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Structural Subsumption for $\mathcal{ALN}$

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1 Introduction

Description logics (DLs) and corresponding DL systems can be used to represent the terminological knowledge of a problem domain in a structured and well-defined way. Relevant concepts of the domain are described by concept descriptions, which are formed from atomic concepts (unary predicates) and roles (binary predicates) using concept forming operators provided by the DL. One of the most important inference services of a DL system is to arrange the represented concepts of the domain in a superconcept/subconcept hierarchy. This reasoning task is based on the subsumption relation between concept descriptions. Intuitively, a concept D subsumes a concept C if the set of individuals represented by D is a superset of the one represented by C.

In the literature several approaches to subsumption have been investigated (see [DLNS96] for an overview). In order to decide subsumption for very expressive languages we can employ tableaux-based algorithms [BBH94, BS96]. The automata theoretic approach has been proposed in order to gain a more profound understanding of the semantics as well as the subsumption relation in cyclic terminologies for rather small languages [Neb90, Baa96, Küs98]. On the other hand, structural subsumption algorithms are efficient methods for deciding subsumption of concept descriptions that do not use full negation, disjunction or existential restrictions.

The structural subsumption algorithm employed by the system CLASSIC [BP94, CH94] is based on a specific data structure for representing concept descriptions, called *description graphs*. The idea behind is as follows: given two concepts C and D, we translate the concepts into equivalent description graphs $\mathcal{G}_C$ and $\mathcal{G}_D$. A normalization of $\mathcal{G}_C$ yields the *canonical* description graph $\widehat{\mathcal{G}}_C$ of C. Thereafter, one can decide $C \sqsubseteq D$ by a structural comparison of $\widehat{\mathcal{G}}_C$ and $\mathcal{G}_D$.

In this paper, we reuse the representation formalism 'description graph' in order to characterize subsumption of $\mathcal{ALN}$-concepts. The description logic $\mathcal{ALN}$ allows for conjunction, value restrictions, number restrictions, and primitive negation. Since CLASSIC allows for more constructors than $\mathcal{ALN}$, e.g., equality restrictions an attribute chains by the constructor SAME-AS, we can confine the notion of description graphs from [BP94]. On the other hand, $\mathcal{ALN}$ explicitly allows for primitive negation which yields another possibility – besides conflicting number restrictions – to express inconsistency. Thus, we have to modify the notion of canonical description graphs in order to cope with inconsistent concepts in the structural characterization of subsumption.

It turns out that the description graphs obtained from $\mathcal{ALN}$-concepts are in fact trees. A canonical graph is a deterministic tree. The conditions required by the structural characterization of subsumption on these trees can be tested by an efficient algorithm, i.e., we obtain an algorithm deciding subsumption of C and D in time polynomial in the size of C and D.

The report is structured as follows. In the preliminaries, we define syntax and semantics of the description logic $\mathcal{ALN}$ as well as the inference problem of subsumption. In Section 3, we introduce description graphs, the data structure our structural subsumption algorithm is working on. Besides syntax and semantics also an algorithm for translating $\mathcal{ALN}$-concepts into description graphs is given. Thereafter, we present the main result of this report in Section 6, a characterization of subsumption of $\mathcal{ALN}$-concepts by a structural comparison of corresponding description graphs. Furthermore, a structural subsumption algorithm can be found in Section 6.2. In the last section we summarize our results and give an outlook to further applications of structural subsumption in terminological knowledge representation systems.

2 Preliminaries

We first introduce syntax and semantics of the description logic $\mathcal{ALN}$ as well as the inference problem of subsumption. *Concept descriptions* are inductively defined by means of a set $\mathcal{C}$ of *primitive concepts* (unary predicates), a set $\mathcal{R}$ of *role names* (binary predicates), and a set of *constructors*. The semantics of a concept description is also inductively defined whereby primitive concepts are interpreted as subsets of a domain Δ and role names are interpreted as binary relations on $\Delta \times \Delta$.

Definition 1 (Syntax and Semantics).
Let $\mathcal{C}$ be a set of *concept names* and $\mathcal{R}$ a set of *role names*. $\mathcal{ALN}$*-concepts* are inductively defined as follows.

- P and $\neg P$ are concepts for each $P \in \mathcal{C}$.
- Let C, D be concepts, $R \in \mathcal{R}$ a role name and $n \in \mathbb{N}$. Then
  - $C \sqcap D$ (conjunction),
  - $\forall R.C$ (value restriction),
  - $(\leq n\, R)$ and $(\geq n\, R)$ (number restrictions)

  are concepts as well.

Concepts of the form P or $\neg P$, $P \in \mathcal{C}$, are called *literals*. Literals and number restrictions are called *atomic concepts*.

An interpretation $\mathcal{I} = (\Delta, \cdot^I)$ exists of a set of individuals Δ and a function $\cdot^I$ that maps each concept name $P \in \mathcal{C}$ to a subset $P^I \subseteq \Delta$ and each role name $R \in \mathcal{R}$ to a subset $R^I \subseteq \Delta \times \Delta$. The extension of $\cdot^I$ to arbitrary $\mathcal{ALN}$-concepts is inductively defined as shown in Table 1. The constructor $\top$ describes the entire domain whereas $\perp$ represents the empty set. $\diamond$

Syntax	Semantics
$\top$	Δ
$\bot$	$\emptyset$
P	$P^I \subseteq \Delta$
$\neg P$	$\Delta \setminus P^I$
$C \sqcap D$	$C^I \cap D^I$
$\forall R.C$	$\{x \in \Delta \mid \forall y : (x, y) \in R \longrightarrow y \in C^I\}$
$(\leq n\, R)$	$\{x \in \Delta \mid \|\{y \mid (x, y) \in R^I\}\| \leq n\}$
$\geq nR$	$\{x \in \Delta \mid \|\{y \mid (x, y) \in R^I\}\| \geq n\}$

Table 1: Semantics of $\mathcal{ALN}$-concepts

Notice that both constructors $\top$ and $\bot$ are expressible in $\mathcal{ALN}$ because of $\top \equiv (\geq 0\, R)$ and $\bot \equiv (P \sqcap \neg P)$. W.l.o.g. we will use them only as abbriviations in some considerations and do not allow for these constructors in $\mathcal{ALN}$-concepts explicitly.

One of the most important inference tasks in description logics is computing the *subsumption relation* between concepts, i.e., deciding the question if one concept is more specific than another.

Definition 2 (Subsumption for $\mathcal{ALN}$).
Let C, D be $\mathcal{ALN}$-concepts.
D *subsumes* C (for short $C \sqsubseteq D$) iff $C^I \subseteq D^I$ for all interpretations $\mathcal{I}$.
C is *equivalent* to D (for short $C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$, i.e., $C^I = D^I$ for all interpretations $\mathcal{I}$. $\diamond$

3 Syntax and semantics of description graphs

Description graphs were introduced in [BP94, CH94] for deciding subsumption of concepts in the terminological knowledge representation system Classic. Classic allows for more constructors than $\mathcal{ALN}$, e.g., equality restrictions on attribute chains by the constructor SAME-AS. Therefore, we confine the notion of description graphs presented in [CH94]. It will turn out, that the description graph $\mathcal{G}_C$ of an $\mathcal{ALN}$-concept C is in fact a tree. But since we are concerned with a sublanguage of Classic, we will reuse most of the notations from [BP94] and [CH94].

Cohen and Hirsh introduce a labeling function l_E for edges in a description graph. The label $l_E(v, w, R)$ of an edge form v to w by the role name R is used to represent number restrictions and individuals given by a FILLS-concept on a role R. Intuitively, the concept (FILLS $R\, A_1 \ldots A_k$) denotes the set of all individuals $x \in \Delta$ such that there exist k R-successors $y_1, \ldots, y_k$ of x with $y_i \in A_i^I$, $1 \leq i \leq k$. For example, the set of all individuals who are employees at IBM can be described by the Classic-concept (FILLS *employer IBM*).

$\mathcal{ALN}$ only allows for number restrictions on role names. Motivated by the alternative approach to characterize subsumption for $\mathcal{ALN}$ presented in [Baa96, Küs98] and the comparison between both approaches [BKM98], we dispense with labeled edges. Instead, we allow for number restrictions in the labels of nodes.

