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Franz Baader Silvio Ghilardi Cesare Tinelli

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Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
http://lat.inf.tu-dresden.de

Hans-Grundig-Str. 25
01062 Dresden
Germany

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Franz Baader
Institut für Theoretische Informatik, TU Dresden, Germany
baader@tcs.inf.tu-dresden.de

Silvio Ghilardi
Dipartimento di Scienze dell'Informazione
Università degli Studi di Milano, Italy
ghilardi@dsi.unimi.it

Cesare Tinelli
Department of Computer Science, The University of Iowa, USA
tinelli@cs.uiowa.edu

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Abstract

Previous results for combining decision procedures for the word problem in the non-disjoint case do not apply to equational theories induced by modal logics—which are not disjoint for sharing the theory of Boolean algebras. Conversely, decidability results for the fusion of modal logics are strongly tailored towards the special theories at hand, and thus do not generalize to other types of equational theories.

In this paper, we present a new approach for combining decision procedures for the word problem in the non-disjoint case that applies to equational theories induced by modal logics, but is not restricted to them. The known fusion decidability results for modal logics are instances of our approach. However, even for equational theories induced by modal logics our results are more general since they are not restricted to so-called normal modal logics.

Contents

1 Introduction

The combination of decision procedures for logical theories arises in many areas of logic in computer science, such as constraint solving, automated deduction, term rewriting, modal logics, and description logics. In general, one has two first-order theories T_1 and T_2 over the signatures Σ_1 and Σ_2, for which validity of a certain type of formulae (e.g., universal, existential positive, etc.) is decidable. The question is then whether one can combine the decision procedures for T_1 and T_2 into one for their union $T_1 \cup T_2$. The problem is usually much easier (though not at all trivial) if the theories do not share symbols, i.e., if $\Sigma_1 \cap \Sigma_2 = \emptyset$. For non-disjoint signatures, the combination of theories can easily lead to undecidability, and thus one must find appropriate restrictions on the theories to be combined

In automated deduction, the Nelson-Oppen combination procedure [NO79, Nel84] as well as the problem of combining decision procedures for the word problem [Pig74, Tid86, SS89, Nip91, BT97] have drawn considerable attention. The Nelson-Oppen method combines decision procedures for the validity of quantifier-free formulae in so-called stably infinite theories. If we restrict the attention to equational theories,[1] then it is easy to see that validity of arbitrary quantifier-free formulae can be reduced to validity of formulae of the form $s_1 \approx t_1 \wedge \ldots \wedge s_n \approx t_n \rightarrow s \approx t$ where $s_1, \ldots, t$ are terms.[2] Thus, in this case the Nelson-Oppen method combines decision procedures for the *conditional word problem* (i.e., for validity of conditional equations of the above form). Though this may at first sight sound surprising, combining decision procedures for the *word problem* (i.e., for validity of equations $s \approx t$) is a harder task: the known combination algorithms for the word problem are more complicated than the Nelson-Oppen method, and the same applies to their proofs of correctness. The reason is that the algorithms for the component theories are then less powerful. For example, if one applies the Nelson-Oppen method to a word problem $s \approx t$, then it will generate as input for the component procedures conditional word problems, not word problems (see [BT97] for a more detailed discussion). Both the Nelson-Oppen method and the methods for combining decision procedures for the word problem have been generalized to the non-disjoint case [DKR94, TR03, BT02, FG03]. The main restriction on the theories to be combined is that they share only so-called constructors.

In modal logics, one is interested in the question of which properties (like decidability) of uni-modal logics transfer to multi-modal logics that are obtained as the fusion of uni-modal logics. For the decidability transfer,[3] one usually consid-

[1]Equational theories are stably infinite if one adds the axiom $\exists x, y.\ x \not\approx y$ that prevents trivial one-element models [BT97].

[2]This is a consequence of the fact that equational theories are convex [Nel84], i.e., a conjunction of equations implies a disjunction of equations iff it implies one of the disjuncts.

[3]To simplify and clarify the matter, in this introduction, we consider only Kripke-complete modal logics (these are the logics arising in most—if not all—concrete applications). Notice,

ers two different decision problems, the *satisfiability* problem (Is there a Kripke structure of the logic that satisfies the formula φ?) and the *relativized satisfiability* problem (Is there a Kripke structure of the logic that satisfies the formula φ and in which every world satisfies ψ?). There are strong combination results that show that in many cases decidability transfers from two modal logics to their fusion [KW91, Spa93, Wol98, Gab99, BLSW02, GKWZ03]. Again, transfer results for the harder decision problem (relativized satisfiability) are easier to show than for the simpler one (satisfiability). In fact, for satisfiability the results only apply to so-called *normal* modal logics,[4] whereas this restriction is not necessary for relativized satisfiability.

There is a close connection between the (conditional) word problem and the (relativized) satisfiability problem in modal logics. In fact, modal formulae can be viewed as terms, on which equivalence of formulae induces an equational theory.[5] The fusion of modal logics then corresponds to the union of the corresponding equational theories, and the (relativized) satisfiability problem to the (conditional) word problem. The union of the equational theories corresponding to two modal logics is over non-disjoint signatures since the Boolean operators are shared.

Unfortunately, in this setting the Boolean operators are not shared constructors in the sense of [TR03, BT02] (see [FG03]), and thus the decidability transfer results for fusions of modal logics cannot be obtained as special cases of the results in [TR03, BT02, FG03].

Recently, a new generalization of the Nelson-Oppen combination method to non-disjoint theories was developed in [Ghi03, GS03]. The main restriction on the theories T_1 and T_2 to be combined is that they are *compatible* with their shared theory T_0, and that their shared theory is *locally finite* (i.e., its finitely generated models are finite). A theory T is compatible with a theory T_0 iff

1. $T_0 \subseteq T$;
2. T_0 has a model completion T_0^*; and
3. every model of T embeds into a model of $T \cup T_0^*$.

It is well-known that the theory *BA* of Boolean algebras is locally finite, and in [Ghi03] it is shown that the equational theories induced by modal logics are

however, that the decidability transfer results we show in this paper do not depend at all on Kripke-completeness assumptions, as it is evident from the definitions and the main theorem of Section 4.

[4] An exception is [BLSW02], where only the existence of "covering normal terms" is required.

[5] To be more precise, equivalence of formulae can only be axiomatized as an equational theory if it is a congruence relation that is closed under substitution, a restriction that is satisfied by most, also non-normal, modal logics. Such logics are usually called *classical* modal logics [Seg71].

compatible with BA. Thus, the combination method in [Ghi03] applies to (equational theories induced by) modal logics. However, since it generalizes the Nelson-Oppen method, it only yields transfer results for decidability of the conditional word problem (i.e., the relativized satisfiability problem).

In the present paper, we address the harder problem of designing a combination method for the word problem in the non-disjoint case that has the known transfer results for decidability of satisfiability in modal logics as instances. In fact, we will see that our approach strictly generalizes these results since it does not require the modal logics to be normal. The question whether such transfer results hold also for non-normal modal logics was a long-standing open problem in modal logics. In addition to the conditions imposed in [Ghi03, GS03] (i.e., compatibility of the component theories with the shared theory T_0, which is locally finite), our method needs the shared theory T_0 to have *local solvers*. Roughly speaking, this is the case if in T_0 one can solve an arbitrary system of equations with respect to any of its variables. Since this allows one to solve systems of equations by an elimination procedure similar to Gaussian elimination know from linear algebra, we call such theories *Gaussian*.

In the next section, we introduce some basic notions for equational theories, and define the restrictions under which our combination approach applies. In Section 3, we describe the new combination procedure, and show that it is sound and complete. Section 4 shows that the restrictions imposed by our procedure are satisfied by all modal logics where equivalence of formulae induces an equational theory. In particular, we show there that the theory of Boolean algebras is Gaussian. This result is obtained as a consequence of results for unification in Boolean rings. In this section, we also analyze the complexity of our combination procedure if applied to modal logics, and illustrate the working of the procedure on two examples.

2 Preliminaries

In this paper we will use standard notions from equational logic, universal algebra and term rewriting (see, e.g., [BN98]). We consider only first-order theories (with equality $\approx$) over a functional signature. A *signature* Σ is a set of *function symbols*, each with an associated *arity*, an integer $n \geq 0$. A *constant* symbol is a function symbol of zero arity. We use the letters Σ, Ω, possibly with subscripts, to denote signatures. Throughout the paper, we fix a countably-infinite set V of *variables* and a countably-infinite set C of *free constants*, both disjoint with any signature Σ. For any $X \subseteq V \cup C$, $T(\Sigma, X)$ denotes the set of Σ-*terms* over X, i.e., first-order terms with variables and free constants in X and function symbols in Σ.[6] First-order Σ-*formulae* are defined in the usual way, using equality as the only predicate

[6]Note that Σ may also contain constants.

symbol. A Σ-*sentence* is a Σ-formula without free variables. An equational theory E over Σ is a set of (implicitly universally quantified) Σ-identities of the form $s \approx t$, where $s, t \in T(\Sigma, V)$. As usual, first-order interpretations of Σ are called Σ-algebras. We denote algebras by calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, $\ldots$), and their carriers by the corresponding Roman letter (A, B, $\ldots$). A Σ-algebra $\mathcal{A}$ is a *model* of a set T of Σ-sentences iff it satisfies every sentence in T. For a set Γ of sentences and a sentence φ, we write $\Gamma \models_E \varphi$ if every model of E that satisfies Γ also satisfies φ. When Γ is the empty set, we write just $\models_E \varphi$, as usual. We denote by $\approx_E$ the equational consequences of E, i.e., the relation $\approx_E = \{(s,t) \in T(\Sigma, V \cup C) \times T(\Sigma, V \cup C) \mid \models_E s \approx t\}$. The *word problem* for E is the problem of deciding the relation $\approx_E$, that is, deciding for any two terms $s, t \in T(\Sigma, V \cup C)$ whether $s \approx_E t$ holds or not.[7]

In this paper we consider two equational theories E_1 and E_2 of respective signatures Σ_1 and Σ_2 with possibly non-empty intersection Σ_0. We want to know under what conditions the decidability of the word problems for E_1 and E_2 implies the decidability of the word problems for $E_1 \cup E_2$. Before we can define these conditions, we must introduce some notation.

If $\mathcal{B}$ is an Ω-algebra and $\Sigma \subseteq \Omega$, we denote by $\mathcal{B}^\Sigma$ the Σ-*reduct* of $\mathcal{A}$, i.e., the algebra obtained from $\mathcal{A}$ by ignoring the symbols in $\Omega \setminus \Sigma$. An *embedding* of a Σ-algebra $\mathcal{A}$ into a Σ-algebra $\mathcal{B}$ is an injective Σ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If such an embedding exists then we say that $\mathcal{A}$ can be embedded into $\mathcal{B}$. It is easy to show that the composition of two embeddings is also an embedding, and that for all subalgebras $\mathcal{A}'$ of an algebra $\mathcal{A}$ the inclusion function is an embedding of $\mathcal{A}'$ into $\mathcal{A}$. If $\mathcal{A}$ is Σ-algebra and $\mathcal{B}$ is an Ω-algebra with $\Sigma \subseteq \Omega$, we say that $\mathcal{A}$ can be Σ-embedded into $\mathcal{B}$ if there is an embedding of $\mathcal{A}$ into $\mathcal{B}^\Sigma$. We call the corresponging embedding a Σ-embedding of $\mathcal{A}$ into $\mathcal{B}$. If this embedding is the inclusion function, then we say that $\mathcal{A}$ is a Σ-subalgebra of $\mathcal{B}$.

Given a signature Σ and a set X disjoint with Σ, we denote by $\Sigma(X)$ the signature obtained by adding the elements of X as constant symbols to Σ. When X is included in the carrier of a Σ-algebra $\mathcal{A}$, we can view $\mathcal{A}$ as a $\Sigma(X)$-algebra by interpreting each $x \in X$ by itself. If X is a set of generators for $\mathcal{A}$, the Σ-*diagram* $\Delta^\Sigma_X(\mathcal{A})$ of $\mathcal{A}$ (w.r.t. X) consists of all ground $\Sigma(X)$-literals (i.e., ground identities $s \approx t$ and negated ground identities $\neg s \approx t$ for terms $s, t \in T(\Sigma(X), \emptyset)$) that hold in $\mathcal{A}$. We write just $\Delta^\Sigma(\mathcal{A})$ when X coincides with the whole carrier of $\mathcal{A}$. By a result known as Robinson's Diagram Lemma [CK90] embeddings and diagrams are related as follows.

Lemma 2.1 *Let $\mathcal{A}$ be a Σ-algebra generated by a set X, and let $\mathcal{B}$ be an Ω-algebra for some $\Omega \supseteq \Sigma(X)$. Then $\mathcal{A}$ can be $\Sigma(X)$-embedded into $\mathcal{B}$ iff $\mathcal{B}$ is a model of*

[7] We have defined the word problem for terms including free constants since we will consider such terms later on. Since free constants behave just like variables, the word problem is decidable for terms in $T(\Sigma, V \cup C)$ iff it is decidable for terms in $T(\Sigma, V)$.

$\Delta^{\Sigma}_{X}(\mathcal{A})$, the Σ-diagram of $\mathcal{A}$ w.r.t. X.

A consequence of the lemma above, which we will use later, is that if two Σ-algebras $\mathcal{A}$, $\mathcal{B}$ are both generated by a set X and if one of them, say $\mathcal{B}$, satisfies the other's diagram w.r.t. X, then they are isomorphic: indeed, if you view $\mathcal{A}$ and $\mathcal{B}$ as $\Sigma(X)$-algebras, then "$\mathcal{B}$ satisfies the diagram of $\mathcal{A}$ (w.r.t. X)" implies that there is a $\Sigma(X)$-embedding of $\mathcal{A}$ into $\mathcal{B}$. This embedding maps X to X and, since X generates $\mathcal{B}$, it is surjective, and thus an isomorphism.

Ground formulae are invariant under embeddings in the following sense.

Lemma 2.2 *Let $\mathcal{A}$ be a Σ-algebra that can be Σ-embedded into an algebra $\mathcal{B}$. For all ground $\Sigma(A)$-formulae φ, $\mathcal{A}$ satisfies φ iff $\mathcal{B}$ satisfies φ where $\mathcal{B}$ is extended to a $\Sigma(A)$-algebra by interpreting $a \in A$ by its image under the embedding.*

We use the notion of model completion from model theory.

