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The Complexity of Reasoning with Boolean Modal Logics (Extended Version)

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The Complexity of Reasoning with Boolean Modal Logics (Extended Version)

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1 Motivation

Since Modal Logics are an extension of Propositional Logic, they provide Boolean operators for constructing complex formulae. However, most Modal Logics do not admit Boolean operators for constructing *complex modal parameters* to be used in the box and diamond operators. This asymmetry is not present in Boolean Modal Logics, in which box and diamond quantify over arbitrary Boolean combinations of atomic modal parameters [9]. Boolean Modal Logics have been considered in various forms and contexts:

1. "Pure" Boolean Modal Logic has been studied in [9]. Negation and intersection of modal parameters occur in some variants of Propositional Dynamic Logic, see, e.g., [7, 16, 22].

2. The modal box operator can be thought of as expressing necessity. More precisely, when employing the usual Kripke Semantics, $\Box\varphi$ holds at a world w iff w' being accessible from w implies that φ holds at w'. Given this, it is obviously quite natural to define a symmetric operator $\boxminus$ (sometimes called "window" operator) such that $\boxminus\varphi$ holds at a world w iff φ holding at a world w' implies that w' is accessible from w. Obviously, the window operator can be thought of as expressing sufficiency. Logics with this operator were investigated from different viewpoints by, e.g., Humberstone, Gargov et al., and Goranko [17, 10, 13, 14]. If negation of modal parameters is available, the window operator comes for free since we can write $\boxminus_R\varphi$ as $[\neg R]\neg\varphi$. For other work related to the window operator see, e.g., [2, 12, 11].

3. There are several Description Logics that provide "negation of roles" which corresponds to the negation of modal parameters, see, e.g. [18]. Union and intersection of modal parameters are also considered in Description Logics and other KR formalisms, as is the window operator [11, 21].

Although—as we just argued—logics involving Boolean operators on modal parameters or the window operator are widely used, to the best of our knowledge, complexity results for this class of logics have never been obtained. In this paper, we close the gap and determine the complexity of the satisfiability and validity problems for many Boolean Modal Logics. In the first part of this paper (Sections 2 and 3), we investigate the logic $\mathbf{K}_\omega$ ($\mathbf{K}$ with a countably infinite number of accessibility relations) enriched with negation of modal parameters and show that the afore mentioned inference problems are ExpTime-complete using an automata-theoretic approach. We then demonstrate the generality of our approach by extending this result to the logic $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, i.e., to the fusion of $\mathbf{K}_\omega$ with $\mathbf{K4}_\omega$ enriched with negation on relations. In the second part of this paper (Sections 4 and 5), we add other Boolean operators on roles. In doing so, one has the choice to either restrict negation to atomic relations or to allow for full negation of relations.

We give a complete list of complexity results for the logics obtained in this way, the central result being that the combination of (atomic) negation with intersection yields a logic whose inference problems are NExpTime-complete. The lower bound is obtained by a reduction of a NExpTime-complete variant of the domino problem. The mentioned result obviously implies that full Boolean Modal Logic $\mathbf{K}_\omega^{\neg,\cap,\cup}$ is also NExpTime-complete. However, the lower bound crucially depends on the number of relations to be unbounded. Inspired by this observation, in Section 5, we supplement our result by showing that, for any fixed finite number of relations, full Boolean Modal Logic is ExpTime-complete. The upper bound is proved by a reduction to multi-modal $\mathbf{K}$ (with finitely many relations) enriched with the universal modality.

To complete our investigation, in Section 6 we show that $\mathbf{K}_\omega$ with union and intersection of roles and without negation is of the same complexity as pure $\mathbf{K}_\omega$, i.e., PSpace-complete. Summing up, we thus have tight complexity bounds for $\mathbf{K}_\omega$ extended with any combination of Boolean operators on roles.

2 Preliminaries

In this section, we define syntax and semantics of $\mathbf{K}_\omega^\neg$ and discuss some model- and complexity-theoretic properties of this logic.

Definition 1 Given a countably infinite set of *propositional variables* Φ and a countably infinite set of *atomic modal parameters* $R_1, R_2, \ldots$, the set of $\mathbf{K}_\omega^\neg$-*formulae* is the smallest set that

- contains the propositional variables in Φ,
- is closed under boolean connectives $\wedge$, $\vee$, and $\neg$, and
- if it contains φ, then it also contains $\langle R_i \rangle \varphi$, $[R_i]\varphi$, $\langle \neg R_i \rangle \varphi$, and $[\neg R_i]\varphi$ for $i \geq 1$.

The set of $\mathbf{K}_\omega^\neg$-modal parameters is the smallest set containing all atomic modal parameters and their negations (i.e., expressions of the form $\neg R_i$).

The semantics of $\mathbf{K}_\omega^\neg$-formulae is given by *Kripke structures*

$$\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \mathcal{R}_2, \ldots \rangle,$$

where W is a set of *worlds*, π is a mapping from the set of propositional variables into sets of worlds (i.e., for each $p \in \Phi$, $\pi(p)$ is the set of worlds in which p holds), and $\mathcal{R}_i$ is a binary relation on the worlds W, the so-called *accessibility relation* for the atomic modal parameter R_i.

The semantics is then given as follows, where, for a $\mathbf{K}_\omega^\neg$-formula φ, a Kripke structure $\mathcal{M}$, and a world $w \in W$; the expression $\mathcal{M}, w \models \varphi$ is read as "φ holds in $\mathcal{M}$ in world w".

$\mathcal{M}, w \models p_i$	iff	$w \in \pi(p_i)$ for $p \in \Phi$
$\mathcal{M}, w \models \varphi_1 \wedge \varphi_2$	iff	$\mathcal{M}, w \models \varphi_1$ and $\mathcal{M}, w \models \varphi_2$
$\mathcal{M}, w \models \varphi_1 \vee \varphi_2$	iff	$\mathcal{M}, w \models \varphi_1$ or $\mathcal{M}, w \models \varphi_2$
$\mathcal{M}, w \models \neg\varphi$	iff	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \langle R_i \rangle \varphi$	iff	there exists $w' \in W$ with $(w, w') \in \mathcal{R}_i$ and $\mathcal{M}, w' \models \varphi$
$\mathcal{M}, w \models [R_i] \varphi$	iff	for all $w' \in W$, if $(w, w') \in \mathcal{R}_i$, then $\mathcal{M}, w' \models \varphi$
$\mathcal{M}, w \models \langle \neg R_i \rangle \varphi$	iff	there exists $w' \in W$ with $(w, w') \notin \mathcal{R}_i$ and $\mathcal{M}, w' \models \varphi$
$\mathcal{M}, w \models [\neg R_i] \varphi$	iff	for all $w' \in W$, if $(w, w') \notin \mathcal{R}_i$, then $\mathcal{M}, w' \models \varphi$

A $\mathbf{K}_\omega^\neg$-formula φ is *satisfiable* iff there is a Kripke structure $\mathcal{M}$ with a set of worlds W and a world $w \in W$ such that $\mathcal{M}, w \models \varphi$. Such a structure is called a *model* of φ. Two $\mathbf{K}_\omega^\neg$-formulae φ and ψ are *equivalent* (written $\varphi \equiv \psi$) iff $\mathcal{M}, w \models \varphi \iff \mathcal{M}, w \models \psi$ for all Kripke structures $\mathcal{M}$ with set of worlds W and worlds $w \in W$. Let R be a modal parameter. We write $\mathcal{M}, (w, w') \models R$ to express that (i) $(w, w') \in \mathcal{R}_i$ if R is an atomic modal parameter R_i and (ii) $(w, w') \notin \mathcal{R}_i$ if $R = \neg R_i$ for an atomic modal parameter R_i.

▲

Throughout this paper, we denote modal parameters by R and S. For the sake of brevity, we will often omit the word "modal" when talking about modal parameters. As usual, we write $\varphi \to \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. The semantics of the window operator discussed in the introduction can formally be defined as follows:

$$\mathcal{M}, w \models \boxminus_{R_i} \varphi \quad \text{iff} \quad \text{for all } w' \in W, \text{ if } \mathcal{M}, w' \models \varphi, \text{ then } (w, w') \in \mathcal{R}_i$$

Obviously, we have $\boxminus_{R_i} \varphi \equiv [\neg R_i] \neg\varphi$, and, hence, the window operator is available in $\mathbf{K}_\omega^\neg$.

It is not hard to see that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is ExpTime-hard and in NExpTime: (i) satisfiability of $\mathbf{K}^u$-formulae, where $\mathbf{K}^u$ is uni-modal $\mathbf{K}$ enriched with the universal modality, can be reduced to the satisfiability of $\mathbf{K}_\omega^\neg$-formulae: Just replace

- every occurrence of $[u] \varphi$ by $[R] \varphi \wedge [\neg R] \varphi$ and
- every occurrence of $\langle u \rangle \varphi$ by $\langle R \rangle \varphi \vee \langle \neg R \rangle \varphi$

where $[u]$ and $\langle u \rangle$ denote the universal modality, and R is an arbitrary atomic modal parameter. This translation may clearly lead to an exponential blowup in the formula. However, in the class of formulae used to prove the ExpTime-hardness of $\mathbf{K}^u$ [25], $[u]$ occurs only once, and $\langle u \rangle$ does not occur. In this case, the translation is linear, and, thus, satisfiability of $\mathbf{K}_\omega^\neg$-concepts is ExpTime-hard; (ii) when using the standard translation of modal formulae into first order formulae (see, e.g, [5, 3]), $\mathbf{K}_\omega^\neg$-formulae are translated to first-order formulae with at most 2 variables. Since L^2, the two-variable fragment of first-order logic, is decidable in NExpTime [15], this implies that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is also in NExpTime. However, these two complexity bounds are obviously not tight. One main contribution of this paper is to give an ExpTime-algorithm for the satisfiability of $\mathbf{K}_\omega^\neg$-formulae, thus tightening the complexity bounds.

For devising a satisfiability algorithm, it is interesting to know what kind of models need to be considered. In [10], it is proved that $\mathbf{K}_\omega^\neg$ has the finite model property. $\mathbf{K}_\omega^\neg$ does not have the tree model property since, e.g., the formula $p \wedge [\neg R]\neg p$ has no tree model. However, we will show that there exists a one-to-one correspondence between models and so-called Hintikka-trees which we then use to decide satisfiability (and thus validity) of $\mathbf{K}_\omega^\neg$-formulae. We do this by building, for each $\mathbf{K}_\omega^\neg$-formula φ, a looping automaton $\mathcal{A}_\varphi$ which accepts the empty (tree-) language iff φ is unsatisfiable. Hence we introduce trees, looping automata, and the language they accept here.

Definition 2 Let M be a set and $k \geq 1$. A *k-ary M-tree* is a mapping $T : \{1, \ldots, k\}^* \mapsto M$ that labels each node $\alpha \in \{1, \ldots, k\}^*$ with $T(\alpha) \in M$. Intuitively, the node αi is the i-th child of α. We use ϵ to denote the empty word (corresponding to the root of the tree).

A *looping automaton* $\mathcal{A} = (Q, M, I, \Delta)$ for k-ary M-trees is defined by a set Q of states, an alphabet M, a subset $I \subseteq Q$ of initial states, and a transition relation $\Delta \subseteq Q \times M \times Q^k$.
A *run* of $\mathcal{A}$ on an M-tree T is a mapping $r : \{1, \ldots, k\}^* \mapsto Q$ with

$$(r(\alpha), T(\alpha), r(\alpha 1), \ldots, r(\alpha k)) \in \Delta$$

for each $\alpha \in \{1, \ldots, k\}^*$.

A looping automaton accepts all those M-trees for which a run exists, i.e., the language $\mathcal{L}(\mathcal{A})$ of M-trees accepted by $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) = \{T \mid \text{There is a run from } \mathcal{A} \text{ on } T\}.$$

▲

Since looping automata are special Büchi automata, emptiness of their language can effectively be tested using the well-known (quadratic) emptiness test for Büchi-automata [26]. However, for looping tree automata, this algorithm can be specialized into a simpler (linear) one.