Definition 3 (Description Graph).
Let $\mathcal{C}$ be a set of concept names and $\mathcal{R}$ a set of role names. A *description graph over $\mathcal{C}$ and $\mathcal{R}$* is a tuple $\mathcal{G} = (V, E, v_0, l)$ where $V = \{v_0, \ldots, v_n\}$ is a set of nodes, $E \subseteq V \times \mathcal{R} \times V$ is a set of edges and $v_0 \in V$ is the root of $\mathcal{G}$ such that

- there exists no edge vRv_0 in E,
- for each $v \in V \setminus \{v_0\}$ exists exactly one $v' \in V$ and exactly one $R \in \mathcal{R}$ with $v'Rv \in E$,
- each $v \in V$ is reachable from v_0, i.e., it exists a path $v_0R_1v_1 \ldots v_{n-1}R_nv$ in E, and
- the label $l(v)$ of a node $v \in V$ is a finite set of atomic concepts, i.e., a finite subset of the set

$$\mathcal{C} \cup \{\neg P \mid P \in \mathcal{C}\} \cup \{(\geq n\, R) \mid n \in \mathbb{N},\ R \in \mathcal{R}\} \cup \{(\leq n\, R) \mid n \in \mathbb{N},\ R \in \mathcal{R}\}.$$

$\diamond$

In the sequel, we will use the following notions referring to paths and subgraphs. $\mathsf{Lit}(v) := \{P \in \mathcal{C} \mid P \in l(v)\} \cup \{\neg P \mid P \in \mathcal{C}, \neg B \in l(v)\}$ denotes the set of all literals in the label of the node $v \in V$. $p = w_0R_1w_1 \ldots w_{n-1}R_nw_n$ is called *path* from w_0 to w_n with label $W = R_1 \ldots R_n$ in $\mathcal{G}$ iff $w_{i-1}R_iw_i \in E$ for all $1 \leq i \leq n$. p is called *rooted path*, if $w_0 = v_0$, i.e., p starts at the root of $\mathcal{G}$. $\mathcal{G}|_v$ denotes the *subgraph* of $\mathcal{G}$ with root $v \in V$, i.e., $\mathcal{G}|_v = (V', E', v, l')$ with $V' := \{w \in V \mid \text{ exists path from } v \text{ to } w \text{ in } \mathcal{G}\}$, $E' := E \cap (V' \times \mathcal{R} \times V')$, and $l'(w) := l(w)$ for $w \in V'$. The *size* of a description graph $\mathcal{G}_C = (V, E, v_0, l)$ is defined as the sum of the number of nodes and edges and the sum of the size of all labels, i.e.,

$$|\mathcal{G}_C| := |V| + |E| + \sum_{v \in V} |l(v)|.$$

After introducing the syntax of description graphs and thus the data structure our structural subsumption algorithm is working on, we now have to define the semantics: which set of individuals $\mathcal{G}^I$ is determined by a description graph $\mathcal{G}$ and an interpretation $\mathcal{I}$.

Definition 4 (Extension of Description Graphs).
Let $\mathcal{I} = (\Delta, \cdot^I)$ be an interpretation of $\mathcal{C}$ and $\mathcal{R}$, $\mathcal{G} = (V, E, v_0, l)$ a description graph over $\mathcal{C}$ and $\mathcal{R}$.
The *extension* of a node $v \in V$ is recursively defined by $x \in v^I$ iff

- $x \in P^I$ for all $P \in l(v)$,
- $x \notin P^I$ for all $\neg P \in l(v)$,
- $|\{y \mid (x, y) \in R^I\}| \leq n$ for all $(\leq n\, R) \in l(v)$,
- $|\{y \mid (x, y) \in R^I\}| \geq n$ for all $(\geq n\, R) \in l(v)$, and
- for all $vRv' \in E$, for all $y \in \Delta$ with $(x, y) \in R^I$ it is $y \in v'^I$.

The *extension* of $\mathcal{G}$ is defined as $\mathcal{G}^I := v_0^I$. $\diamond$

Remark 5.
It is not hard to see that the five conditions in Definition 4 are equivalent to $x \in v^I$ iff

- $x \in \bigcap_{C \in l(v)} C^I$ and
- for all $(x, y) \in R^I$ it is $y \in \bigcap_{1 \leq i \leq n} v_i^I$, where $\{v_1, \ldots, v_n\}$ is the set of all R-successors of v in G.

$\diamond$

In order to prove completeness of our structural characterization of subsumption we will need the alternative characterization of the semantics of description graphs given in the next lemma. Intuitively, each instance x of $\mathcal{G}$ must satisfy all restrictions given by the labels of the nodes in $\mathcal{G}$, i.e., each W-successor of x satisfies each atomic concept in the label $l(v)$ of a W-successor node v of v_0.

Lemma 6.
Let $\mathcal{I}$ be an interpretation and $\mathcal{G}$ a description graph. Then $x_0 \in \mathcal{G}^I$ iff for each rooted path $v_0 R_1 v_1 \ldots v_{n-1} R_n v_n$ in $\mathcal{G}$ and each $(R_1 \circ \ldots \circ R_n)^I$-successor x_n of x_0 in $\mathcal{I}$ it is $x_n \in \bigcap_{C \in l(v_n)} C^I$.
Proof.
"$\Rightarrow$" Assume that there exists a rooted path $v_0 R_1 v_1 \ldots v_{n-1} R_n v_n$ in $\mathcal{G}$ and an $(R_1 \circ \ldots \circ R_n)^I$-successor x_n of x_0 with $x_n \notin \bigcap_{C \in l(v_n)} C^I$. Let $(x_{i-1}, x_i) \in R_i^I$ for $1 \leq i \leq n$. By Remark 5 it is $x_n \notin v_n^I$ and since $(x_{n-1}, x_n) \in R_n^I$ also $x_{n-1} \notin v_{n-1}^I$. Analogously, it follows $x_i \notin v_i^I$ for $0 \leq i < n-1$ which is a contradiction to $x_0 \in G^I = v_0^I$.
"$\Leftarrow$" By induction on the maximal role depth $\mathsf{depth}(\mathcal{G})$ of $\mathcal{G}$, i.e., the length of the longest rooted path in $\mathcal{G}$, $\mathsf{depth}(\mathcal{G}) := \mathsf{max}\{|W| \mid \text{exists a rooted path } p \text{ with label } W \text{ in } \mathcal{G}\}$.

$\mathsf{depth}(\mathcal{G}) = 0$**:** There are no R-successors of v_0 in $\mathcal{G}$ that yield to a restriction on an x in $\mathcal{G}^I$. So, $x_0 \in \bigcap_{C \in l(v_0)} C^I$ implies $x_0 \in \mathcal{G}^I$.

$\mathsf{depth}(\mathcal{G}) > 0$**:** We have to show $x_0 \in \bigcap_{C \in l(v_0)} C^I$ and for each $v_0 R_1 v_1$ in E and each $(x_0, x_1) \in R_1^I$ it is $x_1 \in v_1^I$.
The preconditions of Lemma 6 imply $x_0 \in \bigcap_{C \in l(v_0)} C^I$ for $n = 0$, i.e., for the path v_0 of length 0 in $\mathcal{G}$. Furthermore, it follows that for each rooted path $v_1 R_2 v_2 \ldots v_{n-1} R_n v_n$ in $\mathcal{G}|_{v_1}$ and each $(R_2 \circ \ldots \circ R_n)$-successor x_n of x_1 it is $x_n \in \bigcap_{C \in l(v_n)} C^I$. By induction it is $x_1 \in (\mathcal{G}|_{v_1})^I = v_1^I$. So, $x_0 \in \mathcal{G}^I = v_0^I$.

$\square$

We have introduced syntax and semantics of description graphs. Our aim is to use this representation formalism to characterize subsumption of $\mathcal{ALN}$-concepts. Therefore, we first have to translate $\mathcal{ALN}$-concepts into (equivalent) description graphs.

Input: An $\mathcal{ALN}$-concept $C = P_1 \sqcap \ldots \sqcap P_n \sqcap \neg Q_1 \sqcap \ldots \sqcap \neg Q_k \sqcap (\leq \nu_1\, S_1) \sqcap \ldots \sqcap (\leq \nu_l\, S_l) \sqcap (\geq \mu_1\, T_1) \sqcap \ldots \sqcap (\geq \mu_r\, T_r) \sqcap \forall R_1.C_1 \sqcap \ldots \sqcap \forall R_m.C_m)$

Output: The corresponding description graph $\mathcal{G}_C = (V, E, v_0, l)$

$|C|_\forall = 0$: $\mathcal{G}_C := (\{v_0\}, \emptyset, v_0, l)$ where
$l(v_0) := \{P_1, \ldots, P_n, \neg Q_1 \ldots, \neg Q_k (\leq \nu_1\, S_1), \ldots, (\leq \nu_l\, S_l), (\geq \mu_1\, T_1), \ldots, (\geq \mu_r\, T_r)\}$.

$|C|_\forall > 0$: Let $\mathcal{G}_{C_i} = (V_i, E_i, v_{0i}, l_{V_i})$ be the recursively defined description graph for C_i, $1 \leq i \leq k$ where w.l.o.g. the V_i are pairwise disjoint and $v_0 \notin \bigcup_{1 \leq i \leq k} V_i$. $\mathcal{G}_C := (V, E, v_0, l)$ is defined by

- $V := \{v_0\} \cup \bigcup\limits_{1 \leq i \leq k} V_i$,
- $E := \{v_0 R_i v_{0i} \mid 1 \leq i \leq k\} \cup \bigcup\limits_{1 \leq i \leq k} E_i$,
- $l(v) := \begin{cases} \{P_1, \ldots, P_n, \neg Q_1 \ldots, \neg Q_k, & \\ (\leq \nu_1\, S_1), \ldots, (\leq \nu_l\, S_l), (\geq \mu_1\, T_r), \ldots, (\geq \mu_k\, T_r)\} & , v = v_0 \\ l_{V_i}(v) & , v \in \bigcup\limits_{1 \leq i \leq k} V_i \end{cases}$

Figure 1: Translating concepts into description graphs

4 Translating $\mathcal{ALN}$-Concepts to Description Graphs

The translation of $\mathcal{ALN}$-concepts into description graphs is formalized by the algorithm in Figure 1. In the sequel, $\mathcal{G}_C$ denotes the description graph of C where C is an $\mathcal{ALN}$-concept and $\mathcal{G}_C$ is obtained from C by the algorithm in Figure 1.

The description graph of C is inductively defined. If C has role depth 0, i.e., there exists no value-restriction in C, $\mathcal{G}_C$ only consists of a root node labeled with the set of all subconcepts of C. Otherwise, the description graph $\mathcal{G}_{C_i}$ of each concept C_i occuring in a top-level value-restriction $\forall R_i.C_i$ in C is defined recursively. Thereafter, each such subgraph $\mathcal{G}_{C_i}$ is appended to the root node by an edge labeled R_i.