Definition 2.3 (Model Completion) *Let E be an equational Σ-theory and let E^* be a first-order Σ-theory entailing every identity in E.*[8] *Then E^* is a* model completion *of E iff for every model $\mathcal{A}$ of E*

1. *$\mathcal{A}$ can be embedded into a model of E^*, and*
2. *$E^* \cup \Delta^{\Sigma}(\mathcal{A})$ is a complete $\Sigma(A)$-theory.*[9]

One can show that when it exists, the model completion of a theory is unique [CK90].

Given the equational theories E_1 and E_2 to be combined, we want to define conditions under which the decidability of the word problem for E_1 and E_2 implies decidability of the word problem for their union.

First condition: Our first restriction is that both E_1 and E_2 are compatible with a shared subtheory E_0 over the shared signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$ in the following sense.

Definition 2.4 (Compatibility) *Let E be an equational theory over the signature Σ, and let E_0 be an equational theory over a subsignature $\Sigma_0 \subseteq \Sigma$. We say that E is E_0-*compatible *iff*

[8] The notion of model completion applies more generally to first-order theories, but we are interested only in the equational case. Notice that E^* is usually not equational, even if E is so.

[9] A first-order Σ-theory T is *complete* iff for every Σ-sentence φ, either φ or $\neg\varphi$ is entailed by T.

1. $\approx_{E_0} \subseteq \approx_E$*;*

2. E_0 *has a model completion* E_0^**;*

3. every model of E *embeds into a model of* $E \cup E_0^*$*.*

Examples of theories that satisfy this definition can be found in [Ghi03, GS03] and in Section 4. The intuition underlying the definition is also explained there. Here we just show two consequences that will be important when proving completeness of our combination procedure.

Lemma 2.5 *Assume that* E_1 *and* E_2 *are both* E_0-compatible *for some equational theory* E_0 *with signature* $\Sigma_0 = \Sigma_1 \cap \Sigma_2$*. For* $i = 0, 1, 2$*, let* $\mathcal{A}_i$ *be a model of* E_i *such that* $\mathcal{A}_0$ *is a* Σ_0*-subalgebra of both* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*. Then there are a model* $\mathcal{A}$ *of* $E_1 \cup E_2$ *and* Σ_i*-embeddings* f_i *of* $\mathcal{A}_i$ *into* $\mathcal{A}$ *whose restrictions to* A_0 *coincide.*

Proof. To simplify the notation, let us assume that Σ_0 contains all the elements of A_0 as constants, and that $\mathcal{A}_0$ interprets each such constant by itself, i.e., $a^{\mathcal{A}_0} = a$ for all $a \in A_0$. Otherwise we can always add those elements to all the signatures in question.[10] Let $i \in \{1, 2\}$ and let $\mathcal{A}_0$ be a Σ_0-subalgebra of $\mathcal{A}_i$.

By the E_0-compatibility of E_i, there is a model $\mathcal{B}_i$ of $E_i \cup E_0^*$ and a Σ_i-embedding $h_i : \mathcal{A}_i \longrightarrow \mathcal{B}_i$. We can assume without loss of generality that A_i is contained in B_i and that h_i is the inclusion mapping, i.e., $\mathcal{A}_i$ is a Σ_i-subalgebra of $\mathcal{B}_i$. Otherwise, we could just rename the carrier of $\mathcal{B}_i$ appropriately. Let T_i be the set of all first-order $\Sigma_i(B_i)$-sentences satisfied by $\mathcal{B}_i$. We claim that $E_0^* \cup \Delta^{\Sigma_0}(\mathcal{A}_0) \subseteq T_i$. The inclusion is immediate for E_0^* as $\mathcal{B}_i$ is a model of $E_i \cup E_0^*$. To see that $\Delta^{\Sigma_0}(\mathcal{A}_0) \subseteq T_i$, note that $\mathcal{A}_0$ is a Σ_0-subalgebra of $\mathcal{B}_i$. Since $A_0 \subseteq \Sigma_0$, this implies by Lemma 2.1 that $\mathcal{B}_i$ satisfies $\Delta^{\Sigma_0}(\mathcal{A}_0)$, hence $\Delta^{\Sigma_0}(\mathcal{A}_0) \subseteq T_i$.

We have then that $E_0^* \cup \Delta^{\Sigma_0}(\mathcal{A}_0)$, which is a complete theory by Definition 2.3, is included in both T_1 and T_2. It follows by Robinson's Joint Consistency Lemma [CK90] that $T_1 \cup T_2$ is satisfiable. Therefore, let $\mathcal{A}$ be any model of $T_1 \cup T_2$ and let $i \in \{1, 2\}$. By construction of T_i, $\mathcal{A}$ satisfies $\Delta^{\Sigma_i}(\mathcal{B}_i)$, therefore, by Lemma 2.1, there is a $\Sigma_i(B_i)$-embedding h'_i of $\mathcal{B}_i$ into $\mathcal{B}$. Let f_i be the restriction of h'_i to $A_i \subseteq B_i$.

Finally, to see that f_1 coincides with f_2 on A_0, note that for $a \in A_0 \subseteq \Sigma_0$ we have $f_1(a) = f_1(a^{\mathcal{A}_0}) = f_1(a^{\mathcal{B}_1}) = a^{\mathcal{A}} = f_2(a^{\mathcal{B}_2}) = f_2(a^{\mathcal{A}_0}) = f_2(a)$. □

In the following, we call conjunctions of Σ-identities *e-formulae*. We will write $\varphi(\boldsymbol{x})$ to denote an e-formula φ all of whose variables are included in the tuple $\boldsymbol{x}$. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ we will write $\varphi(\boldsymbol{a})$ to denote that $\boldsymbol{a}$ is a tuple of constant symbols of the form $(a_1, \ldots, a_n)$ and $\varphi(\boldsymbol{a})$ is the formula obtained from φ by replacing every occurrence of x_i by a_i for $i = 1, \ldots, n$.

[10] This causes no loss of generality because a Σ-embedding is a Σ'-embedding for all $\Sigma' \subseteq \Sigma$.

Lemma 2.6 *Let* E_1 *be* E_0-compatible *where* E_1 *and* E_0 *are equational theories over the respective signatures* Σ_1 *and* Σ_0 *with* $\Sigma_1 \supseteq \Sigma_0$. *Let* $\psi_1(\boldsymbol{x}, \boldsymbol{y})$ *be an e-formula in the signature* Σ_1 *and* $\psi_2(\boldsymbol{y}, \boldsymbol{z})$ *an e-formula in the signature* Σ_0 *such that* $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0) \models_{E_1} \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$, *where* $\boldsymbol{a}_1$, $\boldsymbol{a}_0$ *and* $\boldsymbol{a}_2$ *are tuples of fresh constants. Then, there is an e-formula* $\psi_0(\boldsymbol{y})$ *in the signature* Σ_0, *such that* $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0) \models_{E_1} \psi_0(\boldsymbol{a}_0)$ *and* $\psi_0(\boldsymbol{a}_0) \models_{E_0} \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$.

Proof. Let Γ_0 be the set of ground e-formulae $\gamma_0(\boldsymbol{a}_0)$ in the signature $\Sigma_0' = \Sigma_0 \cup \boldsymbol{a}_0$[11] such that $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0) \models_{E_1} \gamma_0(\boldsymbol{a}_0)$ or, equivalently stated, such that $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\} \models \gamma_0(\boldsymbol{a}_0)$. By compactness, it is enough to show that $E_0 \cup \Gamma_0 \models \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$.

Let E_2 be the equational theory E_0 in the signature $\Sigma_2 = \Sigma_0$. By definition of model completion, E_2 is trivially E_0-compatible. Consider now the algebras $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ where $\mathcal{A}_0$ is an initial model of the equational theory $E_0 \cup \Gamma_0$ over the signature Σ_0',[12] $\mathcal{A}_1$ is an initial model of the equational theory $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\}$ over the signature $\Sigma_1' = \Sigma_1 \cup \boldsymbol{a}_1 \cup \boldsymbol{a}_0$, and $\mathcal{A}_2$ is an initial model of the equational theory $E_2 \cup \Gamma_0$ over the signature $\Sigma_2' = \Sigma_2 \cup \boldsymbol{a}_0 \cup \boldsymbol{a}_2$. We claim that $\mathcal{A}_0$ can be Σ_0'-embedded into $\mathcal{A}_1$ and into $\mathcal{A}_2$, from which it follows that $\mathcal{A}_0^{\Sigma_0}$, a model of E_0, can be Σ_0-embedded into $\mathcal{A}_1^{\Sigma_1}$, a model of E_1 and into $\mathcal{A}_2^{\Sigma_2}$, a model of E_2.

To see that $\mathcal{A}_0$ can be Σ_0'-embedded into $\mathcal{A}_1$, by Lemma 2.1 it is enough to show that $\mathcal{A}_1$ satisfies $\Delta_{\emptyset}^{\Sigma_0'}(\mathcal{A}_0)$. So let P be a positive ground Σ_0'-literal satisfied by $\mathcal{A}_0$. Since $\mathcal{A}_0$ is an initial model of $E_0 \cup \Gamma_0$, we have that $E_0 \cup \Gamma_0 \models P$. But then, $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\} \models P$ because $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\} \models E_0 \cup \Gamma_0$. Since $\mathcal{A}_1$ is a model of $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\}$, we can conclude that $\mathcal{A}_1$ satisfies P as well. Now let $\neg P$ be a negative ground Σ_0'-literal satisfied by $\mathcal{A}_0$ and assume by contradiction that $\mathcal{A}_1$ satisfies P. Then, since $\mathcal{A}_1$ is an initial model of $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\}$, we have that $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\} \models P$. It follows that $P \in \Gamma_0$ and so it must be satisfied by $\mathcal{A}_0$, against the assumption that $\mathcal{A}_0$ satifies $\neg P$.

To see that $\mathcal{A}_0$ can be Σ_0'-embedded into $\mathcal{A}_2$ it is enough to observe that $\mathcal{A}_2^{\Sigma_0'}$ is a free model of $E_0 \cup \Gamma_0$ over the generators $\boldsymbol{a}_2$. Since $\mathcal{A}_0$ is an initial model of $E_0 \cup \Gamma_0$, it follows by well-known results on free algebras that $\mathcal{A}_0$ can be Σ_0'-embedded into $\mathcal{A}_2^{\Sigma_0'}$.

Thus, let g_1 and g_2 be Σ_0-embeddings of $\mathcal{A}_0^{\Sigma_0}$ into $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$, respectively. By renaming the carriers A_1 and A_2 appropriately, we can make sure that these embeddings are in fact inclusion mappings. It follows from Lemma 2.5 that there is a model $\mathcal{B}$ of $E_1 \cup E_2$ and embeddings f_1 and f_2 of $\mathcal{A}_1^{\Sigma_1}$ and $\mathcal{A}_2^{\Sigma_2}$ into $\mathcal{B}$, such that f_1 coincides with f_2 on A_0. Let $\mathcal{A}$ be the expansion of $\mathcal{B}$ to the signature

[11] In this paper, by abuse of notation, we will consider tuples such as $\boldsymbol{a}_0$ also as a sets.

[12] Since Γ_0 is a the set of ground e-formulae, it can be equivalently seen as a set of ground identities, and thus such an initial model exists.

$\Sigma_1' \cup \Sigma_2'$ defined by interpreting each constant a in $\boldsymbol{a}_1$ as $f_1(a^{\mathcal{A}_1})$, each constant a in $\boldsymbol{a}_0$ as $f_1(a^{\mathcal{A}_0})$, and each constant a in $\boldsymbol{a}_2$ as $f_2(a^{\mathcal{A}_2})$. It is not difficult to see that this expansion is well defined and that f_i is a Σ_i'-embedding of $\mathcal{A}_i$ into $\mathcal{A}$ for $i = 1, 2$.

Now consider the ground formula $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)$, which is true in $\mathcal{A}_1$. Since $\mathcal{A}_1$ is Σ_1'-embedded in $\mathcal{A}$, $\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)$ is true in $\mathcal{A}$ as well by Lemma 2.2. Since $\mathcal{A}$ is a model of $E_1 \cup E_2$ it follows that it is also a model of $E_1 \cup \{\psi_1(\boldsymbol{a}_1, \boldsymbol{a}_0)\}$. Therefore, by our assumptions, $\mathcal{A}$ must be a model of $\psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$. By Lemma 2.2 and the fact that $\mathcal{A}_2$ is Σ_2'-embedded in $\mathcal{A}$, we then have that $\psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$ is true in $\mathcal{A}_2$. Since $\psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$ is a conjunction of ground identities over the signature Σ_2', and $\mathcal{A}_2$ is an initial model of $E_2 \cup \Gamma_0 = E_0 \cup \Gamma_0$ over this signature, it follows that $E_0 \cup \Gamma_0 \models \psi_2(\boldsymbol{a}_0, \boldsymbol{a}_2)$. □

Second condition: The second restriction is that the theory E_0 is locally finite, i.e., all of its finitely generated models are finite. From a more syntactical point of view this means that if C_0 is a finite subset of C, then there are only finitely many E_0-equivalence classes of terms in $T(\Sigma_0, C_0)$. For our combination procedure to be effective, we must be able to compute representatives of these equivalence classes.

Definition 2.7 *An an equational theory E_0 over the signature Σ_0 is* effectively locally finite *iff for every (finite) tuple $\boldsymbol{c}$ of constants from C we can effectively compute a finite set of terms $R_{E_0}(\boldsymbol{c}) \subseteq T(\Sigma_0, \boldsymbol{c})$ such that*

1. *$s \not\approx_{E_0} t$ for all distinct $s, t \in R_{E_0}(\boldsymbol{c})$;*
2. *for all terms $s \in T(\Sigma_0, \boldsymbol{c})$, there is some $t \in R_{E_0}(\boldsymbol{c})$ such that $s \approx_{E_0} t$.*

Example 2.8 A well-known example of an effectively locally finite theory is the theory BA of Boolean algebras, that is, the equational theory over the signature $\{\cap, \cup, \overline{(\_)}, 1, 0\}$ given by the following identities.

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
x \cap y & \approx & y \cap x & x \cup y & \approx & y \cup x \\
x \cap (y \cap z) & \approx & (x \cap y) \cap z & x \cup (y \cup z) & \approx & (x \cup y) \cup z \\
(x \cap y) \cup y & \approx & y & (x \cup y) \cap y & \approx & y \\
x \cap (y \cup z) & \approx & (x \cap y) \cup (x \cap z) & x \cup (y \cap z) & \approx & (x \cup y) \cap (x \cup z) \\
x \cap x & \approx & x & x \cup x & \approx & x \\
x \cap 0 & \approx & 0 & x \cup 0 & \approx & x \\
x \cap 1 & \approx & x & x \cup 1 & \approx & 1 \\
x \cap \overline{x} & \approx & 0 & x \cup \overline{x} & \approx & 1
\end{array}
$$

In fact, if $\boldsymbol{c} = (c_1, \ldots, c_n)$, every ground Boolean term over the constants in $\boldsymbol{c}$ is equivalent in BA to a term in "conjunctive normal form," a meet of terms of the kind $d_1 \cup \cdots \cup d_n$, where each d_i is either c_i or $\overline{c}_i$. It is easy to see that the set $R_{BA}(\boldsymbol{c})$ of such normal forms is isomorphic to the powerset of the powerset of $\boldsymbol{c}$, which is effectively computable and has cardinality 2^{2^n}.

