is to apply certain transformations such as flattening to the terms before comparing them with one of the standard path orderings. A major drawback of these approaches is that they impose rather strong restrictions on the precedence orderings on function symbols that may be used. One consequence of these restrictions is that the obtained AC-compatible orderings are not total on ground terms if more than one AC-symbol is present. This problem has finally been overcome in [21, 22], where an AC-compatible reduction ordering total on ground terms is defined that is based on a recursive path ordering (with status). In [20] it was shown that this approach can even be used to construct reduction orderings total on ground terms that are compatible with theories that axiomatize several associative, commutative, associative-commutative, and free symbols.

The present paper proposes a different way of attacking the problem of how to construct E-compatible orderings that are total on ground terms. It was motivated by the observation that it is very easy to define an AC-compatible reduction ordering total on ground terms if there is only one AC-symbol in the signature. Instead of directly defining an AC-compatible ordering total on ground terms for the case of more than one AC-symbol, we try to obtain such an ordering by combining the orderings that exist for the case of one AC-symbol.[1] To be more precise, assume that AC_1 axiomatizes associativity-commutativity of the symbol $+ \in \Sigma_1$ and that AC_2 axiomatizes associativity-commutativity of the symbol $* \in \Sigma_2$, where Σ_1 and Σ_2 are disjoint signatures that may contain additional free function symbols. For $i = 1, 2$, let $\succ_i$ be an AC_i-compatible reduction ordering

[1] This should not be confused with Rubio's approach for combining orderings on disjoint signatures [20]. To obtain his combined ordering, which extends given orderings on terms over the single signatures to an ordering on terms over the union of the signatures, he presupposes the existence of a compatible reduction ordering total on ground terms for the combined signature. In the present paper, the main goal is to show that such an ordering exists.

that is total on the AC_i-equivalence classes of ground terms, i.e., $\succ_i$ can be seen as a total ordering on $\mathcal{T}(\Sigma_i, \emptyset)/_{=_{AC_i}}$. In order to define a reduction ordering that is total on $\mathcal{T}(\Sigma_1 \cup \Sigma_2, \emptyset)/_{=_{AC_1 \cup AC_2}}$ from the given orderings $\succ_1$ and $\succ_2$, we utilize the fact that this combined algebra can be represented as the free amalgamated product of the single algebras $\mathcal{T}(\Sigma_i, \emptyset)/_{=_{AC_i}}$. This product was introduced in [1] in the context of combining unification algorithms. The construction of the amalgamated product represents the universe of $\mathcal{T}(\Sigma_1 \cup \Sigma_2, \emptyset)/_{=_{AC_1 \cup AC_2}}$ as a (possibly infinite) tower of layers. In principle, the combined ordering compares elements of the combined algebra first with respect to the layers they are in: elements in higher layers are larger than elements in lower ones. If two elements are in the same layer, then one of the original orderings ($\succ_1$ or $\succ_2$) is used to compare them.

This combination approach is, of course, not restricted to AC-theories. It can be used to combine arbitrary compatible reduction orderings that are total on ground terms, provided that the single theories are over disjoint signatures and satisfy some additional properties that will be introduced below. For example, theories that axiomatize associativity, commutativity, or associativity-commutativity of a binary function symbol satisfy these properties.

2 Compatible reduction orderings

Let Σ be a signature, and let $T(\Sigma, X)$ denote the terms over Σ with variables in X. A *reduction ordering* on $T(\Sigma, X)$ is a strict partial ordering $\succ$ that is Noetherian, stable under Σ-operations (i.e., $s \succ t$ implies $f(\ldots, s, \ldots) \succ f(\ldots, t, \ldots)$ for all $f \in \Sigma$), and stable under substitutions (i.e., $s \succ t$ implies $\sigma(s) \succ \sigma(t)$ for all Σ-substitutions σ). In the following, we will restrict our attention to reduction orderings on ground terms, which means that stability under substitutions can be dispensed with. However, the ground terms that will be considered may contain additional *free constants* from a set of constants C with $C \cap \Sigma = \emptyset$. By a slight abuse of notation, the set of these ground terms will be written as $T(\Sigma, C)$. The only difference between variables and free constants is the fact that constants cannot be replaced by substitutions, and thus it is possible to order them with a reduction ordering. The set of free constants occurring in a term t is denoted by $C(t)$.

Let E be a set of identities over Σ, and let $=_E$ denote the equational theory induced by E. A reduction ordering $\succ$ is *E-compatible* iff $s \succ t$, $s =_E s'$, and $t =_E t'$ imply $s' \succ t'$. Thus, an E-compatible reduction ordering induces a well-defined ordering on the set of $=_E$-equivalence classes. For a set of free constants C, the E-free algebra with generators C, i.e., $\mathcal{T}(\Sigma, C)/_{=_E}$, will be denoted by $\langle C \rangle_{\Sigma,E}$. We call a reduction ordering *total on* $\langle C \rangle_{\Sigma,E}$ (or simply "total on ground terms," if the set of ground terms is clear from the context) iff it induces a total ordering on $\langle C \rangle_{\Sigma,E}$, i.e., iff for all $s, t \in T(\Sigma, C)$ we have $s \succ t$, or $s =_E t$, or

$s \prec t$.

An E-compatible reduction ordering on $T(\Sigma, C)$ can also be seen as an ordering on $\langle C \rangle_{\Sigma,E}$ (whose elements are the $=_E$-equivalence classes). This ordering is Noetherian and stable under (the interpretation of) the Σ-operations in $\langle C \rangle_{\Sigma,E}$. Conversely, one can define an E-compatible reduction ordering on $T(\Sigma, C)$ by directly introducing a Noetherian and stable ordering on $\langle C \rangle_{\Sigma,E}$. Two terms s, t are then compared by considering their interpretation in the algebra $\langle C \rangle_{\Sigma,E}$, i.e, their image under the canonical homomorphism from $T(\Sigma, C)$ onto $\langle C \rangle_{\Sigma,E}$.

If E is a non-trivial equational theory (i.e., admits models of cardinality greater than 1), then we have $c \neq_E c'$ for every pair of distinct free constants $c, c' \in C$. Thus, an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$ yields a total Noetherian ordering on C. We say that an E-compatible reduction ordering *extends* a total Noetherian ordering $>$ on C iff its restriction to C coincides with $>$. In the following, we consider *only non-trivial equational theories* (without mentioning it explicitly as a condition).

We close this section by proving some properties of equational theories and reduction orderings compatible with equational theories that will be important for the proof of our combination result:

Lemma 2.1 *1. If there exists a non-empty E-compatible reduction ordering, then E is a regular equational theory, that is, we have for all terms $s, t \in T(\Sigma, C)$ that $s =_E t$ implies $C(s) = C(t)$.*

2. If there exists a non-empty E-compatible reduction ordering, then for any free constant $c \in C$ and term $t \in T(\Sigma, C)$ we can have $c =_E t$ only if c occurs exactly once in t.

3. If $\succ$ is an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$, then $c \in C(t)$ for a free constant $c \in C$ and a term $t \neq_E c$ implies $t \succ c$.

4. Let $\succ$ be an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$, and assume that $0 \in \Sigma$ is a signature constant and $c \in C$ is a free constant. If there exists a term s containing 0 *such that $s =_E c$, then 0 is the smallest element of $\langle C \rangle_{\Sigma,E}$ with respect to $\succ$.*

Proof. (1) Since the reduction ordering $\succ$ is non-empty, there exist terms $l, r \in T(\Sigma, C)$ such that $l \succ r$. If E is non-regular, then there exist terms s, t and a free constant c such that $s =_E t$ and $c \in C(s) \setminus C(t)$. Let ϕ_l be the endomorphism on $T(\Sigma, C)$ that replaces c by l and leaves all other free constants invariant, and let ϕ_r be defined correspondingly, with r in place of l. Since c was assumed to be a free constant, $s =_E t$ implies $\phi_l(s) =_E \phi_l(t)$ and $\phi_r(s) =_E \phi_r(t)$. In addition, since c does not occur in t, we have $\phi_l(t) = \phi_r(t)$. This shows that $\phi_l(s) =_E \phi_r(s)$. However, since $\phi_r(s)$ is obtained from $\phi_l(s)$ by replacing every occurrence of l by

r, the assumption $l \succ r$ implies $\phi_l(s) \succ \phi_r(s)$, which yields a contradiction (to $\succ$ being Noetherian and E-compatible).

(2) Assume that $c =_E t$ and that $l \succ r$. Because of the first part of the lemma, we know that c must occur at least once in t. Now, assume that c has at least two occurrences in t, which we indicate by writing $t(c, \ldots, c)$. Since c is a free constant, $c =_E t(c, \ldots, c)$ implies $l =_E t(l, \ldots, l)$. The following infinitely descending chain is a contradiction to the assumption that $\succ$ is an E-compatible reduction ordering:

$$l =_E t(l, \ldots, l) \succ t(l, \ldots, r) =_E t(t(l, \ldots, l), \ldots, r) \succ t(t(l, \ldots, r), \ldots, r) =_E \cdots$$

(3) Since $\succ$ is assumed to be total, $t \neq_E c$ and $t \not\succ c$ imply $c \succ t$. If c occurs in t, this yields an infinitely descending chain:

$$c \succ t(c) \succ t(t(c)) \succ \cdots$$

(4) Assume that 0 is not the smallest element with respect to $\succ$, i.e., there exists a term $t \neq_E 0$ such that $0 \succ t$. Then $s =_E c$ and the fact that s contains 0 imply that we can write $s = s(c, 0)$, and since c is a free constant, this yields the following infinitely descending chain:

$$t =_E s(t, 0) \succ s(t, t) =_E s(t, s(t, 0)) \succ s(t, s(t, t)) =_E s(t, s(t, s(t, 0))) \succ \cdots$$

$\square$

3 Construction of compatible reduction orderings

In this section, we show how given reduction orderings total on ground terms can be used to construct new ones satisfying certain additional properties. First, we proof a proposition that will serve as the main tool in these constructions.

We have seen that an E-compatible reduction ordering can be seen as an ordering on the E-free algebra $\langle C \rangle_{\Sigma,E}$. More generally, we may consider an arbitrary (not necessarily free) Σ-algebra $\mathcal{A}$, and call a Noetherian ordering $\succ_A$ that is stable under (the interpretation of) the Σ-operations in $\mathcal{A}$ a reduction ordering on $\mathcal{A}$. If $\phi : \mathcal{T}(\Sigma, C) \to \mathcal{A}$ is a homomorphism such that $s =_E t$ implies $\phi(s) = \phi(t)$, then $\succ_A$ induces an E-compatible reduction ordering $\succ$ on $T(\Sigma, C)$:

$$s \succ t \ \text{ iff } \ \phi(s) \succ_A \phi(t).$$

The next proposition generalizes this construction in two ways. First, it considers an arbitrary Σ-algebra (instead of the term algebra) as domain of the homomorphism. Second, it introduces an additional comparison, which applies when the homomorphism yields the same image. This is necessary to obtain a total ordering.