Example 7.
In the sequel, we will use the $\mathcal{ALN}$-concepts C and D to illustrate some notions and algorithms.

$$\begin{aligned} C &= \forall R.(P \sqcap Q \sqcap \forall S > ((\geq 1\, S) \sqcap \forall S.(Q \sqcap \neg Q))) \sqcap \\ &\quad\ \forall S.Q \sqcap \forall S.(\geq 3\, S) \\ D &= \forall R.(P \sqcap \forall S.Q) \sqcap \\ &\quad\ \forall S.(Q \sqcap (\geq 1\, S)) \end{aligned}$$

The description graphs of C and D, respectively, are depicted in Figure 2. $\diamond$

The translation is correct in the sense that the semantics of a concept is equal to the extension of its description graph for all interpretations. Formally, we prove

Lemma 8 (Equivalence of concepts and description graphs).
Let C be an arbitrary $\mathcal{ALN}$-concept and $\mathcal{G}_C$ the description graph of C. Then $C^I = \mathcal{G}_C^I$ for all interpretations $\mathcal{I}$.

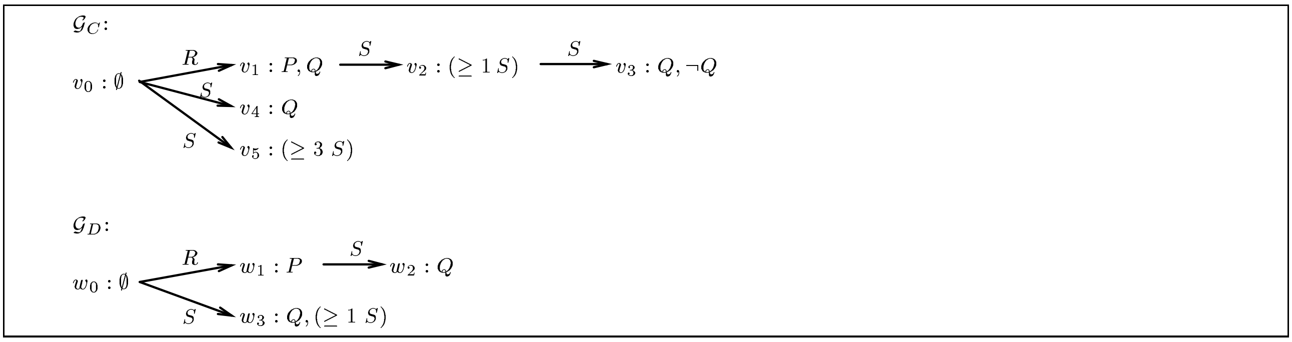

Figure 2: The $\mathcal{ALN}$-description graphs of C and D.

Proof.
By induction on the number of all-quantifiers in C.

$|C|_\forall = 0$: Let $\mathcal{I}$ be an interpretation, $x \in \Delta$, $\mathcal{G}_C = (\{v_0\}, \emptyset, v_0, l)$ and
$C = P_1 \sqcap \ldots \sqcap P_n \sqcap \neg Q_1 \sqcap \ldots \sqcap \neg Q_k \sqcap (\leq \nu_1\, S_1) \sqcap \ldots \sqcap (\leq \nu_l\, S_l) \sqcap (\geq \mu_1\, T_1) \sqcap \ldots \sqcap (\geq \mu_r\, T_r)$.

As an easy consequence of the definitions of C^I, $\mathcal{G}_C$ and the extension of description graphs it follows $x \in C^I$ iff $x \in v_0^I$. So, $C^I = \mathcal{G}_C^I$.

$|C|_\forall > 0$: Let $\mathcal{I}$ be an interpretation, $x \in \Delta$, $\mathcal{G}_C = (V, E, v_0, l)$ and

$$\begin{array}{rcl} C & = & P_1 \sqcap \ldots \sqcap P_n \sqcap \neg Q_1 \sqcap \ldots \sqcap \neg Q_k \sqcap \\ & & (\leq \nu_1\, S_1) \sqcap \ldots \sqcap (\leq \nu_l\, S_l) \sqcap (\geq \mu_1\, T_1) \sqcap \ldots \sqcap (\geq \mu_r\, T_r) \sqcap \\ & & \forall R_1.C_1 \sqcap \ldots \sqcap \forall R_m.C_m. \end{array}$$

We have to show $x \in C^I$ iff $x \in v_0^I$.

Since $l(v_0) = \{P_1, \ldots, P_n, \neg Q_1 \ldots, \neg Q_k, (\leq \nu_1\, S_1), \ldots, (\leq \nu_l\, S_l), (\geq \mu_1\, T_1), \ldots, (\geq \mu_r\, T_r)\}$, it is

$$x \in \bigcap_{1 \leq i \leq n} P_i^I \cap \bigcap_{1 \leq i \leq k} (\neg Q_i)^I \cap \bigcap_{1 \leq i \leq l} (\leq \nu_i\, S_i)^I \cap \bigcap_{1 \leq i \leq r} (\geq \mu_i\, T_i)^I \text{ iff } x \in \bigcap_{C' \in l(v_0)} C'^I \qquad (*)$$

Let $x \in (\forall R_i.C_i)^I$, $(x, y) \in R_i^I$. By definition of $\mathcal{G}_C$ it exists $w \in V$ such that $v_0 R w \in E$ and w is the root of the recursively defined description graph for C_i. It is $|C|_\forall > |C_i|_\forall$ and $y \in C_i^I$. It follows by induction $y \in \mathcal{G}_{C_i}$. By $(*)$ and since $\forall R_i.C_i$ in C and y have been chosen arbitrarily it follows $x \in C^I \Longrightarrow x \in v_0^I$.

Conversely, let $x \in v_0^I$, $v_0 R w \in E$ and $(x, y) \in R^I$. By definition of $\mathcal{G}_C$ it exists $\forall R_i.C_i$ in C such that $R_i = R$ and w is the root of the recursively defined description graph for C_i. It is $|C|_\forall > |C_i|_\forall$ and $y \in w^I = G_{C_i}^I$. It follows by induction $y \in C_i^I$. By $(*)$ and since $v_0 R w$ in E and y have been chosen arbitrarily, it follows $x \in v_0^I \Longrightarrow x \in C^I$.

$\square$

Notice that the size of the description graph $\mathcal{G}_C$ of an $\mathcal{ALN}$-concept C is linear in the length of C.

$\mathcal{G}$:

$$v_0 : P \xrightarrow{R} v_1 : (\leq 0\ S) \xrightarrow{S} v_2 : Q$$

$\widehat{\mathcal{G}}$ obtained from rule 4.:

$$v_0 : P \xrightarrow{R} v_1 : (\leq 0\ S)$$

Figure 3: Normalizing number restrictions of the form $(\leq 0\, R)$.

5 Canonical Description Graphs

The subsumption test for $\mathcal{ALN}$-concepts C and D introduced in Section 6 is based on syntactical conditions on the corresponding description graphs. To obtain a complete subsumption test, we will need some normal form for the graph corresponding to C in order to abstain from different descriptions of equivalent concepts, e.g., $\forall R.(P \sqcap \neg P) \equiv (\leq 0\, R)$. More precisely, we apply some normalization rules as long as possible to the description graph $\mathcal{G}_C$ of the subsumee C. The graph obtained this way is called *canonical description graph* of C and is used in the structural comparison with $\mathcal{G}_D$.

The normalization rules for $\mathcal{ALN}$-description graphs are summarized in Figure 4. These rules can be divided into three groups.

The first group consists only of rule 1 and is based on the equivalence $\forall R.C \sqcap \forall R.D \equiv \forall R.(C \sqcap D)$. Its application merges several different R-successors of a node v to one R-successor of v. This leads to a deterministic description graph.

The second group of rules (the rules 2, 3, 4, and 5) copes with nodes labeled by inconsistent sets, i.e., nodes v with $\{P, \neg P\} \subseteq l(v)$ or $\{(\geq l\, S), (\leq r\, S)\} \subseteq l(v)$, $l > r$. Intuitively speaking, nodes labeled by inconsistent sets and the edges leading to these nodes are removed. In addition, if there was an edge labeled R from node v to the inconsistent node, the label of v is extended by $(\leq 0\, R)$. This is due to the equivalence $\forall R.\bot \equiv (\leq 0\, R)$, i.e., if each R-successor must satisfy the empty concept $\bot$, there is at least no R-successor. For the same reason, we have to remove each subgraph with root v if the label of the R-predecessor of v contains $(\leq 0\, R)$.

For example, in the description graph of $P \sqcap \forall R.((\leq 0\, S) \sqcap \forall S.Q)$, the RS-successor of the root is removed because the R-successor node is labeled by $\{(\leq 0\, S)\}$ (see Figure 3).

If the root v_0 is labeled by an inconsistent set, then the whole concept is inconsistent. In this case, we remove all nodes except the root and all edges and label v_0 by $\bot$ (rule 5).

The last group of rules ensure that the canonical description graph of an $\mathcal{ALN}$-concept is unique. Therefore, we have to deal with number restrictions (rules 6 and 7) and with the equivalences $\forall R.\top \equiv \top$ and $(\geq 0\, R) \equiv \top$. The concept $\top$ corresponds to the empty label.

For example, the description graph $\mathcal{G} = (\{v_0, v_1\}, \{v_0 R v_1\}, v_0, l)$ with $l(v_0) = l(v_1) = \emptyset$ corresponds to $\forall R.\top$. Notice that though we do not allow for the constructor $\top$ explicitely, a graph similar to $\mathcal{G}$ may occur after applying Rule 8.

