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Completeness of E-unification with eager Variable Elimination

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Abstract. The paper contains a proof of completeness of a goal-directed inference system for general E-unification with eager Variable Elimination. The proof is based on an analysis of a concept of ground, equational proof. The theory of equational proofs is developed in the first part. Solving variables in a goal is then shown to be reflected in defined transformations of an equational proof. The termination of these transformations proves termination of inferences with eager Variable Elimination.

1 Introduction

E-unification is concerned with finding a set of solutions for a given equation in a given equational theory E. The problem of E-unification arises in many areas of computer science like formal verification, theorem proving and logic programming. In general the E-unification problem, i.e. the problem of finding a set of solutions for a given equation in a non-empty equational theory E is undecidable, unlike in the case of the syntactic unification problem, i.e. in the case of searching for a solution for an equation in the context of the empty equational theory. Nevertheless, the E-unification problem is semi-decidable and there are complete algorithms for solving it.

Goal-directed algorithms for E-unification are based on the idea of transforming goal equations into a solved form which will allow easily to define a solution. Such an inference system was presented first in [2], and is displayed here in a different notation in Figure 1. Consider the rule Variable Elimination in this set of inference rules. If applied to an equation of the form $x \approx v$ in the goal, it will eliminate x from all other equations in the goal and thus solve the equation $x \approx v$.[1] The Variable Elimination is forced to be applied eagerly here, because there is no other rule to deal with equations of the form $x \approx v$, where x is not a variable in v.

There was no proof up to now that this system of inferences is complete for E-unification. It is complete, when we allow other rules to apply to an equation $x \approx v$, but then Variable Elimination cannot be applied eagerly. The problem

[1] Formal definition of a solved equation is in the section 6.

was first discovered and called *the Eager Variable Elimination Problem* by Gallier and Snyder in [2].

Eager Variable Elimination is justified in the context of syntactic unification because it decreases the number of unsolved variables in the goal, while preserving a set of solutions. The number of unsolved variables is not increased by any other rule and hence we may be sure that the inferences will terminate.

In the context of E-unification we must use another rule called Mutate.[2] Notice that we have here conflicting results of applications of Mutate and Variable Elimination to the goal: Variable Elimination decreases the number of unsolved variables in the goal, but Mutate increases this number, and while Mutate decreases the length of a ground proof of an instance of a goal, Variable Elimination may increase this length.

In [3] Gallier and Snyder proved completeness of their system without eager Variable Elimination. In [5] (p. 207) the authors stated that Mutate (*replacement*) and eager Variable Elimination (*merging*) do not preserve the form of the proof.

In this paper we prove that Variable Elimination may be applied eagerly without destroying the completeness of the E-unification procedure. The fact that Variable Elimination can be applied eagerly decreases non-determinism in the inherently non-deterministic general E-unification algorithms. It may reduce redundancy of inferences and limit the search space for a solution to a given equation. This was pointed out e.g. in [7], [6], [4].

The main idea in the proof of completeness of our inference rules (Figure 1), is to consider a ground equational proof for a goal. If $x \approx v$ is part of this goal, we know that there must be subproof for a ground instance of this equation. We then can discover how the proof of the ground instance of a goal is changed, when Variable Elimination is applied to $x \approx v$. The main problem is to show that eliminating variables from the goal will not lead us into infinite loops of inferences. Hence we must show, what is decreasing in the ground proof after a variable is solved. Here we use the idea of *a path* in a proof, i.e. any composition of subproofs in the ground proof, which starts with an occurrence of an unsolved variable. We show that the multiset of lengths of the paths in a ground proof of an instance of a goal will be smaller after Variable Elimination is applied.

Most of the paper is concerned with a description of a theory of equational ground proofs (definitions in Section 3) and a construction of new equational proof which reflects effects of eager Variable Elimination (Section 4). We present then the concept of *paths* in an equational proof (Section 5) and this enables us to define a measure of a goal and prove the result by induction on this measure (Section 6).

2 Preliminaries

We use standard definitions as in [1].

[2] In [3] this rule is called *Root Rewriting*. The name *Mutate* came from [5], where it was used for E-unification in Syntactic Theories.

We will consider equations of the form $s \approx t$, where s and t are terms. Please note that throughout this paper these equations are considered to be oriented, so that $s \approx t$ is a different equation than $t \approx s$. Let E be a set of equations, and $u \approx v$ be an equation, then we write $E \models u \approx v$ (or $u =_E v$) if $u \approx v$ is true in any model containing E. We call E an equational theory, and assume that E is closed under symmetry. A goal (E-unification problem) is usually denoted by G and it is a set of equations. $E \models G$ means that $E \models e$ for all e in G.

We will be considering ground terms as ground objects that may or may not have the same syntactic form. In other words we will be concerned with the occurrences of the terms more than their values. A term may be identified by its address in a proof sequence and a position of it as a subterm in a term in the proof. Hence the equality sign between ground terms is treated in a special way. If u, v are ground terms, by $u = v$, u is understood to be an object identical with v, whereas when syntactic equality is sufficient, it will be denoted by $u == v$. Syntactic inequality will be denoted by $u \neq= v$. The difference between identity and syntactic identity is that the first involves *objects* and the second involves *names*.