3 Negation of Modal Parameters

We show that satisfiability of $\mathbf{K}_\omega^\neg$-formulae is decidable in exponential time. For this purpose, we first abstract from models of $\mathbf{K}_\omega^\neg$-formulae to Hintikka-trees, and then show how to construct a looping automaton that accepts exactly Hintikka-trees.

Notation: We assume all formulae to be in negation normal form (NNF), i.e., negation occurs only in front of atomic parameters and propositional variables. Each formula can easily be transformed into an equivalent one in NNF by pushing negation inwards, employing de Morgan's law and the duality between $[R]$ and $\langle R\rangle$ and between $[\neg R]$ and $\langle \neg R\rangle$. We use $\bar{\varphi}$ to denote the NNF of $\neg\varphi$.

Since we treat modalities with negated and unnegated modal parameters symmetrically, we introduce the notion

$$\langle \bar{R}\rangle\,\varphi = \begin{cases} \langle \neg R\rangle\,\varphi & \text{if } R \text{ is atomic,} \\ \langle S\rangle\,\varphi & \text{if } R = \neg S \text{ for some atomic parameter } S \end{cases}$$

and analogously $[\bar{R}]\,\varphi$.

Let $\mathsf{cl}(\varphi)$ denote the set of φ's subformulae and the NNFs of their negations, i.e.,

$$\mathsf{cl}(\varphi) := \{\psi \mid \; \psi \text{ is a subformula of } \varphi \text{ or } \psi = \bar{\rho} \text{ for a subformula } \rho \text{ of } \varphi\}.$$

We assume that diamond-formulae $\langle R\rangle\,\psi$ in $\mathsf{cl}(\varphi)$ are linearly ordered, and that $\mathcal{D}(i)$ yields the i-th diamond-formula in $\mathsf{cl}(\varphi)$.

Definition 3 (Hintikka-set and Hintikka-tree)
Let φ be a $\mathbf{K}_\omega^\neg$-formula and k the number of diamond-formulae in $\mathsf{cl}(\varphi)$.
A set $\Psi \subseteq \mathsf{cl}(\varphi)$ is a *Hintikka-set* iff it satisfies the following conditions:

(H1) if $\varphi_1 \wedge \varphi_2 \in \Psi$, then $\{\varphi_1, \varphi_2\} \subseteq \Psi$,

(H2) if $\varphi_1 \vee \varphi_2 \in \Psi$, then $\{\varphi_1, \varphi_2\} \cap \Psi \neq \emptyset$,

(H3) $\{\psi, \bar{\psi}\} \not\subseteq \Psi$ for all $\mathbf{K}_\omega^\neg$-formulae ψ.

A k-ary $2^{\mathsf{cl}(\varphi)}$-tree T is a *Hintikka-tree for* φ iff $T(\alpha)$ is a Hintikka-set for each node α in T, and T satisfies, for all nodes $\alpha, \beta \in \{1, \ldots, k\}^*$, the following conditions:

(H4) $\varphi \in T(\epsilon)$,

(H5) if $\{\langle R\rangle\,\psi, [R]\,\rho_1, \ldots, [R]\,\rho_m\} \subseteq T(\alpha)$ and $\mathcal{D}(i) = \langle R\rangle\,\psi$, then $\{\psi, \rho_1, \ldots, \rho_m\} \subseteq T(\alpha i)$

(H6) if $\mathcal{D}(i) \notin T(\alpha)$, then $T(\alpha i) = \emptyset$,

(H7) if $[R]\,\rho \in T(\alpha)$, then $\rho \in T(\beta)$, $\bar{\rho} \in T(\beta)$, or $T(\beta) = \emptyset$,

(H8) if $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$, then $\psi \in T(\beta)$.

For **(H5)**, **(H7)**, and **(H8)**, recall that R denotes atomic parameters and also negations of atomic parameters.

Lemma 4 A $\mathbf{K}_\omega^\neg$-formula φ is satisfiable iff φ has a Hintikka-tree.

Proof: Let φ be a $\mathbf{K}_\omega^\neg$-formula and let there be k diamond-formulae in $\mathsf{cl}(\varphi)$.

"$\Leftarrow$" Let T be a Hintikka-tree for φ. We define a Kripke structure $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \rangle$ as follows:

$$\begin{array}{rcl} W &=& \{\alpha \in \{1,\ldots,k\}^* \mid T(\alpha) \neq \emptyset\} \\ \pi(p) &=& \{\alpha \mid p \in T(\alpha)\} \text{ for all } p \in \Phi \\ \mathcal{R}_i &=& \{(\alpha,\beta) \mid \beta = \alpha j \text{ and } \mathcal{E}(j) = \langle R_i \rangle\,\psi \in T(\alpha)\} \,\cup \\ && \{(\alpha,\beta) \mid [\neg R_i]\,\psi \in T(\alpha) \text{ and } \bar{\psi} \in T(\beta)\} \end{array}$$

To show that there exists a $w \in W$ such that $\mathcal{M}, w \models \varphi$, we first prove the following claim:

Claim: $\psi \in T(\alpha)$ implies $\mathcal{M}, \alpha \models \psi$ for all $\alpha \in W$ and $\psi \in \mathsf{cl}(\varphi)$.

The claim is proved by induction over the structure of ψ. The induction start, i.e., the case that ψ is a propositional variable, is an immediate consequence of the definition of $\mathcal{M}$. For the induction step, we make a case distinction according to the topmost operator in ψ. Assume $\psi \in T(\alpha)$.

- $\psi = \neg\rho$. Since φ is in NNF (by the definition of Hintikka-sets and cl), ρ is a propositional variable. By definition of $\mathcal{M}$ and since $T(\alpha)$ is a Hintikka-set and thus satisfies **(H3)**, we have $\mathcal{M}, \alpha \models \neg\rho$.

- $\psi = \varphi_1 \wedge \varphi_2$ or $\psi = \varphi_1 \vee \varphi_2$. Straightforward by **(H1)** and **(H2)** of Hintikka-sets and by induction hypothesis.

- $\psi = \langle R \rangle\,\rho = \mathcal{E}(j)$ for a j with $1 \leq j \leq k$. First assume that $R = R_i$, i.e., R is atomic. By definition of $\mathcal{R}_i$, we have $(\alpha, \alpha j) \in \mathcal{R}_i$. By **(H5)**, $\langle R_i \rangle\,\rho \in T(\alpha)$ implies $\rho \in T(\alpha j)$. By induction, $\mathcal{M}, \alpha j \models \rho$, and, hence, $\mathcal{M}, \alpha \models \langle R_i \rangle\,\rho$.

  Now assume that $R = \neg R_i$ for an atomic parameter R_i. We show that $(\alpha, \alpha j) \notin \mathcal{R}_i$, for, if we have done this, $\mathcal{M}, \alpha \models \langle R \rangle\,\rho$ follows as in the previous case (where R is atomic). Assume to the contrary that $(\alpha, \alpha j) \in \mathcal{R}_i$. Then, by definition of $\mathcal{R}_i$, we have either

  1. $\mathcal{E}(j) = \langle R_i \rangle\,\rho' \in T(\alpha)$, or
  2. $[\neg R_i]\,\rho' \in T(\alpha)$ and $\bar{\rho}' \in T(\alpha j)$

where $\rho' \in \mathsf{cl}(\varphi)$. In the first case, we have a contradiction to the assumption $\mathcal{E}(j) = \langle \neg R_i \rangle \rho$. In the second case, we have $\{\langle \neg R_i \rangle \rho, [\neg R_i] \rho'\} \subseteq T(\alpha)$ which, by **(H5)**, implies $\{\rho, \rho'\} \subseteq T(\alpha j)$. Since we also know that $\bar{\rho}' \in T(\alpha j)$, we obtain a contradiction to **(H3)** of Hintikka-sets and conclude that $(\alpha, \alpha j) \notin \mathcal{R}_i$.

- $\psi = [R] \rho$. First assume that $R = R_i$, i.e., R is atomic, and fix a β such that $(\alpha, \beta) \in \mathcal{R}_i$. By definition of $\mathcal{R}_i$, we have to distinguish two cases:

  1. $\beta = \alpha j$ and $\mathcal{E}(j) = \langle R_i \rangle \rho' \in T(\alpha)$, or
  2. $[\neg R_i] \rho' \in T(\alpha)$ and $\bar{\rho}' \in T(\beta)$

  In the first case, we have $\{\langle R_i \rangle \rho', [R_i] \rho\} \subseteq T(\alpha)$ which, by **(H5)**, implies $\{\rho, \rho'\} \subseteq T(\alpha j)$. By induction, we obtain $\mathcal{M}, \beta \models \rho$. In the second case, we have $\{[R_i] \rho, [\neg R_i] \rho'\} \subseteq T(\alpha)$ and $\bar{\rho}' \in T(\beta)$. By **(H8)**, we have $\rho \in T(\beta)$, and, by induction, $\mathcal{M}, \beta \models \rho$. Since this holds independently of the choice of β, we conclude $\mathcal{M}, \alpha \models [R_i] \rho$.

  Now assume that $R = \neg R_i$ for an atomic parameter R_i. Fix a β such that $(\alpha, \beta) \notin \mathcal{R}_i$. Since $\beta \in W$, we have that $T(\beta) \neq \emptyset$. Hence, by **(H7)**, we have $\rho \in T(\beta)$ or $\bar{\rho} \in T(\beta)$. However, $\bar{\rho} \in T(\beta)$ would imply $(\alpha, \beta) \in \mathcal{R}_i$ by definition of $\mathcal{R}_i$, which is a contradiction to our choice of β. Hence we deduce $\rho \in T(\beta)$. By induction, we obtain $\mathcal{M}, \beta \models \rho$. Since this holds independently of the choice of β, we conclude $\mathcal{M}, \alpha \models [\neg R_i] \rho$.

This completes the proof of the claim. Since $\varphi \in T(\epsilon)$ by **(H4)**, it is an immediate consequence of the claim that $\mathcal{M}$ is a model of φ.

"⇒" Let $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \rangle$ be a model of φ, i.e., there exists a $w_0 \in W$ with $\mathcal{M}, w_0 \models \varphi$. We define a Hintikka-tree for φ (i.e., a Hintikka-set label $T(\alpha)$ for each $\alpha \in \{1, \ldots, k\}^*$) that satisfies **(H4)** to **(H8)**. To do this, we inductively define a mapping τ from $\{1, \ldots, k\}^*$ to $W \cup \{\bot\}$ in such a way that

$$T(\alpha) = \begin{cases} \{\psi \in \mathsf{cl}(\varphi) \mid \mathcal{M}, \tau(\alpha) \models \psi\} & \text{if } \tau(\alpha) \neq \bot \\ \emptyset & \text{otherwise} \end{cases} \qquad (*)$$

For the induction start, set

$$\begin{array}{rcl} \tau(\epsilon) & := & w_0 \\ T(\epsilon) & := & \{\psi \in \mathsf{cl}(\varphi) \mid \mathcal{M}, w_0 \models \psi\} \end{array}$$

Now for the induction step. Let $\alpha \in \{1, \ldots, k\}^*$ such that $\tau(\alpha)$ is already defined, and let $i \in \{1, \ldots, k\}$. We make a case distinction as follows:

1. $\tau(\alpha) \neq \bot$ and $\mathcal{E}(i) = \langle R \rangle \psi \in T(\alpha)$. By $(*)$, we have $\mathcal{M}, \tau(\alpha) \models \langle R \rangle \psi$ which implies the existence of a world $w \in W$ such that $\mathcal{M}, (\tau(\alpha), w) \models R$ and $\mathcal{M}, w \models \psi$. Choose such a w and define $\tau(\alpha i) := w$ and $T(\alpha i) := \{\rho \in \mathsf{cl}(\varphi) \mid \mathcal{M}, w \models \rho\}$.