Proposition 3.1 *Let Σ be a signature, and $\mathcal{A}, \mathcal{B}$ be Σ-algebras. Assume that $\succ_A$ is a reduction ordering total on $\mathcal{A}$, $\succ_B$ is a reduction ordering total on $\mathcal{B}$, and $\phi : \mathcal{A} \to \mathcal{B}$ is a Σ-homomorphism. Then the relation $\succ$ defined as*

$$\begin{aligned} a \succ b \quad \textit{iff} \quad & \phi(a) \succ_B \phi(b) \ \textit{or} \\ & \phi(a) = \phi(b) \ \textit{and} \ a \succ_A b \end{aligned}$$

is a reduction ordering that is total on $\mathcal{A}$.

Proof. (1) Transitivity of $\succ$ can be shown by a simple case distinction. For example, assume that $a \succ b$ is due to $\phi(a) \succ_B \phi(b)$ and that $b \succ c$ is due to $\phi(b) = \phi(c)$ and $b \succ_A c$. Obviously, this yields $\phi(a) \succ_B \phi(c)$, and thus $a \succ c$. The other cases are similar.

(2) It is easy to see that an infinitely descending $\succ$-chain yields an infinitely descending $\succ_B$-chain, if the first case in the definition applies infinitely often. Otherwise, it yields an infinitely descending $\succ_A$-chain. Thus, $\succ$ must be Noetherian.

(3) Stability under Σ-operations is an easy consequence of the fact that the orderings $\succ_A$ and $\succ_B$ satisfy this property, and that ϕ is a Σ-homomorphism.

(4) Assume that $a \neq b$ are distinct elements of $\mathcal{A}$. If $\phi(a) \neq \phi(b)$, then totality of $\succ_B$ implies $\phi(a) \succ_B \phi(b)$ or $\phi(b) \succ_B \phi(a)$, and thus $a \succ b$ or $b \succ a$. Otherwise, we have $\phi(a) = \phi(b)$, and totality of $\succ_A$ yields the desired comparison. □

The constant dominance condition

The following "constant dominance condition" will be an important prerequisite for our combination method to apply:

Definition 3.2 Let $\succ$ be an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$. Then $\succ$ satisfies the *constant dominance condition (CDC)* iff for all $t \in T(\Sigma, C)$ and $c \in C$ such that $c \succ c'$ for all $c' \in C(t)$, we have $c \succ t$.

Intuitively, this means that large constants dominate terms containing only small constants. An arbitrary E-compatible reduction ordering total on ground terms need not satisfy this property. For certain equational theories, which we will call "strongly regular," the existence of an arbitrary E-compatible reduction ordering total on ground terms implies the existence of such an ordering that also satisfies the CDC.

Let C be a countably infinite set of free constants. For a term $t \in T(\Sigma, C)$ and a free constant $c \in C$, let $|t|_c$ denote the number of occurrences of c in t. We say that the equational theory E is *strongly regular* iff $s =_E t$ implies $|s|_c = |t|_c$ for all terms $s, t \in T(\Sigma, C)$ and free constants c. For example, theories axiomatizing commutativity, associativity, or associativity-commutativity of a binary function symbol are obviously strongly regular.

Proposition 3.3 *Let E be strongly regular. If there exists an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$, then there also exists such an ordering that additionally satisfies the CDC.*

Proof. We denote by $MM(C)$ the set of all finite multisets with elements in C [8]. The Σ-algebra $\mathcal{MM}(C)$ with carrier set $MM(C)$ is obtained by interpreting all Σ-operations as union "$\cup$" of multisets:

$$f^{\mathcal{MM}(C)}(M_1, \ldots, M_n) := M_1 \cup \cdots \cup M_n.$$

For a given total Noetherian ordering $>$ on C, the induced multiset ordering $>_{MM}$ (see [8]) is a total Noetherian ordering on $MM(C)$, and it is easy to see that it is stable under the operations $f^{\mathcal{MM}(C)}$. Thus, it is a reduction ordering total on $\mathcal{MM}(C)$. By definition, this ordering satisfies: $\{c\} >_{MM} \{c_1, \ldots, c_n\}$ iff $c > c_i$ for all $i, 1 \leq i \leq n$.

Now, assume that E is strongly regular, and that $\succ$ is an E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$. The ordering $\succ$ induces a total Noetherian ordering $>$ on C, and thus a reduction ordering $>_{MM}$ that is total on $\mathcal{MM}(C)$.

Obviously, every term $t \in T(\Sigma, C)$ can be mapped to the multiset $MM(t)$ of the free constants in t; for instance, $t = f(c, f(d, f(c, f(d, d))))$ yields the multiset $MM(t) = \{c, c, d, d, d\}$ with 2 occurrences of c, 3 of d, and 0 for all other free constants. Since E is strongly regular, $=_E$-equivalent terms yield the same multiset, and thus we may consider this mapping as a mapping from $\langle C \rangle_{\Sigma,E}$ into $\mathcal{MM}(C)$. It is easy to see that this mapping is in fact a homomorphism.

We can now use the construction of Proposition 3.1 to define a new E-compatible reduction ordering total on $\langle C \rangle_{\Sigma,E}$:

$$\begin{aligned} s \succ_{CDC} t \quad \text{iff} \quad & MM(s) >_{MM} MM(t) \text{ or} \\ & MM(s) = MM(t) \text{ and } s \succ t. \end{aligned}$$

If t is a term such that $c > c'$ for all $c' \in C(t)$, then we obviously have $MM(c) = \{c\} >_{MM} MM(t)$, which shows that $\succ_{CDC}$ satisfies the CDC. $\square$

The layer ordering

Assume that C_∞ is a set of free constants that is obtained as a disjoint union of (not necessarily non-empty) sets C_i, that is, $C_\infty := \bigcup_{i=0}^{\infty} C_i$. For $n \geq 0$, let A_n

$$\begin{array}{c} \dots \\ \\ C_{n+1} \\ A_n \setminus (A_{n-1} \cup C_n) \\ C_n \\ \\ \dots \\ \\ C_2 \\ A_1 \setminus (A_0 \cup C_1) \\ C_1 \\ A_0 \end{array}$$

Figure 1: Partitioning of A_∞ into layers.

denote the carrier set of the E-free algebra $\langle \bigcup_{i=0}^{n} C_i \rangle_{\Sigma,E}$. The carrier set A_∞ of $\langle C_\infty \rangle_{\Sigma,E}$ can then be partitioned into the layers shown in Figure 1.

Assume that $\succ$ is an E-compatible reduction ordering that is total on $\langle C_\infty \rangle_{\Sigma,E}$ and that extends a total ordering $>$ on C_∞ that satisfies

$$(*) \quad c \in C_i, c' \in C_j, i > j \;\Rightarrow\; c > c'.$$

Our goal is to modify $\succ$ into an E-compatible reduction ordering $\succ_\ell$ that is total on $\langle C_\infty \rangle_{\Sigma,E}$, extends $>$, and additionally respects the layers, that is, elements of higher layers are larger with respect $\succ_\ell$ than elements of lower layers. Formally, this means that the new ordering should satisfy

$$c \in C_1, a \in A_0 \;\Rightarrow\; c \succ_\ell a,$$

and for all $n \geq 1$

$$\begin{aligned} c \in C_{n+1}, a \in A_n \setminus (A_{n-1} \cup C_n) &\;\Rightarrow\; c \succ_\ell a, \\ c \in C_n, a \in A_n \setminus (A_{n-1} \cup C_n) &\;\Rightarrow\; a \succ_\ell c. \end{aligned}$$

In this case, we say that $\succ_\ell$ *respects the layers induced by the partition* $C_\infty = \bigcup_{i=0}^{\infty} C_i$.

Proposition 3.4 *Let $>$ be a total ordering on $C_\infty = \bigcup_{i=0}^{\infty} C_i$ that satisfies the condition $(*)$, and assume that there exists an E-compatible reduction ordering that is total on $\langle C_\infty \rangle_{\Sigma,E}$, extends $>$, and satisfies the CDC. Then there exists an E-compatible reduction ordering that is total on $\langle C_\infty \rangle_{\Sigma,E}$, extends $>$, and respects the layers induced by the partition $C_\infty = \bigcup_{i=0}^{\infty} C_i$.*

Proof. For every non-empty set C_i, let $\widehat{c}_i$ be an element of C_i. The endomorphism ϕ of $\langle C_\infty \rangle_{\Sigma,E}$ is defined by mapping every free constant c on the representative $\widehat{c}_i$ of the set C_i with $c \in C_i$. The homomorphism ϕ is now used to define $\succ_\ell$ with the help of the construction of Proposition 3.1:

$$a \succ_\ell b \quad \text{iff} \quad \begin{array}{l} \phi(a) \succ \phi(b) \text{ or} \\ \phi(a) = \phi(b) \text{ and } a \succ b. \end{array}$$

(1) By Proposition 3.1, $\succ_\ell$ is an E-compatible reduction ordering that is total on $\langle C_\infty \rangle_{\Sigma,E}$.

(2) To show that $\succ_\ell$ extends $>$, assume that $a, b \in C_\infty$ satisfy $a > b$. Let i, j be such that $a \in C_i$ and $b \in C_j$. Because $>$ satisfies $(*)$, we have $i \geq j$. By definition of ϕ, $\phi(a) = \widehat{c}_i$ and $\phi(b) = \widehat{c}_j$. If $i > j$, condition $(*)$ for $>$ and the fact that $\succ$ extends $>$ imply $\phi(a) \succ \phi(b)$, and thus $a \succ_\ell b$. If $i = j$, then $\phi(a) = \phi(b)$, and since $\succ$ extends $>$, $a > b$ implies $a \succ b$.

(3) Next we show that $c \in C_{n+1}$ and $a \in A_n \setminus (A_{n-1} \cup C_n)$ (for $n \geq 1$) imply $c \succ_\ell a$. By definition of ϕ, we have $\phi(c) = \widehat{c}_{n+1}$, and since $a \in A_n \setminus (A_{n-1} \cup C_n)$, both a and $\phi(a)$ "contain" only constants in $\bigcup_{i=0}^n C_i$.[2] Because $\succ$ satisfies the CDC, this implies $\phi(c) \succ \phi(a)$, and thus $c \succ_\ell a$.

(4) The case where $c \in C_1$ and $a \in A_0$ can be treated analogously. One should note that the case where a does not contain any free constants also yields $\phi(c) \succ \phi(a)$ by the CDC.

(5) Finally, we prove that $c \in C_n$ and $a \in A_n \setminus (A_{n-1} \cup C_n)$ (for $n \geq 1$) imply $a \succ_\ell c$. By definition, $\phi(c) = \widehat{c}_n$. Let $a = [t]$, i.e., a is the $=_E$-equivalence class of the term t. Now $a \in A_n \setminus (A_{n-1} \cup C_n)$ implies that

- $C(t) \subseteq \bigcup_{i=0}^n C_i$ since $a \in A_n$,
- $C(t) \cap C_n \neq \emptyset$ since $a \notin A_{n-1}$,
- $t \neq_E c$ for all $c \in C_n$ since $a \notin C_n$.