By rules 8 and 9 we remove all concepts of the form $(\geq 0\, R)$ and all leaves in the graph labeled with $\emptyset$. Furthermore, using $(\geq n\, R) \sqsubseteq (\geq m\, R)$ iff $n \geq m$ and $(\leq n\, R) \sqsubseteq (\leq m\, R)$ iff $n \leq m$, we can reduce all $\geq$-restrictions and all $\leq$-restrictions for an $R \in \mathcal{R}$ to one $\geq$-restriction and one $\leq$-restriction, respectively, in the label of a node v (rules 6 and 7).

As an easy consequence of the preconditions of the normalization rules it follows that $|\mathcal{G}| > |\mathcal{G}'|$ if $\mathcal{G}'$ is obtained from $\mathcal{G}$ by one of the rules. Therefore, each iterated application of the rules to an $\mathcal{ALN}$-description graph terminates. Furthermore, since each normalization rule is based on an equivalence between concept descriptions, e.g., $(\geq m\, R) \sqcap (\geq n\, R) \equiv (\geq \mathsf{max}\{n, m\}\, R)$, it is not hard to see that the rules are sound. Formally, we can prove

Lemma 9.
1. The normalization rules in Figure 4 are sound, i.e., if $\mathcal{G}'$ is obtained from $\mathcal{G}$ by applying one of the rules then $\mathcal{G}'^{I} = \mathcal{G}^{I}$ for all interpretations $\mathcal{I}$.
2. Every iterated application of the rules to a given description graph $\mathcal{G}$ terminates. $\square$

Thus, the normalization rules lead to an equivalent, role based normal form for $\mathcal{ALN}$-concepts that will be used by the structural subsumption test.

Definition 10 (Canonical description graphs).
Let C be an $\mathcal{ALN}$-concept description and $\mathcal{G}_C$ the description graph of C. The *canonical description graph of* $\mathcal{G}_C$ is defined as the description graph $\widehat{\mathcal{G}}_C$ that is obtained from $\mathcal{G}_C$ by an iterated application of the normalization rules in Figure 4 such that no rule is applicable to $\widehat{\mathcal{G}}_C$. $\diamond$

Example 11 (continuous Example 7).
Consider the description graph $\mathcal{G}_C = (V, E, v_0, l)$ of C in Figure 2. First, $\mathcal{G}_C$ is made deterministic by merging the two S-successors v_4, v_5 to one S-successor of v_0 labeled with $\{Q, (\geq 3\, S)\}$. Because of $\{Q, \neg Q\} \subseteq l(v_3)$, the node v_3 and the edge $v_2 S v_3$ are removed, and $(\leq 0\, S)$ is added to the label of v_2. Now, v_2 is labeled with $\{(\geq 1\ S), (\leq 0\ S)\}$, which is again inconsistent. Consequently, it is removed, and $(\leq 0\, S)$ is added to the label of the S-predecessor node v_1. The description graph obtained this way is depicted in Figure 5. $\diamond$

In the next section we give a characterization of subsumption of two $\mathcal{ALN}$-concepts C and D by means of a structural comparison of the corresponding description graphs. We introduced canonical description graphs in order to cope with inconsistencies occuring in the subsumee C. On the other hand, since $\top$ is expressible in $\mathcal{ALN}$ by $(\geq 0\, R)$, concepts equivalent to $\top$, e.g., $\forall S.(\geq 0\, R)$, must be taken into account.

It is easy to see, that D is equivalent to $\top$ iff each atomic concept D' in the labels of the nodes in $\mathcal{G}_D$ is of the form $(\geq 0\, R)$. Intuitively, if there is an atomic concept D' in $\mathcal{G}_D$ not of the form $(\geq 0\, R)$, then there is at most one non-trivial restriction to an instance of D and therefore, D is not equivalent to $\top$.

In order to decide $D \equiv \top$, we can apply the normalization rules 8. and 9. from Figure 4 as long as possible to the description graph $\mathcal{G}_D$. Let $\mathcal{G}'_D$ be the description graph obtained this way. Now, it is $D \equiv \top$ iff $\mathcal{G}'_D = (\{v_0\}, \emptyset, v_0, l')$ with $l'(v_0) = \emptyset$.

Consequently, $D \equiv \top$ can be decided in time polynomial in the size of D. In order to simplify the structural characterization of subsumption as well as the presentation of our structural subsumption algorithm, we assume that no subconcept of the form $(\geq 0\, R)$ occurs in the subsumer D. Thus, w.l.o.g. we reduce our attention to subsumers that are not equivalent to $\top$.

Let $\mathcal{G} = (V, E, v_0, l)$ be a description graph. $\mathcal{G}' = (V', E', v_0, l')$ denotes the description graph that is obtained from $\mathcal{G}$ by applying one of the following rules.

1. Let $v \in V$ with $n > 1$ R-successors $v_1, \ldots, v_n$ in E and v' a new node not occuring in V. Then $\mathcal{G}'$ is defined by
   - $V' := V \setminus \{v_1, \ldots, v_n\} \cup \{v'\}$ and $E' := E[v_i/v' \mid i = 1 \ldots n]$ (each v_i is replaced by v' in E),
   - $l(v') := \bigcup_{i=1\ldots n} l(v_i)$ and $l(w) := l(w)$, $w \in V' \setminus \{v'\}$.
2. Let $v, w \in V$ with $vRw \in E$ and $\{P, \neg P\} \subseteq l(w)$. Then $\mathcal{G}'$ is defined by
   - $V' := V \setminus (\{w\} \cup \{w' \mid \text{exists a path from } v \text{ to } w'\})$ and $E' := E \cap V' \times \mathcal{R} \times V'$,
   - $l'(v') := l(v')$, $v' \in V' \setminus \{v\}$ and $l'(v) := l(v) \cup \{(\leq 0\, R)\}$.
3. Let $v, w \in V$ with $vRw \in E$, $\{(\geq n\, S), (\leq m\, S)\} \subseteq l(w)$, and $m > n$. Then $\mathcal{G}'$ is defined by
   - $V' := V \setminus (\{w\} \cup \{w' \mid \text{exists a path from } w \text{ to } w'\})$ and $E' := E \cap V' \times \mathcal{R} \times V'$,
   - $l'(v') := l(v')$, $v' \in V' \setminus \{v\}$ and $l'(v) := l(v) \cup \{(\leq 0\, R)\}$.
4. Let $v \in V$ with $(\leq 0\, R) \in l(v)$ and $vRw \in E$. Then $\mathcal{G}'$ is defined by
   - $V' := V \setminus (\{w\} \cup \{w' \mid \text{exists a path from } w \text{ to } w'\})$ and $E' := E \cap V' \times \mathcal{R} \times V'$,
   - $l'(v') := l(v')$, $v' \in V'$.
5. If $\{P, \neg P\} \subseteq l(v_0)$ or $\{(\geq n\, R), (\leq m\, R)\} \subseteq l(v_0)$, $m > n$, then $\mathcal{G}'$ is defined by
   - $V' := \{v_0\}$ and $E' := \emptyset$ and
   - $l'(v_0) := \bot$.
6. Let $v \in V$ with $k > 1$ number restrictions $\{(\leq n_1\, R), \ldots, (\leq n_k\, R)\} \subseteq l(v)$. Then $\mathcal{G}'$ is defined by
   - $V' := V$ and $E' := E$,
   - $l'(v) := l(v) \setminus \{(\leq n_1\, R), \ldots, (\leq n_k\, R)\} \cup e\{(\leq \mathsf{min}\{n_1, \ldots, n_k\}\, R)\}$, $l'(w) := l(w)$, $w \in V' \setminus \{v\}$.
7. Let $v \in V$ with $k > 1$ number restrictions $\{(\geq n_1\, R), \ldots, (\geq n_k\, R)\} \subseteq l(v)$. Then $\mathcal{G}'$ is defined by
   - $V' := V$ and $E' := E$,
   - $l'(v) := l(v) \setminus \{(\geq n_1\, R), \ldots, (\geq n_k\, R)\} \cup \{(\geq \mathsf{max}\{n_1, \ldots, n_k\}\, R)\}$, $l'(w) := l(w)$, $w \in V' \setminus \{v\}$.
8. Let $v \in V$ with $(\geq 0\, R) \in l(v)$. Then $\mathcal{G}'$ is defined by
   - $V' := V$ and $E' := E$,
   - $l'(v) := l(v) \setminus \{(\geq 0\, R)\}$ and $l'(w) := l(w)$, $w \in V' \setminus \{v\}$.
9. Let $vRw \in E$, $l(w) = \emptyset$ and w a leaf in $\mathcal{G}$. Then $\mathcal{G}'$ is defined by
   - $V' := V \setminus \{w\}$ and $E' := E \cap V' \times \mathcal{R} \times V'$,
   - $l'(v') := l(v')$, $v' \in V'$.

Figure 4: Normalization rules for $\mathcal{ALN}$-description graphs

$\widehat{\mathcal{G}}_C$:

$$\begin{tikzcd} & \widehat{v}_1 : P, Q, (\leq 0\ S) \\ \widehat{v}_0 : \emptyset \arrow[ur, "R"] \arrow[dr, "S"'] & \\ & \widehat{v}_2 : Q, (\geq 3\ S) \end{tikzcd}$$

Figure 5: The canonical $\mathcal{ALN}$-description graph of C.

6 A Structural Subsumption Algorithm for $\mathcal{ALN}$

Before we can characterize subsumption of $\mathcal{ALN}$-concepts by some kind of structural comparison of description graphs, we need the following notions [CH94].