We can say that a variable x points to its occurrences in a term u, where each of these occurrences under some ground substitution γ, is identical with some subterm of $u\gamma$ at a position α ($x\gamma = u\gamma|_\alpha$). Different occurrences of the same variables are different objects, though they have the same syntactic form (each one is of the form $x\gamma$). In order to distinguish between different occurrences of the same variable, we will use superscript numbers, usually numbering the occurrences from left to right in order of their appearances in an equational proof. Hence $x\gamma^1$ and $x\gamma^2$ are different occurrences of x in a proof.

Sometimes we will want to state that some subterm has a form (or value) of $x\gamma$, but is not identical to $x\gamma$ (hence is not pointed to by a variable x). This will be indicated by quote marks. Hence $w[\text{“}x\gamma\text{”}]_\alpha$ is different from $w[x\gamma]_\alpha$ since in the second term $x\gamma$ actually occurs at position α, while in the first one there is only a subterm that has the value of $x\gamma$.

If γ is a ground substitution, γ_x means the restriction of this substitution to a variable x. Hence if $\gamma = [x \mapsto a, y \mapsto b, z \mapsto c]$, $\gamma_x = [x \mapsto a]$.

3 Equational proofs

Given an equational theory E, we define an equational proof as a pair (Π, γ) such that Π is a series of ground terms and γ is a ground substitution.

Definition 1. *(equational proof)*
Let E be a set of equations. An equational proof of an equation $u \approx v$ is a pair (Π, γ) where $\Pi = (w_1, w_2, \ldots, w_n)$ is series of ground terms, called proof sequence*, such that:*

1. *$u\gamma = w_1$, $v\gamma = w_n$,*
2. *for each pair (w_i, w_{i+1}) for $1 \leq i \leq (n-1)$, there is an equation $s \approx t \in E$ and a matcher ρ, such that there is a subterm $w_i|_\alpha$ of w_i and a subterm $w_{i+1}|_\alpha$ of w_{i+1}, and $w_i|_\alpha = s\rho$, $w_{i+1}|_\alpha = t\rho$.*

We can write the equational proof as
$u\gamma = w_1 \approx_{[\alpha_1, s_1 \approx t_1, \rho_1]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \rho_2]} \cdots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \rho_{n-1}]} w_n = v\gamma$
where $[\alpha_i, s_i \approx t_i, \rho_i]$ indicates at what position α_i is the matching subterm, which equation from E was used ($s_i \approx t_i$), and how the variables in this equation were substituted (ρ). Each w_i in the above sequence is called a term in the proof, as distinct from any proper subterms of w_i, which are not counted as terms in the proof. Since an equational proof is a sequence of ground terms, we will sometimes use the notation borrowed from that for arrays, and $\Pi[i]$ will mean the i'th term in Π.

Let γ be a ground substitution, and G a set of equations such that $E \models G\gamma$. Hence by Birkhoff's theorem, there must be an equational proof for each $u\gamma \approx v\gamma$, where $u \approx v \in G$: $u\gamma = w_1 \approx w_2 \approx \ldots \approx w_n = v\gamma$.

Since every matcher at each step uses a renamed version of an equation from E, the domain of the matcher is disjoint from the domain of γ and the domains of matchers at all other steps in the proof, we extend γ to γ' such that: $\gamma' = \gamma \cup \rho_1 \cup \ldots \cup \rho_n$. From now on we will assume that γ is an extended version of itself.

In order to be able to identify new variables introduced by a possible application of Variable Decomposition (Figure 1), we have to extend γ even more.[3] A general extension of γ will add variables for each subterm of a term v if $\gamma_x = [x \mapsto v]$. We will call these new variables *subterm variables.*

Definition 2. *(general extension of γ)*
Let γ be a ground substitution. A general extension of γ, $ex(\gamma)$, is defined recursively as follows:

1. *if $\gamma_x = [x \mapsto v]$ and $|v| = 1$ (v is a constant), then $ex(\gamma_x) = \gamma_x$,*
2. *if $\gamma_x = [x \mapsto f(v_1, \ldots, v_n)]$, and $n \geq 1$, then let $\gamma_{y_i} = [y_i \mapsto v_i]$, for $1 \leq i \leq n$, and $ex(\gamma_x) = \gamma_x \cup ex(\gamma_{y_1}) \cup \cdots \cup ex(\gamma_{y_n})$,*
3. *$ex(\gamma) = \bigcup_{x \in Dom(\gamma)} ex(\gamma_x)$*

From now on we will consider γ in (Π, γ) as a general extension of itself. We have 3 kinds of variables in $Dom(\gamma)$:

1. the goal variables, i.e. the variables in $Var(u \approx v)$;
2. the system variables, i.e. if there is a step $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \gamma]} \Pi[i+1]$ in (Π, γ), then the variables in $Var(s_i \approx t_i)$ are called *system variables*;
3. the subterm variables in $\Pi[i]$, for each $\Pi[i]$ in the proof, i.e. variables that are introduced by general extension of γ ;

We will see that each variable occurrence starts or ends some subproof in an equational proof. In order to define this subproof, we will use a notion of *orientation* of a variable occurrence. We define it for each variable occurrence in the following way:

[3] The following definition is similar to the definition of general extension of a substitution in [3]. It was introduced there with a similar purpose: to accommodate the Variable Decomposition rule.