Third condition: The third restriction on our theories E_1 and E_2 is that they are both a *conservative extensions* of E_0, i.e., for $i = 1, 2$ and for all $s, t \in T(\Sigma_0, V)$,

$$s \approx_{E_0} t \quad \text{iff} \quad s \approx_{E_i} t.$$

Fourth condition: The final restriction is that the theory E_0 has local solvers, in the sense that any finite set of equations can be *solved* with respect to any of its variables. Since this means that finite sets of equations can be solved by something similar to the Gaussian elimination procedure known from linear algebra, we call a theory like that *Gaussian*.

Definition 2.9 (Gaussian) *The equational theory E_0 is* Gaussian *iff for every e-formula $\varphi(\boldsymbol{x}, y)$ it is possible to compute an e-formula $C(\boldsymbol{x})$ and a term $s(\boldsymbol{x}, \boldsymbol{z})$ with fresh variables $\boldsymbol{z}$ such that*

$$\models_{E_0} \varphi(\boldsymbol{x}, y) \Leftrightarrow (C(\boldsymbol{x}) \wedge \exists \boldsymbol{z}.(y = s(\boldsymbol{x}, \boldsymbol{z}))) \tag{1}$$

We call the formula C the solvability condition *of φ w.r.t. y, and the term s a* (local) solver *of φ w.r.t. y in E_0.*

The precise connection between the above definition and Gaussian elimination is explained in the following example.

Example 2.10 Let K be a fixed field (e.g., the field of rational or real numbers). We consider the theory of vector spaces over K whose signature consists of a symbol for addition, a symbol for additive inverse and, for every scalar $k \in K$, a unary function symbol $k \cdot (-)$. Axioms are the usual vector spaces axioms (namely, the Abelian group axioms plus the axioms for scalar multiplication). In this theory, terms are equivalent to linear homogeneous polynomials (with non-zero coefficients) over K. Every e-formula $\varphi(\boldsymbol{x}, y)$ can be transformed into a homogeneous system

$$t_1(\boldsymbol{x}, y) = 0 \wedge \cdots \wedge t_k(\boldsymbol{x}, y) = 0$$

of linear equations with unknowns $\boldsymbol{x}, y$. If y does not occur in φ, then φ is its own solvability condition and z is a local solver.[13] If y occurs in φ, then (modulo easy algebraic transformations) we can assume that φ contains an equation of the

[13]Note that φ is trivially equivalent to $\varphi \wedge \exists z.(y = z)$.

form $y = t(\boldsymbol{x})$; this equation gives the local solver, which is $t(\boldsymbol{x})$ (the sequence of existential quantifiers $\exists \boldsymbol{z}$ in (1) is empty), whereas the solvability condition is the e-formula obtained from φ by eliminating y, i.e., replacing y by $t(\boldsymbol{x})$ everywhere in φ.

Another example of a Gaussian theory is the theory of equality over the empty signature.

Example 2.11 The pure equality theory (that is, the empty theory in the empty signature) is also Gaussian. To show that one can argue as in the previous example. Specifically, if $\varphi(\boldsymbol{x}, y)$ does not contain y, then it is its own solvability condition. Otherwise, φ contains an equation like $y = x_i$, and so on. If φ contains only trivial equations $y = y$ involving y, the local solver is again z and in order to get the solvability condition, we can just remove all such trivial equations from φ, reducing the solvability condition to the tautology $\top$ if no equation survives.

Another Gaussian theory will be discussed Section 4. Specifically, we will show there that the theory of Boolean algebras is Gaussian. This is a more sophisticated example, in which the string of existential quantifiers $\exists \boldsymbol{z}$ in (1) can be both not empty and applied to a non-trivial solver (on the contrary, in the above examples, we always have that either there are no parameters $\boldsymbol{z}$, or that the solver is the trivial term z).

We close this section by observing that the theories introduced in Examples 2.10 and 2.11 also satisfy our other restrictions. It is easy to see that the theory in the latter example is *effectively locally finite* since there are no proper terms, and that the theory in the former example is effectively locally finite if the field K is finite.

Notice also that the theories in the two examples above *admit model completions*, which in both cases are axiomatized by saying that models are infinite. In fact, every set embeds into an infinite set, and every vector space V_1 embeds into an infinite vector space V_2 (take, e.g., V_2 equal to the biproduct $V_1 \oplus V'$, where V' has an infinite basis). Thus, the first condition of Definition 2.3 is satisfied.

To show that the second condition of Definition 2.3 is satisfied, it is sufficient to observe that both the theory of an infinite set (over the empty signature) and the theory of an infinite vector space admit quantifier-elimination. Once quantifier-elimination for E^* is achieved, Condition 2 of Definition 2.3 becomes immediate. To see this, let $\mathcal{A}$ be a model of E and let φ be a $\Sigma(A)$-sentence. Now, it is not possible to find models $\mathcal{A}_1, \mathcal{A}_2$ of $E^* \cup \Delta^{\Sigma}(\mathcal{A})$ such that φ is true in $\mathcal{A}_1$ and false in $\mathcal{A}_2$. By Lemma 2.1, $\mathcal{A}$ can be $\Sigma(A)$-embedded into both $\mathcal{A}_1$ and $\mathcal{A}_2$. By quantifier elimination, φ is equivalent (modulo E^*) to a ground $\Sigma(A)$-formula φ'. It follows that φ' is true in $\mathcal{A}_1$, hence in $\mathcal{A}$ (by Lemma 2.2), and finally also in $\mathcal{A}_2$

(again by Lemma 2.2). Thus φ is true in $\mathcal{A}_2$, which contradicts our assumption that φ is false in $\mathcal{A}_2$.

Finally, let us sketch why the theory of Examples 2.10 extended by axioms that say that all models are infinite *admits quantifier-elimination* (the argument for the theory of Example 2.11 is similar).

With no loss of generality we can consider only formulae of the form $\exists x.\varphi$, where φ is a conjunction of literals each inequivalent to $\bot$ and to $\top$ in the original theory. To eliminate the quantifier $\exists x$ we can proceed as follows. If φ contains an identity involving x, by solving with respect to x with the usual Gaussian elimination algorithm, we can convert φ into a conjunction of the form $x \approx t \wedge \varphi'$ where neither t nor φ' contain x. The resulting formula $\exists x.\ (x \approx t\ \wedge\ \varphi')$ to which $\exists x.\varphi$ is equivalent in the original theory, is in turn logically equivalent to φ'. If φ contains no (positive) identities involving x, we can rewrite each negated identity in φ containing x into one of the form $x \not\approx t$, with x not occurring in t. The resulting formula, which is equivalent to $\exists x.\varphi$ in the original theory, has the form

$$\exists x.\ (x \not\approx t_1\ \wedge\ \cdots\ \wedge\ x \not\approx t_k\ \wedge\ \varphi')$$

where $t_1, \ldots, t_k$, and φ' do not contain x. This formula is equivalent to φ' in the extended theory since all the models of that theory are infinite.

3 The combination procedure

In the following, we assume that E_1, E_2 are equational theories over the signatures Σ_1, Σ_2 with decidable word problems, and that there exists an equational theory E_0 over the signature $\Sigma_0 := \Sigma_1 \cap \Sigma_2$ such that

- E_0 is Gaussian and effectively locally finite;
- for $i = 1, 2$, E_i is E_0-compatible and a conservative extension of E_0.

3.1 Abstraction rewrite systems

Our combination procedure works on the following data structure.

Definition 3.1 *An* abstraction rewrite system (ARS) *R is a finite ground rewrite system that can be partitioned into $R = R_1 \cup R_2$ so that*

- *for $i = 1, 2$, the rules of R_i are of the form $a \to t$ where $a \in C$ and $t \in T(\Sigma_i, C)$, and every constant a occurs at most once as a left-hand side in R_i;*

- $R = R_1 \cup R_2$ *is terminating.*

The ARS R is an initial *ARS iff every constant a occurs at most once as a left-hand side in the whole R.*

Since R is terminating, we can find a strict total ordering $>$ on the left-hand side constants of R such that for all $a \to t \in R$, the term t contains only left-hand side constants smaller than a. In particular, for $i = 1, 2$, R_i is also terminating, and the restriction that every constant occurs at most once as a left-hand side in R_i implies that R_i is confluent. We denote the unique normal form of a term s w.r.t. R_i by $s{\downarrow}_{R_i}$.

Given a ground rewrite system R, an equational theory E, and an e-formula ψ, we write $R \models_E \psi$ to express that $\{l \approx r \mid l \to t \in R\} \models_E \psi$.

Lemma 3.2 *Let $R = R_1 \cup R_2$ be an ARS, and $s, t \in T(\Sigma_i, C)$ for some $i \in \{1, 2\}$. Then $R_i \models_{E_i} s \approx t$ iff $s{\downarrow}_{R_i} \approx_{E_i} t{\downarrow}_{R_i}$.*

Proof. Let $i \in \{1, 2\}$. Let $a_n > a_{n-1} > \cdots > a_1$ be a total ordering of the left-hand side (lhs) constants of $R_i = \{a_j \to t_j \mid j = 1, \ldots, n\}$ such that t_j contains only lhs constants smaller than a_j.

($\Leftarrow$) Obviously, $s{\downarrow}_{R_i} \approx_{E_i} t{\downarrow}_{R_i}$ implies $R_i \models_{E_i} s \approx t$.

($\Rightarrow$) Assume that $R_i \models_{E_i} s \approx t$. Since $R_i \models_{E_i} s \approx s{\downarrow}_{R_i}$ and $R_i \models_{E_i} t \approx t{\downarrow}_{R_i}$, this yields $R_i \models_{E_i} s{\downarrow}_{R_i} \approx t{\downarrow}_{R_i}$. Now assume that $s{\downarrow}_{R_i} \not\approx_{E_i} t{\downarrow}_{R_i}$, i.e., there is a model $\mathcal{A}$ of E_i in which the identity $s{\downarrow}_{R_i} \approx t{\downarrow}_{R_i}$ does not hold. Since the terms $s{\downarrow}_{R_i}, t{\downarrow}_{R_i}$ do not contain the constants $a_1, \ldots, a_n$, we may assume that $\mathcal{A}$ does not interpret these constants. We show below that we can expand $\mathcal{A}$ to a model $\widehat{\mathcal{A}}$ of E_i that also interprets the constants $a_1, \ldots, a_n$ and satisfies $\{a_j \approx t_j \mid j = 1, \ldots, n\}$. This will imply that $R_i \not\models_{E_i} s{\downarrow}_{R_i} \approx t{\downarrow}_{R_i}$, a contradiction.

We define expansions $\mathcal{A}_j$ of $\mathcal{A}$ that interpret the lhs constants $a_1, \ldots, a_j$ by induction on j:

- The algebra $\mathcal{A}_1$ interprets a_1 by the interpretation of t_1 in $\mathcal{A}$, i.e., $a_1^{\mathcal{A}_1} := t_1^{\mathcal{A}}$. Note that $t_1^{\mathcal{A}}$ is well-defined since t_1 does not contain any of the constants $a_1, \ldots, a_n$.
- The algebra $\mathcal{A}_j$ extends $\mathcal{A}_{j-1}$ by interpreting a_j by the interpretation of t_j in $\mathcal{A}_{j-1}$, i.e., $a_j^{\mathcal{A}_j} := t_j^{\mathcal{A}_{j-1}}$. Note that $t_j^{\mathcal{A}_{j-1}}$ is well-defined since t_j does not contain any of the constants $a_j, \ldots, a_n$.

Now, $\widehat{\mathcal{A}}$ is defined to be $\mathcal{A}_n$. It is easy to see that this algebra has the required properties, i.e., it is a model of $E_i \cup \{a_j \approx t_j \mid j = 1, \ldots, n\}$ in which the identity $s{\downarrow}_{R_i} \approx t{\downarrow}_{R_i}$ does not hold. □

If we want to decide the word problem in $E_1 \cup E_2$, it is sufficient to consider ground terms with free constants, i.e., terms $s, t \in T(\Sigma_1 \cup \Sigma_2, C)$. Given such terms s, t we can employ the usual abstraction procedures that replace subterms by new constants in C (see, e.g., [BT02]) to generate terms $u, v \in T(\Sigma_0, C)$ and an initial ARS $R = R_1 \cup R_2$ such that

$$s \approx_{E_1 \cup E_2} t \quad \text{iff} \quad R \models_{E_1 \cup E_2} u \approx v.$$

For example, assume that $\Sigma_1 = \{f, g\}$ and $\Sigma_2 = \{f, h\}$, and consider the terms $s = f(h(c_1), g(h(c_1)))$ and $t = g(f(h(c_1), c_2))$. Then we can take $u = f(a_1, a_2)$, $v = a_3$, $R_1 = \{a_1 \to h(c_1)\}$, and $R_2 = \{a_2 \to g(a_1), a_3 \to g(f(a_1, c_2))\}$.

Thus, to decide the word problem in $E_1 \cup E_2$, it is sufficient to devise a procedure that can solve problems of the form "$R \models_{E_1 \cup E_2} u \approx v$?" where R is an initial ARS and $u, v \in T(\Sigma_0, C)$. We present this procedure next.

3.2 The combination procedure

The input of the procedure is an initial ARS $R = R_1 \cup R_2$ and two terms $u, v \in T(\Sigma_0, C)$. Let $>$ be a total ordering of the left-hand side (lhs) constants of R such that for all $a \to t \in R$, t contains only lhs constants smaller than a. Given this ordering, we can assume that $R = \{a_i \to t_i \mid i = 1, \ldots, n\}$ for some $n \geq 0$ where $a_n > a_{n-1} > \cdots > a_1$.

Note that u, v and each t_i may also contain free constants from C that are not left-hand side constants. In the following, we use $\boldsymbol{c}$ to denote a tuple of all these constants. Furthermore, for $j = 1, 2$ and $i = 0, \ldots, n$, we denote by $R_j^{(i)}$ the restriction of R_j to the rules whose left-hand sides are smaller or equal to a_i—where, by convention, $R_j^{(0)}$ is the empty system.