Definition 12 (More Specific Nodes, Paths, and Graphs).
Let $\mathcal{G} = (E, V, v_0, l)$ and $\mathcal{G}' = (V', E', v_0', l')$ be description graphs.
A node $v \in V$ is *more specific* than a node $v' \in V'$ iff

- $\mathsf{Lit}(v) \subseteq \mathsf{Lit}(v')$,
- for each $(\geq \mu'\, S) \in l'(v')$ exists $(\geq \mu\, S) \in l(v)$ with $\mu \geq \mu'$, and
- for each $(\leq \nu'\, S) \in l'(v')$ exists $(\leq \nu\, S) \in l(v)$ with $\nu \leq \nu'$.

A rooted path $p = v_0 R_1 v_1 \dots v_{n-1} R_n v_n$ in $\mathcal{G}$ is *more specific* than a rooted path $p' = v_0' R_1' v_1' \dots v_{m-1}' R_m' v_m'$ in $\mathcal{G}'$ iff

- $R_i = R_i'$ for $1 \leq i \leq \mathsf{min}(m, n)$,
- for all $0 \leq i \leq \mathsf{min}(m, n)$ it is v_i more specific than v_i', and
- if $n < m$, then $(\leq 0\, R_{n+1}') \in l(v_n)$.

The description graph $\mathcal{G}$ is *more specific* than the description graph $\mathcal{G}'$ iff $\mathcal{G}$ corresponds to $\bot$, i.e., $l(v_0) = \bot$, or if for each rooted path p' in $\mathcal{G}'$ there exists a more specific rooted path p in $\mathcal{G}$. $\diamond$

The conditions on more specific nodes v and v' ensure that the conditions given by atomic concepts in the label of v' are satisfied by each instance of v. As an example consider the node $\widehat{v}_1$ in $\widehat{\mathcal{G}}_C$ from Figure 5 and a node v labeled with $l(v) = \{P, (\leq 1\, S)\}$. Obviously, $\widehat{v}_1$ is more specific than v and it holds that

$$\bigsqcap_{C \in \widehat{l}(\widehat{v}_1)} C = P \sqcap Q \sqcap (\leq 0\, S) \;\sqsubseteq\; P \sqcap (\leq 1\, S) = \bigsqcap_{C \in l(v)} C.$$

More generally, it is not hard to show that the conjunction of all atomic concepts in $l'(v')$ subsumes the conjunction of all atomic concepts in $l(v)$ if v is more specific than v'.

Due to number restrictions of the form $(\leq 0\, R)$, a path, which is more specific than a path p', can be shorter than p'. To be more precise, let $\mathcal{G} = (V, E, v_0, l)$ and $\mathcal{G}' = (V', E', v_0', l')$ be description graphs. If v is the W-successor node of the root v_0 and $(\leq 0\, R) \in l(v)$, then each instance x of v_0 has no WR-successor. Thus, all conditions on WR-successors v' of v_0' in $\mathcal{G}'$ are satisfied trivially and the rooted path with label W in $\mathcal{G}$ is more specific than a rooted path p' in $\mathcal{G}'$ with label WRW', $W' \in \mathcal{R}^*$.

As an example, consider the rooted path $p' = w_0Rw_1Sw_2$ in $\mathcal{G}_D$ from Figure 2. The rooted path $\widehat{v}_0R\widehat{v}_1$ in $\widehat{\mathcal{G}}_C$ from Figure 5 is more specific than p' since $(\leq 0\, S) \in \widehat{l}(\widehat{v}_1)$, $\widehat{v}_0$ is more specific than w_0, and $\widehat{v}_1$ is more specific than w_1.

For canonical description graphs the property of being more specific even holds for some subgraphs. This property will be used later in an induction on the size of description graphs.

Lemma 13.
Let $\widehat{\mathcal{G}} = (\widehat{V}, \widehat{E}, \widehat{v}_0, \widehat{l})$ be a canonical description graph and $\mathcal{G}' = (V', E', v_0', l')$ a description graph. Let further $\widehat{v}$ be the W-successor node of $\widehat{v}_0$ in $\widehat{\mathcal{G}}$ and v' a W-successor node in $\mathcal{G}'$. If $\widehat{\mathcal{G}}$ is more specific than $\mathcal{G}'$ then $\widehat{\mathcal{G}}|_{\widehat{v}}$ is more specific than $\mathcal{G}'|_{v'}$.
Proof.
We have to show that for each rooted path $p_{v'}$ in $\mathcal{G}'|_{v'}$ there is a more specific rooted path $\widehat{p}_{\widehat{v}}$ in $\widehat{\mathcal{G}}|_{\widehat{v}}$.

Let $\widehat{v}_0R_1\widehat{v}_1 \dots \widehat{v}_{n-1}R_n\widehat{v}$ be the path with label $W = R_1 \dots R_n$ in $\widehat{\mathcal{G}}$, $p' = v_0'R_1v_1' \dots v_{n-1}'R_nv'$ the path from v_0' to v' with label W in $\mathcal{G}'$. Let further $p_{v'} = v'S_1'v_1' \dots v_{m-1}'S_m'v_m'$ be a rooted path in $\mathcal{G}'|_{v'}$. Since $p = v_0'R_1v_1' \dots v_{n-1}'R_nv'S_1'w_1' \dots w_{m-1}'S_m'w_m'$ is a rooted path in $\mathcal{G}'$ and $\widehat{\mathcal{G}}$ is more specific than $\mathcal{G}'$, there exists a rooted path $\widehat{p} = \widehat{v}_0R_1\widehat{v}_1' \dots \widehat{v}_{n-1}'R_n\widehat{v}_n'S_1\widehat{v}_{n+1}' \dots \widehat{v}_{m'-1}S_{m'}\widehat{v}_{m'}$, $m' \leq m$, in $\widehat{\mathcal{G}}$ such that $\widehat{p}$ is more specific than p. Since $\widehat{\mathcal{G}}$ is canonical, it is $\widehat{v}_i = \widehat{v}_i'$ for $1 \leq i < n$ and $\widehat{v} = \widehat{v}_n'$. Therefore, $\widehat{p}_{\widehat{v}} = \widehat{v}S_1\widehat{v}_{n+1}' \dots \widehat{v}_{m'-1}S_{m'}\widehat{v}_{m'}$ is a rooted path in $\widehat{\mathcal{G}}|_{\widehat{v}}$. Now it is easy to see, that $\widehat{p}_{\widehat{v}}$ is more specific than $p_{v'}$. $\square$

Now we are equipped to characterize subsumption of $\mathcal{ALN}$-concepts by a structural comparison of description graphs. Let C be an $\mathcal{ALN}$-concept and $\mathcal{G}_C = (V, E, v_0, l)$ the description graph of C. Intuitively speaking, if the label of a W-successor node v of v_0 contains the atomic concept C', then for each instance x of C all W-successors of x must be in the extension of C'. Thus, C is subsumed by $\forall W.C'$[1]. More generally, C is subsumed by D iff the conditions to W-successors of instances of D are subsets of the conditions to W-successors of instances of C for each label $W \in \mathcal{R}^*$. Since these conditions to W-successors are represented by the labels of W-successor nodes in the corresponding description graphs, we can decide $C \sqsubseteq D$ by testing wether the label of each W-successor node in $\mathcal{G}_D$ is a subset of the sum of the labels of all W-successor nodes in $\mathcal{G}_C$ or not for each $W \in \mathcal{R}^*$. Notice that in the canonical description graph $\widehat{\mathcal{G}}_C$ all restrictions to W-successors are already summarized in one W-successor node of the root.

Theorem 14 (Structural subsumption for $\mathcal{ALN}$).
Let C, D be $\mathcal{ALN}$-concepts, $\widehat{\mathcal{G}}_C = (\widehat{V}, \widehat{E}, \widehat{v}_0, \widehat{l})$ the canonical description graph of C and $\mathcal{G}_D = (V, E, v_0, l)$ the description graph of D. Then $C \sqsubseteq D$ iff $\widehat{\mathcal{G}}_C$ is more specific than $\mathcal{G}_D$, i.e., $\widehat{l}(\widehat{v}_0) = \bot$, or for each rooted path p in $\mathcal{G}_D$ there exists a more specific rooted path $\widehat{p}$ in $\widehat{\mathcal{G}}_C$. $\square$

The proof of Theorem 14 is lengthy and technical. The "if"-direction can be proved by induction on the size of description graphs. To prove the "only if"-direction, we will show that $\widehat{\mathcal{G}}_C$ is not more specific than $\mathcal{G}_D$ implies $C \not\sqsubseteq D$ by construction of an interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$

[1] $\forall W.C'$ is used as abbreviation for $\forall R_1. \dots .\forall R_n.C'$ where $W = R_1 \dots R_n$.

such that $x \in \widehat{\mathcal{G}}_C^I$ and $x \notin \mathcal{G}_D^I$ for some $x \in \Delta$. A formal proof of Theorem 14 is given in Section 6.1. In Section 6.2, we will introduce a structural subsumption algorithm deciding $C \sqsubseteq D$ by testing wether $\mathcal{G}_D$ is more specific than $\widehat{\mathcal{G}}_C$.

At this point, we should illustrate Theorem 14 by some examples.

Consider the canonical description graph $\widehat{\mathcal{G}}_C$ of C in Figure 5 and the description graph $\mathcal{G}_D$ in Figure 2. Obviously, the rooted path $\widehat{v}_0 R \widehat{v}_1$ is more specific than the rooted paths with label ε, R, and RS in $\mathcal{G}_D$. Furthermore, the rooted path $\widehat{v}_0 S \widehat{v}_2$ is more specific than $w_0 S w_3$. Consequently, $\widehat{\mathcal{G}}_C$ is more specific than $\mathcal{G}_D$ and hence, $C \sqsubseteq D$.