2. if α, i do not match the above case, set $\tau(\alpha i) = \bot$ and $T(\alpha i) = \emptyset$.

By definition, T and τ satisfy $(*)$. We need to prove that the k-ary $2^{\mathsf{cl}(\varphi)}$-tree T just defined is a Hintikka-tree for φ. From the semantics of $\mathbf{K}_\omega^\neg$ and the definition of cl, it is clear that $T(\alpha)$ is a Hintikka-set for each $\alpha \in \{1, \ldots, k\}^*$. Hence, it remains to show that T satisfies **(H4)** to **(H8)**.

(H4) Satisfied by definition of T (see induction start).

(H5) Let $\{\langle R\rangle\,\psi, [R]\,\rho_1, \ldots, [R]\,\rho_m\} \subseteq T(\alpha)$ and $\mathcal{E}(i) = \langle R\rangle\,\psi$. By $(*)$, we have $\tau(\alpha) \neq \bot$ and $\mathcal{M}, \tau(\alpha) \models \langle R\rangle\,\psi \wedge [R]\,\rho_1 \wedge \cdots \wedge [R]\,\rho_m$. By definition of τ (induction step, first case), we have $\tau(\alpha i) = w$ for some $w \in W$, with $\mathcal{M}, (\tau(\alpha), w) \models R$, and $\mathcal{M}, w \models \psi$. Moreover, the semantics of $\mathbf{K}_\omega^\neg$ implies $\mathcal{M}, w \models \rho_1 \wedge \cdots \wedge \rho_m$, and, by $(*)$, we thus have $\{\psi, \rho_1, \ldots, \rho_m\} \subseteq T(\alpha i)$.

(H6) Satisfied by definition of T (see induction step, second case).

(H7) Let $[R]\,\psi \in T(\alpha)$ and fix a $\beta \in \{1, \ldots, k\}^*$. If $\tau(\beta) = \bot$, then we have $T(\beta) = \emptyset$ by $(*)$ and **(H7)** is satisfied. If $\tau(\beta) \neq \bot$, then $\tau(\beta) \in W$ and we have either $\mathcal{M}, \tau(\beta) \models \psi$ or $\mathcal{M}, \tau(\beta) \models \bar{\psi}$. Again, $(*)$ implies that **(H7)** is satisfied.

(H8) Assume $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$. By $(*)$, we have $\mathcal{M}, \tau(\alpha) \models [R]\,\rho \wedge [\bar{R}]\,\psi$ and $\mathcal{M}, \tau(\beta) \models \bar{\rho}$. This implies $\mathcal{M}, (\tau(\alpha), \tau(\beta)) \models \bar{R}$ since

1. we have either $\mathcal{M}, (\tau(\alpha), \tau(\beta)) \models R$ or $\mathcal{M}, (\tau(\alpha), \tau(\beta)) \models \bar{R}$ and
2. $\mathcal{M}, (\tau(\alpha), \tau(\beta)) \models R$ is not possible since $\mathcal{M}, \tau(\alpha) \models [R]\,\rho$ and $\mathcal{M}, \tau(\beta) \models \bar{\rho}$.

Hence, due to the semantics of $\mathbf{K}_\omega^\neg$, we have $\mathcal{M}, \tau(\beta) \models \psi$, which, by $(*)$, implies $\psi \in T(\beta)$.

❏

Thus, we have that Hintikka-trees are appropriate abstractions of models of $\mathbf{K}_\omega^\neg$-formulae. Hintikka-trees enjoy the nice property that they are trees, and we can thus define, for a $\mathbf{K}_\omega^\neg$-formula φ, a tree-automaton $\mathcal{A}_\varphi$ that accepts exactly the Hintikka-trees for φ.

Definition 5 For a $\mathbf{K}_\omega^\neg$-formula φ with k diamond-formulae in $\mathsf{cl}(\varphi)$, the looping automaton $\mathcal{A}_\varphi = (Q, 2^{\mathsf{cl}(\varphi)}, \Delta, I)$ is defined as follows:

- Let $P = \{\{[R]\,\psi, [\bar{R}]\,\rho\} \mid [R]\,\psi, [\bar{R}]\,\rho \in \mathsf{cl}(\varphi)\}$,
  $S = \{[R]\,\psi \mid [R]\,\psi \in \mathsf{cl}(\varphi)\}$,
  Q is the set of all those elements (Ψ, p, s) of

  $$\{\Psi \in 2^{\mathsf{cl}(\varphi)} \mid \Psi \text{ is a Hintikka-set}\} \times 2^P \times 2^S$$

  satisfying the following conditions:

  1. if $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$ and $\bar{\rho} \in \Psi$, then $\psi \in \Psi$,

2. if $[R]\,\rho \in s$, then $\Psi = \emptyset$ or $\{\rho, \bar{\rho}\} \cap \Psi \neq \emptyset$,
3. if $[R]\,\rho \in \Psi$, then $[R]\,\rho \in s$, and
4. if $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq \Psi$, then $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$.

- $I = \{(\Psi, p, s) \mid \varphi \in \Psi\}$.
- $((\Psi, p, s), \Psi', (\Psi_1, p_1, s_1), \ldots, (\Psi_k, p_k, s_k)) \in \Delta$ iff

  $\Psi = \Psi'$, $\quad p_i = p$, $\quad s_i = s$ for all $1 \leq i \leq k$, and
  if $\mathcal{D}(i) = \langle R \rangle\,\psi \in \Psi$, then $\psi \in \Psi_i$ and $\rho \in \Psi_i$ for each $[R]\,\rho \in \Psi$ and
  if $\mathcal{D}(i) = \langle R \rangle\,\psi \notin \Psi$, then $\Psi_i = \emptyset$.

▲

Note that, since $\mathcal{A}_\varphi$ is a looping automata, every run is accepting. As a consequence of the following lemma and Lemma 4, we can reduce satisfiability of $\mathbf{K}_\omega^\neg$-formulae to the emptyness of the language accepted by looping automata.

Lemma 6 T is a Hintikka-tree for a $\mathbf{K}_\omega^\neg$-formula φ iff $T \in \mathcal{L}(\mathcal{A}_\varphi)$.

Proof: Let φ be a $\mathbf{K}_\omega^\neg$-formula and k, $\mathcal{A}_\varphi$ as in Definition 5.

"⇒" Let T be Hintikka-tree for φ. We prove that there is an accepting run of $\mathcal{A}_\varphi$ on T. First, define

$$p \;:=\; \{\{[R]\,\psi, [\bar{R}]\,\rho\} \mid \text{ There is a node } \alpha \text{ in } T \text{ with } \{[R]\,\psi, [\bar{R}]\,\rho\} \subseteq T(\alpha)\}$$
$$s \;:=\; \{[R]\,\psi \mid \text{ There is a node } \alpha \text{ in } T \text{ with } [R]\,\psi \in T(\alpha)\}$$

Next, we show that $r(\alpha) = (T(\alpha), p, s)$ is an accepting run of $\mathcal{A}_\varphi$ on T. By definition, r is defined for each $\alpha \in \{1, \ldots, k\}^*$. We have to show that, for each node α in T, r satisfies the following three conditions.

(i) $r(\alpha) \in Q$. Let α be a node in T. Since T is a Hintikka-tree, $T(\alpha)$ is a Hintikka-set. It remains to prove that $(T(\alpha), p, s)$ satisfies the four properties of states Q in Definition 5.

1. If $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$, then there is some node β with $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq T(\beta)$. Hence if, additionally, $\bar{\rho} \in T(\alpha)$, then **(H8)** ensures that $\psi \in T(\alpha)$.

2. If $[R]\,\rho \in s$, then there is some node β with $[R]\,\rho \in T(\beta)$, and **(H7)** ensures that $T(\alpha) = \emptyset$, $\rho \in T(\alpha)$, or $\bar{\rho} \in T(\alpha)$.

3. & 4. are satiesfied by definition of p and s.

(ii) $r(\epsilon) \in I$. Since T is a Hintikka-tree for φ, **(H4)** ensures that $\varphi \in T(\epsilon)$, hence $r(\epsilon) = (T(\epsilon), p, s) \in I$.

(iii) $((T(\alpha), p, s), T(\alpha), (T(\alpha 1), p, s), \ldots, (T(\alpha_k), p, s)) \in \Delta$. There are only two conditions to prove: Firstly, if $\mathcal{D}(i) = \langle R \rangle\,\psi \in T(\alpha)$, then **(H5)** ensures that $\psi \in T(\alpha i)$ and, for each $[R]\,\rho \in T(\alpha)$, **(H5)** ensures that $\rho \in T(\alpha i)$.

Secondly, if $\mathcal{D}(i) = \langle R \rangle\,\psi \notin T(\alpha)$, then **(H6)** ensures that $T(\alpha i) = \emptyset$.

"$\Leftarrow$" Let $T \in \mathcal{L}(\mathcal{A}_\varphi)$ and r be an accepting run of $\mathcal{A}_\varphi$ on T. We prove that T is a Hintikka-tree for φ.

- By definition of $\mathcal{A}_\varphi$, T is a k-ary $2^{\mathsf{cl}(\varphi)}$-tree, and $r(\alpha) = (\Psi_\alpha, p_\alpha, s_\alpha)$ implies $\Psi_\alpha = T(\alpha)$ by definition of Δ. Hence, by definition of Q, each node in T is labelled with a Hintikka-set. Let $r(\epsilon) = (T(\epsilon), p, s)$. Then, by definition of Δ, for each node α, we have $p_\alpha = p$ and $s_\alpha = s$.

- Let $r(\epsilon) = (\Psi_\epsilon, p, s)$, then $\varphi \in \Psi_\epsilon$ by definition of I and, since $\Psi_\epsilon = T(\epsilon)$, we have that T satisfies **(H4)**.

- For **(H5)**, let $\{\langle R\rangle\,\psi, [R]\,\rho_1, \ldots, [R]\,\rho_m\} \subseteq T(\alpha)$ and $\mathcal{D}(i) = \langle R\rangle\,\psi$. Again, we have $r(\alpha) = (T(\alpha), p, s)$, and $r(\alpha i) = (T(\alpha i), p, s)$. Since r is a run of $\mathcal{A}_\varphi$ on T, we have

  $$((T(\alpha), p, s), T(\alpha), (T(\alpha 1), p, s), \ldots, (T(\alpha k), p, s)) \in \Delta,$$

  which implies $\{\psi, \rho_1, \ldots, \rho_m\} \subseteq T(\alpha i)$ by Definition of Δ, and thus T satisfies **(H5)**.

- T satisfies **(H6)** due to the last implication in the definition of Δ and since $r(\alpha) = (T(\alpha), p, s)$.

- For **(H7)**, let $[R]\,\rho \in T(\alpha)$. Since $r(\alpha) = (T(\alpha), p, s)$ and, due to 3. in the definition of Q, we have $[R]\,\rho \in s$. Then, for a node β, we have $r(\beta) = (T(\beta), p, s)$, and, due to 2. in the definition of Q, $T(\beta) = \emptyset$ or $\{\rho, \bar{\rho}\} \cap T(\beta) \neq \emptyset$.

- For **(H8)**, let $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$. Since $r(\alpha) = (T(\alpha), p, s)$ and, due to 4. in the definition of Q, we have $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$. Now $r(\beta) = (T(\beta), p, s)$ and, due to 1. in the definition of Q, we have $\psi \in T(\beta)$.