The combination procedure is described in Figure 1. First, note that all of the steps of the procedure are effective. Step 1 of the for loop is trivially effective; Step 2 is effective because E_0 is effectively locally finite by assumption. Step 3 is effective because the test that $R_j^{(i)} \models_{E_j} t \approx t'$ can be reduced by Lemma 3.2 to testing that $t{\downarrow}_{R_j^{(i)}} \approx_{E_j} t'{\downarrow}_{R_j^{(i)}}$. The latter test is effective because, (i) the word problem in E_j is decidable by assumption and (ii) $R_j^{(i)}$ is confluent and terminating at each iteration of the loop. Now, in Step 4 the formula φ can be computed because T is finite and the local solver in Step 5 can be computed by the algorithm provided by the definition of a Gaussian theory. Step 6 is trivial and for the final test after the loop, the same observations as for Step 3 apply.

A few more remarks on the procedure are in order. In the fifth step of the loop, $\boldsymbol{d}$ is a tuple of new constants introduced by the solver s. In the definition of a local solver, we have used variables instead of constants, but this difference will turn out to be irrelevant since free constants behave like variables. One may

Input: an initial ARS $R = R_1 \cup R_2 = \{a_i \to t_i \mid i = 1, \ldots, n\}$ and terms $u, v \in T(\Sigma_0, C)$.

Let $\boldsymbol{c}$ collect the free constants in R, u, v that are not in $\{a_1, \ldots, a_n\}$.

for $i = 1$ to n do

1. Let j be such that $a_i \to t_i \in R_j$ and k such that $\{j, k\} = \{1, 2\}$.
2. Let $T = R_{E_0}(a_1, \ldots, a_i, \boldsymbol{c})$ (see Definition 2.7).
3. For each pair of distinct terms $t, t' \in T$, test whether $R_j^{(i)} \models_{E_j} t \approx t'$.
4. Let $\varphi(a_1, \ldots, a_i, \boldsymbol{c})$ be the conjunction of those identities $t \approx t'$ for which the test succeeds.
5. Let $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ be a local solver of φ w.r.t. a_i in E_0.
6. Add to R_k the new rule $a_i \to s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$.

done

Output: "yes" if $R_1 \models_{E_1} u \approx v$, and "no" otherwise.

Figure 1: The combination procedure.

wonder why the procedure ignores the solvability condition for the local solver. The reason is that this condition follows from both R_1 and R_2, as will be shown in the proof of completeness.

Adding the new rule to R_k in the sixth step of the loop does not destroy the property of $R_1 \cup R_2$ being an ARS—although it will make it non-initial. In fact, $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ contains only lhs constants smaller than a_i, and R_k before did not contain a rule with lhs a_i because the input was an *initial* ARS.

The test after the loop is performed using R_1, E_1. The choice R_1 and E_1 versus R_2 and E_2 is arbitrary. As it will be made clear by the completeness proof for the procedure, using R_2, E_2 instead would produce the same results.

3.3 The correctness proof

Since the combinations procedure obviously terminates on any input, it is sufficient to show soundness and completeness. In this proof, we will use the notation $R_{1,i}, R_{2,i}$ to denote the updated rewrite systems obtained after step i in the loop ($R_{1,0}$ and $R_{2,0}$ are the input systems R_1 and R_2).

Proposition 3.3 (Soundness) *If the combination procedure yields the answer "yes", then $R_1 \cup R_2 \models_{E_1 \cup E_2} u \approx v$.*

Proof. Let $i \in \{1, \ldots, n\}$. We start by showing that

$$R_{1,i} \cup R_{2,i} \models_{E_1 \cup E_2} u \approx v \quad \text{implies} \quad R_{1,i-1} \cup R_{2,i-1} \models_{E_1 \cup E_2} u \approx v. \tag{2}$$

First observe that

$$R_{1,i} \cup R_{2,i} \quad = \quad R_{1,i-1} \cup R_{2,i-1} \cup \{a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})\} \tag{3}$$

where (i) the term $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ is a local solver of $\varphi(a_1, \ldots, a_i, \boldsymbol{c})$ w.r.t. the free constant a_i in E_0, and (ii) $\varphi(a_1, \ldots, a_i, \boldsymbol{c})$ is an e-formula such that $R_{j,i-1} \models_{E_i} \varphi(a_1, \ldots, a_i, \boldsymbol{c})$ for some $j \in \{1, 2\}$. Now assume that $R_{1,i} \cup R_{2,i} \models_{E_1 \cup E_2} u \approx v$. By (3) above and the fact that the constants $\boldsymbol{d}$ occur only in the solver s, we have that

$$R_{1,i-1} \cup R_{2,i-1} \cup \{\exists \boldsymbol{z}.(a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z}))\} \quad \models_{E_1 \cup E_2} \quad u \approx v.$$

To prove that $R_{1,i-1} \cup R_{2,i-1} \models_{E_1 \cup E_2} u \approx v$ it is obviously enough to show that $R_{1,i-1} \cup R_{2,i-1} \models_{E_1 \cup E_2} \exists \boldsymbol{z}.(a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z}))$. To show this, first observe that $R_{1,i-1} \cup R_{2,i-1} \models_{E_1 \cup E_2} \varphi(a_1, \ldots, a_i, \boldsymbol{c})$ by monotonicity of $\models$ and (ii) above. Second, by construction of s (see Definition 2.9) and the fact that $E_1 \cup E_2$ extends E_0 it follows, again by monotonicity, that $R_{1,i-1} \cup R_{2,i-1} \models_{E_1 \cup E_2} \exists \boldsymbol{z}.(a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z}))$.

To prove the proposition now, assume that procedure answers "yes". Then it must be that $R_{1,n} \models_{E_1} u \approx v$ which implies that $R_{1,n} \cup R_{2,n} \models_{E_1 \cup E_2} u \approx v$. But then, by a repeated application of Property (2) above, we have that $R_1 \cup R_2 = R_{1,0} \cup R_{2,0} \models_{E_1 \cup E_2} u \approx v$. $\square$

The following lemma will be useful to prove the completeness of the combination procedure.

Lemma 3.4 *For every $i = 1, \ldots, n$ and every ground e-formula $\psi(a_1, .., a_i, \boldsymbol{c})$ in the signature $\Sigma_0 \cup \{a_1, .., a_i\} \cup \boldsymbol{c}$*

$$R_{1,i}^{(i)} \models_{E_1} \psi \quad \textit{iff} \quad R_{2,i}^{(i)} \models_{E_2} \psi.$$

In particular, for $i = n$ we have that $R_{1,n} \models_{E_1} \psi$ iff $R_{2,n} \models_{E_2} \psi$ for every ground e-formula $\psi(a_1, \ldots, a_n, \boldsymbol{c})$ in the signature $\Sigma_0 \cup \{a_1, .., a_n\} \cup \boldsymbol{c}$.

Proof. We prove the lemma by induction on i. The base case $i = 0$ is trivial since $R_{1,0}^{(0)}, R_{2,0}^{(0)}$ are empty and E_1, E_2 are conservative extensions of the same theory E_0 over Σ_0.

Let $i > 0$ and assume that the lemma holds for $i - 1$. Let $j, k, t_i, \varphi(a_1, \ldots, a_i, \boldsymbol{c})$, and $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ be defined as in the i-th iteration of the loop in the combination procedure. Then we have $R_{j,i} = R_{j,i-1}$ and $R_{k,i} = R_{k,i-1} \cup \{a_i \rightarrow s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})\}$.

First, we show that $R_{j,i}^{(i)} \models_{E_j} \psi$ implies $R_{k,i}^{(i)} \models_{E_k} \psi$. Observe that $R_{j,i}^{(i)}$ is equal to $R_{j,i-1}^{(i)}$ and that $R_{k,i}^{(i)}$ is equal to to $R_{k,i-1}^{(i-1)} \cup \{a_i \rightarrow s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})\}$. From $R_{j,i}^{(i)} \models_{E_j} \psi$ it follows that $\varphi \models_{E_0} \psi$ (since, modulo E_0, every conjunct of ψ occurs as a conjunct in φ by the definition of φ). Thus, it is sufficient to show that $R_{k,i}^{(i)} \models_{E_k} \varphi$. Because $a_i \rightarrow s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ belongs to $R_{k,i}^{(i)}$ and since $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ is a local solver of φ w.r.t. a_i, it is sufficient to show that the corresponding solvability condition $C(a_1, \ldots, a_{i-1}, \boldsymbol{c})$ follows from E_k and $R_{k,i}^{(i)}$. However, this formula does not contain a_i, and thus we can argue as follows. Since φ implies its own solvability condition (in E_0, and thus also in E_j), $R_{j,i}^{(i)} \models_{E_j} \varphi$ implies $R_{j,i}^{(i)} \models_{E_j} C(a_1, ..., a_{i-1}, \boldsymbol{c})$. Because $C(a_1, ..., a_{i-1}, \boldsymbol{c})$ does not contain a_i and since $R_{j,i} = R_{j,i-1}$, this implies that $R_{j,i-1}^{(i-1)} \models_{E_j} C(a_1, ..., a_{i-1}, \boldsymbol{c})$ by Lemma 3.2.[14] Thus, the induction hypothesis yields $R_{k,i-1}^{(i-1)} \models_{E_k} C(a_1, ..., a_{i-1}, \boldsymbol{c})$. Since $R_{k,i-1}^{(i-1)} \subseteq R_{k,i}^{(i)}$, this finally implies $R_{k,i}^{(i)} \models_{E_k} C(a_1, ..., a_{i-1}, \boldsymbol{c})$. In conclusion, we have shown that $R_{k,i}^{(i)} \models_{E_k} \psi$.

Second, we show that $R_{k,i}^{(i)} \models_{E_k} \psi$ implies $R_{j,i}^{(i)} \models_{E_j} \psi$. Since $R_{k,i} := R_{k,i-1} \cup \{a_i \rightarrow s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})\}$, we know (again by Lemma 3.2) that $R_{k,i}^{(i)} \models_{E_k} \psi$ implies that $R_{k,i-1}^{(i-1)} \models_{E_k} \psi_2(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$ where ψ_2 is obtained from ψ by replacing every occurrence of a_i by $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$. Let $\psi_1(a_1, \ldots, a_i, \boldsymbol{c})$ be the conjunction of all the identities denoted by $R_{k,i-1}^{(i-1)}$. Applying Lemma 2.6 to $\psi_1(a_1, \ldots, a_i, \boldsymbol{c}) \models_{E_k} \psi_2(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$, we then obtain an e-formula $\psi_0(x_1, \ldots, x_{i-1}, \boldsymbol{y})$ in the shared signature Σ_0 such that

1. $R_{k,i-1}^{(i-1)} \models_{E_k} \psi_0(a_1, \ldots, a_{i-1}, \boldsymbol{c})$ and
2. $\psi_0(a_1, \ldots, a_{i-1}, \boldsymbol{c}) \models_{E_0} \psi_2(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d})$.

By induction hypothesis on the first entailment, we then have

$$R_{j,i-1}^{(i-1)} \models_{E_j} \psi_0(a_1, ..., a_{i-1}, \boldsymbol{c}),$$

and so, since $R_{j,i-1} = R_{j,i}$, also $R_{j,i}^{(i)} \models_{E_j} \psi_0(a_1, ..., a_{i-1}, \boldsymbol{c})$. By the substitutivity property of equality and the construction of ψ_2, the second entailment implies that $\psi_0(a_1, ..., a_{i-1}, \boldsymbol{c}) \wedge a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{d}) \models_{E_0} \psi$, which is equivalent to

$$\psi_0(a_1, ..., a_{i-1}, \boldsymbol{c}) \wedge \exists \boldsymbol{z}.\ (a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z})) \quad \models_{E_0} \quad \psi,$$

[14] Lemma 3.2 applies here because $C(a_1, ..., a_{i-1}, \boldsymbol{c})$ is a conjunction of identities, and so it is entailed by a set of formulae iff each of its identities is.

as the constants $\boldsymbol{d}$ do not occur in ψ. Given that $s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z})$ is a local solver for $\varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{z})$, we have by Definition 2.9 that $\varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{z}) \models_{E_0} \exists \boldsymbol{z}.\, (a_i \approx s(a_1, \ldots, a_{i-1}, \boldsymbol{c}, \boldsymbol{z}))$. It follows that $\{\psi_0, \varphi\} \models_{E_0} \psi$.

Recalling that $R_{j,i}^{(i)} \models_{E_j} \varphi$ by construction of φ and that $R_{j,i}^{(i)} \models_{E_j} \psi_0$ as just seen, we can conclude that $R_{j,i}^{(i)} \models_{E_j} \psi$. □

Proposition 3.5 (Completeness) *If $R_1 \cup R_2 \models_{E_1 \cup E_2} u \approx v$, then the combination procedure answers "yes".*

Proof. Since the procedure is terminating, it is enough to show that $R_{1,0} \cup R_{2,0} \not\models_{E_1 \cup E_2} u \approx v$ whenever the combination procedure answer "no". We do that by building a model of $R_{1,0} \cup R_{2,0} \cup E_1 \cup E_2$ that falsifies $u \approx v$.

Assume then that the combination procedure answer "no", let $\boldsymbol{a} := (a_1, \ldots, a_n)$ and let $k \in \{1, 2\}$. Where $\boldsymbol{c}$ is defined as in Figure 1 and $\boldsymbol{d}_k$ is a tuple collecting all the new constants introduced in the rewrite system R_k during execution of the procedure (see Step 4 of the loop), let $\mathcal{A}_{k,0}$ be the initial model of E_k over the signature $\Sigma_k \cup \boldsymbol{c} \cup \boldsymbol{d_k}$.

Observe that the final rewrite system $R_{k,n}$ contains (exactly) one rule of the form $a_i \to u_i$ for all $i = 1, \ldots, n$. This is because either the rule $a_i \to t_i$ was already in $R_{k,0}$ to begin with (then $u_i = t_i$), or a rule of the form $a_i \to s_i$ for some solver s_i was added to $R_{k,i-1}$ at step i to produce $R_{k,i}$ (in which case $u_i = s_i$).

Now, as in the proof of Lemma 3.2, we can use the rewrite rules of $R_{k,n}$ to define by induction on $i = 1, \ldots, n$ an expansion $\mathcal{A}_{k,i}$ of $\mathcal{A}_{k,0}$ to the constants $a_1, \ldots, a_i$. Specifically, $\mathcal{A}_{k,i}$ is defined as the expansion of $\mathcal{A}_{k,i-1}$ that interprets a_i as $u_i^{\mathcal{A}_{k,i-1}}$ where u_i is the term such that $a_i \to u_i \in R_{k,n}$. Note that $u_i^{\mathcal{A}_{k,i-1}}$ is well defined because u_i does not contain any of the constants $a_i, \ldots, a_n$. As a consequence, all the $\mathcal{A}_{k,i}$ are well defined.