Let $\widehat{t}$ be a term in the class $\phi(a)$, i.e., $\phi(a) = [\widehat{t}]$. Because E is regular (by Lemma 2.1), and t contains an element of C_n, we know that $\widehat{t}$ contains $\widehat{c}_n$. By Lemma 2.1, we can deduce $\phi(a) \succ \phi(c)$ (and thus $a \succ_\ell c$) as soon as we have shown that $\widehat{t} \neq_E \widehat{c}_n$. Assume to the contrary that $\widehat{t} =_E \widehat{c}_n$. Since $t \neq_E c$ for all $c \in C_n$, this is only possible if ϕ has identified two different free constants in t. Consequently, $\widehat{c}_n$ occurs more than once in $\widehat{t}$, which together with $\widehat{t} =_E \widehat{c}_n$ is a contradiction (by Lemma 2.1). $\square$

[2] Since E is regular by Lemma 2.1, it makes sense to say that an $=_E$-equivalence class contains a free constant since the terms in a class contain exactly the same free constants.

If the signature contains constant symbols, A_0 contains elements a that do not "contain" free constants. In this case, the argument used in part (5) of the proof cannot be adapted to show that $a \in A_0 \setminus C_0$ and $c \in C_0$ imply $a \succ_\ell c$. If the signature does not contain constant symbols, however, any element of A_0 contains at least one constant from C_0, and thus the argument used in part (5) of the above proof applies.

Corollary 3.5 *If Σ does not contain constant symbols, then the ordering $\succ_\ell$ constructed above also satisfies*

$$c \in C_0, a \in A_0 \setminus C_0 \;\Rightarrow\; a \succ_\ell c.$$

4 Combination of compatible reduction orderings

In principle, we want to solve the following combination problem: Let Σ_1, Σ_2 be disjoint signatures and E_1, E_2 be equational theories over the respective signature. Assume that, for $i = 1, 2$ and any set C of free constants, there exists an E_i-compatible reduction ordering $\succ_i$ that is total on $\langle C \rangle_{\Sigma_i, E_i}$. Can the orderings $\succ_1, \succ_2$ be used to construct an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$?

The next example demonstrates that this is not always possible.

Example 4.1 Let $\Sigma_1 := \{+, 0\}$, $\Sigma_2 := \{*, 1\}$, $E_1 := \{x + 0 = x\}$, and $E_2 := \{x * 1 = x\}$. It is easy to see that there exist E_i-compatible reduction orderings $\succ_i$ that are total on $\langle C \rangle_{\Sigma_i, E_i}$. In fact, any term in $T(\Sigma_1, C)$ is either $=_{E_1}$-equivalent to a term in $T(\{+\}, C)$ or to 0. Since $=_{E_1}$ is the syntactic equality on $T(\{+\}, C)$, one can simply take a lexicographic path ordering that is induced by a well-ordering of C to order the terms equivalent to a term in $T(\{+\}, C)$. The terms equivalent to 0 are then made smaller than all the other terms. The same argument applies to E_2.

However, assume that $\succ$ is an $(E_1 \cup E_2)$-compatible reduction ordering total on $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$. Obviously, we have $c + 0 =_{E_1 \cup E_2} c$ and $c * 1 =_{E_1 \cup E_2} c$. By Property 4 of Lemma 2.1, both 0 and 1 must be the smallest element in $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$, which is a contradiction since $0 \neq_{E_1 \cup E_2} 1$.

In our general combination result, this kind of problem is avoided by restricting the attention to theories whose signatures do not contain constant symbols, that is, the only constants that may occur are free constants. This restriction will allow us to use Corollary 3.5.[3] The second restriction will be that the orderings to be combined must satisfy the CDC.

[3]Actually, it would be sufficient to apply the restriction to only one of the two theories to be combined since Corollary 3.5 is only needed for one of the two theories.

$$
\begin{array}{ccc}
\dots & & \dots \\
 & & \\
A_{n+1} \setminus (A_n \cup C_{n+1}) & \xrightarrow{h_{n+1}} & D_{n+1} \\
C_{n+1} & \xleftarrow{g_{n+1}} & B_{n+1} \setminus (B_n \cup D_n) \\
A_n \setminus (A_{n-1} \cup C_n) & \xrightarrow{h_n} & D_n \\
C_n & \xleftarrow{g_n} & B_n \setminus (B_{n-1} \cup D_{n-1}) \\
 & & \\
\dots & & \dots \\
 & & \\
C_2 & \xleftarrow{g_2} & B_2 \setminus (B_1 \cup D_1) \\
A_1 \setminus (A_0 \cup C_1) & \xrightarrow{h_1} & D_1 \\
C_1 & \xleftarrow{g_1} & B_1 \setminus D_0 \\
A_0 & \xrightarrow{h_0} & D_0
\end{array}
$$

Figure 2: The double tower of the amalgamation construction.

Our method for combining compatible reduction orderings depends on the representation of $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$ as the free amalgamated product of $\langle C \rangle_{\Sigma_1, E_1}$ and $\langle C \rangle_{\Sigma_2, E_2}$, as introduced in [1].[4]

The free amalgamated product

The free amalgamated product of $\langle C \rangle_{\Sigma_1, E_1}$ and $\langle C \rangle_{\Sigma_2, E_2}$ is defined using two ascending towers of the following form: We consider disjoint sets of free constants $C_\infty = \bigcup_{i=0}^{\infty} C_i$ and $D_\infty = \bigcup_{i=0}^{\infty} D_i$ such that $C_0 = C$. In addition, for $n \geq 0$, let A_n be the carrier set of $\langle \bigcup_{i=0}^{n} C_i \rangle_{\Sigma_1, E_1}$, and let B_{n+1} be the carrier set of $\langle \bigcup_{i=0}^{n} D_i \rangle_{\Sigma_2, E_2}$. The partitioning of C_∞ and D_∞ into the sets C_i and D_i is such that sets on corresponding floors of the double tower shown in Figure 2 have the same cardinality. Thus, there are bijections $h_0 : A_0 \to D_0$, $g_1 : B_1 \setminus D_0 \to C_1$, and for all $n \geq 1$, bijections $h_n : A_n \setminus (A_{n-1} \cup C_n) \to D_n$ and $g_{n+1} : B_{n+1} \setminus (B_n \cup D_n) \to C_{n+1}$.

Let A_∞ be the carrier set of $\langle C_\infty \rangle_{\Sigma_1, E_1}$, i.e., the union of all set in the left tower, and let B_∞ be the carrier set of $\langle D_\infty \rangle_{\Sigma_2, E_2}$, i.e., the union of all set in the right tower. The above bijections can be used in the obvious way to define

[4]It should be noted, however, that we use a slightly modified construction, which is not as symmetric as the original one, but more easy to adapt to our purposes.

bijections h_∞ and g_∞, which are inverse to each other:

$$h_\infty := \bigcup_{i=0}^{\infty} h_i \cup g_{i+1}^{-1} : A_\infty \to B_\infty \quad \text{and} \quad g_\infty := \bigcup_{i=0}^{\infty} h_i^{-1} \cup g_{i+1} : B_\infty \to A_\infty.$$

By definition, A_∞ is equipped with a Σ_1-structure and B_∞ with a Σ_2-structure. The bijections h_∞ and g_∞ can be used to carry the Σ_2-structure on B_∞ to A_∞ and the Σ_1-structure on A_∞ to B_∞. Let f_1 be an n-ary symbol in Σ_1, f_2 an n-ary symbol in Σ_2, $a_1, \ldots, a_n \in A_\infty$, and $b_1, \ldots, b_n \in B_\infty$. We define

$$\begin{aligned} f_2^{\mathcal{A}_\infty}(a_1, \ldots, a_n) &:= g_\infty(f_2^{\mathcal{B}_\infty}(h_\infty(a_1), \ldots, h_\infty(a_n))), \\ f_1^{\mathcal{B}_\infty}(b_1, \ldots, b_n) &:= h_\infty(f_1^{\mathcal{A}_\infty}(g_\infty(b_1), \ldots, g_\infty(b_n))). \end{aligned}$$

Thus, we obtain a $(\Sigma_1 \cup \Sigma_2)$-algebra $\mathcal{A}_\infty$ and a $(\Sigma_1 \cup \Sigma_2)$-algebra $\mathcal{B}_\infty$. By definition, the bijections h_∞ and g_∞ are inverse $(\Sigma_1 \cup \Sigma_2)$-isomorphisms between these algebras. The algebra $\mathcal{A}_\infty \simeq \mathcal{B}_\infty$ is the *free amalgamated product* of $\langle C \rangle_{\Sigma_1, E_1}$ and $\langle C \rangle_{\Sigma_2, E_2}$. As shown in [1], this algebra is isomorphic to $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$.

An ordering on the free amalgamated product

As mentioned above, we assume that the signatures Σ_1 and Σ_2 do not contain constant symbols, i.e., the only constants are free constants. In addition, assume that, for $i = 1, 2$, there is a mechanism for constructing E_i-compatible reduction orderings that satisfies the following properties:

1. For any finite or countably infinite set of free constants C and any total Noetherian ordering $>$ on C, the mechanism yields an E_i-compatible reduction ordering $\succ_{C,>}^{(i)}$ that extends $>$, is total on $\langle C \rangle_{\Sigma_i, E_i}$, and satisfies the CDC.

2. The mechanism is monotone in the following sense: Let $C_1 \subseteq C_2$, let $>_1$ be a total Noetherian ordering on C_1, and let $>_2$ be a total Noetherian ordering on C_2 such that $>_1 \subseteq >_2$. Then $\succ_{C_1,>_1}^{(i)} \subseteq \succ_{C_2,>_2}^{(i)}$.

3. The mechanism is invariant under monotone renaming of free constants. To be more precise, let $>_1$ be a total Noetherian ordering on C_1, $>_2$ be a total Noetherian ordering on C_2, and let $\pi : C_1 \to C_2$ be an order isomorphism. Then $s \succ_{C_1,>_1}^{(i)} t$ implies $\pi(s) \succ_{C_2,>_2}^{(i)} \pi(t)$, where the terms $\pi(s), \pi(t)$ are obtained from s, t by replacing the free constants in these terms by their π-images.

Theorem 4.2 *Assume that Σ_1 and Σ_2 are disjoint signatures that do not contain constant symbols, and that, for $i = 1, 2$, there exist mechanisms for constructing E_i-compatible reduction orderings total on ground terms satisfying the three conditions from above. Then there exists an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$.*

Before presenting a formal proof of the theorem, we give an intuitive description of how this ordering looks like. Its definition depends on the representation of $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$ as the free amalgamated product $\mathcal{A}_\infty$ of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$. Going from bottom to top, one simultaneously defines an ordering on A_∞ and B_∞ by induction. Elements that belong to different levels of one of the towers are compared according to their height in the tower. Elements in a level $A_n \setminus (A_{n-1} \cup C_n)$ are compared with respect to the E_1-compatible ordering on A_n obtained by the mechanism (assuming that the precedence ordering on $\bigcup_{i=0}^n C_i$ is already defined). Elements in a level C_n are ordered using the bijection $g_n : B_n\setminus(B_{n-1}\cup D_{n-1}) \to C_n$ (assuming that the ordering on $B_n \setminus (B_{n-1} \cup D_{n-1})$ is already defined). The right tower is treated analogously.