The following example illustrates that the notion of more specific graphs must be modified if we allow for atomic concepts $(\geq 0\, R)$ in the description graph of the subsumer D.

Example 15.
Let $C' = \forall R.P$ and $D' = \forall S.(\geq 0\, R)$. Obviously, it is $D' \equiv \top$ and therefore, $C' \sqsubseteq D'$. But since there exists no rooted path with label S in the canonical description graph $\widehat{\mathcal{G}}_{C'}$ of C', there is no more specific path for the rooted path with label S in the description graph $\mathcal{G}_{D'}$ of D'. Thus, by Definition 12 $\widehat{\mathcal{G}}_{C'}$ is not more specific than $\mathcal{G}_{D'}$ though $C' \sqsubseteq D'$. $\diamond$

6.1 Proof of the Theorem 14

Proof of Theorem 14, "$\Leftarrow$":

Let $\mathcal{I} = (\Delta, \cdot^I)$ be an interpretation of $\mathcal{C}$ and $\mathcal{R}$. We have to show $C^I \subseteq D^I$.

$\widehat{l}(\widehat{v}_0) = \{\bot\}$ implies $\widehat{\mathcal{G}}_C^I = C^I = \emptyset \subseteq D^I$. Otherwise, let $\widehat{l}(\widehat{v}_0) \neq \{\bot\}$ and $x \in C^I = \widehat{\mathcal{G}}_C^I$. We have to show $x \in \mathcal{G}_D^I = D^I$.

It is $\widehat{v}_0$ more specific than v_0. Since $\mathsf{Lit}(v_0) \subseteq \mathsf{Lit}(\widehat{v}_0)$, it is $x \in P^I$ for $P \in l(v_0)$ and $x \notin Q^I$ for $\neg Q \in l(v_0)$.
Let $(\geq \mu'\, S) \in l(v_0)$, $\mu' > 0$. It exists $(\geq \mu\, S) \in \widehat{l}(\widehat{v}_0)$ with $\mu \geq \mu'$. So, $|\{y \mid (x,y) \in S^I\}| \geq \mu \geq \mu'$.
Let $(\leq \nu'\, T) \in l(v_0)$. It exists $(\leq \nu\, T) \in \widehat{l}(\widehat{v}_0)$ with $\nu \leq \nu'$. So, $|\{y \mid (x,y) \in T^I\}| \leq \nu \leq \nu'$.

Now, let $v_0 R v_1 \in E$. It exists a more specific rooted path in $\widehat{\mathcal{G}}_C$.
Case 1: $(\leq 0\, R) \in \widehat{l}(\widehat{v}_0)$. There exists no $y \in \Delta$ such that $(x,y) \in R^I$, because $x \in \widehat{v}_0^I$.
Case 2: $(\leq 0\, R) \notin \widehat{l}(\widehat{v}_0)$. Then there exists $\widehat{v} \in \widehat{V}$ such that the rooted path $\widehat{v}0 R \widehat{v}$ is more specific than $v_0 R v_1$. By Lemma 13 it follows that $\widehat{\mathcal{G}}_C|_{\widehat{v}}$ is more specific than $\mathcal{G}_D|_{v_1}$. By induction on the size of the description graphs, it follows $\widehat{v}_0^I \subseteq v_1^I$. This implies $y \in v_1^I$ for $x \in \widehat{v}_0^I$ and $(x,y) \in R^I$.

So, all conditions of Definition 4 are satisfied and it follows $x \in v_0^I$ and hence $C^I = \widehat{\mathcal{G}}_C^I \subseteq \mathcal{G}_D^I = D^I$.

Proof of Theorem 14, "$\Rightarrow$":

If $\widehat{l}(\widehat{v}_0) = \{\bot\}$, nothing has to be shown. Otherwise, it is $\widehat{l}(\widehat{v}_0) \neq \{\bot\}$ and $\widehat{\mathcal{G}}_C$ is not more specific then $\mathcal{G}_D$. We show $C^I \not\subseteq D^I$ by construction of an interpretation $\mathcal{I} = (\Delta, \cdot^I)$ such that $x \in \widehat{\mathcal{G}}_C^I = C^I$ and $x \notin \mathcal{G}_D^I = D^I$ for some $x \in \Delta$. In the sequel, assume that $\widehat{l}(\widehat{v}_0) \neq \{\bot\}$.

First, we define the *canonical interpretation* $\mathcal{I} = (\Delta, \cdot^I)$ of $\widehat{\mathcal{G}}_C = (\widehat{V}, \widehat{E}, \widehat{v}_0, \widehat{l})$ inductively by

- $\mathcal{I}_0 := (\Delta_0, \cdot^{I_0})$ where

- $\Delta_0 := \{x_0\}$ and
- $P^{I_0} := \{x_0\}$ if $P \in \widehat{l}(\widehat{v}_0)$ and $P^{I_0} := \emptyset$ otherwise and
- $R^{I_0} := \emptyset$ for all role names R in $\widehat{\mathcal{G}}_C$.

- $\mathcal{I}_{i+1} := (\Delta_{i+1}, \cdot^{I_{i+1}})$ is defined as follows: If there exists $x \in \Delta_i$ such that x is a W-successor of x_0, and $\widehat{v}_n$ is the W-successor node of $\widehat{v}_0$ in $\widehat{\mathcal{G}}_C$ with $(\geq \mu\, R) \in \widehat{l}(\widehat{v}_n)$ and $x \notin (\geq \mu\, R)^{I_i}$, then let $\mu' := \mu - |\{y \mid (x, y) \in R^{I_i}\}|$ and define
  - $\Delta_{i+1} := \Delta_i \cup \{y_1, \ldots, y_{\mu'}\}$ for new variables y_j and
  - $R^{I_{i+1}} := R^{I_i} \cup \{(x, y_j) \mid 1 \leq j \leq \mu'\}$ and
  - $P^{I_{i+1}} := \{y \in \Delta_i \mid P \in \widehat{l}(v), \widehat{v}$ the W-successor node of $\widehat{v}_0$ in $\widehat{\mathcal{G}}_C$, and y is a W-successor of x_0 in $\mathcal{I}_i\}$.

  Else, $\mathcal{I}_{i+1} := \mathcal{I}_i$.

The *canonical interpretation* $\mathcal{I} = (\Delta, \cdot^I)$ of $\widehat{\mathcal{G}}_C$ is defined by $\Delta := \bigcup\limits_{i\geq 0} \Delta_i$, $P^I := \bigcup\limits_{i\geq 0} P^{I_i}$ and $R^I := \bigcup\limits_{i\geq 0} R^{I_i}$.

Remark 16.
The above definition of a canonical interpretation of a canonical description graph $\widehat{\mathcal{G}}_C$ leads to a tree of variables. The depth of the tree is bounded by the maximal role depth of $\widehat{\mathcal{G}}_C$ plus 1.[2] For each $(R_1 \ldots R_n)^I$-successor x of x_0 exists at most one path $\widehat{v}_0 R_1 \widehat{v}_1 \ldots \widehat{v}_{n-1} R_n \widehat{v}_n$ in $\widehat{\mathcal{G}}_C$ and $\widehat{l}(\widehat{v}_n)$ contains only a finite number of $\geq$-concepts. Therefore, each x in $\mathcal{I}$ has only a finite number of direct successors and the tree is finite. Consequently, there exists an N such that $\mathcal{I}_k = \mathcal{I}_N$ for all $k \geq N$ and $\mathcal{I} = \bigcup\limits_{0\leq i\leq N} \mathcal{I}_i$. $\diamond$

In the next step, we prove that the canonical interpretation is a model of $\widehat{\mathcal{G}}_C$, i.e., $x_0 \in \widehat{\mathcal{G}}_C^I$. The prove is based on the characterization of the extension of a description graph given in Lemma 6.

Lemma 17.
Let $\widehat{\mathcal{G}}$ be a canonical description graph with $\widehat{l}(\widehat{v}_0) \neq \{\bot\}$ and $\mathcal{I}$ its canonical interpretation. It holds that $x_0 \in \widehat{\mathcal{G}}^I$.
Proof.
Let x_n be an $(R_1 \ldots R_n)^I$-successor of x_0 and $\widehat{v}_0 R_1 \widehat{v}_1 \ldots \widehat{v}_{n-1} R_n \widehat{v}_n$ the unique rooted path with label $R_1 \ldots R_n$ in $\widehat{\mathcal{G}}$.

Case 1: $n = 0$**:** By construction it is $x_0 \in P^I$ for $P \in \widehat{l}(\widehat{v}_0)$, $x_0 \notin P^I$ for $\neg P \in \widehat{l}(\widehat{v}_0)$ and for each $(\geq \mu\, R) \in \widehat{l}(\widehat{v}_0)$ exist μ R^I-successors of x_0, so $x_0 \in (\geq \mu\, R)^I$. Further, $x_0 \in (\leq \nu\, S)^I$ for all $(\leq \nu\, S) \in \widehat{l}(\widehat{v}_0)$. Otherwise, $\widehat{l}(\widehat{v}_0)$ would contain conflicting number restrictions and hence, $\widehat{\mathcal{G}}$ would not be canonical. So, $x_0 \in \bigcap_{C \in \widehat{l}(\widehat{v}_0)} C^I$.

[2] An $R_1 \ldots R_{n+1}$-successor x of x_0 may occur in $\mathcal{I}$ if the label of a leaf v_n of G contains a concept $(\geq \mu\, R_{n+1})$ and if there already exists an $(R_1 \ldots R_n)^I$-successor x_n of x_0 in $\mathcal{I}$.