By induction on i we can show that for every ground e-formula $\varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k)$ in the signature $\Sigma_k \cup \{a_1, \ldots, a_i\} \cup \boldsymbol{c} \cup \boldsymbol{d}_k$, we have that

$$\mathcal{A}_{k,i} \text{ satisfies } \varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k) \quad \text{iff} \quad R_{k,n}^{(i)} \models_{E_k} \varphi(a_1, \ldots, a_i, \boldsymbol{c}, \boldsymbol{d}_k). \tag{4}$$

In fact, let $i = 0$ and observe that $R_{k,n}^{(0)} = \emptyset$. If $\mathcal{A}_{k,0}$ satisfies φ, since φ is a conjunction of ground identities and $\mathcal{A}_{k,0}$ is an initial model of E_k, we have immediately that $\emptyset \models_{E_k} \varphi$. Conversely, if $\emptyset \models_{E_k} \varphi$ then φ is satisfied by every model of E_k, and so in particular by $\mathcal{A}_{k,0}$. For $i > 0$ we have that $\mathcal{A}_{k,i}$ satisfies φ iff $\mathcal{A}_{k,i-1}$ satisfies $\varphi[u_i/a_i]$[15] iff (by induction) $R_{k,n}^{(i-1)} \models_{E_k} \varphi[u_i/a_i]$ iff $R_{k,n}^{(i-1)} \cup \{a_i \approx u_i\} \models_{E_k} \varphi$ iff $R_{k,n}^{(i)} \models_{E_k} \varphi$.

[15] Where $\varphi[u_i/a_i]$ denotes the formula obtained from φ by replacing every occurrence of a_i by u_i.

Let $\mathcal{A}_k = \mathcal{A}_{k,n}^{\Omega_k}$ where $\Omega_k = \Sigma_k \cup \boldsymbol{a} \cup \boldsymbol{c}$. As a special case of (4) above, we have that for every ground e-formula $\varphi(\boldsymbol{a}, \boldsymbol{c})$ in the signature $\Sigma_0 \cup \boldsymbol{a} \cup \boldsymbol{c}$,

$$\mathcal{A}_k \text{ satisfies } \varphi \quad \text{iff} \quad R_{k,n} \models_{E_k} \varphi. \tag{5}$$

For $k = 1, 2$ let $\mathcal{B}_k$ be the subalgebra of $\mathcal{A}_k^{\Sigma_0}$ generated by (the interpretations in $\mathcal{A}_k$ of) the constants $\boldsymbol{a} \cup \boldsymbol{c}$. We claim that the algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy each other's diagram. To see that, let ψ be a ground identity of signature $\Sigma_0 \cup \boldsymbol{a} \cup \boldsymbol{c}$. Then,

$$\begin{array}{llll} \psi \in \Delta^{\Sigma_0}_{\boldsymbol{a}\cup\boldsymbol{c}}(\mathcal{B}_k) & \text{iff} & \mathcal{B}_k \text{ satisfies } \psi & [\text{by definition of } \Delta^{\Sigma_0}_{\boldsymbol{a}\cup\boldsymbol{c}}(\mathcal{B}_k)] \\ & \text{iff} & \mathcal{A}_k \text{ satisfies } \psi & [\text{by construction of } \mathcal{B}_k \text{ and Lemma 2.2}] \\ & \text{iff} & R_{k,n} \models_{E_k} \psi & [\text{by (5) above}]. \end{array}$$

By Lemma 3.4, we can conclude that $\psi \in \Delta^{\Sigma_0}_{\boldsymbol{a}\cup\boldsymbol{c}}(\mathcal{B}_1)$ iff $\psi \in \Delta^{\Sigma_0}_{\boldsymbol{a}\cup\boldsymbol{c}}(\mathcal{B}_2)$. It follows from the observation after Lemma 2.1 that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Σ_0-isomorphic, hence they can be identified with no loss of generality. Therefore, let $\mathcal{A}_0 = \mathcal{B}_1 = \mathcal{B}_2$ and observe that (i) $\mathcal{A}_k^{\Sigma_k}$ is a model of E_k by construction; (ii) $\mathcal{A}_0$ is a Σ_0-subalgebra of $\mathcal{A}_k^{\Sigma_k}$; and (iii) $\mathcal{A}_0$ is a model of E_0 because $\mathcal{A}_k^{\Sigma_0}$ is a model of E_0 and the set of models of an equational theory is closed under subalgebras.

By Lemma 2.5 it follows that there is a model $\mathcal{A}$ of $E_1 \cup E_2$ such that for $k = 1, 2$ there is a Σ_k-embedding f_k of $\mathcal{A}_k^{\Sigma_k}$ into $\mathcal{A}$. By the same lemma we also have that $f_1(c^{\mathcal{A}_1}) = f_2(c^{\mathcal{A}_2})$ for all $c \in \boldsymbol{a} \cup \boldsymbol{c}$. Let then $\mathcal{A}'$ be the expansion of $\mathcal{A}$ to the signature $\Sigma_1 \cup \Sigma_2 \cup \boldsymbol{a} \cup \boldsymbol{c}$ such that $c^{\mathcal{A}'} = f_1(c^{\mathcal{A}_1})$ for every $c \in \boldsymbol{a} \cup \boldsymbol{c}$. It is not difficult to see that f_k is an Ω_k-embedding of $\mathcal{A}_k$ into $\mathcal{A}'$ for $k = 1, 2$.

Observe that $\mathcal{A}'$, which is clearly a model of $E_1 \cup E_2$, is also a model of $R_{1,0} \cup R_{2,0}$. In fact, by construction of $R_{1,n}$ and $R_{2,n}$, for all $a \to t \in R_{1,0} \cup R_{2,0}$, there is a $k \in \{1, 2\}$ such that $a \to t \in R_{k,n}$. It follows immediately that $R_{k,n} \models_{E_k} a \approx t$, which implies by (5) above that $\mathcal{A}_k$ satisfies $a \approx t$. But then $\mathcal{A}'$ satisfies $a \approx t$ as well by Lemma 2.2.

In conclusion, we have that $\mathcal{A}'$ is a model of $R_{1,0} \cup R_{2,0} \cup E_1 \cup E_2$. All we need to show then is that $\mathcal{A}'$ falsifies $u \approx v$. Now, since the procedure returns "no" by assumption, it must be that $R_{1,n} \not\models_{E_1} u \approx v$. We then have that $\mathcal{A}_1$ falsifies $u \approx v$ by (5) above and $\mathcal{A}'$ falsifies $u \approx v$ by Lemma 2.2. $\square$

Note that in the last paragraph of the proof above we could have given a completely symmetrical argument if the final test in the procedure had been on whether $R_{2,n} \models_{E_2} u \approx v$. In other words, the procedure's completeness does not depend on which component theory is used for the final test.

From the total correctness of the combination procedure, we then obtain the following modular decidability result.

Theorem 3.6 *Let E_0, E_1, E_2 be three equational theories of respective signature $\Sigma_0, \Sigma_1, \Sigma_2$ such that*

- $\Sigma_0 = \Sigma_1 \cap \Sigma_2$;
- E_0 *is Gaussian and effectively locally finite;*
- *for* $i = 1, 2$, E_i *is* E_0*-compatible and a conservative extension of* E_0.

If the word problem in E_1 *and in* E_2 *is decidable, then the word problem in* $E_1 \cup E_2$ *is also decidable.*

We conclude this section by pointing out that, modulo a minor technicality explained in the following, the results above can be seen as a generalization of the corresponding combination result for the disjoint case. This known result (see, e.g., [Pig74]) states that, if E_1 and E_2 are two non-trivial[16] equational theories with disjoint signatures and decidable word problem, then $E_1 \cup E_2$ has a decidable word problem.

To recast this result in terms of Theorem 3.6, one needs to show that for every two non-trivial equational theories E_1 and E_2 with disjoint signatures, there is a theory E_0 in the empty signature which is Gaussian, effectively locally finite, and such that both E_1 and E_2 conservatively extend E_0 and are E_0-compatible. Bar the compatibility requirement, such a theory E_0 is the pure equality theory (see Example 2.11). As discussed in Section 2, this theory is effectively locally finite and Gaussian, and admits as its model completion E_0^* the theory of an infinite set. It is immediate that any non-trivial equational theory E is a conservative extension of E_0. Furthermore, it is almost true that any non-trivial E is E_0-compatible. Specifically, while points 1 and 2 of Definition 2.4 are always satisfied, point 3 (requiring that every model of E be embeddable in a model of $E \cup E_0^*$) is always satisfied only by non-trivial models of E. The reason is that every algebra $\mathcal{A}$ is embedded into the infinite direct product $\mathcal{A}^\omega$ of $\mathcal{A}$ with itself by the diagonal function.[17] If $\mathcal{A}$ is a model of E, then $\mathcal{A}^\omega$ is a model of E because the set of models of an equational theory is closed under direct products. If $\mathcal{A}$ is also non-trivial, then $\mathcal{A}^\omega$ is infinite and so it is also a model of $E \cup E_0^*$. If $\mathcal{A}$ is trivial, depending on E, $\mathcal{A}$ may or may not be embeddable into a model of $E \cup E_0^*$.

Now, the problem with the trivial models can be eliminated by considering the combination not of equational theories E_1 and E_2, but of theories of the form $E_i' = E_i \cup \{\exists x, y.\, x \not\approx y\}$ for $i = 1, 2$, where E_i is a non-trivial equational theory. These theories admit only non-trivial models. Moreover, for the purposes of deciding the word problem, there is no real loss of generality in considering E_i' instead of E_i because the word problems for E_i' and for E_i coincide. While strictly speaking E_1' and E_2' are not equational, they satisfy all results stated in this paper for the equational theories E_1 and E_2 (although some of the proofs need some minor adjustments). Hence Theorem 3.6 applies to them as well.

[16] A theory is non-trivial if it admits non-trivial models, that is, models of cardinality greater than 1.

[17] The function that maps every $a \in A$ to the infinite tuple of all a's.

4 Fusion decidability in modal logics

In this section we define the modal logics to which our combination procedure applies. Basically, these are modal logics that corresponds to equational extensions of the theory of Boolean algebras; for this reason, our definition is very liberal and covers most modal systems considered in the literature (with few exceptions, as we will see).

4.1 Equational theories induced by modal logics

A *modal signature* Σ_M is a set of operation symbols endowed with corresponding arities; from Σ_M propositional formulae are built up using countably many propositional variables, the operation symbols in Σ_M, the Boolean connectives, and the constant $\top$ for truth and $\bot$ for falsity. We use letters $x, x_1, \ldots, y, y_1, \ldots$ for propositional variables and letters $t, t_1, \ldots, u, u_1, \ldots$ as metavariables for propositional formulae.

The following definition is taken from [Seg71], pp. 8–9:[18]

Definition 4.1 *A* classical modal logic L *based on a modal signature* Σ_M *is a set of propositional formulae that*

(i) *contains all classical tautologies;*

(ii) *is closed under uniform substitution of propositional variables by propositional formulae;*

(iii) *is closed under the modus ponens rule ('from t and $t \Rightarrow u$ infer u');*

(iv) *is closed under the replacement rules, which are specified as follows. We have one such rule for each n-ary $o \in \Sigma_M$, namely:*

$$\frac{t_1 \Leftrightarrow u_1, \ \ldots, \ t_n \Leftrightarrow u_n}{o(t_1, \ldots, t_n) \Leftrightarrow o(u_1, \ldots, u_n)}$$

As classical modal logics (based on a given modal signature) are closed under intersections, it makes sense to speak of the least classical modal logic $[S]$ containing a certain set of propositional formulae S. If $L = [S]$, we say that S is a set of axiom schemata for L and write $S \vdash t$ for $t \in [S]$.

[18]Strictly speaking, K. Segerberg in [Seg71] considers only modal signatures consisting of a single unary modal operator (i.e., unary unimodal logics; more general multimodal systems became quite popular only later on). The least classical modal logic with a single unary operator is usually called **E**.

Notice that giving a set of axiom schemata for L is not the only way to introduce a classical modal logic L: for instance, one can introduce L just by specifying a certain (e.g. Kripke, neighborhood, algebraic, etc.) semantics and saying that L is the set of formulae that are valid in that semantics.

We say that a classical modal logic L is *decidable* iff L is a recursive set of propositional formulae; the *decision problem* for L is just the membership problem for L.

A classical modal logic L is said to be *normal* iff for every n-ary modal operator o in the signature of L and every $i = 1, \dots, n$, L contains the formula

$$o(\boldsymbol{x}, \top, \boldsymbol{x}')$$

and also the formula

$$o(\boldsymbol{x}, (y \Rightarrow z), \boldsymbol{x}') \Rightarrow (o(\boldsymbol{x}, y, \boldsymbol{x}') \Rightarrow o(\boldsymbol{x}, z, \boldsymbol{x}')).$$

where $\boldsymbol{x}$ abbreviates the tuple $(x_1, \dots, x_{i-1})$ and $\boldsymbol{x}'$ abbreviates $(x_{i+1}, \dots, x_n)$. The latter schema is called the "Aristotle law".[19] The least normal (classical modal, unary, unimodal) logic is the modal logic usually called **K** [CZ97].

Most well-known modal logics considered in the literature (both normal and non-normal) fit Definition 4.1: these include standard unary unimodal systems like **K**, **T**, **K4**, **S4**, **S5** and so on [CZ97], tense systems like $\mathbf{K}_t$ and other temporal logics [GHR94], the propositional dynamic logic *PDL* [Pra76], common knowledge systems [HM92], computational tree logic *CTL* [CE82],[20] and the propositional μ-calculus [Koz83]. Modal logics with so-called graded modalities [FBDC85, VdHdR95, Tob99] (which correspond to qualified number restrictions in Description Logics [HB91]) are examples of classical modal logics that are not normal [BLSW02].

Let us call an equational theory *Boolean-based* if its signature includes the signature Σ_{BA} of Boolean algebras and its axioms include the Boolean algebras axioms BA (see Example 2.8). For notational convenience, we will assume that Σ_{BA} also contains the binary symbol $\supset$, defined by the axiom

$$x \supset y \quad \approx \quad \overline{x} \cup y.$$

Given a classical modal logic L we can associate with it a Boolean-based equational theory E_L. Conversely, given a Boolean-based equational theory E we can associate with it a classical modal logic L_E. The constructions are the obvious ones and are recalled in the following.

[19] The axiom schema $o(\boldsymbol{x}, \top, \boldsymbol{x}')$ can be dropped in favor of the necessitation rule: 'from t infer $o(\boldsymbol{x}, t, \boldsymbol{x}')$'; in that case, thanks to the Aristotle laws, the replacement rules become redundant.