Formally, the combined ordering is defined by induction on the level of the double tower. We will make use of the following "layer endomorphisms" (cf. Proposition 3.4): α is the endomorphism on $\langle C_\infty\rangle_{\Sigma_1,E_1}$ that maps every element c of C_∞ to the representative $\widehat{c}_i$ of the set C_i with $c \in C_i$. Accordingly, β is the endomorphism on $\langle D_\infty\rangle_{\Sigma_2,E_2}$ that maps every element d of D_∞ to the representative $\widehat{d}_i$ of the set D_i with $d \in D_i$.

Induction base: Here we consider the first two layers of the double tower.

(1) Let $>_{C,0}$ be an arbitrary total Noetherian ordering on $C = C_0$, and let $\succ^{(1)}_{C_0,>_{C,0}}$ be the E_1-compatible reduction ordering total on A_0 produced by the mechanism. We define the ordering $\succ^{(1)}_0$ on A_0 by

$$\begin{aligned} a \succ^{(1)}_0 a' \quad \text{iff} \quad & \alpha(a) \succ^{(1)}_{C_0,>_{C,0}} \alpha(a') \text{ or} \\ & \alpha(a) = \alpha(a') \text{ and } a \succ^{(1)}_{C_0,>_{C,0}} a'. \end{aligned}$$

The bijection $h_0 : A_0 \to D_0$ is now used to transfer this ordering (which is a total and E_1-compatible reduction ordering by construction) to D_0:

$$d >_{D,0} d' \;\text{ iff }\; h_0^{-1}(d) \succ^{(1)}_0 h_0^{-1}(d').$$

Obviously, the ordering $>_{D,0}$ is a total Noetherian ordering on D_0.

(2) Using $>_{D,0}$, the mechanism produces an E_2-compatible reduction ordering $\succ^{(2)}_{D_0,>_{D,0}}$ that extends $>_{D,0}$ and is total on B_1. We define the ordering $\succ^{(2)}_1$ as

$$\begin{aligned} b \succ^{(2)}_1 b' \quad \text{iff} \quad & \beta(b) \succ^{(2)}_{D_0,>_{D,0}} \beta(b') \text{ or} \\ & \beta(b) = \beta(b') \text{ and } b \succ^{(2)}_{D_0,>_{D,0}} b'. \end{aligned}$$

The bijection $g_1 : B_1 \setminus D_0 \to C_1$ is now used to transfer this ordering to C_1:

$$c >_{C,1} c' \;\text{ iff }\; g_1^{-1}(c) \succ^{(2)}_1 g_1^{-1}(c').$$

The ordering $>_{C,1}$ on C_1 is extended to $C_0 \cup C_1$ as follows: on C_0 it coincides with $>_{C,0}$, and for $c \in C_1$ and $c' \in C_0$ we set $c >_{C,1} c'$.

Induction step: Here we consider the layer of D_n and C_{n+1} ($n \geq 1$). By induction, we already have a total Noetherian ordering $>_{C,n}$ on $\widehat{C}_n := \bigcup_{i=0}^{n} C_i$ that satisfies

$$(*) \quad c \in C_i, c' \in C_j, i > j \;\Rightarrow\; c >_{C,n} c',$$

an E_2-compatible reduction ordering $\succ_n^{(2)}$ total on B_n, a total Noetherian ordering $>_{D,n-1}$ on $\widehat{D}_{n-1} := \bigcup_{i=0}^{n-1} D_i$ that satisfies a condition analogous to $(*)$, and an E_1-compatible reduction ordering $\succ_{n-1}^{(1)}$ that is total on A_{n-1}. In addition, we may assume that these orderings are related to each other just like the corresponding orderings constructed in the base case (for $n = 1$).

(1) Let $\succ^{(1)}_{\widehat{C}_n,>_{C,n}}$ be the E_1-compatible reduction ordering total on A_n produced by the mechanism. We define the ordering $\succ_n^{(1)}$ on A_n by

$$\begin{aligned} a \succ_n^{(1)} a' \quad \text{iff} \quad & \alpha(a) \succ^{(1)}_{\widehat{C}_n,>_{C,n}} \alpha(a') \text{ or} \\ & \alpha(a) = \alpha(a') \text{ and } a \succ^{(1)}_{\widehat{C}_n,>_{C,n}} a'. \end{aligned}$$

By construction and part (2) of the proof of Proposition 3.4, $\succ_n^{(1)}$ extends $>_{C,n}$. In addition, $\succ_n^{(1)}$ extends $\succ_{n-1}^{(1)}$. In fact, $>_{C,n}$ extends $>_{C,n-1}$ (by construction), and thus monotonicity of the mechanism yields that $\succ^{(1)}_{\widehat{C}_n,>_{C,n}}$ extends $\succ^{(1)}_{\widehat{C}_{n-1},>_{C,n-1}}$.

The bijection $h_n : A_n \setminus (A_{n-1} \cup C_n) \to D_n$ is now used to transfer this ordering to D_n:

$$d >_{D,n} d' \quad \text{iff} \quad h_n^{-1}(d) \succ_n^{(1)} h_n^{-1}(d').$$

The ordering $>_{D,n}$ on D_n is extended to $\widehat{D}_n := \widehat{D}_{n-1} \cup D_n$ as follows: on $\widehat{D}_{n-1}$ it coincides with $>_{D,n-1}$, and for $d \in D_n$ and $d' \in \widehat{D}_{n-1}$ we set $d >_{D,n} d'$.

(2) Using $>_{D,n}$, the mechanism produces an E_2-compatible reduction ordering $\succ^{(2)}_{\widehat{D}_n,>_{D,n}}$ that extends $>_{D,n}$ and is total on B_{n+1}. We define the ordering $\succ_{n+1}^{(2)}$ as

$$\begin{aligned} b \succ_{n+1}^{(2)} b' \quad \text{iff} \quad & \beta(b) \succ^{(2)}_{\widehat{D}_n,>_{D,n}} \beta(b') \text{ or} \\ & \beta(b) = \beta(b') \text{ and } b \succ^{(2)}_{\widehat{D}_n,>_{D,n}} b'. \end{aligned}$$

This ordering extends $>_{D,n}$ and $\succ_n^{(2)}$.

The bijection $g_{n+1} : B_{n+1} \setminus (B_n \cup D_n) \to C_{n+1}$ is now used to transfer this ordering to C_{n+1}:

$$c >_{C,n+1} c' \quad \text{iff} \quad g_{n+1}^{-1}(c) \succ_{n+1}^{(2)} g_{n+1}^{-1}(c').$$

This ordering is extended to $\widehat{C}_n \cup C_{n+1}$ as follows: on $\widehat{C}_n$ it coincides with $>_{C,n}$, and for $c \in C_{n+1}$ and $c' \in \widehat{C}_n$ we set $c >_{C,n+1} c'$. This completes the description of the induction step.

In the limit, we thus obtain a total Noetherian ordering $>_{C,\infty}$ on C_∞ that satisfies the "layer condition" $(*)$:

$$c >_{C,\infty} c' \text{ iff } c >_{C,n} c' \text{ for some } n.$$

The total Noetherian ordering $>_{D,\infty}$ on D_∞ is defined analogously.

Using the ordering $>_{C,\infty}$, the mechanisms produces an E_1-compatible reduction ordering $\succ^{(1)}_{C_\infty,>_{C,\infty}}$ that is total on A_∞. Correspondingly, $>_{D,\infty}$ yields an E_2-compatible reduction ordering $\succ^{(2)}_{D_\infty,>_{D,\infty}}$ that is total on B_∞. The endomorphisms α, β are now used to obtain orderings respecting the layers:

$$\begin{aligned} a \succ^{(1)}_\infty a' \quad \text{iff} \quad & \alpha(a) \succ^{(1)}_{C_\infty,>_{C,\infty}} \alpha(a') \text{ or} \\ & \alpha(a) = \alpha(a') \text{ and } a \succ^{(1)}_{C_\infty,>_{C,\infty}} a', \\ b \succ^{(2)}_\infty b' \quad \text{iff} \quad & \beta(b) \succ^{(2)}_{D_\infty,>_{D,\infty}} \beta(b') \text{ or} \\ & \beta(b) = \beta(b') \text{ and } b \succ^{(2)}_{D_\infty,>_{D,\infty}} b'. \end{aligned}$$

Since the tower on the right-hand side distinguishes between $B_1 \setminus D_0$ and D_0, we need Corollary 3.5 (and thus the restriction that Σ_2 does not contain constants) to obtain an ordering that respects these two layers.

By construction, $\succ^{(1)}_\infty$ is stable under the interpretation of the Σ_1-operations in $\mathcal{A}_\infty$, and $\succ^{(2)}_\infty$ is stable under the interpretation of the Σ_2-operations in $\mathcal{B}_\infty$. Below we shall show that $\succ^{(1)}_\infty$ is also stable under the interpretation of the Σ_2-operations in $\mathcal{A}_\infty$ (as introduced in the definition of the free amalgamated product), and $\succ^{(2)}_\infty$ is stable under the interpretation of the Σ_1-operations in $\mathcal{B}_\infty$.

Lemma 4.3 *1. $\succ^{(1)}_\infty$ extends $>_{C,\infty}$ and $\succ^{(1)}_n$ (for all $n \geq 0$).*

2. $\succ^{(2)}_\infty$ extends $>_{D,\infty}$ and $\succ^{(2)}_n$ (for all $n \geq 1$).

Proof. (1) By construction and part (2) of the proof of Proposition 3.4, $\succ^{(1)}_\infty$ extends $>_{C,\infty}$. In addition, $>_{C,\infty}$ extends $>_{C,n}$ (for all $n \geq 0$). Monotonicity of the construction mechanism thus implies that $\succ^{(1)}_{C_\infty,>_{C,\infty}}$ extends $\succ^{(1)}_{\widehat{C}_n,>_{C,n}}$. It is easy to see that this implies that $\succ^{(1)}_\infty$ extends $\succ^{(1)}_n$.

(2) can be proved analogously. □

Lemma 4.4 *$h_\infty : A_\infty \to B_\infty$ and $g_\infty : B_\infty \to A_\infty$ are inverse order isomorphisms.*

Proof. Without loss of generality, we restrict our attention to h_∞. Let $a, a' \in A_\infty$ be such that $a \succ^{(1)}_\infty a'$. In order to show that $h_\infty(a) \succ^{(2)}_\infty h_\infty(a')$, we distinguish the following cases:

Case 1: a and a' are in different layers. Since the ordering $\succ_\infty^{(1)}$ respects the layers, this means that a is in a higher layer than a'. Consequently, $h_\infty(a)$ is in a higher layer than $h_\infty(a')$, and since $\succ_\infty^{(2)}$ also respects the layers, this implies $h_\infty(a) \succ_\infty^{(2)} h_\infty(a')$.