Summing up, $\mathcal{A}_\varphi$ accepts each Hintikka-tree for φ and, vice versa, each Hintikka-tree for φ is accepted by $\mathcal{A}_\varphi$. ❏

What is the size of looping automata $\mathcal{A}_\varphi = (Q_\varphi, M_\varphi, I_\varphi, \Delta_\varphi)$? Obviously, the cardinality of $\mathsf{cl}(\varphi)$ is linear in the length of φ. Hence, by definition of $\mathcal{A}_\varphi$, the cardinality of Q_φ and M_φ are exponential in the length of φ. Again by definition of $\mathcal{A}_\varphi$, this implies that the cardinalities of I_φ and Δ_φ are also exponential in the length of φ. Hence, the size of $\mathcal{A}_\varphi$ is exponential in the length of φ. This fact together with Lemma 4, Lemma 6, and the fact that emptiness of the language accepted by a looping automaton $\mathcal{A}_\varphi$ can be tested in time polynomial in the size of $\mathcal{A}_\varphi$, we have that satisfiability of $\mathbf{K}^{\neg}_{\omega}$-formulae is in ExpTime. In Section 2, we already noted that satisfiability of $\mathbf{K}^{\neg}_{\omega}$-formulae is ExpTime-hard, and, hence, we obtain the following theorem:

Theorem 7 Satisfiability of $\mathbf{K}^{\neg}_{\omega}$-formulae is ExpTime-complete.

3.1 $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ is also in ExpTime

In this section, we show that the same technique as in the previous section can be used to prove that $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, i.e., the fusion of $\mathbf{K}_\omega$ with $\mathbf{K4}_\omega$ extended with the negation of modal parameters, is also in ExpTime.

$(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ provides two disjoint sets of atomic modal parameters $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$, where the latter are called *transitive modal parameters*. The syntax of $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ is the same as the one of $\mathbf{K}_\omega^\neg$ except that, in $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, transitive modal parameters may be used anywhere were modal parameters are allowed in $\mathbf{K}_\omega^\neg$. For the semantics, we restrict Kripke structures to those where accessibility relations $\mathcal{S}_i$ corresponding to transitive atomic parameters S_i are transitive.

Again, w.l.o.g., we assume that φ is in NNF.

Definition 8 A $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-Hintikka-tree is a Hintikka-tree as in Definition 3 extended by the following two conditions:[1]

(H5b) if, for a transitive parameter S_i, we have $\{\langle S_i \rangle \psi, [S_i]\rho_1, \ldots, [S_i]\rho_m\} \subseteq T(\alpha)$ and $\mathcal{D}(i) = \langle S_i \rangle \psi$, then $\{\psi, \rho_1, \ldots, \rho_m, [S_i]\rho_1, \ldots, [S_i]\rho_m\} \subseteq T(\alpha i)$

(H8b) if, for a transitive parameter S_i, we have $\{[S_i]\psi, [\neg S_i]\rho\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$, then $\{[S_i]\psi, \psi\} \subseteq T(\beta)$.

▲

We can now "lift" Lemma 4 to the $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ case.

Lemma 9 A $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ is satisfiable iff φ has a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-Hintikka-tree.

Proof: The proof is analogous to the one for Lemma 4. Let φ be a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula and let there be k diamond-formulae in $\mathsf{cl}(\varphi)$.

"⇐" Let T be a Hintikka-tree for φ. For each $S \in \{R_1, \ldots, S_1, \ldots\}$, define relations $\mathcal{K}_S$ as follows:

$$\begin{aligned} \mathcal{K}_S \;=\; & \{(\alpha, \beta) \mid \mathcal{E}(j) = \langle S \rangle \psi \in T(\alpha) \text{ and } \beta = \alpha j\} \cup \\ & \{(\alpha, \beta) \mid [\neg S]\psi \in T(\alpha) \text{ and } \bar{\psi} \in T(\beta)\} \end{aligned}$$

Based on the relations $\mathcal{K}_S$, we define a Kripke structure $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots, \mathcal{S}_1, \ldots \rangle$ as follows:

$$\begin{aligned} W &= \{\alpha \in \{1, \ldots, k\}^* \mid T(\alpha) \neq \emptyset\} \\ \pi(p) &= \{\alpha \mid p \in T(\alpha)\} \text{ for all } p \in \Phi\} \\ \mathcal{R}_i &= \mathcal{K}_{R_i} \text{ for all } i \geq 1 \\ \mathcal{S}_i &= \mathcal{K}_{S_i}^+ \text{ for all } i \geq 1 \end{aligned}$$

[1] Note that "R" in Definition 3 now denotes both standard and transitive modal parameters and negations thereof.

where $\mathcal{K}^+$ denotes the transitive closure of the relation $\mathcal{K}$. As for $\mathbf{K}_\omega^\neg$, the "only if" direction is now an immediate consequence of the following claim:

Claim: $\psi \in T(\alpha)$ implies $\mathcal{M}, \alpha \models \psi$ for all $\alpha \in W$ and $\psi \in \mathsf{cl}(\varphi)$.

The claim is proved by induction over the structure of ψ. The induction start and all but one case in the induction step are identical to the $\mathbf{K}_\omega^\neg$ case and omitted here. The only interesting case is the following (note that the complement of a transitive relation does not need to be transitive, hence we need to consider only the positive case here):

- $\psi = [S_i]\,\rho$ for a transitive atomic parameter S_i with corresponding accessibility relation $\mathcal{S}_i$. Fix a β with $(\alpha, \beta) \in \mathcal{S}_i$. We need to show that $\mathcal{M}, \beta \models \rho$. By definition of $\mathcal{M}$, there exists a sequence $\gamma_1, \ldots, \gamma_r$ with $r \geq 2$ such that

  - $(\gamma_\ell, \gamma_{\ell+1}) \in \mathcal{K}_{S_i}$ for $1 \leq \ell < r$, and
  - $\gamma_1 = \alpha$ and $\gamma_r = \beta$.

  We show that $[S_i]\,\rho \in T(\gamma_\ell)$ implies $[S_i]\,\rho \in T(\gamma_{\ell+1})$ for each $1 \leq \ell < r$. By definition of $\mathcal{K}_{S_i}$, we have to distinguish two cases:

  1. $\gamma_{\ell+1} = \gamma_\ell j$ and $\mathcal{E}(j) = \langle S_i \rangle\,\rho' \in T(\gamma_\ell)$, or
  2. $[\neg S_i]\,\rho' \in T(\gamma_\ell)$ and $\bar{\rho}' \in T(\gamma_{\ell+1})$

  In the first case, we have $\{\langle S_i \rangle\,\rho', [S_i]\,\rho\} \subseteq T(\gamma_\ell)$ which, by **(H5b)**, implies $\{\rho, \rho', [S_i]\,\rho\} \subseteq T(\gamma_\ell j)$. In the second case, we have $\{[S_i]\,\rho, [S_i]\,\rho'\} \subseteq T(\gamma_\ell)$ and $\bar{\rho}' \in T(\gamma_{\ell+1})$. By **(H8b)**, we have $\{[S_i]\,\rho, \rho\} \subseteq T(\gamma_{\ell+1})$.

  Hence $[S_i]\,\rho \in T(\gamma_{r-1})$ because $[S_i]\,\rho \in T(\gamma_1)$. We can then use the same arguments as in the proof of Lemma 4 to show that $\rho \in T(\gamma_r)$, and thus we have $\mathcal{M}, \gamma_r \models \rho$ by induction.

"$\Leftarrow$" Let $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots, \mathcal{S}_1, \ldots \rangle$ be a model of φ, i.e., there exists a $w_0 \in W$ with $\mathcal{M}, w_0 \models \varphi$. Define a Hintikka-tree T based on $\mathcal{M}$ as in the proof of Lemma 4. We need to show that T satisfies the additional properties **(H5b)** and **(H8b)**.

(H5b) Let $\{\langle S_i \rangle\,\psi, [S_i]\,\rho_1, \ldots, [S_i]\,\rho_m\} \subseteq T(\alpha)$ and $\mathcal{E}(i) = \langle S_i \rangle\,\psi$ for a transitive parameter S_i. By $(*)$, we have $\tau(\alpha) \neq \bot$ and $\mathcal{M}, \tau(\alpha) \models \langle S_i \rangle\,\psi \wedge [S_i]\,\rho_1 \wedge \cdots \wedge [S_i]\,\rho_m$. By definition of τ (induction step, first case), we have $\tau(\alpha i) = w$ for a w with $\mathcal{M}, (\tau(\alpha), w) \models S_i$, and $\mathcal{M}, w \models \psi$. By semantics of $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, we also have $\mathcal{M}, w \models \rho_1 \wedge \cdots \wedge \rho_m$.

Now let $w' \in W$ such that $\mathcal{M}, (\tau(\alpha i), w') \models S_i$. Since $\mathcal{S}_i$ is transitive, we have $\mathcal{M}, (\tau(\alpha), w') \models S_i$ and hence $\mathcal{M}, w' \models \rho_1 \wedge \cdots \wedge \rho_m$. Since this holds independently of the choice of w', we have that $\mathcal{M}, \tau(\alpha i) \models [S_i]\,\rho_1 \wedge \cdots \wedge [S_i]\,\rho_m$.

Summing up and applying $(*)$, we obtain $\{\psi, \rho_1, \ldots, \rho_m, [S_i]\,\rho_1, \ldots, [S_i]\,\rho_m\} \subseteq T(\alpha i)$.

(H8b) Assume $\{[S_i]\,\psi, [\neg S_i]\,\rho\} \subseteq T(\alpha)$ and $\bar{\rho} \in T(\beta)$ for a transitive parameter S_i. By $(*)$, we have $\mathcal{M}, \tau(\alpha) \models [S_i]\,\psi \wedge [\neg S_i]\,\rho$ and $\mathcal{M}, \tau(\beta) \models \bar{\rho}$. Analogously to the corresponding case in the proof of Lemma 4, we deduce $\mathcal{M}, (\tau(\alpha), \tau(\beta)) \models S_i$ and $\mathcal{M}, \tau(\beta) \models \psi$. As in the case **(H5b)**, we obtain $\mathcal{M}, \beta \models [S_i]\,\psi$, and, by $(*)$, we conclude $\{\psi, [S_i]\,\psi\} \subseteq T(\beta)$.

❏

It remains to construct a looping automaton that accepts exactly the Hintikka-trees for a given $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ. This constructions is a simple extension of the one for $\mathbf{K}_\omega^\neg$-formulae with the approriate translations of the additional properties **(H5b)** and **(H8b)**. More precisely, the construction is the same as the one in Defintion 5, with an additional fifth condition in the definition of Q as a translation of **(H8b)**, and an additional implication in the definition of Δ as a translation of **(H5b)**.

Definition 10 For a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ with k diamond-formulae in $\mathsf{cl}(\varphi)$, the looping automaton $\mathcal{A}_\varphi = (Q, 2^{\mathsf{cl}(\varphi)}, \Delta, I)$ is defined as follows:

- Let $P = \{\{[R]\,\psi, [\bar{R}]\,\rho\} \mid [R]\,\psi, [\bar{R}]\,\rho \in \mathsf{cl}(\varphi)\}$,
  $S = \{[R]\,\psi \mid [R]\,\psi \in \mathsf{cl}(\varphi)\}$,
  Q is the set of all those elements (Ψ, p, s) of

  $$\{\Psi \in 2^{\mathsf{cl}(\varphi)} \mid \Psi \text{ is a Hintikka-set}\} \times 2^P \times 2^S$$

  satisfying the following conditions:

  1. if $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$ and $\bar{\rho} \in \Psi$, then $\psi \in \Psi$,
  2. if $[R]\,\rho \in s$, then $\Psi = \emptyset$ or $\{\rho, \bar{\rho}\} \cap \Psi \neq \emptyset$,
  3. if $[R]\,\rho \in \Psi$, then $[R]\,\rho \in s$,
  4. if $\{[R]\,\rho, [\bar{R}]\,\psi\} \subseteq \Psi$, then $\{[R]\,\rho, [\bar{R}]\,\psi\} \in p$, and
  5. if $\{[S_i]\,\psi, [\neg S_i]\,\rho\} \in p$ and $\bar{\rho} \in \Psi$ for a transitive parameter S_i, then $\{\psi, [S_i]\,\psi\} \in \Psi$.