Definition 3. *(orientation of variable occurrences)*
Let (Π, γ) *be an equational proof and* $x \in Dom(\gamma)$*.*

1. *If* $x\gamma$ *is a system variable occurrence in* $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \gamma]} \Pi[i+1]$ *and* $x\gamma = \Pi[i]|_\alpha$ *for some position* α*, then* $x\gamma$ *has left orientation. If* $x\gamma = \Pi[i+1]|_\alpha$*, then* $x\gamma$ *has right orientation.*
2. *if* $x\gamma$ *is a goal variable occurrence in* $\Pi[1]$ *(*$x\gamma = \Pi[1]|_\alpha$*), then* $x\gamma$ *has right orientation, and if* $x\gamma = \Pi[n]|_\alpha$*, where* $\Pi[n]$ *is the last term in the proof, then* $x\gamma$ *has left orientation.*
3. *if* $x\gamma$ *is a subterm variable, hence* $x\gamma = y\gamma|_\alpha$*, then* $x\gamma$ *has the same orientation as* $y\gamma$*.*

3.1 Part of equational proof and subproof

Now we define subproofs in an equational proof as proofs embedded at some position in *parts* of this proof.

Definition 4. *(part of proof for depth* α*)*
Let (Π, γ) *be an equational proof*
$w_1 \approx_{[\alpha_1, s_1 \approx t_1, \gamma]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \gamma]} \cdots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w_n$.
Let α *be one of* $\alpha_1, \ldots, \alpha_{n-1}$*, which are the positions at which the steps in the proof are performed. A part of the proof* (Π, γ) *for depth* α *is a sequence:* $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \gamma]} \cdots \approx_{[\alpha_{i+j-1}, s_{i+j-1} \approx t_{i+j-1}, \gamma]} \Pi[i+j]$*, such that for* $i \leq k \leq j-1$*,* $\alpha_k \geq \alpha$ *or* $\alpha_k || \alpha$*.*

Hence part of a proof is a subsequence of steps in the proof, such that each step is performed at a position α, lower or at a parallel position in the subsequent terms of the proof. If $j = 0$, the part of the proof is composed of one term only. Now we will define a subproof in an equational proof as a sequence of subterms of terms in a part of the original proof.

Definition 5. *(subproof)*
Let (Π, γ) *be an equational proof.*

Let $\Pi[i] \approx_{[\alpha_i, s'_1 \approx t'_1, \gamma]} \cdots \approx_{[\alpha_{i+k-1}, s'_{i+k-1} \approx t'_{i+k-1}, \gamma]} \Pi[i+k]$ *be a part of the proof* (Π, γ) *for depth* α*, and let* α_n *be a such that* $\alpha \leq \alpha_n$*.*

Then a pair (Σ, γ)*, where* Σ *is a sequence of terms (called* subproof sequence*):* $\Pi[i]|_{\alpha_n}, \Pi[i+1]|_{\alpha_n}, \ldots, \Pi[i+k]|_{\alpha_n}$ *is called a subproof of* (Π, γ)*.*

In the next sections, we want to be able to use a copy of a subproof in creating new proofs. In this copy only some variables, called *internal variables*, will be renamed.

Definition 6. *(internal/external variables in a subproof)*
Let (Π, γ) *be an equational proof and* $(\Sigma_{w \approx w'}, \gamma)$ *a subproof in* (Π, γ)*. If there is a step in* $(\Sigma_{w \approx w'}, \gamma)$*:* $w_i \approx_{[\alpha, s \approx t, \gamma]} w_{i+1}$*,* $y \in Var(s \approx t)$*,* y *is called an internal variable in* $(\Sigma_{w \approx w'}, \gamma)$*. If* y *has occurrences in* $(\Sigma_{w \approx w'}, \gamma)$*, but is not internal variable in this subproof, it is called an external variable in* $(\Sigma_{w \approx w'}, \gamma)$*.*

Definition 7. *(renaming of a subproof)*
Let (Π, γ) be an equational proof and $(\Sigma_{w \approx w'}, \gamma)$ a subproof in (Π, γ). $(\Sigma'_{w \approx w'}, \gamma')$ is a renaming of $(\Sigma_{w \approx w'}, \gamma)$ if $(\Sigma'_{w \approx w'}, \gamma')$ is exactly like $(\Sigma_{w \approx w'}, \gamma)$, with all internal variables renamed.

Example 1. Let $E := \{ffx \approx fgx\}$ and the equational proof (Π, γ) is the following:
$fgfa \approx_{[\epsilon, ffx_1 \approx fgx_1, [x_1 \mapsto fa]]} fffa \approx_{[<1>, ffx_2 \approx fgx_2, [x_2 \mapsto a]]} ffga \approx_{[\epsilon, ffx_3 \approx fgx_3, [x_3 \mapsto ga]]} fgga$.

Obviously, (Π, γ) is its own subproof. We have also one more subproof: $ffa \approx_{[\epsilon, ffx_2 \approx fgx_2, [x_2 \mapsto a]]} fga$, where $ffa = \Pi[2]|_{<1>}$. A renaming of this subproof would have the following form: $ffa \approx_{[\epsilon, ffx_4 \approx fgx_4, [x_4 \mapsto a]]} fga$, where x_4 is a new variable.

Further analysis of subproofs and their normal forms may be found in Appendix A.