Case 2: a and a' are in the same layer.

(2.1) Assume that $a, a' \in C_n$. Since $\succ_\infty^{(1)}$ extends $>_{C,\infty}$, we have $a >_{C,\infty} a'$, and thus $a >_{C,n} a'$.

For $n = 0$, $a >_{C,0} a'$ implies $a \succ_0^{(1)} a'$, which in turn implies $h_\infty(a) = h_0(a) >_{D,0} h_0(a') = h_\infty(a')$. This yields $h_\infty(a) \succ_\infty^{(2)} h_\infty(a')$ since $\succ_\infty^{(2)}$ extends $>_{D,0}$.

For $n > 0$, $a >_{C,n} a'$ and the definition of $>_{C,n}$ yield $g_n^{-1}(a) \succ_n^{(2)} g_n^{-1}(a')$. By definition of h_∞, we have $h_\infty(a) = g_n^{-1}(a)$ and $h_\infty(a') = g_n^{-1}(a')$, which shows $h_\infty(a) \succ_n^{(2)} h_\infty(a')$. Now, the previous lemma yields $h_\infty(a) \succ_\infty^{(2)} h_\infty(a')$.

(2.2) Assume that $a, a' \in A_n \setminus (A_{n-1} \cup C_n)$. From $a \succ_\infty^{(1)} a'$ we can infer $a \succ_n^{(1)} a'$, by the previous lemma. The definition of $>_{D,n}$ then yields $h_n(a) >_{D,n} h_n(a')$. Since $\succ_\infty^{(2)}$ extends $>_{D,n}$, we obtain $h_\infty(a) = h_n(a) \succ_\infty^{(2)} h_n(a') = h_\infty(a')$. $\square$

Lemma 4.5 *$\succ_\infty^{(1)}$ is also stable under the Σ_2-operations, and $\succ_\infty^{(2)}$ is stable under the Σ_1-operations.*

Proof. Without loss of generality, we restrict our attention to $\succ_\infty^{(1)}$. Let $f_2 \in \Sigma_2$ be n-ary, and assume that $a_1, \ldots, a_n, a'_i \in A_\infty$ are given such that $a_i \succ_\infty^{(1)} a'_i$. Because h_∞ is an order isomorphism, this implies $h_\infty(a_i) \succ_\infty^{(2)} h_\infty(a'_i)$, and since $\succ_\infty^{(2)}$ is by definition stable under Σ_2-operations, we obtain

$$f_2^{\mathcal{B}_\infty}(h_\infty(a_1), \ldots, h_\infty(a_i), \ldots, h_\infty(a_n)) \succ_\infty^{(2)} f_2^{\mathcal{B}_\infty}(h_\infty(a_1), \ldots, h_\infty(a'_i), \ldots, h_\infty(a_n)).$$

By definition of $f_2^{\mathcal{A}_\infty}$, and since g_∞ is an order isomorphism, this implies

$$f_2^{\mathcal{A}_\infty}(a_1, \ldots, a_i, \ldots, a_n) \succ_\infty^{(1)} f_2^{\mathcal{A}_\infty}(a_1, \ldots, a'_i, \ldots, a_n).$$

$\square$

To sum up, we have shown that $\succ_\infty^{(1)}$ is a total reduction ordering (with respect to the combined signature) on the free amalgamated product $\mathcal{A}_\infty$ of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$. Since $\mathcal{A}_\infty$ is (isomorphic to) the $(E_1 \cup E_2)$-free algebra $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$, this completes the proof of Theorem 4.2.

In the inductive construction of $\succ_\infty^{(1)}$, the induction base is given by an arbitrary total Noetherian ordering on C. The combined ordering obtained this way depends on the set C and on the ordering on C used for starting the inductive construction. Thus, we again obtain a construction mechanism that transforms a given total Noetherian ordering on a set of free constants C into an $(E_1 \cup E_2)$-compatible reduction ordering that is total on $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$. The combined ordering does not satisfy the CDC. However, if E_1 and E_2 are strongly regular, then so is $E_1 \cup E_2$. Thus, Proposition 3.3 can be used to modify the combined ordering into one satisfying the CDC. It can be shown that the mechanism satisfies the other properties required in Theorem 4.2. Consequently, the construction can be applied iteratedly, provided that the involved theories are strongly regular.

5 A decision procedure for the combined ordering

If we want to use the combined ordering in an application, we must be able to effectively work with it, that is, for given mixed terms s, t we must be able to decide whether $s \succ^{(1)}_{\infty} t$ holds or not.

Theorem 5.1 *If the word problem for E_i and the orderings $\succ^{(i)}_{C,>}$ are decidable for $i = 1, 2$, then the combined ordering $\succ^{(1)}_{\infty}$ is also decidable.*

The decision procedure for the combined ordering depends on a method that is similar to the approach used to show that the word problem for $E_1 \cup E_2$ is decidable, provided that the word problems for the single theories E_1, E_2 are decidable (see, e.g., [19, 13]). Note that a decision procedure for the combined ordering also yields a decision procedure for $=_{E_1\cup E_2}$ since $s =_{E_1\cup E_2} t$ iff neither $s \succ^{(1)}_{\infty} t$ nor $t \succ^{(1)}_{\infty} s$ holds.

The free amalgamated product is an algebraic description of how the combined algebra $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$ can be obtained from $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$. In order to obtain a decision procedure for $=_{E_1\cup E_2}$ and for $\succ^{(1)}_{\infty}$, we will look at the combined algebra from a more syntactic point of view: $\langle C\rangle_{\Sigma_1\cup\Sigma_2,E_1\cup E_2}$ can also be obtained as the quotient of the term algebra $T(\Sigma_1 \cup \Sigma_2, C)$ by the congruence $=_{E_1\cup E_2}$. But first, we must introduce some notation.

As above, we assume that Σ_1 and Σ_2 are disjoint signatures that do not contain constant symbols. For a term $t \in T(\Sigma_1 \cup \Sigma_2, C)$, the *top symbol* $top(t)$ is $f \in \Sigma_1 \cup \Sigma_2 \cup C$ iff t starts with f, i.e., t is of the form $f(t_1, \ldots, t_n)$ for $n \geq 0$ and terms $t_1, \ldots, t_n$. The elements of $C \cup \Sigma_1$ are called *1-symbols*, and the elements of Σ_2 are called *2-symbols*. An *i-term* ($i = 1, 2$) is a term whose top symbol is an i-symbol. Let t be an i-term and s a j-term for $i \neq j$. Then s is called an *alien subterm* of t iff s is a subterm of t such that all its superterms in t are i-terms. For a term $t \in T(\Sigma_1 \cup \Sigma_2, C)$, the depth $depth(t)$ is the maximal number of signature changes in t. To be more precise, $depth(t) := 0$ if $t \in T(\Sigma_1, C)$. Otherwise, let t be of the form $t = \hat{t}(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are the alien subterms of t. Then $depth(t) := 1 + \max\{depth(t_1), \ldots, depth(t_n)\}$.

The decision procedure for $=_{E_1 \cup E_2}$ and for $\succ_\infty^{(1)}$ depends on computing a layer-reduced form and on variable abstraction. Let Π be a bijection between the combined algebra $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$ and a set of variables Y of appropriate cardinality. Obviously, Π induces a mapping $\pi : T(\Sigma_1 \cup \Sigma_2, C) \to Y : t \mapsto \Pi([t])$, where $[t]$ denotes the $=_{E_1 \cup E_2}$-equivalence class of the term t. This mapping satisfies $\pi(s) = \pi(t)$ iff $s =_{E_1 \cup E_2} t$. The *variable abstractions* π_1, π_2 induced by π are defined by induction on the depth of the term t. If $t \in T(\Sigma_1, C)$, then $t^{\pi_1} := t$ and $t^{\pi_2} := \pi(t)$. Otherwise, let t be an i-term of the form $t = \widehat{t}(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are the alien subterms of t. Then $t^{\pi_i} := t(\pi(t_1), \ldots, \pi(t_n))$ and $t^{\pi_j} := \pi(t)$ for $j \neq i$. Thus, the variable abstraction π_i replaces j-terms and alien subterms of i-terms by variables such that $=_{E_1 \cup E_2}$-equivalent terms are replaced by the same variable.

The *layer-reduced form* $t\Downarrow$ of $t \in T(\Sigma_1 \cup \Sigma_2, C)$ is also defined by induction on the depth of the term t. If $t \in T(\Sigma_1, C)$, then $t\Downarrow := t$. Otherwise, let t be an i-term of the form $t = \widehat{t}(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are the alien subterms of t. By induction, we may assume that $t_1\Downarrow, \ldots, t_n\Downarrow$ are already defined. Let $\widehat{t}(t_1\Downarrow, \ldots, t_n\Downarrow) = \widehat{s}(s_1, \ldots, s_m)$, where $s_1, \ldots, s_m$ are the alien subterms of $\widehat{t}(t_1\Downarrow, \ldots, t_n\Downarrow)$. If there is a $\nu \in \{1, \ldots, m\}$ such that $\widehat{s}(s_1^{\pi_i}, \ldots, s_m^{\pi_i}) =_{E_i} s_\nu^{\pi_i}$, then $t\Downarrow := s_\nu$.[5] Otherwise, $t\Downarrow := \widehat{t}(t_1\Downarrow, \ldots, t_n\Downarrow) = \widehat{s}(s_1, \ldots, s_m)$. Note that the terms s_i are layer-reduced (i.e., $s_i = s_i\Downarrow$) since they are subterms of the layer-reduced terms $t_j\Downarrow$. By construction, we have $t =_{E_1 \cup E_2} t\Downarrow$ for all terms $t \in T(\Sigma_1 \cup \Sigma_2, C)$.

With this notation, the relation $=_{E_1 \cup E_2}$ can be characterized as follows (see, e.g., [19, 13] for a proof):

Lemma 5.2 *Let $t, t' \in T(\Sigma_1 \cup \Sigma_2, C)$. Then $t =_{E_1 \cup E_2} t'$ iff $top(t\Downarrow) = i = top(t'\Downarrow)$ (for some $i \in \{1, 2\}$) and $(t\Downarrow)^{\pi_i} =_{E_i} (t'\Downarrow)^{\pi_i}$.*

At first sight, it might not be obvious that this lemma can be used to construct a decision procedure for $=_{E_1 \cup E_2}$ from decision procedures for $=_{E_1}$ and $=_{E_2}$. In particular, it is not clear how to obtain an appropriate bijection Π without already having a decision procedure for $=_{E_1 \cup E_2}$. However, one can construct the necessary parts of such a bijection by induction on the depth of terms. In the following, assume that terms $t, t' \in T(\Sigma_1 \cup \Sigma_2, C)$ are given.