- $I = \{(\Psi, p, s) \mid \varphi \in \Psi\}$.
- $((\Psi, p, s), \Psi', (\Psi_1, p_1, s_1), \ldots, (\Psi_k, p_k, s_k)) \in \Delta$ iff

  $\Psi = \Psi'$, $p_i = p$, $s_i = s$ for all $1 \leq i \leq k$, and
  if $\mathcal{D}(i) = \langle R\rangle\,\psi \in \Psi$, then $\psi \in \Psi_i$ and $\rho \in \Psi_i$ for each $[R]\,\rho \in \Psi$
  if $\mathcal{D}(i) = \langle S_i\rangle\,\psi \in \Psi$ for a transitive parameter S_i, then $[S_i]\,\rho \in \Psi_i$ for each $[S_i]\,\rho \in \Psi$ and
  if $\mathcal{D}(i) = \langle R\rangle\,\psi \notin \Psi$, then $\Psi_i = \emptyset$.

▲

With the remarks above, the proof of the following lemma is completely analogous to the one of Lemma 6, and thus omitted.

Lemma 11 T is a Hintikka-tree for a $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formula φ iff $T \in \mathcal{L}(\mathcal{A}_\varphi)$.

Analogously to Theorem 7, we obtain the following theorem:

Theorem 12 Satisfiability of $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$-formulae is ExpTime-complete.

4 Adding Intersection and Union of Modal Parameters

In this section, we investigate the complexity of adding intersection and union of modal parameters to the logic $\mathbf{K}_\omega^\neg$. In doing this, one has the choice to either restrict the applicability of negation to atomic modal parameters or allowing for full negation w.r.t. modal parameters. In the latter case, adding union is obviously equivalent to adding intersection or both.

We start with the smallest extension, i.e., we add either intersection or union on modal parameters while restricting negation to atomic parameters.

Definition 13 A $\mathbf{K}_\omega^{(\neg),\cup}$-formula ($\mathbf{K}_\omega^{(\neg),\cap}$-formula) is a $\mathbf{K}_\omega^\neg$-formula which, additionally, allows for modal parameters of the form $S_1 \cup \cdots \cup S_k$ ($S_1 \cap \cdots \cap S_k$), where each S_i is an atomic or a negated atomic parameter. The semantics of the new modal operators is defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \models \langle S_1 \cup \cdots \cup S_k \rangle \varphi & \text{iff} & \exists w' \in W \text{ with } \mathcal{M}, (w, w') \models S_i \text{ for} \\
& & \text{some } i \in \{1, \ldots, k\} \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [S_1 \cup \cdots \cup S_k] \varphi & \text{iff} & \forall w' \in W, \text{ if } \mathcal{M}, (w, w') \models S_i \text{ for} \\
& & \text{some } i \in \{1, \ldots, k\}, \text{ then } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models \langle S_1 \cap \cdots \cap S_k \rangle \varphi & \text{iff} & \exists a w' \in W \text{ with } \mathcal{M}, (w, w') \models S_i \\
& & \text{for all } 1 \leq i \leq k \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models [S_1 \cap \cdots \cap S_k] \varphi & \text{iff} & \forall w' \in W, \text{ if } \mathcal{M}, (w, w') \models S_i \\
& & \text{for all } 1 \leq i \leq k, \text{ then } \mathcal{M}, w' \models \varphi
\end{array}
$$

▲

Let us first investigate the logic $K_\omega^{(\neg),\cup}$. It is not hard to see that

$$
\begin{array}{rcl}
[S_1 \cup \cdots \cup S_k] \varphi & \equiv & [S_1] \varphi \wedge \cdots \wedge [S_k] \varphi \text{ and} \\
\langle S_1 \cup \cdots \cup S_k \rangle \varphi & \equiv & \langle S_1 \rangle \varphi \vee \cdots \vee [S_k] \varphi,
\end{array}
$$

i.e., satisfiability of $K_\omega^{(\neg),\cup}$-formulae can be reduced to satisfiability of $\mathbf{K}_\omega^\neg$-formulae. However, this naive reduction might lead to an exponential blow-up of the formula. In order to avoid this blow-up, we can proceed as follows to transform a $K_\omega^{(\neg),\cup}$-formula ψ into an equivalent $\mathbf{K}_\omega^\neg$-formula $\widehat{\psi}$ whose length is linear in the length of ψ: As the first step, recursively apply the following substitutions to ψ from the inside to the outside (i.e., no union on modal parameters occurs in φ)

$$\begin{array}{lcl}[S_1 \cup \cdots \cup S_k]\,\varphi & \rightsquigarrow & [S_1]\,p_\varphi \wedge \cdots \wedge [S_k]\,p_\varphi \text{ and} \\ \langle S_1 \cup \cdots \cup S_k\rangle\,\varphi & \rightsquigarrow & \langle S_1\rangle\,p_\varphi \vee \cdots \vee [S_k]\,p_\varphi\end{array}$$

where p_φ is a new propositional variable. Call the result of these substitutions ψ'. Secondly, use a new modal parameter R and define

$$\widehat{\psi} \quad := \quad \psi' \wedge \bigwedge_{p_\varphi \text{ occurs in } \psi'} [R](p_\varphi \leftrightarrow \varphi) \wedge [\neg R](p_\varphi \leftrightarrow \varphi)$$

It can easily be seen that this gives the following result.

Theorem 14 Satisfiability of $K_\omega^{(\neg),\cup}$-formulae is ExpTime-complete.

Next, we show that the satisfiability of $K_m^{(\neg),\cap}$-formulae is NExpTime-hard. The proof is given by a reduction of a NExpTime-complete variant of the well-known, undecidable domino problem.

A domino problem [4, 19] is given by a finite set of *domino types*. All domino types are of the same size, each type has a quadratic shape and colored edges. Of each type, an unlimited number of dominoe is available. The problem in the original domino problem is to arrange these dominoe to cover the plane without holes or overlapping, such that adjacent dominoe have identical colors on their touching edges (rotation of the dominoe is not allowed). In the NExpTime-complete variant of the domino problem that we use, the task is not to tile the whole plane, but to tile a $2^{n+1} \times 2^{n+1}$-torus, i.e., a $2^{n+1} \times 2^{n+1}$-rectangle whose edges are “glued” together. See, e.g., [4, 19] for undecidable versions of the domino problem and [6] for bounded variants. We now formally introduce bounded domino systems.

Definition 15 Let $\mathcal{D} = (D, H, V)$ be a *domino system*, where D is a finite set of *domino types* and $H, V \subseteq D \times D$ represent the horizontal and vertical matching conditions. For $s, t \in \mathbb{N}$, let $U(s,t)$ be the torus $\mathbb{Z}_s \times \mathbb{Z}_t$, where $\mathbb{Z}_n$ denotes the set $\{0, \ldots, n-1\}$. Let $a = a_0, \ldots, a_{n-1}$ be an n-tuple of dominoe (with $n \leq s$). We say that $\mathcal{D}$ *tiles* $U(s,t)$ *with initial condition* a iff there exists a mapping $\tau : U(s,t) \rightarrow D$ such that, for all $(x,y) \in U(s,t)$:

- if $\tau(x,y) = d$ and $\tau(x \oplus_s 1, y) = d'$, then $(d, d') \in H$
- if $\tau(x,y) = d$ and $\tau(x, y \oplus_t 1) = d'$, then $(d, d') \in V$
- $\tau(i, 0) = a_i$ for $0 \leq i < n$.

where $\oplus_n$ denotes addition modulo n. Such a mapping τ is called a *solution* for $\mathcal{D}$ w.r.t. a. ▲

These bounded domino systems are capable of expressing the computational behaviour of restricted, so-called simple, Turing Machines (TMs). This restriction is non-essential in the following sense: Every language accepted in time $T(n)$ and space $S(n)$ by some one-tape TM is accepted within the same time and space bounds by a simple TM, provided that $S(n), T(n) \geq 2n$ [6].

Theorem 16 [[6], Theorem 6.1.2] Let M be a simple TM with input alphabet Σ. Then there exists a domino system $\mathcal{D} = (D, H, V)$ and a linear time reduction which takes any input $x \in \Sigma^*$ to an n-tuple a of dominoe with $|x| = n$ such that

- If M accepts x in time t_0 with space s_0, then $\mathcal{D}$ tiles $U(s,t)$ with initial condition a for all $s \geq s_0 + 2, t \geq t_0 + 2$;
- if M does not accept x, then $\mathcal{D}$ does not tile $U(s,t)$ with initial condition a for any $s, t \geq 2$.

Corollary 17 There exists a domino system $\mathcal{D}$ such that the following is a NExpTime-hard problem: Given an initial condition $a = a_0 \cdots a_{n-1}$ of length n, does $\mathcal{D}$ tile the torus $U(2^{n+1}, 2^{n+1})$ with initial condition a?

Proof: Let M be a (w.l.o.g. simple) non-deterministic TM with time- (and hence space-) bound 2^n deciding an arbitrary NExpTime-complete language over the alphabet Σ. Let $\mathcal{D}$ be the corresponding domino system and trans the reduction from Theorem 16. The function trans is a linear reduction from $\mathcal{L}(M)$ to the problem above: For $b \in \Sigma^*$ with $|b| = n$, it holds that $b \in \mathcal{L}(M)$ iff M accepts b in time and space $2^{|b|}$ iff $\mathcal{D}$ tiles $U(2^{n+1}, 2^{n+1})$ with initial condition $\mathsf{trans}(b)$. ❑

We reduce the NExpTime-complete variant of the domino problem from Corollary 17 to the satisfiability of $K_m^{(\neg),\cap}$-formulae. Given a domino system $\mathcal{D} = (D, H, V)$ and an initial condition $a = a_0, \ldots, a_{n-1}$, we define a reduction formula $\varphi_{(\mathcal{D},a)}$ such that $\varphi_{(\mathcal{D},a)}$ is satisfiable iff $\mathcal{D}$ tiles the torus $U(2^{n+1}, 2^{n+1})$ with initial condition a. The reduction formula $\varphi_{(\mathcal{D},a)}$ can be found in Figure 1. In this figure, $[u]\varphi$ is an abbreviation for $[R]\,\varphi \wedge [\neg R]\,\varphi$, where R is an arbitrary atomic modal parameter. Obviously, in each model of $[u]\varphi$, each world satisfies φ. In *Init*, we write $[R]^n\varphi$ for the n-fold nesting of $[R]$, i.e., for

$$\underbrace{[R] \cdots [R]}_{n \text{ times}} \varphi.$$

Before we formally prove the correctness of the reduction, we discuss the underlying intuition.

The general strategy is to define the reduction formula $\varphi_{(\mathcal{D},a)}$ such that, for every model $\mathcal{M}$ of $\varphi_{(\mathcal{D},a)}$ with set of worlds W,

1. there exists a propositional variable p_d for every domino type $d \in D$ such that each $w \in W$ is in the extension of p_d for exactly one $d \in D$ (see the first line of *Tiling*),

2. for each point (i, j) in the torus $U(2^{n+1}, 2^{n+1})$, there exists a corresponding set of worlds $\{w_1, \ldots, w_k\} \subseteq W$ with $k \geq 1$ and a $d \in D$ such that all $w_1, \ldots, w_k$ are in the extension of p_d,

3. the horizontal and vertical conditions V and H are satisfied w.r.t. sets of worlds representing points in the plane (see the second and third line of *Tiling*), and

4. the initial condition is satisfied (see *Init*).