Case 2: $n > 0$**:** It exists $i > 0$ with $x_n \in \Delta_i \setminus \Delta_{i-1}$. By construction of $\mathcal{I}_j$ it is $x_n \in P^{I_j}$ for each $P \in \widehat{l}(\widehat{v}_n)$ and $x_n \notin P^{I_j}$ for each $\neg P \in \widehat{l}(v_n)$ for $j \geq i$ and for $j > i$ it is $x_n \in (\geq \mu\, R)^{I_j}$ for each $(\geq \mu\, R) \in \widehat{l}(\widehat{v}_n)$. Analogous to Case 1, it follows $x_n \in (\leq \nu\, S)^{I_j}$ for each $(\leq \nu\, S) \in \widehat{l}(\widehat{v}_n)$ and $j \geq i$. So, $x_n \in \bigcap_{C \in \widehat{l}(\widehat{v}_n)} C^I$.

By Lemma 6 it follows $x_0 \in \widehat{\mathcal{G}}^I$. □

Let $\mathcal{I}$ be the canonical interpretation of $\widehat{\mathcal{G}}$. $x_0 \notin \mathcal{G}_D^I$ implies $C \not\sqsubseteq D$ and nothing more has to be shown. Otherwise, we have to define an extension $\mathcal{J}'$ of the canonical interpretation $\mathcal{I}$ such that $x_0 \in \widehat{\mathcal{G}}_C^{J'}$ and $x_0 \notin \mathcal{G}_D^{J'}$.

Assume that $\widehat{l}(\widehat{v}_0) \neq \{\bot\}$ and $\widehat{\mathcal{G}}_C$ is not more specific than $\mathcal{G}_D$. Thus, there is a rooted path $p = v_0 R_1 v_1 \ldots v_{n-1} R_n v_n$ in $\mathcal{G}_D$ such that there exists no more specific rooted path $\widehat{p}$ in $\widehat{\mathcal{G}}_C$. W.l.o.g. we can choose p such that $l(v_n) \neq \emptyset$. By our assumption from Section 5, there occurs no atomic concept of the form $(\geq 0\, R)$ in $\mathcal{G}_D$. Thus, $l(v_n)$ contains an atomic concept of the form P, $\neg P$, $(\leq \nu\, R)$, or $(\geq \mu\, R)$, $\mu > 0$.

We define the *extended canonical interpretation* $\mathcal{J} = (\Delta_J, \cdot^J)$ of $\widehat{\mathcal{G}}_C$ and p such that $\Delta \subseteq \Delta_J$, $P^J \cap \Delta = P^I$, $R^J \cap \Delta \times \Delta = R^I$, and $x_0 \in \widehat{\mathcal{G}}_C^J$.

Therefore, let $\widehat{p} = \widehat{v}_0 R_1 \widehat{v}_1 \ldots \widehat{v}_{m-1} R_m \widehat{v}_m$, $m \leq n$, be the unique rooted path in $\widehat{\mathcal{G}}_C$ of maximal length such that the label of $\widehat{p}$ is a prefix of $R_1 \ldots R_n$. By assumption, $\widehat{p}$ is not more specific than p and since $\widehat{\mathcal{G}}_C$ is canonical, it is $(\leq 0\, R_k) \notin \widehat{l}(\widehat{v}_{k-1})$ for $1 \leq k \leq m$.

To define $\mathcal{J}$, we consider two cases.

Case 1: There exists an index i such that $0 \leq i \leq m$ and $\widehat{v}_i$ is not more specific then v_i. Let N be the minimal index with these properties.

Case 2: For each $0 \leq i \leq m$, $\widehat{v}_i$ is more specific than v_i. Let $N := n$. It is $m < n$ and $(\leq 0\, R_{m+1}) \notin \widehat{l}(\widehat{v}_m)$. Otherwise, $\widehat{p}$ would be more specific than p.

Roughly speaking, we want to extend $\mathcal{I}$ by an $R_1 \ldots R_N$-successor x_N such that

- x_N satisfies each restriction on $R_1 \ldots R_N$- successors given by an $R_1 \ldots R_N$-successor node in $\widehat{\mathcal{G}}_C$ and
- at least one atomic concept in the label $l(v_N)$ in $\mathcal{G}_D$ is violated by x_N.

If there exists an $(R_1 \ldots R_N)^I$-successor x_N of x_0 in $\mathcal{I}$, then $\mathcal{J} := \mathcal{I}$. Otherwise, let $1 \leq j \leq N$ be the maximal index such that there exists an $(R_1 \ldots R_{j-1})^I$-successor x_{j-1} of x_0 in $\mathcal{I}$. Extend $\mathcal{I}$ to $\mathcal{I}' := (\Delta', \cdot^{I'})$ by

- $\Delta' := \Delta \cup \{x_j, \ldots, x_N\}$ for new variables $x_j, \ldots, x_N$ and
- $R_k^{I'} := R_k^I \cup \{(x_{k-1}, x_k)\}$ for $j \leq k \leq N$.

Notice that no $\leq$-restriction of the form $(\leq 0\, R)$ is violated by the new variables $x_j, \ldots, x_N$. As already mentioned above, it is $(\leq 0\, R_k) \notin \widehat{l}(\widehat{v}_{k-1})$ for $1 \leq k \leq m$. Thus, if $N \leq m$ nothing has to be shown. Otherwise, Case 2 from above holds. It is $m < N = n$ and $(\leq 0\, R_{m+1}) \notin \widehat{l}(\widehat{v}_m)$. Since $\widehat{p}$ has been chosen by maximal length, there exists no $R_1 \ldots R_k$-successor node in $\widehat{\mathcal{G}}_C$ for

$m < k \le n$. Thus, there exist no restrictions $(\le 0\, R_k)$ in $\widehat{l}(\widehat{v}_{k-1})$, $1 \le k \le N$, that would be violated by $x_j, \ldots, x_N$.

The extended canonical interpretation $\mathcal{J}$ is inductively defined in the same way as the canonical interpretation but starting with $\mathcal{J}_0 := \mathcal{I}'$.

It is not hard to see, that the proof of Lemma 17 even holds if the induction is started with an interpretation $\mathcal{I}_0 = (\Delta_0, \cdot^{I_0})$ such that the following conditions are satisfied:

1. Δ_0 is a finite set of variables and the interpretation of the role names in $\widehat{\mathcal{G}}$ under $\mathcal{I}_0$ yield a tree with root $x_0 \in \Delta_0$.

2. None of the atomic concepts in the labels of $\widehat{\mathcal{G}}$ is hurt by $\mathcal{I}_0$, i.e., for each $(R_1 \ldots R_n)^{I_0}$-successor x_n of x_0 and each rooted path $\widehat{v}_0 R_1 \widehat{v}_1 \ldots \widehat{v}_{n-1} R_n \widehat{v}_n$ in $\widehat{\mathcal{G}}$ it holds $x_n \notin \Delta_0 \setminus C^{I_0}$ for all $C \in \widehat{l}(\widehat{v}_n)$.

Since $\mathcal{I}'$ satisfies the conditions 1. and 2., it follows $x_0 \in \widehat{\mathcal{G}}_C^J$.

Now, we can complete the proof of Theorem 14. Let $\mathcal{J}$ be the extended canonical interpretation of $\widehat{\mathcal{G}}_C$ and p. Let further p and N be determined as in the definition of $\mathcal{J}$. If $x_0 \notin \mathcal{G}_D^J$, nothing more has to be shown. Otherwise, we define an interpretation $\mathcal{J}' = (\Delta_{J'}, \cdot^{J'})$ such that $\mathcal{J}'$ is an extension of $\mathcal{J}$ and $x_0 \in \widehat{\mathcal{G}}_C^{J'}$ and $x_0 \notin \mathcal{G}_D^{J'}$.

By construction of $\mathcal{J}$ there exists an $(R_1 \ldots R_N)^J$-successor x_N of x_0. We have to consider the two cases from above.

Case 1: $N \le m$ and $\widehat{v}_N$ is not more specific than v_N. By Lemma 6, $x_0 \in \mathcal{G}_D^J$ implies $x_N \in (\ge \mu'\, R)^J$ for each $(\ge \mu'\, R) \in l(v_N)$. As an easy consequence of the construction of $\mathcal{J}$ it follows that for each $(\ge \mu'\, R) \in l(v_N)$ there exists $(\ge \mu\, R) \in \widehat{l}(\widehat{v}_N)$ with $\mu \ge \mu'$. Since $\widehat{v}_N$ is not more specific than v_N, one of the cases 1.1 or 1.2 must hold.

Case 1.1: $\mathsf{Lit}(v_N) \not\subseteq \mathsf{Lit}(\widehat{v}_N)$. There exists no $P \in \mathsf{Lit}(v_N) \setminus \mathsf{Lit}(\widehat{v}_N)$. Otherwise, $x_0 \notin \mathcal{G}_D^J$ because of $x_N \in P^J$ iff $P \in \widehat{l}(\widehat{v}_N)$ (see Lemma 6). So, there exists $\neg P \in \mathsf{Lit}(v_N) \setminus \mathsf{Lit}(\widehat{v}_N)$. Because of $x_0 \in \mathcal{G}_D^J$ it is $x_N \notin P^J$ and $P \notin \widehat{l}(\widehat{v}_n)$. Extend $\mathcal{J}$ to $\mathcal{J}'$ by $P^{J'} := P^J \cup \{x_N\}$. Consequently, $x_0 \notin \mathcal{G}_D^{J'}$ but still $x_0 \in \widehat{\mathcal{G}}_C^{J'}$.