In order to compute $t\Downarrow$ and $t'\Downarrow$, we determine the alien subterms $t_1, \ldots, t_n$ of $t = \widehat{t}(t_1, \ldots, t_n)$ and $t'_1, \ldots, t'_{n'}$ of $t' = \widehat{t}'(t'_1, \ldots, t'_{n'})$. By induction, we may assume that we already know how to compute $t_1\Downarrow, \ldots, t_n\Downarrow, t'_1\Downarrow, \ldots, t'_{n'}\Downarrow$. Thus, we can also compute the alien subterms $s_1, \ldots, s_m$ of $\widehat{t}(t_1\Downarrow, \ldots, t_n\Downarrow) = \widehat{s}(s_1, \ldots, s_m)$ and $s'_1, \ldots, s'_{m'}$ of $\widehat{t}'(t'_1\Downarrow, \ldots, t'_n\Downarrow) = \widehat{s}'(s'_1, \ldots, s'_{m'})$. Since these alien subterms have a smaller depth than the original terms, we may assume that we can decide $=_{E_1 \cup E_2}$ on them. For this reason, we can effectively construct a mapping

[5] If there is more than on such ν, then one arbitrarily takes one of them. If $s_\nu^{\pi_i} =_{E_i} \widehat{s}(s_1^{\pi_i}, \ldots, s_m^{\pi_i}) =_{E_i} s_\mu^{\pi_i}$, then $s_\nu^{\pi_i} = s_\mu^{\pi_i}$ (since E_i is non-trivial), and thus $s_\nu =_{E_1 \cup E_2} s_\mu$.

$\pi' : \{s_1, \ldots, s_m, s'_1, \ldots, s'_{m'}\} \to Y'$, for an appropriate set of variables Y', such that $\pi'(u) = \pi'(v)$ iff $u =_{E_1 \cup E_2} v$ holds for all $u, v \in \{s_1, \ldots, s_m, s'_1, \ldots, s'_{m'}\}$. In order to compute the necessary variable abstractions, it is sufficient to use this mapping π'. (Note that $=_{E_i}$ is invariant under renaming of variables.) Assume that π'_1, π'_2 are the corresponding abstraction mappings. Since $=_{E_i}$ was assumed to be decidable on pure terms, we can decide $\widehat{s}(s_1^{\pi'_i}, \ldots, s_m^{\pi'_i}) =_{E_i} s_\nu^{\pi'_i}$ and $\widehat{s}'(s_1'^{\pi'_i}, \ldots, s_{m'}'^{\pi'_i}) =_{E_i} s_\nu'^{\pi'_i}$. This shows that $t\Downarrow$ and $t'\Downarrow$ can be computed.

Now, $top(t\Downarrow) = top(t'\Downarrow)$ is obviously decidable, and $(t\Downarrow)^{\pi_i}$ and $(t'\Downarrow)^{\pi_i}$ can be computed since induction again yields that we can decide $=_{E_1 \cup E_2}$ on the alien subterms of $t\Downarrow$ and $t'\Downarrow$. Finally, $(t\Downarrow)^{\pi_i} =_{E_i} (t'\Downarrow)^{\pi_i}$ is decidable by assumption.

In order to decide $\succ_\infty^{(1)}$, we must be able to determine to which layer of the amalgamated product (the equivalence class of) a given term in $T(\Sigma_1 \cup \Sigma_2, C)$ belongs. The following theorem shows that there is a close connection between this layer and the depth of the layer-reduced form of the term. For a given term $t \in T(\Sigma_1 \cup \Sigma_2, C)$, we denote its interpretation in $\mathcal{A}_\infty$ (resp. B_∞) by $t^{\mathcal{A}_\infty}$ (resp. t^{B_∞}).

Theorem 5.3 *Assume that E_1 and E_2 are regular theories over disjoint signatures Σ_1 and Σ_2, and that the free amalgamated product of $\langle C\rangle_{\Sigma_1,E_1}$ and $\langle C\rangle_{\Sigma_2,E_2}$ is constructed as described in Section 4. Let $t \in T(\Sigma_1 \cup \Sigma_2, C)$.*

(A_0) *$t^{\mathcal{A}_\infty} \in A_0$ iff $top(t\Downarrow) \in \Sigma_1 \cup C$ and $depth(t\Downarrow) = 0$.*

(D_0) *$t^{\mathcal{B}_\infty} \in D_0$ iff $top(t\Downarrow) \in \Sigma_1 \cup C$ and $depth(t\Downarrow) = 0$.*

(A_n) *For $n > 0$, $t^{\mathcal{A}_\infty} \in A_n \setminus (A_{n-1} \cup C_n)$ iff $top(t\Downarrow) \in \Sigma_1$ and $depth(t\Downarrow) = 2n$.*

(D_n) *For $n > 0$, $t^{\mathcal{B}_\infty} \in D_n$ iff $top(t\Downarrow) \in \Sigma_1$ and $depth(t\Downarrow) = 2n$.*

(B_n) *For $n > 0$, $t^{\mathcal{B}_\infty} \in B_n \setminus (B_{n-1} \cup D_{n-1})$ iff $top(t\Downarrow) \in \Sigma_2$ and $depth(t\Downarrow) = 2n - 1$.*

(C_n) *For $n > 0$, $t^{\mathcal{A}_\infty} \in C_n$ iff $top(t\Downarrow) \in \Sigma_2$ and $depth(t\Downarrow) = 2n - 1$.*

Proof. First, note that 1-terms, i.e., terms with top symbol in $\Sigma_1 \cup C$, have an even depth, whereas 2-terms, i.e., terms with top symbol in Σ_2 have an odd depth. This is so because elements of C are counted as 1-symbols and $\Sigma_1 \cup \Sigma_2$ does not contain constants.

(A_0) Assume that $t^{\mathcal{A}_\infty} \in A_0 = T(\Sigma_1, C)/{=_{E_1}}$, i.e., there exists a term $s \in T(\Sigma_1, C)$ such that $t^{\mathcal{A}_\infty} = s^{\mathcal{A}_\infty}$. Since $t =_{E_1 \cup E_2} t\Downarrow$ and $s = s\Downarrow$, and since $\mathcal{A}_\infty$ is the $(E_1 \cup E_2)$-free algebra with generators C, this implies $t\Downarrow =_{E_1 \cup E_2} s\Downarrow$. Consequently, $top(t\Downarrow) = top(s\Downarrow) \in \Sigma_1 \cup C$, and $(t\Downarrow)^{\pi_1} =_{E_1} (s\Downarrow)^{\pi_1}$. Now, $s \in T(\Sigma_1, C)$ implies that $(s\Downarrow)^{\pi_1}$ does not contain abstraction variables (i.e., variables that replace alien subterms). Since E_1 is regular, this implies that $(t\Downarrow)^{\pi_1}$ does not contain abstraction variables, and thus $depth(t\Downarrow) = 0$.

Conversely, assume that $top(t\Downarrow) \in \Sigma_1 \cup C$ and $depth(t\Downarrow) = 0$. In this case, $t\Downarrow \in T(\Sigma_1, C)$, and thus $t^{\mathcal{A}_\infty} = (t\Downarrow)^{\mathcal{A}_\infty} \in A_0 = T(\Sigma_1, C)/{=_{E_1}}$.

(D_0) Assume that $t^{\mathcal{B}_\infty} \in D_0$. By construction of the free amalgamated product, this implies $t^{\mathcal{A}_\infty} = g_\infty(t^{\mathcal{B}_\infty}) \in A_0$. Thus, $top(t\Downarrow) \in \Sigma_1 \cup C$, and $depth(t\Downarrow) = 0$ follows from (A_0).

Conversely, $top(t\Downarrow) \in \Sigma_1 \cup C$ and $depth(t\Downarrow) = 0$ yields $t^{\mathcal{A}_\infty} \in A_0$ by (A_0), and thus $t^{\mathcal{B}_\infty} = h_\infty(t^{\mathcal{A}_\infty}) \in D_0$ by construction of the free amalgamated product.

In the following, we assume that $n > 0$. We prove (A_n), (D_n), (B_n), and (C_n) by *induction on n* for the "only-if" direction and by *induction on depth*$(t\Downarrow)$ for the "if" direction.

(B_n) Assume that $t^{\mathcal{B}_\infty} \in B_n \setminus (B_{n-1} \cup D_{n-1})$. This means that there exists a term $\widehat{s} \in T(\Sigma_2, \widehat{D}_{n-1})$ such that

- $\widehat{s}^{\mathcal{B}_\infty} = t^{\mathcal{B}_\infty} = (t\Downarrow)^{\mathcal{B}_\infty}$,
- $\widehat{s}$ contains at least one element of D_{n-1} since otherwise $\widehat{s}^{\mathcal{B}_\infty} \in B_{n-1}$,
- $\widehat{s} \neq_{E_2} d$ for all $d \in D_{n-1}$ since otherwise $\widehat{s}^{\mathcal{B}_\infty} \in D_{n-1}$. Note that $\widehat{s} =_{E_2} d$ for some $d \in \widehat{D}_{n-2}$ is impossible because E_2 was assumed to be regular.

(Recall that $\widehat{D}_{n-1} = \bigcup_{i=0}^{n-1} D_i$.) Assume that $d_1, \ldots, d_m$ are all the elements of $\widehat{D}_{n-1}$ that occur in $\widehat{s}$, i.e., we can write $\widehat{s} = \widehat{s}(d_1, \ldots, d_m)$. Without loss of generality, we may assume that $d_1 \in D_{n-1}$. Considered as a $(\Sigma_1 \cup \Sigma_2)$-algebra, $\mathcal{B}_\infty$ is generated by C, and thus there exist terms $s_1, \ldots, s_m \in T(\Sigma_1 \cup \Sigma_2, C)$ such that $(s_i\Downarrow)^{\mathcal{B}_\infty} = s_i^{\mathcal{B}_\infty} = d_i$ for $i = 1, \ldots, m$. Let us consider the corresponding layer-reduced terms $s_i\Downarrow$ more closely. We know that $d_i \in D_{m_i}$ for some $m_i \leq n-1$:

$m > 0$: By induction, $(s_i\Downarrow)^{\mathcal{B}_\infty} = d_i \in D_{m_i}$ yields $top(s_i\Downarrow) \in \Sigma_1$ and $depth(s_i\Downarrow) = 2m_i$.

$m = 0$: (D_0) yields $top(s_i\Downarrow) \in \Sigma_1 \cup C$ and $depth(s_i\Downarrow) = 0$.