3.2 Embedding a proof into a term

Embedding a proof into a term is a way to construct a proof from a given subproof.

Definition 8. *(embedding of a proof)*
If w is a ground term, (Π, γ) is a proof of the form:

$$w_1 \approx_{[\alpha_1, s_1 \approx t_1, \gamma]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \gamma]} \cdots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w_n$$

and there is a position β in w such that $w|_\beta == w_1$, then there is a proof (Π', γ) of the form:

$$w[w_1]_\beta \approx_{[\beta\alpha_1, s_1 \approx t_1, \gamma]} w[w_2]_\beta \approx_{[\beta\alpha_2, s_2 \approx t_2, \gamma]} \cdots \approx_{[\beta\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w[w_n]_\beta$$

We say that (Π', γ) is the **embedding of the proof** (Π, γ) **in the term** w.

We can attach a proof to a given equational proof (Π, γ) by embedding it into the last term of (Π, γ), if the conditions of the definition are met.

If (Π, γ) is a proof such that it is composed from (Σ_1, γ_1) and (Σ_2, γ_2) by embedding (Σ_2, γ_2) into the last term of (Σ_1, γ_1), we say that (Π, γ) is a *composition* of (Σ_1, γ_1) and (Σ_2, γ_2).

3.3 Contracting

Definition 9. *(non-redundant equational proof)*
An equational proof Π is non-redundant if there are no two terms $\Pi[i]$ and $\Pi[j]$ such that $i \neq j$ and $\Pi[i] == \Pi[j]$, and all proper subproofs of Π are non-redundant.

A simple procedure (called **contraction**) of cutting out loops out of subproof sequences in a proof sequence, allows us to obtain a non-redundant proof from any redundant one.[4]

3.4 Associated subproofs, associated terms and a hierarchy of variable occurrences

In this section, for each occurrence of a variable x in $Dom(\gamma)$, we define a ground term associated with this occurrence. The intuition is that a term associated with a given occurrence of a variable, $x\gamma$, is the term on the opposite end of a longest subproof with starts with $x\gamma$. If $x \approx v$ is an equation in our goal G, and $E \models G\gamma$, then $v\gamma$ is a term associated with $x\gamma$.

First, we define ground subproofs associated with each occurrence of x in an equational proof.

Definition 10. *(subproof associated with an occurrence of a variable)*
Let (Π, γ) be an equational proof, $x \in Dom(\gamma)$ and $x\gamma$ is an occurrence of x in (Π, γ).

1. *If $x\gamma$ has a left orientation and $x\gamma = \Pi[i]|_\alpha$, then there is the longest subproof* $\Pi[i-k]|_\alpha \approx \cdots \approx \Pi[i]|_\alpha$
   We reverse the order of the terms in this subproof:
   $\Pi[i]|_\alpha \approx \cdots \approx \Pi[i-k]|_\alpha$
   and we call this subproof **a subproof associated with this** *$x\gamma$. We say that the subproof associated with $x\gamma$ is left-oriented.*

2. *If $x\gamma$ has right orientation and $x\gamma = \Pi[i]|_\alpha$, then there is the longest subproof* $\Pi[i+1]|_\alpha \approx \cdots \approx \Pi[i+l]|_\alpha$
   We call this subproof **a subproof associated with this** *$x\gamma$ and we say that it is right-oriented.*

Notice that if (Π, γ) is an equational proof of $u\gamma \approx v\gamma$, then the external variables in this proof are only variables in $Var(u)$ and $Var(v)$. By the definition of subproofs associated with variable occurrences, if $(\Sigma_{x\gamma \approx v}, \gamma)$ is such a subproof, external variables in this subproof have their occurrences only in $x\gamma$ (x and its subterm variables are external variables in this subproof) and v. The external variable occurrences in v have opposite orientation to that of $x\gamma$. We will sometimes indicate an orientation of an occurrence of a variable by an arrow, like in $\overrightarrow{x}\gamma$, which denotes an occurrence of x with right orientation. Similarly, if (Σ, γ) is a subproof in (Π, γ), $(\overrightarrow{\Sigma}, \gamma)$ indicates that this subproof has right orientation.

[4] In the case of proofs in normal form, it is enough to require that there are no identical terms in the proof, to show that it is non-redundant. The definition of normal form for a proof is in Appendix A.

Definition 11. *(term associated with an occurrence of x)*
Let (Π, γ) be an equational proof, $x \in Dom(\gamma)$ and $x\gamma$ is an occurrence of x in (Π, γ). Let a subproof $(\Sigma_{x\gamma \approx v}, \gamma)$ be a subproof associated with $x\gamma$, then we define a term associated with $x\gamma$, $ass(x\gamma)$, in the following way:

1. *if no occurrence of x appears in v, then $ass(x\gamma) = v$,*
2. *if an occurrence of x appears in v, then*
   (a) *if there is a step at the root in $(\Sigma_{x\gamma \approx v}, \gamma)$, we will choose the rightmost such step: $w_i \approx_{[\epsilon, s_i \approx t_i, \gamma]} w_{i+1}$ and define $ass(x\gamma) = w_i$,*
   (b) *if there is no step at the root in $(\Sigma_{x\gamma \approx v}, \gamma)$, we define $ass(x\gamma) = x\gamma$.*

The point of this analysis is the observation that if we want to perform eager Variable Elimination with a goal equation $x \approx w$, where $x \notin Var(w)$, knowing that there is a ground proof of $x\gamma \approx w\gamma$, even if $x\gamma \neq= w\gamma$, $w\gamma = ass(x\gamma)$. In this situation, we will show how to construct an equational proof of the goal with the ground substitution changed to γ', such that $\gamma'_x = [x \mapsto w\gamma]$.