Case 1.2: $(\le \nu'\, R) \in l(v_N)$ and there is no $(\le \nu\, R) \in \widehat{l}(\widehat{v}_N)$ with $\nu \le \nu'$. Extend $\mathcal{J}$ to $\mathcal{J}'$ by $\Delta_{J'} := \Delta_J \cup \{y_1, \ldots, y_{\nu'+1}\}$, $R^{J'} := R^J \cup \{(x_N, y_k) \mid 1 \le k \le \nu' + 1\}$ and satisfy all atomic concepts in $\widehat{\mathcal{G}}_C$ for the new variables $y_1, \ldots y_{\nu+1}$ inductively as in the definition of the canonical interpretation of $\widehat{\mathcal{G}}_C$. It follows $x_0 \in \widehat{\mathcal{G}}_C^{J'}$ and obviously, it is $x_0 \notin \mathcal{G}_D^{J'}$.

Case 2: $N = n > m$. By assumption, there exists an atomic concept $D' \in l(v_N)$ not of the form $(\ge 0\, R)$. As an easy consequence of the construction of $\mathcal{J}$ there exist no successors of x_N in $\mathcal{J}$ because there is no $R_1 \ldots R_N$-successor $\widehat{v}_N$ of $\widehat{v}_0$ in $\widehat{\mathcal{G}}_C$ and hence no $\ge$-restriction requiring successors of x_N. Consequently, there exists no $\ge$-concept in $l(v_N)$ (otherwise, $x_0 \notin \mathcal{G}_D^J$). Analogous to Case 1.1 it follows that there is no $P \in l(v_N)$. So, one of the cases 2.1 or 2.2 must hold.

Case 2.1: $\neg P \in l(v_N)$. Extend $\mathcal{J}$ to $\mathcal{J}'$ by $P^{J'} := P^J \cup \{x_N\}$. Consequently, $x_0 \notin \mathcal{G}_D^{J'}$ but still $x_0 \in \widehat{\mathcal{G}}_C^{J'}$ because $\widehat{\mathcal{G}}_C$ yields no restrictions on $(R_1 \ldots R_N)^J$- successors of x_0.

Case 2.2: $(\leq \nu' R) \in l(v_N)$. Extend $\mathcal{J}$ to $\mathcal{J}'$ by $\Delta_{J'} := \Delta_J \cup \{y_1, \ldots y_{\nu'+1}\}$ and $R^{J'} := R^J \cup \{(x_N, y_k) \mid 1 \leq k \leq \nu' + 1\}$ for new variables $y_1, \ldots, y_{\nu'+1}$. Analogous to Case 2.1 it follows $x_0 \in \widehat{\mathcal{G}}_C^{J'}$ and $x_0 \notin \mathcal{G}_D^{J'}$.

This completes the proof of Theorem 14.

6.2 The algorithm

An algorithm deciding $C \sqsubseteq D$ by Theorem 14 considers the (canonical) description graphs $\widehat{\mathcal{G}}_C$ and $\mathcal{G}_D$, respectively, and tests wether there exists a more specific rooted path $\widehat{p}$ in $\widehat{\mathcal{G}}_C$ for each rooted path p in $\mathcal{G}_D$ or not. In other words, for each $W \in \mathcal{R}^*$ and W-successor node v of v_0 in $\mathcal{G}_D = (V, E, v_0, l)$ we test

1. if there exists a proper prefix W' of W and a W'-successor node $\widehat{v}$ of $\widehat{v}_0$ in $\widehat{\mathcal{G}}_C$ such that
   (a) $W = W'RW''$, $R \in \mathcal{R}$ and $W'' \in \mathcal{R}^*$,
   (b) each node v' on the path labeled with W' in $\widehat{\mathcal{G}}_C$ is more specific than the corresponding node in $\mathcal{G}_D$, and
   (c) $(\leq 0\, R) \in \widehat{l}(\widehat{v})$ or
2. if $\widehat{v}$ is the W-successor node of $\widehat{v}_0$ in $\widehat{\mathcal{G}}_C$, wether $\widehat{v}$ is more specific than v or not.

If (1) and (2) are not satisfied, than $C \not\sqsubseteq D$; otherwise $C \sqsubseteq D$.

The conditions (1) and (2) are tested recursively by the procedure $\mathsf{more-specific?}$ shown in Figure 6. Notice that these conditions do not yield a complete subsumption algorithm if we allow for atomic concepts $(\geq 0\, R)$ in $\mathcal{G}_D$. Consider Example 15. Obviously, $\mathsf{more-specific?}(\widehat{\mathcal{G}}_{C'}, \mathcal{G}_{D'})$ returns $false$, because $(\leq 0\, S) \notin \widehat{l}'(\widehat{v}_0')$ and $\widehat{v}_0'$ has no S-successor in $\widehat{\mathcal{G}}_{C'}$. Thus, a subsumption algorithm based on the procedure $\mathsf{more-specific?}$ would be incomplete if we allow for arbitrary $\mathcal{ALN}$-concepts C, D as inputs.

An algorithm deciding $C \sqsubseteq D$ can be described as follows: Given two $\mathcal{ALN}$-concepts C and D, we compute the description graphs $\mathcal{G}_C$ and $\mathcal{G}_D$, respectively, with the help of the translation algorithm (see Figure 1). An iterated application of the normalization rules from Figure 4 yields the canonical description graph $\widehat{\mathcal{G}}_C$ of C. Now, it is $C \sqsubseteq D$ iff $\mathsf{more-specific?}(\widehat{\mathcal{G}}_C, \mathcal{G}_D)$ returns $true$.

Using Theorem 14 it is not hard to show that the structural subsumption algorithm is sound and complete. The complexity of the algorithm is determined by the complexity of defining the corresponding (canonical) description graphs and by the complexity of the procedure $\mathsf{more-specific?}$. As already mentioned, $\mathcal{G}_C$ as well as $\widehat{\mathcal{G}}_C$ are linear in the size of C, and $\mathcal{G}_D$ is linear in the size of D. Both $\widehat{\mathcal{G}}_C$ and $\mathcal{G}_D$ can be computed in time polynomial in $\max\{|C|, |D|\}$. Testing wether $\widehat{\mathcal{G}}_C$ is more specific than $\mathcal{G}_D$ by the procedure $\mathsf{more-specific?}$ from Figure 6 is polynomial in the size of $\mathcal{G}_D$. If $C \not\equiv \bot$, then each node v in $\mathcal{G}_D$ is reached at most once and since $\widehat{\mathcal{G}}_C$ is canonical, there exists at most one corresponding node $\widehat{v}$ in $\widehat{\mathcal{G}}_C$ for which "$\widehat{v}$ is more specific than v" must be tested. Obviously, this can be done in time polynomial in the size of $l(v)$. Thus, we can decide $D \sqsubseteq D$ using the structural subsumption algorithm described above in time polynomial in the size of $\max\{|C|, |D|\}$.

```
more − specific?(Ĝ_C, G_D)

Input:  Ĝ_C = (V̂, Ê, v̂_0, l̂), a canonical description graph
        G_D = (V, E, v_0, l), a description graph

Output: true        , if Ĝ_C is more specific than G_D
        false       , otherwise

begin
  if l̂(v̂_0) = ⊥
  then return true
  else
    if v̂_0 is not more specific than v_0
    then return false
    else
      for each R-successor v of v_0
        if (≤ 0 R) ∉ l̂(v̂_0) and exists no R-successor of v̂_0
        then return false
        else let v̂ be the R-successor of v̂_0
          if not more − specific?(Ĝ_C|_v̂, G_D|_v)
          then return false
      return true
end
```

Figure 6: More specific graphs.

7 Conclusion and related work

We presented a sound and complete structural subsumption algorithm for deciding subsumption of $\mathcal{ALN}$-concepts based on description graphs. Therefore, the notion of canonical description graphs of Classic-concepts [BP94] was modified in order to cope with conflicting number restrictions as well as primitive negation which both can occur in $\mathcal{ALN}$-concepts.

Note that the algorithm for deciding subsumption of $\mathcal{ALN}$-concepts described in Section 6.2 differs from the algorithm presented in [BP94]. The Classic-algorithm subsumes?$(\mathcal{G}, C)$ for deciding $C \sqsubseteq D$ only computes the canonical description graph $\mathcal{G}$ of C and then tests syntactical conditions on the concept D and the description graph $\mathcal{G}$ recursively. In [CH94], subsumption of Classic-concepts is characterized using description graphs for both the subsumee C and the subsumer D. This is due to the fact that Cohen and Hirsh also introduced an algorithm for computing the *least common subsumer* of Classic-concepts (see also [CBH92]). The least common subsumer of two Classic-concepts C_1, C_2 is the most specific Classic-concept that subsumes both C_1 and C_2. It turned out that computing the LCS of C and D corresponds to some kind of "merging" the canonical description graphs $\widehat{\mathcal{G}}_C$ and $\widehat{\mathcal{G}}_D$. Thus, using the structural characterization of subsumption of $\mathcal{ALN}$-concepts from Theorem 14, it is easy to adapt the LCS-algorithm for Classic to $\mathcal{ALN}$ and to prove its soundness and completeness.

Another, automata theoretic approach to subsumption as well as least common subsumers w.r.t. cyclic $\mathcal{ALN}$-terminologies has been proposed in the literature [Neb90, Baa96, Küs98, BK98]. In the comparison of both approaches [BKM98], it turned out that in the case of

$\mathcal{ALN}$-concepts structural subsumption algorithms can be seen as special implementations of the conditions required by the automata theoretic characterization.

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