Let $s := \widehat{s}(s_1\Downarrow, \ldots, s_m\Downarrow)$. Because the free constants in $\widehat{D}_{n-1}$ behave like abstraction variables for $=_{E_2}$, the fact that $\widehat{s} \neq_{E_2} d$ for all $d \in \widehat{D}_{n-1}$ implies that $s\Downarrow = s$. Now, $s^{\mathcal{B}_\infty} = \widehat{s}^{\mathcal{B}_\infty} = (t\Downarrow)^{\mathcal{B}_\infty}$ yields $s\Downarrow = s =_{E_1 \cup E_2} t\Downarrow$, and thus $top(t\Downarrow) = top(s) \in \Sigma_2$ and $s^{\pi_2} =_{E_2} (t\Downarrow)^{\pi_2}$. Since E_2 is regular, this means that $t\Downarrow$ contains (modulo $=_{E_1 \cup E_2}$) the same alien subterms as s, namely, $t_1\Downarrow =_{E_1 \cup E_2} s_1\Downarrow, \ldots, t_m\Downarrow =_{E_1 \cup E_2} s_m\Downarrow$. Thus, we have $(t_i\Downarrow)^{\mathcal{B}_\infty} = (s_i\Downarrow)^{\mathcal{B}_\infty} = d_i$, which yields $top(t_i\Downarrow) \in \Sigma_1 \cup C$ and $depth(t_i\Downarrow) = 2m_i$, where $m_i \leq n-1$ is such that $d_i \in D_{m_i}$. In addition, we know that $d_1 \in D_{n-1}$. This implies that $depth(t\Downarrow) = 2(n-1)+1 = 2n-1$.

Conversely, assume that $top(t\Downarrow) \in \Sigma_2$ and $depth(t\Downarrow) = 2n-1$. Thus, $t\Downarrow = \widehat{t}(t_1\Downarrow, \ldots, t_n\Downarrow)$, where

- $t_1\Downarrow, \ldots, t_n\Downarrow$ are the alien subterms of $t\Downarrow$,
- $(t\Downarrow)^{\pi_2} = \widehat{t}(y_1, \ldots, y_m)$ for abstraction variables $y_1, \ldots, y_m$,
- $\widehat{t}(y_1, \ldots, y_m) \neq_{E_2} y_i$ for $i = 1, \ldots, m$,
- $top(t_i\Downarrow) \in \Sigma_1 \cup C$.
- $depth(t_i\Downarrow) < 2n-1$ for all $i = 1, \ldots, m$, and there exists a $j, 1 \leq j \leq m$, such that $depth(t_j\Downarrow) = 2(n-1)$,

By definition of the depth of a term, $top(t_i\Downarrow) \in \Sigma_1 \cup C$ implies that $depth(t_i\Downarrow)$ is an even number. Thus, there exist numbers $m_i < n$ such that $depth(t_i\Downarrow) = 2m_i$ (for $i = 1, \ldots, m$). Because $depth(t_i\Downarrow) < depth(t\Downarrow)$ and $top(t_i\Downarrow) \in \Sigma_1 \cup C$, induction or (D_0) yields $(t_i\Downarrow)^{\mathcal{B}_\infty} \in D_{m_i}$. This shows that $(t\Downarrow)^{\mathcal{B}_\infty} \in B_n$. We have $(t\Downarrow)^{\mathcal{B}_\infty} \notin B_{n-1}$ because there exists a j such that $d_j := (t_j\Downarrow)^{\mathcal{B}_\infty} \in D_{n-1}$ and E_2 is regular. Finally, $(t\Downarrow)^{\mathcal{B}_\infty} \notin D_{n-1}$ holds since E_2 is non-trivial and $\widehat{t}(y_1, \ldots, y_m) \neq_{E_2} y_i$ for $i = 1, \ldots, m$.

(C_n) Assume that $t^{\mathcal{A}_\infty} \in C_n$. By construction of the free amalgamated product, this implies $t^{\mathcal{B}_\infty} = h_\infty(t^{\mathcal{A}_\infty}) \in B_n \setminus (B_{n-1} \cup D_{n-1})$. Thus, (B_n) yields $top(t\Downarrow) \in \Sigma_2$ and $depth(t\Downarrow) = 2n-1$.

Conversely, $top(t\Downarrow) \in \Sigma_2$ and $depth(t\Downarrow) = 2n-1$ implies $t^{\mathcal{B}_\infty} \in B_n \setminus (B_{n-1} \cup D_{n-1})$, and thus $t^{\mathcal{A}_\infty} = g_\infty(t^{\mathcal{B}_\infty}) \in C_n$.

(A_n) can be treated like (B_n). The only difference is that in some places where the proof of (B_n) used an induction argument (for (D_m) with $m < n$), the proof of (A_n) makes use of the already proved (C_n).

(D_n) can be treated like (C_n). $\square$

The following examples shows that the requirement "E_1 and E_2 regular" is necessary for the theorem to hold. For our purposes, this is not a real restriction since the existence of a non-empty E_i-compatible reduction ordering implies that E_i is regular.

Example 5.4 Let $E_1 := \{f(x,y) = h(y)\}$ and $E_2 := \{g(x) = g(x)\}$, and $c \in C$. We have $f(g(c),c)\Downarrow = f(g(c),c)$ since $g(c)\Downarrow = g(c)$ and $f(y_1,y_2) \neq_{E_1} y_i$ for $i = 1,2$. Thus, $depth(f(g(c),g(c))\Downarrow) = 2$, since $depth(g(c)) = 1$. However, $f(g(c),c)^{\mathcal{A}_\infty} = h(c)^{\mathcal{A}_\infty} \in A_0$. The reason is that, because of the non-regularity of E_1, the layer-reduced term $f(g(c),c)$ is $=_{E_1 \cup E_2}$-equivalent to a term of smaller depth.

If $=_{E_1}$ and $=_{E_2}$ are decidable, then the layer-reduced form of a term $t \in T(\Sigma_1 \cup \Sigma_2, C)$ can be computed, and thus Theorem 5.3 can be used to decide to which layer of the free amalgamated product $t^{\mathcal{A}_\infty}$ belongs. If $s, t \in T(\Sigma_1 \cup \Sigma_2, C)$ belong to different layers, then this already allows to decide whether $s^{\mathcal{A}_\infty} \succ_\infty^{(1)} t^{\mathcal{A}_\infty}$ (or equivalently, $s^{\mathcal{B}_\infty} \succ_\infty^{(2)} t^{\mathcal{B}_\infty}$) holds or not. The next lemma is concerned with the case in which $s^{\mathcal{A}_\infty}$ and $t^{\mathcal{A}_\infty}$ (and thus also $s^{\mathcal{B}_\infty}$ and $t^{\mathcal{B}_\infty}$) belong to the same layer.

Lemma 5.5 *Let $s, t \in T(\Sigma_1 \cup \Sigma_2, C)$.*

(A_0) *If $s^{\mathcal{A}_\infty}, t^{\mathcal{A}_\infty} \in A_0$, then $s^{\mathcal{A}_\infty} \succ_\infty^{(1)} t^{\mathcal{A}_\infty}$ is decidable.*

(D_0) *If $s^{\mathcal{B}_\infty}, t^{\mathcal{B}_\infty} \in D_0$, then $s^{\mathcal{B}_\infty} \succ_\infty^{(2)} t^{\mathcal{B}_\infty}$ is decidable.*

(A_n) *If $s^{\mathcal{A}_\infty}, t^{\mathcal{A}_\infty} \in A_n \setminus (A_{n-1} \cup C_n)$ for $n > 0$, then $s^{\mathcal{A}_\infty} \succ_\infty^{(1)} t^{\mathcal{A}_\infty}$ is decidable.*

(D_n) *If $s^{\mathcal{B}_\infty}, t^{\mathcal{B}_\infty} \in D_n$ for $n > 0$, then $s^{\mathcal{B}_\infty} \succ_\infty^{(2)} t^{\mathcal{B}_\infty}$ is decidable.*

(B_n) *If $s^{\mathcal{B}_\infty}, t^{\mathcal{B}_\infty} \in B_n \setminus (B_{n-1} \cup D_{n-1})$ for $n > 0$, then $s^{\mathcal{B}_\infty} \succ_\infty^{(2)} t^{\mathcal{B}_\infty}$ is decidable.*

(C_n) *If $s^{\mathcal{A}_\infty}, t^{\mathcal{A}_\infty} \in C_n$ for $n > 0$, then $s^{\mathcal{A}_\infty} \succ_\infty^{(1)} t^{\mathcal{A}_\infty}$ is decidable.*

Proof. (A_0) Assume that $s^{\mathcal{A}_\infty}, t^{\mathcal{A}_\infty} \in A_0$. We know that $s^{\mathcal{A}_\infty} = (s\Downarrow)^{\mathcal{A}_\infty}$ and $t^{\mathcal{A}_\infty} = (t\Downarrow)^{\mathcal{A}_\infty}$, and Theorem 5.3 yields $s\Downarrow, t\Downarrow \in T(\Sigma_1, C)$. By Lemma 4.3 and the definition of $\succ_0^{(1)}$, we have

$$(s\Downarrow)^{\mathcal{A}_\infty} \succ_\infty^{(1)} (t\Downarrow)^{\mathcal{A}_\infty} \quad \text{iff} \quad \alpha((s\Downarrow)^{\mathcal{A}_\infty}) \succ^{(1)}_{C_0, >_{C,0}} \alpha((t\Downarrow)^{\mathcal{A}_\infty}) \text{ or}$$
$$\alpha((s\Downarrow)^{\mathcal{A}_\infty}) = \alpha((t\Downarrow)^{\mathcal{A}_\infty}) \text{ and } (s\Downarrow)^{\mathcal{A}_\infty} \succ^{(1)}_{C_0, >_{C,0}} (t\Downarrow)^{\mathcal{A}_\infty}.$$

By definition, α replaces every constant $c \in C = C_0$ occurring in $s\Downarrow$ or $t\Downarrow$ by the representative $\widehat{c}_0$ of C_0. Let s', t' be the terms obtained by this replacement. In order to decide $\alpha((s\Downarrow)^{\mathcal{A}_\infty}) = \alpha((t\Downarrow)^{\mathcal{A}_\infty})$, it is sufficient to decide $s' =_{E_1 \cup E_2} t'$ (which is decidable by Lemma 5.2). In addition, we know that $\succ^{(1)}_{C_0, >_{C,0}}$ is decidable by assumption.

(D_0) Assume that $s^{\mathcal{B}_\infty}, t^{\mathcal{B}_\infty} \in D_0$. By construction of the free amalgamated product, $g_\infty(s^{\mathcal{B}_\infty}) = s^{\mathcal{A}_\infty} \in A_0$ and $g_\infty(t^{\mathcal{B}_\infty}) = t^{\mathcal{A}_\infty} \in A_0$. In addition, since g_∞ is an order isomorphism, $s^{\mathcal{B}_\infty} \succ_\infty^{(2)} t^{\mathcal{B}_\infty}$ iff $t^{\mathcal{A}_\infty} \succ_\infty^{(1)} t^{\mathcal{A}_\infty}$. Thus, (A_0) implies (D_0).