There is a hierarchy among occurrences of the variables of an equational proof. In order to display it, we will construct a graph G_Π with occurrences of variables in a given equational proof as nodes and arrows as follows.

1. for each variable x in $Dom(\gamma)$ and for each occurrence $x\gamma$ of this variable, if for any $y \in Dom(\gamma)$, $(\Sigma_{x\gamma \approx w[y\gamma]}, \gamma)$ is a subproof of a proof associated with $x\gamma$ and w is not empty, draw an arrow from $x\gamma$ to $y\gamma$;
2. for each variable x in $Dom(\gamma)$ and for each occurrence $x\gamma$ of this variable: if for any $y \in Dom(\gamma)$, $(\Sigma_{x\gamma \approx y\gamma}, \gamma)$ is a subproof of a proof associated with $x\gamma$:
   (a) if $(\Sigma_{y\gamma \approx x\gamma}, \gamma)$ is a subproof of a proof associated with $y\gamma$, then non-deterministically decide the direction of an arrow between $x\gamma$ and $y\gamma$;
   (b) if $(\Sigma_{y\gamma \approx x\gamma}, \gamma)$ is not a subproof associated with $y\gamma$, then draw an arrow from $x\gamma$ to $y\gamma$;

The parent/child relation defined next, follows the arrows in the graph for an equational proof.

Definition 12. *(parent/child relation) Let (Π, γ) be an equational proof with $x, y \in Dom(\gamma)$ (x may be possibly the same as y) Let $x\gamma$ and $y\gamma$ be any two different occurrences of variables in $Dom(\gamma)$.*

If there is an arrow $x\gamma \to y\gamma$, then $x\gamma$ is called a parent of $y\gamma$ and $y\gamma$ is a child of $x\gamma$.

The graph G_Π, for an equational proof helps us to recognize/decide the parent/child relation. This relation is in some cases determined by the structure of the proof (we cannot discover new variables in the transformation of the goal before solving/eliminating some other variables first), or it is decided by the selection rule and orientation of an equation of the form $x \approx y$. The *maximal* nodes in the graph are just those occurrences of variables that are discovered in the goal and may be selected for eager Variable Elimination.

Definition 13. *(maximal occurrences of variables)*
Let (Π, γ) be an equational proof, and G_Π be a graph for (Π, γ). A set M is a set of maximal nodes in G_Π, if M contains all nodes which have no parents in G_Π.

4 Solving variables in an equational proof

The following construction explains what happens with an equational proof of a goal, if an equation of the type $x \approx t$ is selected for eager Variable Elimination. Notice that in this construction we declare which variables in $Dom(\gamma)$ are solved or unsolved. In the justification of the completeness of the inference system with eager Variable Elimination we start with the equational proof of an instance of a goal with all variables unsolved. Variable Elimination reflects solving variables in a ground equational proof.

Let (Π, γ) be an equational proof with the proof sequence:
$\Pi = (w_1 \approx_{[\alpha_1, s_1 \approx t_1, \gamma]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \gamma]} \dots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w_n)$
and γ be an extended ground substitution.

Let $U = \{x_1, \dots, x_n\}$ be a set of variables called "unsolved" in (Π, γ), G_Π be the graph for (Π, γ) constructed only with respect to unsolved variables (hence we treat all other variables as non-existent in (Π, γ)).

Let $x \in U$ and $x\gamma$ be a maximal node in G_Π and let $ass(x\gamma) = v$.

There is a subproof $(\Sigma_{x\gamma \approx v}, \gamma)$ in (Π, γ), let $(\Sigma'_{x\gamma \approx v}, \gamma')$ be a renaming of this subproof.[5]

If x has no occurrences in v, create a new proof (Π^*, γ^*) that is exactly as (Π, γ) with the proof sequence modified in the following way:

1. **Extension**
Whenever $x\gamma = w_i|_\alpha$ and hence $w_i = w_i[x\gamma]$, and
(a) $x\gamma$ has right orientation, replace w_i (the i'th step in (Π, γ)), by the sequence of steps:

$$w_i[v]_\alpha \approx (\Sigma'_{v \approx \text{``}x\gamma\text{''}}) \approx w_i[\text{``}x\gamma\text{''}]_\alpha$$

where $(\Sigma'_{v \approx \text{``}x\gamma\text{''}})$ means a renaming of $(\Sigma_{\text{``}x\gamma\text{''} \approx v}, \gamma)$ reversed and embedded in w_i at position α leftwards. Note that the renamings of internal occurrences of variables and new occurrences of external variables in the renaming of $(\Sigma_{\text{``}x\gamma\text{''} \approx v}, \gamma)$ have reversed orientation in the new proof.
(b) $x\gamma$ has left orientation, replace w_i (the i'th step in (Π, γ)) by the sequence of steps:

$$w_i[\text{``}x\gamma\text{''}]_\alpha \approx (\Sigma'_{\text{``}x\gamma\text{''} \approx v}) \approx w_i[v]_\alpha$$

where $(\Sigma'_{\text{``}x\gamma\text{''} \approx v})$ means a renaming of $(\Sigma_{\text{``}x\gamma\text{''} \approx v}, \gamma)$ embedded in w_i at position α rightwards. The renamings of internal occurrences of variables and new occurrences of external variables in $(\Sigma'_{\text{``}x\gamma\text{''} \approx v})$ preserve their orientation in the new proof.