[20] On the other hand, the full computational tree logic CTL^* [EH86] is not a classical modal system in the sense of Definition 4.1, as it is not closed under uniform substitution.

Given a logic L with modal signature Σ_M, we define E_L as the theory having as signature $\Sigma_M \cup \Sigma_{BA}$ and as a set of axioms the set

$$BA \cup \{t_{BA} \approx 1 \mid t \in L\}$$

where t_{BA} is obtained from t by replacing t's logical connectives $(\neg, \wedge, \vee, \Rightarrow)$ by the corresponding Boolean algebra operators $(\overline{(\_)}, \cap, \cup, \supset)$,[21] and the logical constants $\top$ and $\bot$ by 1 and 0, respectively. Vice versa, given a Boolean-based equational theory E over the signature Σ, we define L_E as the classical modal logic over the modal signature $\Sigma \setminus \Sigma_{BA}$ axiomatized by the formulae

$$\{t_L \mid \models_E t \approx 1\}$$

where t_L is obtained from t by the inverse of the replacement process above.

Classical modal logics (in our sense) and Boolean-based equational theories are equivalent formalisms, as is well-known from algebraic logic [Ras74]. In particular, for our purposes, the following standard proposition is crucial, as it reduces the decision problem for a classical modal logic L to the word problem in E_L.

Proposition 4.2 *For every classical modal logic L and for every propositional formula t, we have that $t \in L$ iff $\models_{E_L} t_{BA} \approx 1$.*

Proof. The left-to-right direction is immediate from the definition of E_L.

For the opposite direction, we can use the well-known Lindenbaum algebra construction (see e.g. [Ras74]).[22] We define a model $\mathcal{A}_L$ of E_L as follows. Where Σ_L is the signature of L, the carrier of $\mathcal{A}_L$ is defined as the set of all the equivalence classes of Σ_L-formulae with respect to the equivalence relation[23]

$$\equiv \;\; := \;\; \{(t, u) \mid t \Leftrightarrow u \in L\}.$$

It easy to see that, since L is closed under the replacement rules, $\equiv$ is in fact a congruence relation with respect to the modal operators in Σ_L. We define these operators in $\mathcal{A}_L$ as prescribed by L, that is, we interpret each n-ary modal operator o as the n-ary function $o^{\mathcal{A}_L}$ such that

$$o^{\mathcal{A}_L}([t_1]_\equiv, \ldots, [t_n]_\equiv) \;=\; [o(t_1, \ldots, t_n)]_\equiv$$

We then define the Boolean operators in the obvious way, that is, we interpret $\cap$, say, as the binary function $\cap^{\mathcal{A}_L}$ such that $\cap^{\mathcal{A}_L}([t_1]_\equiv, [t_2]_\equiv) = [t_1 \wedge t_2]_\equiv$, and

[21] We can assume without loss of generality that t contains no occurrences of $\Leftrightarrow$, as that connective can be expressed in terms of $\Rightarrow$ and $\wedge$.

[22] Readers familiar with this construction will notice that the closure conditions required by Definition 4.1 are precisely the closure conditions that make the construction work.

[23] That $\equiv$ is in fact an equivalence relation follows from modus ponens and tautologies.

so on. It is a standard exercise to show that $\mathcal{A}_L$ is well-defined. From the closure of L under uniform substitution, we obtain for arbitrary terms t, u that $\mathcal{A}_L \models t_{BA} \approx u_{BA}$ iff $t \Leftrightarrow u \in L$; for $u = 1$, we also get (by modus ponens and tautologies) that $\mathcal{A}_L \models t_{BA} \approx 1$ iff $t \in L$—this shows, in particular, that $\mathcal{A}_L$ is a model of the equational theory E_L. Hence if $\models_{E_L} t_{BA} \approx 1$, we have in particular that $\mathcal{A}_L \models t_{BA} \approx 1$ and finally that $t \in L$, as claimed. □

Given two classical modal logics L_1 and L_2 over two *disjoint* modal signatures Σ_M^1 and Σ_M^2, the *fusion* of L_1 and L_2 is the classical modal logic

$$L_1 \oplus L_2$$

over the signature $\Sigma_M^1 \cup \Sigma_M^2$ defined as $[L_1 \cup L_2]$.[24] As $E_{L_1 \oplus L_2}$ is easily seen to be deductively equivalent to the theory $E_{L_1} \cup E_{L_2}$ (i.e., $\approx_{E_{L_1 \oplus L_2}} = \approx_{E_{L_1} \cup E_{L_2}}$), it is clear that the decision problem $L_1 \cup L_2 \vdash t$ reduces to the word problem $E_{L_1} \cup E_{L_2} \models t_{BA} \approx 1$. Our goal in the remainder of this section is to show that, thanks to the combination result in Theorem 3.6, this combined word problem for $E_{L_1} \cup E_{L_2}$ reduces to the single word problems for E_{L_1} and E_{L_2}, and thus to the decision problems for L_1 and L_2.

Note that, although the modal signatures Σ_M^1 and Σ_M^2 are disjoint, the signatures of E_{L_1} and E_{L_2} are no longer disjoint, because they share the Boolean operators. To show that our combination theorem applies to E_{L_1} and E_{L_2}, we thus must establish that the common subtheory BA of Boolean algebras matches the requirements for our combination procedure. To this end, we will restrict ourselves to component modal logics L_1 and L_2 that are *consistent*, that is, do not include $\bot$ (or, equivalently, do not contain all modal formulae). This is a sensible restriction because when either L_1 or L_2 are inconsistent $L_1 \oplus L_2$ is inconsistent as well, which means that its decision problem is trivial.

We have already shown in Section 2 that BA satisfies one of our requirements, namely effective local finiteness. As for the others, for every consistent classical modal logic L, the theory E_L is guaranteed to be a conservative extension of BA. The main reason is that there are no non-trivial equational extensions of the theory of Boolean algebras. In fact, as soon as one extends BA with an axiom $s \approx t$ for any s and t such that $s \not\approx_{BA} t$, the equation $0 \approx 1$ becomes valid.[25] By Proposition 4.2, this entails that if an equational theory E_L induced by a classical modal logic L is not a conservative extension of BA then $L \vdash \bot$. Hence L cannot be consistent.

In conclusion, all we need to show is that BA is Gaussian and that E_L is BA-compatible for every consistent L.

[24]In other words, $L_1 \oplus L_2$ is just the least classical modal logic extending $L_1 \cup L_2$.

[25]This is can be shown by a proper instantiation of the variables of $s \approx t$ by 0 and 1, followed by simple Boolean simplifications.

4.2 Boolean solved forms

Here we prove that the equational theory of Boolean algebras is Gaussian. Since we will make essential use of results from the Boolean unification literature, we prefer to switch temporarily to a Boolean ring notation, commonly adopted in that literature. It should be recalled anyway that Boolean algebras and Boolean rings are essentially the same theory, expressed in different signatures. The difference is merely a notational question, one can convert terms in the signature of Boolean algebras into terms in the signature of Boolean rings and vice versa, the conversion being bijective modulo the axioms of the respective theories.

The theory *BR* of Boolean rings is the theory in the signature $\Sigma_{BR} = \{+, *, 0, 1\}$ one of whose possible (equivalent) axiomatizations is the following:

$$\begin{array}{rclrcl} x * y & \approx & y * x, & x + y & \approx & y + x, \\ x * (y * z) & \approx & (x * y) * z, & x + (y + z) & \approx & (x + y) + z, \\ x * (y + z) & \approx & (x * y) + (x * z), & x * x & \approx & x, \\ x + x & \approx & 0, & x * 0 & \approx & 0, \\ x + 0 & \approx & x, & x * 1 & \approx & x. \end{array}$$

It is well-known that when working with e-formulae in the theory *BA*, it is enough to consider only e-formulae of the form $t \approx 1$. The reason is that for every e-formula φ of the form $s_1 \approx t_1 \wedge \cdots \wedge s_n \approx t_n$ in the signature of *BA* the following first-order equivalence holds:

$$\models_{BA} \ \varphi \ \Leftrightarrow \ ((s_1 \supset t_1) \cap (t_1 \supset s_1) \cap \cdots \cap (s_n \supset t_n) \cap (t_n \supset s_n)) \approx 1.$$

Note that the symbol $\Leftrightarrow$ here denotes bi-implication on the first order level; it should not be confused with bi-implication on the level of modal logics or of Boolean algebra terms.

In a similar way, when working with e-formulae in the theory *BR*, it is enough to consider only e-formulae of the form $t \approx 0$. The reason now is that, for every e-formula φ of the form $s_1 \approx t_1 \wedge \cdots \wedge s_n \approx t_n$ in the signature of *BR* the following equivalence holds:

$$\models_{BR} \ \varphi \ \Leftrightarrow \ (((s_1 + t_1 + 1) * \cdots * (s_n + t_n + 1)) + 1) \approx 0.$$

We show below that every formula of the form $t(\boldsymbol{x}, y) \approx 0$ can be effectively turned into the conjunction of a solvability condition $c(\boldsymbol{x}) \approx 0$ and of a local solver parametrization $\exists z.\ (y \approx s(\boldsymbol{x}, z))$. It will then follow immediately by Definition 2.9 that *BR* is Gaussian. As a consequence, we will also have that *BA* is Gaussian as well.

We will use the following general result, adapted from [MN89], on the computation of most general unifiers in the theory *BR*.

Proposition 4.3 *Let $t(\boldsymbol{c}, y) \approx 0$ be a BR-unification problem with (free) constants $\boldsymbol{c}$ and (only) variable y. For all unifiers $\{y \mapsto g(\boldsymbol{c})\}$ of $t(\boldsymbol{c}, y) \approx 0$ and fresh variables z, the substitution*

$$\{y \mapsto z + t(\boldsymbol{c}, z) * (z + g(\boldsymbol{c}))\}$$

is a most general BR-unifier of $t(\boldsymbol{c}, y) \approx 0$.

We will also need the next two lemmas.

Lemma 4.4 *Let $t(\boldsymbol{x}, y)$ be any Σ_{BR}-term and let $c(\boldsymbol{x}) = t(\boldsymbol{x}, 1) * t(\boldsymbol{x}, 0)$. Then,*

$$\models_{BR} \; c(\boldsymbol{x}) * (1 + t(\boldsymbol{x}, y)) \approx 0.$$

Proof. To prove the claim we can use the fact that the two-element Boolean ring $\mathcal{B}_2$, with carrier $\{0, 1\}$, generates the whole variety of Boolean rings.[26] Then, it is enough to check that $c(\boldsymbol{x}) * (1 + t(\boldsymbol{x}, y))$ evaluates to 0 for every assignment V of the variables $y, \boldsymbol{x}$ into $\{0, 1\}$.

Let V be such an assignment and for every term u let $V[u]$ be the value denoted by u in $\mathcal{B}_2$ under the assignment V. If $V[t(\boldsymbol{x}, y)] = 1$, the claim follows immediately from the axioms of *BR*. If instead $V[t(\boldsymbol{x}, y)] = 0$, depending on whether $V[y] = 1$ or $V[y] = 0$, we have also $V[t(\boldsymbol{x}, 1)] = 0$ or $V[t(\boldsymbol{x}, 0)] = 0$ and in any case $V[c(\boldsymbol{x})] = 0$. □

Lemma 4.5 *Let $t(\boldsymbol{x}, y)$ be a Σ_{BR}-term and let $c(\boldsymbol{x}) = t(\boldsymbol{x}, 1) * t(\boldsymbol{x}, 0)$. The substitution $\sigma := \{y \mapsto 1 + t(\boldsymbol{x}, 1)\}$ is a BR-unifier of the unification problem*

$$t(\boldsymbol{x}, y) * (1 + c(\boldsymbol{x})) \approx 0$$

in which the elements of $\boldsymbol{x}$ are treated as (free) constants and y is the only variable.

Proof. For notational convenience, let us denote the term $t\sigma$ obtained by applying the substitution σ to t by $t(\boldsymbol{x}, 1 + t(\boldsymbol{x}, 1))$. Let $\mathcal{B}_2$ be again the two-element Boolean ring with carrier $\{0, 1\}$ as in the proof of Lemma 4.4. It is enough to show that the term

$$u \;\; = \;\; t(\boldsymbol{x}, 1 + t(\boldsymbol{x}, 1)) * (1 + c(\boldsymbol{x}))$$

evaluates to 0 for every assignment of the variables $y, \boldsymbol{x}$ into $\{0, 1\}$.

[26] This means that an identity is entailed by *BR* iff it is satisfied by $\mathcal{B}_2$. This may be seen as a consequence e.g. of Stone representation theorem [BD74], saying that any Boolean ring embeds into a cartesian power of $\mathcal{B}_2$.

Let V be such an assignment. If $V[c(\boldsymbol{x})] = 1$, the whole term u trivially evaluates to 0. Therefore, suppose that $V[c(\boldsymbol{x})] = 0$. Then it is enough to show that $V[t(\boldsymbol{x}, 1 + t(\boldsymbol{x}, 1))] = 0$. Since $V[c(\boldsymbol{x})] = 0$, from the definition of $c(\boldsymbol{x})$, it must be that either (i) $V[t(\boldsymbol{x}, 1)] = 0$ or (ii) $V[t(\boldsymbol{x}, 1)] = 1$ and $V[t(\boldsymbol{x}, 0)] = 0$. In the first case, we get that $V[t(\boldsymbol{x}, 1 + t(\boldsymbol{x}, 1))] = V[t(\boldsymbol{x}, 1)] = 0$. In the second case, we get that $V[t(\boldsymbol{x}, 1 + t(\boldsymbol{x}, 1))] = V[t(\boldsymbol{x}, 0)] = 0$. □

We are now ready to prove the existence (and computability) of solvability conditions and local solvers in *BR* for all e-formulae of the form $t(\boldsymbol{x}, y) \approx 0$.

Proposition 4.6 *For every Σ_{BR}-term $t(\boldsymbol{x}, y)$, there exist Σ_{BR}-terms $c(\boldsymbol{x})$ and $s(\boldsymbol{x}, z)$, computable from t in linear time, such that*

$$\models_{BR} \; t(\boldsymbol{x}, y) \approx 0 \;\Leftrightarrow\; (c(\boldsymbol{x}) \approx 0 \wedge \exists z.\ (y \approx s(\boldsymbol{x}, z))).$$

Proof. Let

$$c(\boldsymbol{x}) = t(\boldsymbol{x}, 1) * t(\boldsymbol{x}, 0) \tag{6}$$

as in Lemmas 4.4 and 4.5. We show that we can define a local solver $s(\boldsymbol{x}, z)$ for $t(\boldsymbol{x}, y) \approx 0$ based on the solvability condition $c(\boldsymbol{x}) \approx 0$.