Let us examine the structure of models of $\varphi_{(\mathcal{D},a)}$ in detail. Let

$$\mathcal{M} = (W, \pi, \mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_0, \ldots, \mathcal{R}_n, \mathcal{S}_0, \ldots, \mathcal{S}_n, \ldots)$$

be a model for $\varphi_{(\mathcal{D},a)}$. Every $w \in W$ is associated with a point (i, j) of the torus $U(2^{n+1}, 2^{n+1})$. The number i is binarily coded by the propositional variables $x_0, \ldots, x_n$ while the number j is binarily coded by the propositional variables $y_0, \ldots, y_n$. More precisely, we set

$$\mathsf{xpos}(w) = \Sigma_{i=0}^{n} \alpha_i(w) * 2^i \text{ and } \mathsf{ypos}(w) = \Sigma_{i=0}^{n} \beta_i(w) * 2^i$$

where

$$\alpha_i(w) = \begin{cases} 1 & \text{if } w \in \pi(x_i) \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \beta_i(w) = \begin{cases} 1 & \text{if } w \in \pi(y_i) \\ 0 & \text{otherwise} \end{cases}$$

With $\mathsf{pos}(w)$, we denote the pair $(\mathsf{xpos}(w), \mathsf{ypos}(w))$. The first conjunct of the *Init* formula ensures that $\mathsf{pos}(w) = (0, 0)$ for all w with $\mathcal{M}, w \models \varphi_{(\mathcal{D},a)}$. The *Count*$_x$ and *Count*$_y$ formulae together with the *Stable* formula ensure that, for every $w \in W$, there exist $w_1, w_2 \in W$ such that

(a) $\mathcal{M}, (w, w_1) \models R_x$ and $\mathcal{M}, (w, w_2) \models R_y$,

(b) $\mathsf{xpos}(w_1) = \mathsf{xpos}(w) \oplus_{2^{n+1}} 1$ and $\mathsf{ypos}(w_2) = \mathsf{ypos}(w) \oplus_{2^{n+1}} 1$, and

(c) $\mathsf{xpos}(w_2) = \mathsf{xpos}(w)$ and $\mathsf{ypos}(w_1) = \mathsf{ypos}(w)$.

Here, the *Count*$_x$ and *Count*$_y$ formulae enforce Property (b) by a standard encoding of binary incrementation by 1 modulo 2^{n+1} (see, e.g., [6]). The *Stable* formula enforces Property (c), i.e., *Stable* enforces that the R_x-successors of a world w satisfy the same y_k as w, and analogously for R_y and x_k.

Informally speaking, models of $\varphi_{(\mathcal{D},a)}$ can be thought of as having the form of an infinite binary tree (in which some nodes may coincide) where paths p of $\mathcal{R}_x$ and $\mathcal{R}_y$ relations lead to a world w such that $\mathsf{xpos}(w)$ is the number of $\mathcal{R}_x$ edges in p modulo 2^{n+1} and $\mathsf{ypos}(w)$ is the number of $\mathcal{R}_y$ edges in p modulo 2^{n+1} . The modal parameters R_x and R_y are representing horizontal and vertical successors in the torus.

As already noted, the domino types are represented by propositional variables p_d with $d \in D$, and the (first line of the) *Tiling* formula guarantees that every worlds belongs to p_d for exactly one $d \in D$. We write $\mathsf{dtype}(w)$ to denote

$$
\begin{array}{rcl}
\mathit{Count}_x & = & [u]\Big[\bigwedge_{k=0}^{n}\Big((\bigwedge_{j=0}^{k-1} x_j) \rightarrow (x_k \leftrightarrow [R_x]\,\neg x_k)\Big) \wedge \\
 & & \bigwedge_{k=0}^{n}\Big((\bigvee_{j=0}^{k-1} \neg x_j) \rightarrow (x_k \leftrightarrow [R_x]\,x_k)\Big) \wedge \langle R_x\rangle\,\mathsf{true}\Big] \\
\mathit{Count}_y & & \text{like } \mathit{Count}_x, \text{ replace } R_x \text{ by } R_y,\ x_j \text{ by } y_j, \text{ and } x_k \text{ by } y_k \\
\mathit{Stable} & = & [u]\Big[\bigwedge_{k=0}^{n}(x_k \rightarrow [R_y]\,x_k) \wedge \bigwedge_{k=0}^{n}(\neg x_k \rightarrow [R_y]\,\neg x_k) \wedge \\
 & & \bigwedge_{k=0}^{n}(y_k \rightarrow [R_x]\,y_k) \wedge \bigwedge_{k=0}^{n}(\neg y_k \rightarrow [R_x]\,\neg y_k)\Big] \\
\mathit{Unique} & = & [u]\Big[\bigwedge_{k=0}^{n}\Big((x_k \rightarrow [\neg R_k]\,\neg x_k) \wedge (\neg x_k \rightarrow [\neg R_k]\,x_k)\Big) \wedge \\
 & & \bigwedge_{k=0}^{n}\Big((y_k \rightarrow [\neg S_k]\,\neg y_k) \wedge (\neg y_k \rightarrow [\neg S_k]\,y_k)\Big) \wedge \\
 & & \bigwedge_{d\in D} p_d \rightarrow [R_0 \cap \cdots \cap R_n \cap S_0 \cap \cdots \cap S_n]\,p_d\Big] \\
\mathit{Tiling} & = & [u]\Big[(\bigvee_{d\in D} p_d) \wedge \bigwedge_{d\in D}\ \bigwedge_{d'\in D\setminus\{d\}} \neg(p_d \wedge p_{d'}) \wedge \\
 & & \bigwedge_{d\in D} p_d \rightarrow \big([R_x] \bigvee_{(d,d')\in H} p_{d'}\big) \wedge \\
 & & \bigwedge_{d\in D} p_d \rightarrow \big([R_y] \bigvee_{(d,d')\in G} p_{d'}\big)\Big] \\
\mathit{Init} & = & \bigwedge_{k=0}^{n}(\neg x_i \wedge \neg y_i) \wedge p_{w_0} \wedge [R_x]\,p_{w_1} \wedge \cdots \wedge [R_x]^{n-1} p_{w_{n-1}} \\
C_\varphi & = & \mathit{Count}_x \wedge \mathit{Count}_y \wedge \mathit{Stable} \wedge \mathit{Unique} \wedge \mathit{Tiling} \wedge \mathit{Init}
\end{array}
$$

Figure 1: The $K_m^{(\neg),\cap}$ formula $\varphi_{(\mathcal{D},a)}$ for $\mathcal{D} = (D, H, V)$ and $a = a_0, \ldots, a_{n-1}$.

the $d \in D$ such that $w \in \pi(p_d)$. Since different worlds may be associated with the same point in the torus, we must ensure that $\mathsf{pos}(w) = \mathsf{pos}(w')$ implies $\mathsf{dtype}(w) = \mathsf{dtype}(w')$. This is done by the *Unique* formula: For each $0 \leq i \leq n$, this formula ensures that

$$\mathcal{M},(w,w') \models R_i \text{ if } \begin{array}{l}\text{(i) } w \in \pi(x_i) \text{ and } w' \in \pi(x_i) \text{ or}\\ \text{(ii) } w \notin \pi(x_i) \text{ and } w' \notin \pi(x_i)\ ,\end{array}$$

and similarly for S_i and y_i. Furthermore, due to its last conjunct, *Unique* guarantees that, for worlds $w, w' \in W$, if $\mathcal{M},(w,w') \models R_i$ and $\mathcal{M},(w,w') \models S_i$ for each $0 \leq i \leq n$, then w and w' are labelled by the same domino type, i.e., $\mathsf{dtype}(w) = \mathsf{dtype}(w')$. Recall that, if $\mathcal{M},(w,w') \models R_i$ ($\mathcal{M},(w,w') \models S_i$), then w and w' coincide w.r.t. x_i (y_i). Hence the above relationship w.r.t. to *all* R_i and S_i implies that w and w' coincide on all x_i and y_j and thus, that they represent the same point in the torus.

Before we prove the main proposition of the reduction, we establish some properties of models of $\varphi_{(\mathcal{D},a)}$.

Lemma 18 Let $\mathcal{M}$ be a model for $\varphi_{(\mathcal{D},a)}$ with set of worlds W and let $a = a_0, \ldots, a_{n-1}$. For all $(i,j) \in U(2^{n+1}, 2^{n+1})$, there exists a $w \in W$ such that $\mathsf{pos}(w) = (i,j)$.

Proof: The proof is by induction over $i+j$. For the induction start ($i+j=0$), it suffices to note that

- $\mathsf{pos}(w) = (0,0)$ for all $w \in W$ with $\mathcal{M}, w \models \varphi_{(\mathcal{D},a)}$ by the first conjunct of the *Init* formula and
- such a w exists since $\mathcal{M}$ is a model for $\varphi_{(\mathcal{D},a)}$.

For the induction step, let $(i,j) \in U(2^{n+1}, 2^{n+1})$. First assume $i > 0$. By induction hypothesis, there exists a $w' \in W$ such that $\mathsf{pos}(w') = (i-1, j)$. Since the $Count_x$ formula encodes incrementation by 1 modulo 2^{n+1} w.r.t. R_x successors, there exists a world w such that $\mathcal{M},(w',w) \models R_x$ and $\mathsf{xpos}(w) = \mathsf{xpos}(w') \oplus_{2^{n+1}} 1$. We have $\mathsf{ypos}(w) = \mathsf{ypos}(w')$ since $\mathcal{M}, w' \models$ *Stable*. The case $j > 0$ is analogous, just switch the roles of i and j and replace R_x by R_y and $Count_x$ by $Count_y$. ❑

In the following lemma, we use the dtype function which we already have argued to be well-defined.

Lemma 19 Let $\mathcal{M}$ be a model of $\varphi_{(\mathcal{D},a)}$ with set of worlds W and let $a = a_0, \ldots, a_{n-1}$. For all $w, w' \in W$, $\mathsf{pos}(w) = \mathsf{pos}(w')$ implies $\mathsf{dtype}(w) = \mathsf{dtype}(w')$.

Proof: The choice of w, w' implies that w and w' agree on the interpretation of the propositional variables $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$. By the first two lines of *Unique*, we have that $\mathcal{M},(w,w') \models R_i$ and $\mathcal{M},(w,w') \models S_i$ for $0 \leq k \leq n$. It is then an immediate consequence of the third line of the *Unique* formula that $\mathsf{dtype}(w) = \mathsf{dtype}(w')$. ❑

We are now ready to prove the correctness of the reduction.

Proposition 20 A domino system $\mathcal{D}$ tiles the torus $U(2^{n+1}, 2^{n+1})$ with initial condition $a = a_0, \ldots, a_{n-1}$ iff $\varphi_{(\mathcal{D},a)}$ is satisfiable.

Proof: Let $\mathcal{D} = (D, H, V)$ and $a = a_0, \ldots, a_{n-1}$.

"⇐" First, assume that $\varphi_{(\mathcal{D},a)}$ is satisfiable, i.e., that there exists a Kripke structure

$$\mathcal{M} = (W, \pi, \mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_0, \dots, \mathcal{R}_n, \mathcal{S}_0, \dots, \mathcal{S}_n)$$

and some $w_0 \in W$ such that $\mathcal{M}, w_0 \models \varphi_{(\mathcal{D},a)}$. We show that $\mathcal{D}$ has a solution w.r.t. a. Define a mapping τ from $U(2^{n+1}, 2^{n+1})$ to D by setting $\pi(i,j) := d$ iff $\mathsf{dtype}(w) = d$ for a $w \in W$ with $\mathsf{pos}(w) = (i,j)$. By Lemmas 18 and 19, τ is a well-defined total function. We need to show that τ is a solution for $\mathcal{D}$ with initial condition a.