[5] If x has no occurrences in v, $(\Sigma_{x\gamma \approx v}, \gamma)$ is a subproof associated with $x\gamma$.

2. **Contraction**
For each occurrence of an unsolved variable y in (Π, γ), if $(\Sigma_{y\gamma\approx s}, \gamma)$ is a proper associated subproof of this occurrence in (Π, γ) and there is a subproof sequence: $\Sigma_{s\approx \text{``}y\gamma\text{''}}\,\Sigma_{\text{``}y\gamma\text{''}\approx s}$ in the proof sequence Π^* after extension, contract the subproof sequence to a one-element sequence, s;

The substitution γ^* is defined as follows:
$\gamma^*_x = [x \mapsto v]$,
if $y\gamma|_\alpha = x\gamma$, and $y \notin U$, then $\gamma^*_y = [y \mapsto y\gamma[x\gamma^*]_\alpha]$,
if $z \notin Dom(\gamma)$, z is a renaming of a variable $z' \in Dom(\gamma)$, that appeared in some $(\Sigma'_{x\gamma\approx v}, \gamma')$, then $\gamma^*_z = [z \mapsto z'\gamma]$,
for any other variable, $\gamma^* = \gamma$;
If x has occurrences in v, then $(\Pi^*, \gamma^*) = (\Pi, \gamma)$.

Mark variables
Mark variable x **solved** *in (Π^*, γ^*). If x has no occurrences in v, mark also all subterm variables of x as* **solved** *New variables in $Dom(\gamma^*)$, which did not appear in $Dom(\gamma)$ are marked as unsolved.*

If a proof (Π^*, γ^*) is obtained from (Π, γ) in this way, then we say that (Π^*, γ^*) is generated from (Π, γ) by substitution $[x \mapsto v]$, written $(\Pi, \gamma) \stackrel{[x\mapsto v]}{\rightarrow} (\Pi^*, \gamma^*)$. As a corollary to this construction we notice that:

Corollary 1. *If $(\Pi, \gamma) \stackrel{[x\mapsto v]}{\rightarrow} (\Pi', \gamma')$ and $y \in Dom(\gamma')$, then for each occurrence $y\gamma'$ in (Π', γ'), either*

1. *$y \in Dom(\gamma)$ and $y\gamma'$ is an occurrence of this variable identical with an occurrence in (Π, γ), ($y\gamma'$ is in the part of (Π', γ') not affected by extension and contraction), or*
2. *$y \in Dom(\gamma)$ and $y\gamma'$ is a new occurrence of y, introduced in the effect of extending (Π, γ) with $(\Sigma_{x\gamma\approx v}, \gamma)$, (there was an occurrence $y\gamma^k$ of an external variable y in $(\Sigma_{x\gamma\approx v}, \gamma)$ which generated new occurrences in all places the copy of this subproof was used and not contracted), or*
3. *$y \notin Dom(\gamma)$, (y is a new variable) then $y\gamma'$ may be identified as a renamed version of a variable $y' \in Dom(\gamma)$, where y' was an inner variable in $(\Sigma_{x\gamma\approx v}, \gamma)$.*

Example 2. Let an equational proof be:
$f(a, g(b, b)) \approx_{[<1>, a\approx b, []]} f(b, g(b, b)) \approx_{[\epsilon, f(x, g(x,x))\approx c, [x\mapsto b]]} c$

Then the subproof associated with $\overleftarrow{x\gamma}^{1}$ is $b \approx a$. Notice the left orientation of all occurrences of x in this case. We want to solve x in the proof with $x \mapsto a$. Hence we will use $b \approx a$ for the extension at each occurrence of x.
$f(a, g(b, b)) \approx_{[<1>, a\approx b, []]} f(b, g(b, b)) \approx_{[<1>, b\approx a, []]} f(a, g(b, b))$
$\approx_{[<2>, b\approx a, []]} f(a, g(a, b)) \approx_{[<3>, b\approx a, []]} f(a, g(a, a)) \approx_{[\epsilon, f(x, g(x,x))\approx c, [x\mapsto a]]} c$
Contraction will shorten the proof to:
$f(a, g(b, b)) \approx_{[<2>, b\approx a, []]} f(a, g(a, b)) \approx_{[<3>, b\approx a, []]} f(a, g(a, a))$
$\approx_{[\epsilon, f(x, g(x,x))\approx c, [x\mapsto a]]} c$

Notice that we have a new assignment for x, but now we will treat x as solved.

5 Paths in Equational Proof

A concept of path is a generalization of an associated subproof for an occurrence of a variable. A path is a subproof starting with some variable occurrence, constructed in such a way that it reflects the form of an associated subproof for this variable occurrence assuming that all other variables *involved* in the path were solved first. In order to restrict the definition of a path in a proof (Π, γ), we have to take into consideration solved and unsolved occurrences of variables in $Dom(\gamma)$. We have to remember where the solved variables had their occurrences at the time they were being solved.