In the following, we assume that $n > 0$. We prove (A_n), (D_n), (B_n), and (C_n) by *induction on n.*

(B_n) Assume that $s^{\mathcal{B}_\infty}, t^{\mathcal{B}_\infty} \in B_n \setminus (B_{n-1} \cup D_{n-1})$. Let $s\Downarrow = \widehat{s}(s_1\Downarrow, \ldots, s_k\Downarrow)$ and $t\Downarrow = \widehat{t}(t_1\Downarrow, \ldots, t_\ell\Downarrow)$, where

- $s_1\Downarrow, \ldots, s_k\Downarrow$ are the alien subterms of $s\Downarrow$ and $t_1\Downarrow, \ldots, t_\ell\Downarrow$ are the alien subterms of $t\Downarrow$,

- $top(s\Downarrow) \in \Sigma_2$ and $top(t\Downarrow) \in \Sigma_2$,
- $top(s_1\Downarrow), \ldots, top(t_\ell\Downarrow) \in \Sigma_1 \cup C$,
- $depth(s_1\Downarrow), \ldots, depth(t_\ell\Downarrow) \leq 2(n-1)$,
- $(s\Downarrow)^{\pi_2} = \widehat{s}(y_1, \ldots, y_k)$ and $(t\Downarrow)^{\pi_2} = \widehat{t}(z_1, \ldots, z_\ell)$ for abstraction variables $y_1, \ldots, z_\ell$.

By Lemma 4.3 and the definition of $\succ_n^{(2)}$, we have $(s\Downarrow)^{\mathcal{B}_\infty} \succ_\infty^{(2)} (t\Downarrow)^{\mathcal{B}_\infty}$ iff

$$\begin{array}{l}\beta((s\Downarrow)^{\mathcal{B}_\infty}) \succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}} \beta((t\Downarrow)^{\mathcal{B}_\infty}) \quad \text{or} \\ \beta((s\Downarrow)^{\mathcal{B}_\infty}) = \beta((t\Downarrow)^{\mathcal{B}_\infty}) \text{ and } (s\Downarrow)^{\mathcal{B}_\infty} \succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}} (t\Downarrow)^{\mathcal{B}_\infty}.\end{array}$$

First, note that $\beta((s\Downarrow)^{\mathcal{B}_\infty})$ and $\beta((t\Downarrow)^{\mathcal{B}_\infty})$ can be obtained from $s\Downarrow$ and $t\Downarrow$ by determining the depth $2m_i$ of $s_i\Downarrow$ (for $i = 1, \ldots, k$) and $2n_j$ of $t_j\Downarrow$ (for $j = 1, \ldots, \ell$),and then replacing $s_i\Downarrow$ in $s\Downarrow$ by the representative $\widehat{d}_{m_i}$ of D_{m_i}, and $t_j\Downarrow$ in $t\Downarrow$ by the representative $\widehat{d}_{n_j}$ of D_{n_j}. This yields terms $s', t' \in T(\Sigma_2, \widehat{D}_{n-1})$. We have $\beta((s\Downarrow)^{\mathcal{B}_\infty}) = \beta((t\Downarrow)^{\mathcal{B}_\infty})$ iff $s' =_{E_2} t'$, and $\beta((s\Downarrow)^{\mathcal{B}_\infty}) \succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}} \beta((t\Downarrow)^{\mathcal{B}_\infty})$ iff $s' \succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}} t'$. Note that the ordering $>_{D,n-1}$ on the representatives is given by the layer ordering of the sets D_i. Consequently, to know which of these constants are identical, and how the distinct constants are ordered, we need not really compute these constants. Since $=_{E_2}$ and $\succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}}$ are invariant under order preserving renamings of constants, it is sufficient to replace the alien subterms by constants that satisfy the same ordering and identification relationships as the representatives. Since $=_{E_2}$ and $\succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}}$ were assumed to be decidable, it only remains to be shown that

$$(s\Downarrow)^{\mathcal{B}_\infty} \succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}} (t\Downarrow)^{\mathcal{B}_\infty}$$

is decidable. Each of the alien subterms $s_1\Downarrow, \ldots, t_\ell\Downarrow$ corresponds to a constant $d \in \widehat{D}_{n-1}$. By replacing the alien subterms by these constants, we could obtain terms $s'', t'' \in T(\Sigma_2, \widehat{D}_{n-1})$. It is, however, not quite clear how to determine these constants. But this is not a problem since invariance of $=_{E_2}$ and $\succ^{(2)}_{\widehat{D}_{n-1}, >_{D,n-1}}$ under order preserving renamings allows us to use arbitrary constants, as long as they satisfy the correct ordering and identification relationships. The correct identifications can be determined by using the decision procedure for $=_{E_1 \cup E_2}$. The ordering information can be computed by induction ((D_m) for $0 < m < n$) or (D_0).

(C_n) can be reduced to (B_n) with the help of the order isomorphism h_∞.

(A_n) and (D_n) can be treated similarly. $\square$

This concludes the proof of Theorem 5.1.

6 Conclusion

The aim of this work was to develop a general approach for combining compatible reduction orderings that are total on ground terms. The main motivation was that it is often relatively easy to design such orderings for "small" signatures and theories, whereas it is rather involved to give a direct definition of an appropriate ordering in the case of signatures that contain several symbols axiomatized by equational theories over disjoint subsets of the signature. As an example, we have mentioned the case of signatures containing free symbols and more than one AC-symbol.

The main restrictions that must hold for this combination approach to apply are:

1. The signatures of the single theories must not contain constant symbols, i.e., the only available constants are free constants.
2. Both theories must admit compatible orderings total on ground terms that satisfy the constant dominance condition (CDC).

These restrictions appear to be not overly severe. In fact, we have shown by an example that a violation of the first condition may lead to cases where a compatible reduction ordering total on ground terms does not exist for the combined theory. In addition, for strongly regular theories (such as associativity, commutativity, or associativity-commutativity of a binary function symbol), the existence of a compatible orderings total on ground terms implies the existence such an ordering that also satisfies the CDC.

A major drawback of the presented combination approach is that until now it does not yield a non-trivial ordering for terms with variables. Indeed, we have defined an ordering on $\langle C \rangle_{\Sigma_1 \cup \Sigma_2, E_1 \cup E_2}$, where the elements of C are treated as free *constants*. For an ordering on terms with variables, one must also have stability under substitution. For some applications (e.g., the decision problem for ground equations modulo AC), having an ordering on ground terms is sufficient. For other applications where one works with terms containing variables (such as unfailing completion), this is not quite satisfactory. For example, for unfailing completion, using an ordering where all terms with variables are incomparable would mean that none of the identities can be oriented into a rule, and thus all of them must be used in both directions to compute critical pairs. Thus, an important open problem is to extend the combined ordering in a non-trivial way to a decidable ordering on terms with variables. It might be that this makes additional restrictions on the theories necessary (such as requiring them to be collapse-free).

References

[1] F. Baader and K.U. Schulz. Combination of constraint solving techniques: An algebraic point of view. In *Proceedings of the 6th International Conference on Rewriting Techniques and Applications, Kaiserslautern (Germany)*, volume 914 of *Springer LNCS*, pages 352–366, Berlin, 1995.

[2] L. Bachmair. *Canonical Equational Proofs.* Birkhäuser, Boston, Basel, Berlin, 1991.

[3] L. Bachmair. Associative-commutative reduction orderings. *Information Processing Letters*, 43:21–27, 1992.

[4] L. Bachmair, N. Dershowitz, and D.A. Plaisted. Completion without failure. In H. Aït-Kaci and M. Nivat, editors, *Resolution of Equations in Algebraic Structures*, volume 2: Rewriting Techniques, chapter 1, pages 1–30. Academic Press, New York, 1989.

[5] L. Bachmair and D.A. Plaisted. Associative path orderings. In J. P. Jouannaud, editor, *Proceedings of the International Conference on Rewriting Techniques and Applications*, volume 202 of *Springer LNCS*, pages 241–254, Berlin-Heidelberg-New York, 1986.

[6] C. Delor and L. Puel. Extension of the associative path ordering to a chain of associative commutative symbols. In C. Kirchner, editor, *Proceedings of the Fifth International Conference on Rewriting Techniques and Applications (Montreal, Canada)*, volume 690 of *Springer LNCS*, pages 389–404, Berlin, 1993.

[7] N. Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.

[8] N. Dershowitz and Z. Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476, 1979.

[9] J. Gallier, P. Narendran, D. Plaisted, S. Raatz, and W. Snyder. An algorithm for finding canonical sets of ground rewrite rules in polynomial time. *Journal of the ACM*, 40:1–16, 1993.

[10] J.-P. Jouannaud and H. Kirchner. Completion of a set of rules modulo a set of equations. *SIAM Journal on Computing*, 15:1155–1196, 1986.

[11] S. Kamin and J.-J. Levy. Two generalizations of the recursive path ordering. Univ. of Illinois at Urbana-Champaign. Unpublished manuscript, 1980.

[12] D. Kapur, G. Sivakumar, and H. Zhang. A new method for proving termination of AC-rewrite systems. In *Proceedings of the Tenth International*

Conference of Foundations of Software Technology and Theoretical Computer Science, volume 472 of *Springer LNCS*, pages 133–148, Berlin, 1990.

[13] H. Kirchner and C. Ringeissen. Combining symbolic constraint solvers on algebraic domains. Journal of Symbolic Computation 18, 2 (1994), pages 113-155.

[14] D.E. Knuth and P.B. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–267. Pergamon Press, Oxford, 1970.

[15] D. S. Lankford. Canonical inference. Memo ATP-32, Automatic Theorem Proving Project, University of Texas, Austin, TX, December 1975.

[16] C. Marche. On ground AC-completion. In R. Book, editor, *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications (Como, Italy)*, volume 488 of *Springer LNCS*, pages 411–422, Berlin, 1991.

[17] P. Narendran and M. Rusinowitch. Any ground associative-commutative theory has a finite canonical system. In R. Book, editor, *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications (Como, Italy)*, volume 488 of *Springer LNCS*, pages 423–433, Berlin, 1991.

[18] G. Peterson and M.E. Stickel. Complete sets of reductions for some equational theories. *Journal of the ACM*, 28:223–264, 1981.

[19] T. Nipkow. Combining matching algorithms: The regular case. In N. Dershowitz, editor, *Proceedings of the Third International Conference on Rewriting Techniques and Applications (Chapel Hill, NC)*, volume 355 of *Springer LNCS*, pages 343–358, Berlin, 1989.

[20] A. Rubio. *Automated Deduction with Constrained Clauses.* PhD Thesis, Universita Politècnica de Catalunya, Barcelona, Spain, 1994.

[21] A. Rubio and R. Nieuwenhuis. A precedence-based total AC-compatible ordering. In C. Kirchner, editor, *Proceedings of the Fifth International Conference on Rewriting Techniques and Applications (Montreal, Canada)*, volume 690 of *Springer LNCS*, pages 374–388, Berlin, 1993.

[22] A. Rubio and R. Nieuwenhuis. A total AC-compatible ordering based on RPO. *Theoretical Computer Science*, 142, 1995.