Let $(i,j) \in U(2^{n+1}, 2^{n+1})$ and $i' = i \oplus_{2^{n+1}} +1$. To show that τ satisfies the horizontal matching condition, we need to show that $(\tau(i,j), \tau(i',j)) \in H$. By Lemma 18, there exists a world w such that $\mathsf{pos}(w) = (i,j)$. By Lemma 19 and definition of τ, we have $\mathsf{dtype}(w) = \tau(i,j)$. By the $Count_x$ formula, there exists a $w' \in W$ such that $\mathsf{xpos}(w') = i'$ and $\mathcal{M}, (w, w') \models R_x$. By the *Stable* formula, we have $\mathsf{ypos}(w') = j$. Again, by Lemma 19 and definition of τ, we have $\mathsf{dtype}(w) = \tau(i', j)$. By the second conjunct of the *Tiling* formula, we conclude $(\tau(i,j), \tau(i',j)) \in H$. The proof that the vertical matching condition is satisfied is analogous. Taking into account the $Count_x$, *Stable*, and *Init* formulae and the definition of τ, it is straightforward to prove that the initial condition is satisfied by using induction on n.

"⇒" Let τ be a solution for $\mathcal{D}$ w.r.t. a. For two integers $n, k \in \mathbb{N}$ with $0 \le k \le \log_2 n$, we denote the k'th bit in the binary representation of n by $\mathsf{bit}_k(n)$. We define a Kripke structure

$$\mathcal{M} = (W, \pi, \mathcal{R}_x, \mathcal{R}_y, \mathcal{R}_0, \dots, \mathcal{R}_n, \mathcal{S}_0, \dots, \mathcal{S}_n, \dots)$$

as follows:

$$\begin{array}{rrcl}
 & W & := & \{a_{i,j} \mid 0 \le i, j \le 2^{n+1}\} \\
 & \pi(p_d) & := & \{a_{i,j} \mid \tau(i,j) = d\} \text{ for all } d \in D \\
 & \pi(x_k) & := & \{a_{i,j} \mid \mathsf{bit}_k(i) = 1\} \text{ for } 0 \le k \le n \\
 & \pi(y_k) & := & \{a_{i,j} \mid \mathsf{bit}_k(j) = 1\} \text{ for } 0 \le k \le n \\
 & \mathcal{R}_x & := & \{(a_{i,j}, a_{i',j}) \mid i' = i \oplus_{2^{n+1}} 1\} \\
 & \mathcal{R}_y & := & \{(a_{i,j}, a_{i,j'}) \mid j' = j \oplus_{2^{n+1}} 1\} \\
\text{for } 0 \le k \le n: & \mathcal{R}_k & := & \{(a_{i,j}, a_{i',j'}) \mid \mathsf{bit}_k(i) = \mathsf{bit}_k(i')\} \\
\text{for } 0 \le k \le n: & \mathcal{S}_k & := & \{(a_{i,j}, a_{i',j'}) \mid \mathsf{bit}_k(j) = \mathsf{bit}_k(j')\}
\end{array}$$

It is easy to verify that $\mathcal{M}, a_{0,0} \models \varphi_{(\mathcal{D},a)}$. ❏

Summing up Proposition 20 and Corollary 17, we obtain a NExpTime lower bound for $\mathbf{K}_\omega^{(\neg),\cap}$-formulae. The corresponding upper bound follows from the fact that the translation of $\mathbf{K}_\omega^{\neg}$-formulae to L^2-formulae mentioned in Section 2 can also be applied to $\mathbf{K}_\omega^{(\neg),\cap}$-formulae.

Theorem 21 Satisfiability of $\mathbf{K}_\omega^{(\neg),\cap}$-formulae is NExpTime-complete.

5 Full Boolean Modal Logic

In this section, we investigate the complexity of full Boolean Modal Logic. Let us start with introducing this logic formally.

Definition 22 A *complex modal parameter* is a Boolean formula of atomic modal parameters. We use $\mathbf{K}_\omega^{\neg,\cap,\cup}$ to denote the extension of $\mathbf{K}_\omega$ with complex modal parameters. Let $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \rangle$ be a Kripke structure, and S a (possibly complex) modal parameter. Then the extension $\mathcal{E}(S)$ is inductively defined as follows:

if $S = R_i$ (i.e., S is atomic)	then $\mathcal{E}(S) = \mathcal{R}_i$
if $S = \neg S'$	then $\mathcal{E}(S) = (W \times W) \setminus \mathcal{E}(S')$
if $S = S_1 \cap S_2$	then $\mathcal{E}(S) = \mathcal{E}(S_1) \cap \mathcal{E}(S_2)$
if $S = S_1 \cup S_2$	then $\mathcal{E}(S) = \mathcal{E}(S_1) \cup \mathcal{E}(S_2)$

The semantics of formulae is extended as follows:

$$\mathcal{M}, w \models \langle S \rangle \varphi \quad \text{iff} \quad \exists w' \in W \text{ with } (w, w') \in \mathcal{E}(S) \text{ and } \mathcal{M}, w' \models \varphi$$
$$\mathcal{M}, w \models [S] \varphi \quad \text{iff} \quad \forall w' \in W, \text{ if } (w, w') \in \mathcal{E}(S), \text{ then } \mathcal{M}, w' \models \varphi$$

We write $\mathcal{M}, (w, w') \models S$ iff $(w, w') \in \mathcal{E}(S)$.

▲

From Theorem 21 and the standard tranlation of $\mathbf{K}_\omega^{\neg,\cap,\cup}$ into L^2, we easily obtain the following result:

Theorem 23 Satisfiability of $\mathbf{K}_\omega^{\neg,\cap,\cup}$-formulae is NExpTime-complete.

However, it is interesting to note that the NExpTime reduction used to prove Theorem 21 crucially depends on the fact that an infinite number of modal parameters is available: Since the size of the torus to be tiled is not bounded, there exists no upper bound for the number of the R_i and S_i parameters used for the reduction either. Although Boolean Modal Logics usually provide an infinite number of modal parameters (see, e.g., [9]), the question whether NExpTime-hardness can still be obtained if only a bounded number of modal parameters is available is natural. In the remainder of this section, we answer this question by showing that satisfiability and validity of $\mathbf{K}_m^{\neg,\cap,\cup}$, i.e., full Boolean Modal Logic with a fixed number m of modal parameters, is ExpTime-complete. The upper bound is proved by a reduction to multi-modal **K** enriched with the universal modality.

We show that satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae can be reduced to satisfiability of $\mathbf{K}_n^u$-formulae (i.e., formulae of multi-modal **K** enriched with the universal modality) by giving a series of polynomial reduction steps. We do not introduce $\mathbf{K}_n^u$ formally but refer the reader to, e.g., [25]. The following notions are central to several of the reduction steps.

Definition 24 A Kripke structure $\mathcal{M} = \langle W, \pi, \mathcal{R}_1, \ldots \mathcal{R}_m \rangle$ is called *simple* iff we have $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for all $1 \leq i < j \leq m$. $\mathcal{M}$ is called *complete* iff, for all $w, w' \in W$, there exists a unique i with $1 \leq i \leq m$ such that $(w, w') \in \mathcal{R}_i$. A formula (of any logic defined in this paper) is called *s-satisfiable* iff it has a model which is a simple Kripke structure. Similarly, a formula is called *c-satisfiable* iff it has a model which is a complete Kripke structure. ▲

Note that every complete Kripke structure is also simple. We now describe the reduction steps in detail. Let φ be a $\mathbf{K}_m^{\neg,\cap,\cup}$-formula whose satisfiability is to be decided and let $R_1, \ldots, R_m$ be the modal parameters of $\mathbf{K}_m^{\neg,\cap,\cup}$.

Step 1. Convert all modal parameters in φ to disjunctive normal form using a truth table. If the "empty disjunction" is obtained when converting a modal parameter S, then replace every occurrence of $\langle S \rangle \psi$ with *false* and every occurrence of $[S]\psi$ with *true*. Call the result of the conversion φ_1. The length of φ_1 is linear in the length of φ since the number m of atomic modal parameters is fixed (and the conversion can be done in linear time). It is easy to see that φ_1 is satisfiable iff φ is satisfiable.

Since the conversion to DNF was done using a truth table, each disjunct occurring in a modal parameter in φ_1 is a *relational type*, i.e., of the form

$$S_1 \cap \cdots \cap S_m \text{ with } S_i = R_i \text{ or } S_i = \neg R_i \text{ for } 1 \leq i \leq m$$

Let Γ be the set of all relational types. As is easily seen, if $\mathcal{M}, (w, w') \models S$ for some Kripke structure $\mathcal{M}$ with set of worlds W, $w, w' \in W$, and $S \in \Gamma$, then, for every atomic modal parameters R_i, this determines whether $\mathcal{M}, (w, w') \models R_i$ holds. Hence, for every $w, w' \in W$, we have $\mathcal{M}, (w, w') \models S$ for exactly one $S \in \Gamma$.

Step 2. We reduce satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae of the form of φ_1 (i.e, the modal parameters are in DNF and hence $\cup$ does not appear nested inside other operators) to the satisfiability of $\mathbf{K}_m^{(\neg),\cap}$-formulae in which all modal parameters are relational types. It is not hard to see that this can be done as in Section 4, where $\mathbf{K}_\omega^{(\neg),\cup}$ is reduced to $\mathbf{K}_\omega^{\neg}$: In the reduction, just replace the formula $[R](p_\varphi \leftrightarrow \varphi) \wedge [\neg R](p_\varphi \leftrightarrow \varphi)$ with $\bigwedge_{S \in \Gamma}[S](p_\varphi \leftrightarrow \varphi)$.[2] The reduction can again be done in linear time since m is fixed. The $\mathbf{K}_m^{(\neg),\cap}$-formula obtained by converting φ_1 is called φ_2.

Step 3. We reduce satisfiability of $\mathbf{K}_m^{(\neg),\cap}$-formulae of the form of φ_2 to c-satisfiability of $\mathbf{K}_{2^m}$-formulae. Set $n := 2^m$ and let $K_1, \ldots, K_n$ be the atomic modal parameters of the logic $\mathbf{K}_n$. Let r be some bijection between Γ and the set $\{K_1, \ldots, K_n\}$. The formula φ_3 is obtained from φ_2 by replacing each element S of Γ in φ_2 with $r(S)$. Considering the special syntactic form of φ_2 and the definitions of Γ and of c-satisfiability, it is easy to see that φ_2 is satisfiable iff φ_3 is c-satisfiable. Furthermore, the reduction is obviously linear. Note that

[2]This reduction ensures that all modal parameters in the resulting formula are relational types.

using 2^m instead of m modal parameters does not spoil the reduction since, ultimately, our reduction goes to satisfiability of multi-modal **K** enriched with the universal modality, and this logic is known to be in ExpTime for *any* fixed number of modalities [25].

Step 4. We reduce c-satisfiability of $\mathbf{K}_n$-formulae to s-satisfiability of $\mathbf{K}_n^u$-formulae. Define φ_4 as the conjunction of φ_3 with the formula

$$\chi := [u]\big(\bigwedge_{\psi_1,\ldots,\psi_n \text{ subformulae of } \varphi_3} [K_1]\psi_1 \wedge \cdots \wedge [K_n]\psi_n \rightarrow [u](\psi_1 \vee \cdots \vee \psi_n)\big)$$

Note that the length of φ_4 is polynomial in the length $|\varphi_3|$ of φ_3: The number of subformulae of φ_3 is bounded by $|\varphi_3|$; hence, χ consists of at most $|\varphi_3|^\ell$ conjuncts, where ℓ is a constant since the number of modal parameters is fixed. Let us prove that φ_3 is c-satisfiable iff φ_4 is s-satisfiable. The "only if" direction is straightforward: Let $\mathcal{M}$ be a complete model for φ_3. Obviously, $\mathcal{M}$ is also simple. Moreover, using the fact that $\mathcal{M}$ is complete, it is straightforward to check that $\mathcal{M}$ is a model for φ_4. It remains to prove the "if" direction. Let $\mathcal{M} = \langle W, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n\rangle$ be a simple model for φ_4. We first show that

Claim. For each $w, w' \in W$, there exists an ℓ with $1 \leq \ell \leq n$ such that, for all subformulae ψ of φ_3, $\mathcal{M}, w \models [K_\ell]\psi$ implies $\mathcal{M}, w' \models \psi$.