Since in this section we will deal with compositions of subproofs, in order to simplify notation, we will identify a subproof with its subproof sequence.

Definition 14. *(path starting with a variable occurrence and variables used in a path)*

Let (Π, γ) be an equational proof, U a set of unsolved variables in $Dom(\gamma)$, $x \in U$ and $x\gamma$ a given variable occurrence in (Π, γ). A path in (Π, γ) starting with $x\gamma$ is a composition of subproofs, $\Sigma_1 \dots \Sigma_n$, defined in a recursive way:

1. *if $\Sigma_{x\gamma \approx v}$ is an associated subproof for $x\gamma$, $\Sigma_{x\gamma \approx v}$ is a path starting with $x\gamma$;*
2. *if $x\gamma$ is a parent of $y\gamma$, then $\Sigma_{x\gamma \approx w[y\gamma]}$ is a path starting with $x\gamma$;*
3. (a) *if $\Sigma_1, \dots, \Sigma_n$ is a path in (Π, γ) starting with $x_1\gamma$ and $\Sigma_n = \Sigma_{x_n\gamma \approx v[x_{n+1}\gamma^k]}$, x_{n+1} is an external variable in $\Sigma_{x_n\gamma \approx v[x_{n+1}\gamma^k]}$ different from $x_1\gamma$, $\Sigma'_1, \dots, \Sigma'_m$ is a path in (Π, γ) starting with $x_{n+1}\gamma^i$, and if no variable which is* used *in one path appears as not* used *in the other, then the composition $\Sigma_1 \dots \Sigma_n \Sigma'_1 \dots \Sigma'_m$ is also a path in (Π, γ) starting with $x_1\gamma$ and all variables* used *in the first and second path are* used *in the new path;*
   (b) *if $\Sigma_1, \dots, \Sigma_n$ is a path in (Π, γ) starting with $x_1\gamma$ and $\Sigma_n = \Sigma_{x_n\gamma \approx y\gamma|_\alpha}$, and $\Sigma_{y\gamma^k|_\alpha \approx s}$ is a subproof in (Π, γ) and if no variable which is* used *in one path appears as not* used *in the other, then $\Sigma_1, \dots, \Sigma_n, \Sigma_{y\gamma^k|_\alpha \approx s}$ is also a path in (Π, γ) starting with $x_1\gamma$ and all variables* used *in the first and second path are* used *in the new path;*

Example 3. For example, let our goal be: $G = \{x \approx a, z \approx hx, z \approx c\}$ and an equational theory: $E = \{b \approx a, b \approx fga, hfy \approx c\}$, then the proof (Π, γ) may be:

$$\{ \overset{\substack{x\gamma^1 \\ \downarrow}}{b} \approx_{[\epsilon, b \approx a, []]} a,\ \underset{\substack{\uparrow \\ z\gamma^1}}{\overset{\substack{x\gamma^2 \\ \downarrow}}{hb}},\ \underset{\substack{\uparrow \\ z\gamma^2}}{hb} \approx_{[<1>, b \approx fga, []]} hfga \approx_{[\epsilon, hfy \approx c, [y \mapsto ga]]} c \}$$

1. An example of a path starting with $x\gamma^2$ would be: $\Sigma_{x\gamma^2 \approx z\gamma^1|_{<1>}} \Sigma_{z\gamma^2|_{<1>} \approx fy\gamma}$. z is used and x and y are not used in this path.
2. An example of a path starting with $z\gamma^1$ is: $\Sigma_{z\gamma^1 \approx h(x\gamma^2)} \Sigma_{x\gamma^1 \approx a}$. x is used in this path.

We will prove that if $(\Pi, \gamma) \stackrel{[x \mapsto v]}{\rightarrow} (\Pi', \gamma')$, for an unsolved variable x in $Dom(\gamma)$, then each path in (Π', γ') starting with an unsolved variable in (Π', γ') is identical to a path in (Π, γ) (up to renaming). Hence any new paths will be renamings of the original ones. In order to show that the process of solving variables in (Π, γ) will terminate, we will use a multiset of lengths of paths as a measure, and show that it is decreasing.

Lemma 1. *Let (Π, γ) be an equational proof, $U \subset Dom(\gamma)$ be a set of unsolved variables in (Π, γ), $(\Pi, \gamma) \stackrel{[x \mapsto v]}{\rightarrow} (\Pi', \gamma')$, and U' be a set of unsolved variables in (Π', γ').*

Each path in (Π', γ') starting with a variable occurrence of a variable in U' is identical (up to renaming of some variables) to a path in (Π, γ) starting with a variable occurrence of a variable in U.

If there are many paths in (Π', γ'), which are renamings of one and the same path in (Π, γ), then they are strictly shorter than a path in (Π, γ), starting with a variable occurrence of a variable which is solved in (Π', γ').

Proof. The proof of this lemma is based on the fact that each path starting with an occurrence of an unsolved variable in (Π, γ) is finite. This is the case, because while constructing longer and longer paths, we will have to run out of unused variables.

Hence we can use induction on the lengths of paths.