By Lemma 4.5, the substitution $\{y \mapsto 1+t(\boldsymbol{x}, 1)\}$ is a *BR*-unifier of the unification problem

$$t(\boldsymbol{x}, y) * (1 + c(\boldsymbol{x})) \approx 0 \tag{7}$$

By Proposition 4.3 then, where z is a fresh variable and

$$s(\boldsymbol{x}, z) \;=\; z + t(\boldsymbol{x}, z) * (1 + c(\boldsymbol{x})) * (z + 1 + t(\boldsymbol{x}, 1)), \tag{8}$$

the substitution $\{y \mapsto s(\boldsymbol{x}, z)\}$ is a most general *BR*-unifier of (7), which means in particular that $s(\boldsymbol{x}, z)$ is a solution of (7), i.e.,

$$\models_{BR} \; t(\boldsymbol{x}, s(\boldsymbol{x}, z)) * (1 + c(\boldsymbol{x})) \approx 0. \tag{9}$$

We use (9) to show that

(i) $t(\boldsymbol{x}, y) \approx 0 \;\models_{BR}\; c(\boldsymbol{x}) \approx 0 \wedge \exists z.\ (y \approx s(\boldsymbol{x}, z))$ and

(ii) $c(\boldsymbol{x}) \approx 0 \wedge \exists z.\ (y \approx s(\boldsymbol{x}, z)) \;\models_{BR}\; t(\boldsymbol{x}, y) \approx 0$,

from which the lemma's equivalence immediately follows.

(i) Let $\mathcal{B}$ be any model of *BR* and V any assignment of the variables $\boldsymbol{x}, y$ into $\mathcal{B}$ such that $V[t(\boldsymbol{x}, y)] = 0$.[27] Then extend V to z by letting $V[z] = V[y]$. From

[27]By a slight abuse of notation we denote $0^{\mathcal{B}}$ by 0.

Lemma 4.4 (and the axioms of BR) we can deduce that $V[c(\boldsymbol{x})] = 0$ and that

$$\begin{aligned} V[s(\boldsymbol{x}, z)] &= V[s(\boldsymbol{x}, y)] \\ &= V[y + t(\boldsymbol{x}, y) * (1 + c(\boldsymbol{x})) * (y + 1 + t(\boldsymbol{x}, 1))] \\ &= V[y + 0 * (1 + c(\boldsymbol{x})) * (y + 1 + t(\boldsymbol{x}, 1))] \\ &= V[y + 0] = V[y]. \end{aligned}$$

It follows that $\mathcal{B}$ satisfies $c(\boldsymbol{x}) \approx 0 \wedge \exists z.\ (y \approx s(\boldsymbol{x}, z))$ under the assignment V, which proves claim (i).

(ii) Let $\mathcal{B}$ be any model of BR and V any assignment of $\boldsymbol{x}, y$ into $\mathcal{B}$ such that $\mathcal{B}$ satisfies $c(\boldsymbol{x}) \approx 0 \wedge \exists z.\ (y \approx s(\boldsymbol{x}, z))$. Clearly, it is possible to extend V to z so that $V[c(\boldsymbol{x})] = 0$ and $V[y] = V[s(\boldsymbol{x}, z)]$. Together with (9), we then have that

$$\begin{aligned} V[t(\boldsymbol{x}, y)] &= V[t(\boldsymbol{x}, s(\boldsymbol{x}, z))] \\ &= V[t(\boldsymbol{x}, s(\boldsymbol{x}, z)) * (1 + 0)] \\ &= V[t(\boldsymbol{x}, s(\boldsymbol{x}, z)) * (1 + c(\boldsymbol{x}))] = 0. \end{aligned}$$

It follows that $\mathcal{B}$ satisfies $t(\boldsymbol{x}, y) \approx 0$ under the assignment V, which proves claim (ii).

To conclude the proof, we would need to show that $c(\boldsymbol{x})$ and $s(\boldsymbol{x}, y)$ are computable in linear time from $t(\boldsymbol{x}, y) \approx 0$. This however is immediate from the explicit definitions we have provided for them here. □

Strictly speaking, the result above proves that the theory BR of Boolean rings, not the theory BA of Boolean algebras, is Gaussian. However, given an e-formula $u(\boldsymbol{x}, y) \approx 1$ in the signature Σ_{BA}, one can translate it into a corresponding formula $t(\boldsymbol{x}, y) \approx 0$, compute a satisfiability condition and local solver for $t(\boldsymbol{x}, y) \approx 0$ in BR, and translate those back into a satisfiability condition and local solver for $u(\boldsymbol{x}, y) \approx 1$. Since both translation processes are clearly effective, it follows that, with the possible exception of the linear complexity claim, a result like that in Proposition 4.6 holds for BA as well. If follows that the theory BA of Boolean algebras is Gaussian.

Furthermore, the computational complexity of computing local solvers in BA is indeed linear. This is thanks to the fact that local solvers in BA can be computed directly, without a translation into the signature of BR. In fact, for each e-formula $u(\boldsymbol{x}, y) \approx 1$ (and fresh variable z), the term

$$s'(\boldsymbol{x}, z) = (u(\boldsymbol{x}, 1) \supset u(\boldsymbol{x}, z)) \supset (z \cap (u(\boldsymbol{x}, 0) \supset u(\boldsymbol{x}, z))) \tag{10}$$

is a local solver for $u(\boldsymbol{x}, y) \approx 1$ in BA w.r.t. y. It is immediate that $s'(\boldsymbol{x}, z)$ can be computed in linear time from $u(\boldsymbol{x}, y)$. To see that it is indeed a local solver of $u(\boldsymbol{x}, y)$, one can argue as follows. From formulas (8) and (6), we have that

$$s(\boldsymbol{x}, z) = z + t(\boldsymbol{x}, z) * (1 + t(\boldsymbol{x}, 1) * t(\boldsymbol{x}, 0)) * (z + 1 + t(\boldsymbol{x}, 1)) \tag{11}$$

is a local solver of the formula $t(\boldsymbol{x}, y) \approx 0$ for any Σ_{BR}-term $t(\boldsymbol{x}, y)$. Observing that $u \approx 1$ is equivalent in BA to $\overline{u} \approx 0$, let $t(\boldsymbol{x}, y)$ be the translation of $\overline{u}$ into the signature of BR.[28] Then, modulo the signature translation, t is equivalent to $\overline{u}$. Let u_z, u_0, u_1 abbreviate respectively $u(\boldsymbol{x}, z), u(\boldsymbol{x}, 0), u(\boldsymbol{x}, 1)$. If we replace every occurrence of $t(\boldsymbol{x}, z)$, $t(\boldsymbol{x}, 0)$, $t(\boldsymbol{x}, 1)$ in (11) by $\overline{u}_z$, $\overline{u}_0$, $\overline{u}_1$, respectively, and translate the formula (10) into the the signature of BR, we obtain a formula that is equivalent in BR to (11). To see that, consider the following chains of equalities modulo the signature translation and the axioms of BA and BR.[29]

$$\begin{array}{rcl}
s'(\boldsymbol{x}, z) & \approx & (u_1 \supset u_z) \supset (z \cap (u_0 \supset u_z)) \\
& \approx & \overline{u_1 \supset u_z} \cup (z \cap (u_0 \supset u_z)) \\
& \approx & (u_1 \cap \overline{u}_z) \cup (z \cap (\overline{u}_0 \cup u_z)) \\
& \approx & (u_1\overline{u}_z) \cup (z(\overline{u}_0 + u_z + \overline{u}_0 u_z)) \\
& \approx & (u_1\overline{u}_z) \cup (z(1 + u_0 + u_0 u_z)) \\
& \approx & (u_1\overline{u}_z) \cup (z + u_0 z + u_0 u_z z) \\
& \approx & u_1\overline{u}_z + z + u_0 z + u_0 u_z z + u_1\overline{u}_z z + u_1\overline{u}_z u_0 z + u_1\overline{u}_z u_0 u_z z \\
& \approx & u_1\overline{u}_z + z + u_0 z + u_0 u_z z + u_1\overline{u}_z z + u_0 u_1\overline{u}_z z \\
& \approx & u_1 + u_1 u_z + z + u_0 z + u_0 u_z z + u_1 z + u_1 u_z z + u_0 u_1 z + u_0 u_1 u_z z
\end{array}$$

$$\begin{array}{rcl}
s(\boldsymbol{x}, z) & \approx & z + t(\boldsymbol{x}, z)(1 + t(\boldsymbol{x}, 1)t(\boldsymbol{x}, 0))(z + 1 + t(\boldsymbol{x}, 1)) \\
& \approx & z + \overline{u}_z(1 + \overline{u}_1\overline{u}_0)(z + u_1) \\
& \approx & z + \overline{u}_z(u_1 + u_0 + u_0 u_1)(z + u_1) \\
& \approx & z + \overline{u}_z(u_1 z + u_0 z + u_0 u_1 z + u_1 + u_0 u_1 + u_0 u_1) \\
& \approx & z + (1 + u_z)(u_1 z + u_0 z + u_0 u_1 z + u_1) \\
& \approx & z + u_1 z + u_0 z + u_0 u_1 z + u_1 + u_1 u_z z + u_0 u_z z + u_0 u_1 u_z z + u_1 u_z
\end{array}$$

It is easy to verify at this point that both s and s' reduce to the same Σ_{BR}-term, hence they are equivalent.

4.3 Fusion of modal logics

To apply our combination procedure to the case of fusions of modal logics we still have to show compatibility of Boolean-based equational theories with respect to the theory of Boolean algebras.

Proposition 4.7 *For every classical modal logic L, the equational theory E_L is BA-compatible, where BA is the theory of Boolean algebras.*

[28] This translation can be achieved by the rewrite rules $\overline{x} \rightarrow x + 1$, $x \cap y \rightarrow x * y$, and $x \cup y \rightarrow x + y + x * y$.

[29] To simplify the notation, we omit writing the operator $*$ explicitly, and use the standard precedence rules for $*$ and $+$.

Proof. This is actually a well known fact. There are at least two proofs of it, an algebraic proof from [Ghi03] and a logically-oriented proof that can be adapted from [Wol98]. For the sake of completeness, we report here the latter.

Recall (e.g., from [CK90, GZ02]) that BA admits as a model completion the theory of atomless Boolean algebras (a Boolean algebra $\mathcal{B}$ is said to be atomless iff it does not have atoms, where an atom is a nonzero element $a \in \mathcal{B}$ such that for all $b \in \mathcal{B}$ we have either $a \leq b$ or $a \leq b'$).[30] So we simply need to embed any E_L-algebra $\mathcal{A}$ into an E_L-algebra $\mathcal{B}$ whose Boolean reduct is atomless. Let Σ' be the signature $\Sigma_L \cup A \cup D$, where D is an infinite set (disjoint from A) and let E' be the Σ'-theory obtained from E_L by adding to it, as new ground equations, all positive literals from $\Delta^{\Sigma_L}(\mathcal{A})$, the Σ_L-diagram of $\mathcal{A}$. Consider the initial E'-algebra $\mathcal{B}$ over the signature Σ'.

Recall that, as $\mathcal{B}$ is initial, for every pair of ground terms s, t over the signature Σ', we have that $s^{\mathcal{B}} = t^{\mathcal{B}}$ iff $E' \models s \approx t$. By the definition of E', we have $E' \models s \approx t$ iff there are identities $u_1 \approx v_1, \ldots, u_n \approx v_n$ in the diagram of $\mathcal{A}$ such that the conditional identity

$$u_1 \approx v_1, \ldots, u_n \approx v_n \Rightarrow t \approx s$$

is a logical consequence of E_L. This shows in particular that in $\mathcal{B}$ exactly the $(\Sigma_L \cup A)$-ground identities that are in the diagram of $\mathcal{A}$ are true (and not more), i.e., $\mathcal{B}$ satisfies $\Delta^{\Sigma_L}(\mathcal{A})$. By Lemma 2.1, we consequently have that $\mathcal{A}$ $\Sigma_L(A)$-embeds into $\mathcal{B}$. Thus, we only need to prove that the Boolean reduct of $\mathcal{B}$ is atomless.

Take a candidate atom a in $\mathcal{B}$; clearly $a = t^{\mathcal{B}}$ for some ground term $t \in T(\Sigma_L, A \cup D)$. Pick d from D which does not occur in t (this is possible as D is infinite). For $a = t^{\mathcal{B}}$ to be an atom we must have in $\mathcal{B}$ either $t^{\mathcal{B}} \leq d^{\mathcal{B}}$ or $t^{\mathcal{B}} \leq (d^{\mathcal{B}})'$, but in both cases this yields $t^{\mathcal{B}} = 0$. In fact, in the former case,[31] we have

$$E_L \models u_1 \approx v_1, \ldots, u_n \approx v_n \Rightarrow t \approx t \cap d,$$

for some identities $u_i \approx v_i$ $(i = 1, \ldots, n)$ belonging to the diagram of $\mathcal{A}$. This means that d does not occur in them, so that if we replace d by 0 in the above conditional identity, we get (as $E_L \models t \cap 0 \approx 0$)

$$E_L \models u_1 \approx v_1, \ldots, u_n \approx v_n \Rightarrow t \approx 0$$

proving that in fact $a = t^B = 0$ is not an atom. □

From the results we collected so far, we can immediately conclude that:

Theorem 4.8 *If L_1, L_2 are decidable classical modal logics, so is their fusion $L_1 \oplus L_2$.*

[30]Here b' is the Boolean complement of b; recall also that $a \leq b$ abbreviates the equation $a \cap b = a$.

[31]The latter case is analogous, just use "replace d by 1" in the argument below.

4.4 Complexity issues

The complexity of our combination procedure applied to fusion decidability in modal logic is the same as the complexity of the combination procedures proposed for the classical *normal* modal logics case in [Wol98] and for the classical modal logics with *covering normal terms* in [BLSW02]. In fact, the same remarks as in [BLSW02] apply, as we shall see below.

To begin with, let us recall that

- the preprocessing abstraction procedure[32] takes only linear time;
- the computation of a local solver takes also linear time—although it might be applied to an exponentially long formula, as we will see;
- only linearly many iterations of our procedure's loop (see Fig. 1) need to be executed on any input.

Consequently, the only sources of real complexity in the whole procedure are the tests of Step 3 of the loop (the final test, after the loop, is of the same nature). Hence we have to analyze:

- how many such tests are performed;
- how expensive each of them is.