Assume to the contrary that, for some $w, w' \in W$, there exist no ℓ as in the claim. Hence, for each i with $1 \leq i \leq n$, there exists a subformula ρ_i of φ_3 such that $\mathcal{M}, w \models [K_i]\rho_i$ and $\mathcal{M}, w' \not\models \rho_i$. Since $\mathcal{M}$ is a model for χ, we clearly have

$$\mathcal{M}, w \models [K_1]\rho_1 \wedge \cdots \wedge [K_n]\rho_n \rightarrow [u](\rho_1 \vee \cdots \vee \rho_n).$$

This is obviously a contradiction to the fact that $\mathcal{M}, w \not\models \rho_1 \vee \cdots \vee \rho_n$ which proves the claim.

Extend the Kripke structure $\mathcal{M}$ to $\mathcal{M}' = \langle W, \pi, \mathcal{K}_1', \ldots, \mathcal{K}_n'\rangle$ as follows: For any $w, w' \in W$ with $(w, w') \notin \mathcal{K}_i$ for all i with $1 \leq i \leq n$, augment $\mathcal{K}_\ell$ with the tuple (w, w'), where ℓ is as in the claim. Obviously, $\mathcal{M}'$ is complete. It is now a matter of routine to prove that $\mathcal{M}, w \models \psi$ implies $\mathcal{M}', w \models \psi$ for all subformulae ψ of φ_3. The proof is by induction over the structure of ψ. The only interesting case is:
$\psi = [K_i]\psi'$. Let $(w, w') \in \mathcal{K}_i'$. We need to show that $\mathcal{M}', w' \models \psi'$. First assume that $(w, w') \in \mathcal{K}_i$. Since $\mathcal{M}, w \models \psi$, this implies $\mathcal{M}, w' \models \psi'$. By induction, we have $\mathcal{M}', w' \models \psi'$ and are done. Now assume $(w, w') \in \mathcal{K}_i' \setminus \mathcal{K}_i$. By definition of $\mathcal{K}_i'$, we have that $\mathcal{M}, w \models [K_i]\rho$ implies $\mathcal{M}, w' \models \rho$ for all subformulae ρ of φ_3. Since ψ is a subformula of φ_3, we have $\mathcal{M}, w' \models \psi'$. It remains to apply the induction hypothesis.

Since $\mathcal{M}$ is a model for φ_4, we have that $\mathcal{M}'$ is a model for φ_3. ❏

Step 5. It remains to prove that s-satisfiability of $\mathbf{K}_n^u$-formulae is decidable in ExpTime. This is, however, an easy consequence of the facts that satisfiability of $\mathbf{K}_n^u$-formulae is in ExpTime and that $\mathbf{K}_n^u$ has the tree model property: since every tree model is obviously simple, satisfiability coincides with s-satisfiability.

The sequence of reductions given above yields an ExpTime upper bound for the satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae. Since the lower bound for $\mathbf{K}_\omega^{\neg}$ already holds if we have only a single modal parameter available (again, see [25]), we obtain the following theorem.

Theorem 25 Satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae (i.e., $\mathbf{K}_\omega^{\neg,\cap,\cup}$ with a bounded bumber of modal parameters) is ExpTime-complete.

The sequence of reductions given above immediately yields an upper bound for the satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae. Since the lower bound for $\mathbf{K}_\omega^{\neg}$ already holds if we have only a single modal parameter available (again, see [25]), we obtain the following theorem.

Theorem 26 Satisfiability of $\mathbf{K}_m^{\neg,\cap,\cup}$-formulae (i.e., $\mathbf{K}_\omega^{\neg,\cap,\cup}$ with a bounded bumber of modal parameters) is ExpTime-complete.

6 Boolean Modal Logics without Negation

So far, we have only considered logics with negation of modal parameters. We will complete our investigation by showing that adding intersection and union of modal parameters does not increase the complexity of $\mathbf{K}_\omega$ (and thus neither the complexity of $\mathbf{K}_m$ is increased by this extension). The fact that the extension of $\mathbf{K}_\omega$ with intersection of modal parameters (i.e., $\mathbf{K}_\omega^{\cap}$) is still in PSpace is an immediate consequence of PSpace-completeness of the Description Logic $\mathcal{ALCR}$ [8] and the fact that $\mathcal{ALCR}$ is a notational variant of $\mathbf{K}_\omega^{\cap}$ [24]. Moreover, it is folklore that $\mathbf{K}_\omega$ extended with union of modal parameters (i.e., $\mathbf{K}_\omega^{\cup}$) is also in PSpace (however, the reduction from Section 4 cannot be applied since the universal modality is not available). For both union and intersection, we go into more detail.

With $\mathbf{K}_\omega^{\cap,\cup}$, we denote the variant of $\mathbf{K}_\omega^{\neg,\cap,\cup}$ obtained by disallowing the use of negation of modal parameters. In the following, we will present a slight extension of the standard PSpace tableau algorithm for $\mathbf{K}$, $\mathbf{K}$-*World* [20], to decide satisfiability of $\mathbf{K}_\omega^{\cap,\cup}$-formulae. Please note that we cannot adapt the reduction from the previous section since the disjunctive normal form of a complex modal parameter can yield an exponential blow-up if the number of boolean parameters is not bounded. When started with an input formula φ, $\mathbf{K}$-World decides φ's satisfiability by recursively searching a finite tree-model of φ in a depth-first manner. For each world w in this tree model, it checks whether the set Δ of formulae that w must satisfy is not contradictory. Then, for each $\Diamond\psi$ in Δ, $\mathbf{K}$-World is called recursively with ψ and all ρ with $\Box\rho$ in Δ.

To extend $\mathbf{K}$-World to $\mathbf{K}_\omega^{\cap,\cup}$, it is comfortable to view the semantics of roles in a different way. For S a complex modal parameter and s a set of atomic modal parameters, we say $s \models S$ iff s, when viewed as the valuation that maps each $R_i \in s$ to true and each $R_j \notin s$ to false, evaluates the Boolean expression S

to true. Then $(w, w') \in \mathcal{E}(S)$ iff there is a set s of atomic modal parameters with $s \models S$ and $(w, w') \in \mathcal{R}_i$ for each $R_i \in s$. The only modifications to $\mathbf{K}$-World concern the recursive calls for diamond formulae which are more elaborate in the presence of complex modal parameters. For each $\langle S \rangle \psi$ in the set Δ of formulae currently considered, we guess an s with $s \models S$, and then consider ψ together with all ρ where $[S'] \rho$ is in Δ and $s \models S'$.

For the sake of a succinct presentation, we assume the input formula φ to contain no disjunction and no diamond-formulae. For Δ and S sets of $\mathbf{K}_\omega^{\cap,\cup}$-formulae where S is closed under subformulae and single negations, $\mathbf{K}_\omega^{\cap,\cup}$-World$(\Delta, S)$ returns true iff

- Δ is a maximally propositionally consistent subset of S, i.e.,
  - $\Delta \subseteq S$,
  - for each $\neg\psi \in S$, $\psi \in \Delta$ iff $\neg\psi \notin \Delta$, and
  - for each $\psi_1 \wedge \psi_2 \in S$, $\psi_1 \wedge \psi_2 \in \Delta$ iff $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$.
- For each subformula $\neg [S] \psi \in \Delta$, there exists a set s of modal parameters with $s \models S$ and a set $\Delta_{\psi,s}$ such that
  - $\neg\psi \in \Delta_{\psi,s}$,
  - for each S' and ρ, if $[S'] \rho \in \Delta$ and $s \models S'$, then $\rho \in \Delta_{\psi,s}$,
  - $\mathbf{K}_\omega^{\cap,\cup}-\mathsf{World}(\Delta_{\psi,s}, S')$ returns true, where S' is the closure under subformulae and single negation of $\{\rho \mid [S'] \rho \in \Delta \text{ and } s \models S'\} \cup \{\neg\psi\}$.

Let $\mathsf{cl}(\varphi)$ be the smallest set of formulae containing φ that is closed under subformulae and single negation. The proof that a $\mathbf{K}_\omega^{\cap,\cup}$-formula φ is satisfiable iff there exists a $\Delta \subseteq \mathsf{cl}(\varphi)$ with $\varphi \in \Delta$ such that

$$\mathbf{K}_\omega^{\cap,\cup}\text{-World}(\Delta, \mathsf{cl}(\varphi)\})$$

returns true is analogous to the one for $\mathbf{K}$-World. Just like $\mathbf{K}$-World, $\mathbf{K}_\omega^{\cap,\cup}$-World runs in PSpace (since PSpace = NPSpace [23], the additional non-deterministic guessing of the set of modal parameters s does not matter). Moreover, $\mathbf{K}$ is known to be PSpace-hard [20], and we thus have the following result.

Theorem 27 Satisfiability of $\mathbf{K}_\omega^{\cup,\cap}$-formulae is PSpace-complete.

7 Conclusion

We have given a complete picture of the complexity of Boolean Modal Logics, both with and without a bound on the number of modal parameters. The results for (fragments of) Boolean Modal Logic with an unbounded number of modal parameters are summarised in Figure 2, showing known results in grey.

We have proved that $\mathbf{K}_\omega^{\neg}$ is in ExpTime using looping automata, which turned out to be rather elegent a technique for two reasons. Firstly, we did not need to bound the size of models/Hintikka trees since the looping automata we used work on infinite trees. Secondly, disjunctions were handled simply by introducing non-deterministic transitions of the automaton, which are harmless since the emptyness problem for non-deterministic looping automata is polynomial. Finally, we extended the automata approach to $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$ to show that it is also applicable to similar logics.

NExpTime-hardness of $\mathbf{K}_\omega^{(\neg),\cap}$ was rather surprising since so far, intersection of atomic modal parameters (not of chainings/composition of modal parameters) is mostly considered to be "harmless" w.r.t. complexity. Interestingly, we were able to show that, if a bound m is imposed on the number of atomic modal parameters, then full Boolean Modal Logic $\mathbf{K}_m^{\neg,\cap,\cup}$ becomes ExpTime-complete. For this proof, we did not use the automata-based approach because we considered that extending it to take care of complex modal parameters was more involved than the reduction to $\mathbf{K}_n^u$ that we used.

As future work, it may be interesting to extend our techniques to more expressive logics. For example, one may consider arbitrary combinations of the Boolean operators on modal parameters with composition and converse. Several results for such logics are known from the area of Propositional Dynamic Logics (PDL). For example, Harel proves that PDL extended with negation of modal parameters is undecidable using a reduction to the equivalence problem for relation algebra [16]. It is not hard to see that a similar reduction (of the equivalence problem of boolean algebras of relations with composition only, see, e.g., [1]) can be used to show that Boolean Modal Logic extended with composition of modal parameters is undecidable. On the contrary, it follows from Danecki's results on PDL with intersection that $\mathbf{K}_\omega^{\cap,\cup}$ extended with composition is decidable in double ExpTime [7]. As we demonstrated by extending our results to $(\mathbf{K}_\omega \otimes \mathbf{K4}_\omega)^\neg$, our automata-based approach to proving ExpTime-bounds can be considered flexible. As a first step towards more expressive logics, we hope that our approach can be "married" with the standard automata-based decidability procedure for PDL thus yielding a decidability result for PDL extended with atomic negation of modal parameters.

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	no negation	atomic negation	full negation
–	PSpace-compl.	ExpTime-compl.	
∪	PSpace-compl.	ExpTime-compl.	NExpTime-compl.
∩	PSpace-compl.	NExpTime-compl.	NExpTime-compl.
∩ and ∪	PSpace-compl.	NExpTime-compl.	NExpTime-compl.

Figure 2: Complexity of $\mathbf{K}_\omega$ extended with various role constructors.

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