Suppose that n is the number of the free constants in the procedure's input—the initial ARS R and the shared terms u and v. This number is obviously linear in the size of the input. Let us assume for simplicity that the only free constants in the input are the lhs constants in R: $a_1, \dots, a_n$.[33] Now, as we discussed in Example 2.8, the number of non-equivalent Boolean terms over n constants is 2^{2^n}, hence one might conclude that during the i^{th} iteration of the procedure's loop we will need to execute $O(2^{2^i} \cdot 2^{2^i})$ equivalence tests in Step 3 of the loop. Instead, we can limit ourselves to 2^i tests for the following reason.

Recall that the e-formula φ built at Step 4 of the loop is equivalent in the shared theory BA to an identity of the form $t \approx 1$, where t is a Boolean term. Again as discussed in Example 2.8, this term is in turn equivalent in BA to a term of the form $t_1 \cap \cdots \cap t_m$, where each t_k is a *term-clause*, i.e., a term of the form $b_1 \cup \cdots \cup b_i$ where each b_j is either a_j or $\overline{a}_j$. Now, it is an immediate consequence of BA that

$$\models_{BA} (t_1 \cap \cdots \cap t_m) \approx 1 \quad \text{iff} \quad \models_{BA} t_k \approx 1 \text{ for all } k = 1, \dots, m.$$

[32]The one that converts a formula in the signature of the fusion logic into an initial ARS and two boolean terms u and v.

[33]The complexity analysis does not change if we ignore other possible free constants.

It follows that to generate φ it is enough to consider in the test of Step 3 only identities of the form $t \approx 1$ where t is a term-clause over $a_1, \ldots, a_i$. And we already know that, modulo BA, there are only 2^i such identities. As an additional consequence of the above we have that the size of the e-formula φ is linear in 2^i, which in turn means that the local solver computed in Step 6 of the loop is also linear in 2^i, and so exponential in the size of the input.

Let us now consider the cost of the test $R_j^{(i)} \models_{E_{L_j}} t \approx 1$, where t is any term-clause. This test requires $R_j^{(i)}$-normalization first and then a call to the decision procedure for the input logic L_j. In the worst case $R_j^{(i)}$ is of the form $\{a_1 \to t_1, \ldots, a_i \to t_i\}$ with each right-hand side term being a recursively computed, exponentially long solver. Normalizing the term t with respect to $R_j^{(i)}$ can then raise the length of t from linear to $2^{q(n)}$, where $q(n)$ is a quadratic polynomial. To see this it is helpful to observe that, the way $R_j^{(i)}$ is defined, normalizing t amounts to first replacing every occurrence of a_1 in t by t_1, then replacing every occurrence of a_2 in the resulting term by t_2 and so on. Now let us first consider how the size of the terms $t_1, \ldots, t_i$ grows when we apply the rewrite system to them. First of all, t_1 is irreducible, and so it does not change in size, i.e., its size after rewriting is still $O(2^n)$. The term t_2 is of size $O(2^n)$ and thus may contain at most $O(2^n)$ occurrences of a_1. Thus, by rewriting, its size may grow to $O(2^n + 2^n \cdot 2^n) = O(2^{2n})$. The term t_3 is of size $O(2^n)$ and thus may contain at most $O(2^n)$ occurrences of a_1, a_2. Considering the worst-case that all of them are occurrences of a_2, its size may grow to $O(2^n + 2^n \cdot 2^{2n}) = O(2^{3n})$. If we continue this argument until we reach t_n, we see that indeed t_n may grow by rewriting to size $O(2^{(n^2)})$. Since the size of the term t is linear in n,[34] its size may grow by rewriting (where in the worst case we replace $O(n)$ constants by terms of size $O(2^{(n^2)})$) to size $O(2^{(n^2+1)})$.

In conclusion, the decision procedures for L_1 and for L_2 may have to deal with exponentially many, exponentially long instances of the decision problem in each of the linearly many iterations of the loop. If these procedures are in PSPACE, we get an EXPSPACE combined decision procedure. If instead the procedures are in EXPTIME, we get a 2EXPTIME combined decision procedure. This is the same as the complexity bound given in [BLSW02] for their combination procedure.

4.5 Examples

Here we give two examples of our combination procedure at work in the case of classical modal logics.

Example 4.9 Consider the classical modal logic **KT** with modal signature $\{\Box\}$

[34] Recall that t is a term clause over $\{a_1, \ldots, a_i\}$.

and obtained by adding to **K** the axiom schema

$$\Box x \Rightarrow x.$$

Now let $\mathbf{KT}_1$ and $\mathbf{KT}_2$ be two signature disjoint renamings of **KT** in which $\Box_1$ and $\Box_2$, respectively, replace $\Box$, and consider the fusion logic $\mathbf{KT}_1 \oplus \mathbf{KT}_2$. We can use our combination procedure to show that

$$\mathbf{KT}_1 \oplus \mathbf{KT}_2 \;\vdash\; \Box_2 x \Rightarrow \Diamond_1 x$$

(where as usual $\Diamond_1 x$ abbreviates $\neg \Box_1 \neg x$).
For $i = 1, 2$, let E_i be the equational theory corresponding to $\mathbf{KT}_i$. It is enough to show that

$$\models_{E_1 \cup E_2} (\Box_2(x) \supset \Diamond_1(x)) \approx 1 \tag{12}$$

where now $\Diamond_1 x$ abbreviates $\overline{\Box_1(\overline{x})}$.
After the abstraction process, we get the two rewrite systems:

$$\begin{aligned} R_1 &= \{a_1 \to \Diamond_1(c)\}, \\ R_2 &= \{a_2 \to \Box_2(c)\} \end{aligned}$$

and the goal equation

$$(a_2 \supset a_1) \approx 1.$$

where a_1, a_2 and c are fresh constants.
Recall from our discussion in Section 4.4 that for the test in Step 3 of the procedure's loop we need to consider only identities of the form $t \approx 1$ where t is a term-clause over the set of constants under consideration. During the first execution of the procedure's loop the constants in question are a_1 and c, therefore there are only four identities to consider:

$$\overline{a}_1 \cup \overline{c} \approx 1,\ \overline{a}_1 \cup c \approx 1,\ a_1 \cup \overline{c} \approx 1, \text{ and } a_1 \cup c \approx 1.$$

The only identity for which the test is positive is $a_1 \cup \overline{c}$. In fact, $a_1 \cup \overline{c}$ rewrites to $\Diamond_1(c) \cup \overline{c}$, which is equivalent to $c \supset \Diamond_1(c)$. This is basically the contrapositive of (the translation of) the axiom schema $\Box_1(c) \supset c$.[35]
Using the formula

$$s(\boldsymbol{x}, z) \;=\; (u(\boldsymbol{x}, 1) \supset u(\boldsymbol{x}, z)) \supset (z \cap (u(\boldsymbol{x}, 0) \supset u(\boldsymbol{x}, z))) \tag{13}$$

from Subsection 4.2, we can produce a solver for that identity, which reduces to $c \cup d_1$ after some simplifications, where d_1 is a fresh free constant. Hence, the following rewrite rule is added to R_2 in Step 6 of the loop:

$$a_1 \to c \cup d_1.$$

[35] Another approach for checking this, and also that the tests for the other term-clauses are negative, is to translate the rewritten term-clauses into the corresponding modal formulae, and then check whether their complement is unsatisfiable in all Kripke structures with a reflexive accessibility relation (see [Che80], Fig. 5.1).

Note that at this time we could already quit the loop and provide an output using R_2 and E_2 in the final test instead of R_1 and E_1.[36] If we did that, the final test $R_2 \models_{E_2} (a_2 \supset a_1) \approx 1$ (that is, $\models_{E_2} \Box_2(c) \supset (c \cup d_1) \approx 1$) would succeed because the corresponding modal formula

$$\Box_2\, c \Rightarrow (c \vee d_1)$$

is in fact a theorem of $\mathbf{KT}_2$.
Continuing the execution of the loop with the second—and final—iteration, we get instead the following. Among the eight term-clauses involving a_1, a_2, c, the test in Step 3 is positive for four of them. The conjunction of such term-clauses gives a Boolean e-formula that is equivalent to $(a_2 \supset c) \cap (c \supset a_1) \approx 1$. This e-formula, once solved with respect to a_2, gives (after simplifications) the rewrite rule

$$a_2 \rightarrow d_2 \cap ((c \supset a_1) \supset (d_2 \supset c)).$$

which is added to R_1 before quitting the loop. Using this R_1, the final test of the procedure ($R_1 \models_{E_1} a_2 \supset a_1 \approx 1$) succeeds because the modal formula

$$d_2 \wedge ((c \Rightarrow \Diamond_1 c) \Rightarrow (d_2 \Rightarrow c)) \Rightarrow \Diamond_1 c$$

is a theorem of $\mathbf{KT}_1$.

Example 4.10 Here we consider the fusion $\mathbf{R} \oplus \mathbf{KTB}$, where $\mathbf{KTB}$ is the classical modal logic obtained by adding to $\mathbf{KT}$ the axiom schema

$$\Diamond \Box x \Rightarrow x$$

and $\mathbf{R}$ is obtained from the minimum classical unimodal system $\mathbf{E}$, with modal operator $\bigcirc$, by adding to it the regularity rule:[37]

$$\frac{t \Rightarrow u}{\bigcirc t \Rightarrow \bigcirc u.}$$

Note that $\mathbf{R}$ is classical, but not normal. We can apply our combined procedure to show that

$$\mathbf{R} \oplus \mathbf{KTB} \ \vdash \ \Diamond \Box \bigcirc x \Rightarrow \bigcirc \Diamond x.$$

After purification, we get the ARS

$$\begin{array}{rcl} R_1 & = & \{a_4 \rightarrow \bigcirc a_1,\ a_2 \rightarrow \bigcirc c\}, \\ R_2 & = & \{a_1 \rightarrow \Diamond c,\ a_3 \rightarrow \Diamond \Box a_2\}. \end{array}$$

[36] Recall that it is immaterial whether R_1 and E_1 or R_2 and E_2 are used for the final test.

[37] Instead of the regularity rule, one may equivalently use the axiom schema $\bigcirc(t \wedge u) \Rightarrow \bigcirc u$ to get the logic $\mathbf{R}$ (see [Seg71], page 45).

and the goal identity

$$(a_3 \supset a_4) \approx 1.$$

In the first iteration of the loop, we test the term-clauses over a_1, c,[38] and get $(a_1 \cup \overline{c}) \approx 1$ as the e-formula to be solved with respect to a_1. As in the first step of the previous example, the solver (after simplifications) gives the rewrite rule $a_1 \to (c \cup d_1)$.
In the second iteration, nothing relevant happens because the e-formula to be solved with respect to a_2 is equivalent to an e-formula (namely $(a_1 \cup \overline{c}) \approx 1$ again) in which a_2 does not occur. This entails that using (13) to compute the local solver yields the trivial rewrite rule $a_2 \to d_2$ for some fresh constant d_2. In the third iteration, term-clauses involving a_1, a_2, a_3, c are tested;[39] this results in an e-formula equivalent to $(a_3 \supset a_2) \cap (c \supset a_1) \approx 1$. Solving it with respect to a_3 gives (after simplifications) the rule $a_3 \to d_3 \cap ((c \supset a_1) \supset (d_3 \supset a_2))$.
We can ignore the last iteration of the loop because it modifies R_2, which is not used afterwards. Performing the final test using R_1, the modal formula to be tested for validity in **R** is then

$$(d_3 \wedge ((c \Rightarrow (c \vee d_1)) \Rightarrow (d_3 \Rightarrow \bigcirc c))) \Rightarrow \bigcirc(c \vee d_1).$$

This formula is indeed valid in **R**. To see that, first notice that the subformula $c \Rightarrow (c \vee d_1)$ is a tautology. Therefore it is enough to show the validity of

$$(d_3 \wedge (d_3 \Rightarrow \bigcirc c)) \Rightarrow \bigcirc(c \vee d_1).$$

This follows from the transitivity of implication, because $(d_3 \wedge (d_3 \Rightarrow \bigcirc c)) \Rightarrow \bigcirc c$ and $\bigcirc c \Rightarrow \bigcirc(c \vee d_1)$ are both valid in **R** (for the latter, apply the regularity rule to the tautology $c \Rightarrow (c \vee d_1)$).
As a final remark observe that if we replace in the example the logic **R** by the logic **E**, the execution of the procedure is the same but the final test is negative. To get a falsifying model for the modal propositional formula in the final test, it is sufficient to observe that any Boolean algebra in which the operator $\bigcirc$ is interpreted as the Boolean complement is a model of **E**.[40]

5 Conclusion

In this paper, we have described a new approach for combining decision procedures for the word problem in equational theories over *non-disjoint* signatures.

[38]For instance by checking the complement of the modal formulae obtained after rewriting for unsatisfiability in all Kripke structures with a reflexive and symmetric accessibility relation (see again [Che80], Fig. 5.1).

[39]For instance by checking the complement of the modal formulae obtained after rewriting for unsatisfiability in all neighborhood frames where the set of sets of worlds associated with each world is closed under supersets (see, e.g., [Seg71], page 43.)

[40]It goes without saying that these are not models for **R**, as they violate the regularity rule.

Unlike the previous combination methods for the word problem [BT02, FG03] in the non-disjoint case, this approach has the known decidability transfer results for *satisfiability* in the fusion of modal logics [KW91, Wol98] as consequences. Our combination result is however more general than these transfer results since it applies also to *non-normal* modal logics—thus answering in the affirmative a long-standing open question in modal logics—and to equational theories not induced by modal logics (see, e.g., Example 2.10). Nevertheless, for the modal logic application, the complexity upper-bounds obtained through our approach are the same as for the more restricted approaches [Wol98, BLSW02].

Our results are not consequences of combination results for the conditional word problem (the relativized satisfiability problem) recently obtained by generalizing the Nelson-Oppen combination method [Ghi03, GS03]. In fact, there are modal logics (obtained by translating certain description logics into modal logic notation) for which the satisfiability problem is decidable, but the relativized satisfiability problem is not. This is, e.g, the case for description logics with feature agreements [BBN+93] or with concrete domains [BH92].

Our new combination approach is orthogonal to the previous combination approaches for the word problem in equational theories over non-disjoint signature [BT02, FG03]. On the one hand, the previous results do not apply to theories induced by modal logics [FG03]. On the other hand, there are equational theories satisfying the restrictions imposed by the previous approaches, but they are not locally finite [BT02], and thus do not satisfy our restrictions. Both the approach described in this paper and those in [BT02, FG03] have the combination results for the case of disjoint signatures as a consequence. For the previous approaches, this was already pointed out in [BT02, FG03]. For our approach, the reasons are those given at the end of Section 3.

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