Let $(\Pi, \gamma) \stackrel{[x \mapsto v]}{\rightarrow} (\Pi', \gamma')$, where $\Sigma_{x\gamma^i \approx v}$ was used in construction of (Π', γ'), and $\Sigma_1 \ldots \Sigma_n$ is a path in (Π', γ'), starting with $y_1\gamma$. We can assume that x does not occur in v, because otherwise $ass(x\gamma) = x\gamma$ and then (Π', γ') is identical to (Π, γ) with the only difference that x is solved and does not appear in U'.

We have to consider different cases generated by the possible ways paths are constructed in (Π', γ'), starting with variable occurrences appearing in (Π', γ') of the kinds described in Corollary 1.

1. Let $y_1 \in Dom(\gamma')$ be such that $y_1 \in Dom(\gamma)$ and an occurrence of y_1, $y_1\gamma'$ be as described in Corollary 1. 1.
   Hence $y_1\gamma$ is an occurrence of y_1 in a part of (Π, γ) which is not affected by extension or contraction in the process of constructing (Π', γ').
   Let $\Sigma_1 \ldots \Sigma_n$ be a path in (Π', γ') starting with $y_1\gamma'$.
   (a) If $\Sigma_1 \ldots \Sigma_n$ is a path by Definition 14.1, it is a subproof associated with $y_1\gamma$ in (Π', γ'). The only case, when such a subproof was not already a path in (Π, γ) would be if the composition of shorter paths was prevented by the condition that a variable which is used in one path appears as not used in the other. Hence there would be two paths in (Π, γ): $\Sigma'_{y_1\gamma \approx s[``z\gamma'^k"]} \Sigma'_{``z\gamma'^i" \approx t[x\gamma^i]}$ and $\Sigma'_{x\gamma^i \approx v[z\gamma^k]}$. z is used in the first

one, but not in the second. But we see, that because a variable is solved in all places of its occurrences in a proof, it is impossible that z appears as not used in $\Sigma_{x\gamma^i \approx v}$. The same argument works if we assume that z is used in $\Sigma_{x\gamma^i \approx v}$ and not in $\Sigma'_{y_1'\gamma \approx t[x\gamma^i]}$.
Hence there is a path, $\Sigma'_{y_1\gamma \approx t[x\gamma^i]}\Sigma_{x\gamma \approx v}$ in (Π, γ), such that $\Sigma_1 \dots \Sigma_n$ is a renamed and possibly contracted version of it. Notice that if it is a contracted version of renaming of $\Sigma'_{y_1\gamma \approx t[x\gamma^i]}\Sigma_{x\gamma \approx v}$, then it is shorter of that path which is no longer in (Π', γ').

(b) If $\Sigma_1 \dots \Sigma_n$ is a path in (Π', γ') by Definition 14.2, then it has the form $\Sigma_{y_1\gamma' \approx w[z\gamma']}$. This is a subproof of the subproof associated with $y_1\gamma'$ in (Π', γ').
This subproof is either identical to a subproof $\Sigma_{y_1\gamma \approx w[z\gamma]}$ in (Π, γ) or there must be a composition of subproofs of the form: $\Sigma_{y_1\gamma \approx t[x\gamma^k]}\Sigma_{x\gamma^i \approx v}$ in (Π, γ). (Then $z\gamma$ must be a renamed occurrence of a variable occurrence $z'\gamma$ in $\Sigma_{x\gamma^i \approx v}$. The renaming is identity if z' is external in $\Sigma_{x\gamma^i \approx v}$.)
By the same argument as in the previous case, $\Sigma'_{y_1\gamma \approx t[x\gamma^k]}\Sigma_{x\gamma^i \approx v}$ must be a path in (Π, γ), of which the path $\Sigma_{y_1\gamma' \approx w[z\gamma']}$ in (Π', γ') is a renamed and possibly contracted version.

(c) If $\Sigma_1 \dots \Sigma_n$ is a path in (Π', γ') but is not a subproof of an associated subproof for $y_1\gamma'$, then it must be a composition of paths in (Π', γ'):
$\Sigma_1 \dots \Sigma_k$ and $\Sigma_{k+1} \dots \Sigma_n$.
We will assume that $\Sigma_1 \dots \Sigma_k$ has the form: $\Sigma_1 \dots \Sigma_{z \approx s[``x\gamma^k"]}$, and $\Sigma_{k+1} \dots \Sigma_n$ the form: $\Sigma_{``x\gamma^i" \approx v} \dots \Sigma_n$.
Alternatively the path $\Sigma_1 \dots \Sigma_k$ can have the form: $\Sigma_1 \dots \Sigma_{z \approx ``x\gamma^k|_\alpha"}$, and $\Sigma_{k+1} \dots \Sigma_n$ has the form: $\Sigma_{``x\gamma^i|_\alpha" \approx v} \dots \Sigma_n$. But this case is analyzed in exactly the same way.
Because Variable Elimination affects all occurrences of a given variable, there is no variable with some occurrences "solved" and some "unsolved" in the proof. There is no variable with "unsolved" occurrences in one of the paths, and "solved" in another.
And since these paths are shorter than $\Sigma_1 \dots \Sigma_n$, we can assume that they are renamings of paths in (Π, γ): $\Sigma'_1 \dots \Sigma'_{z \approx s[x\gamma^k]}$ and $\Sigma'_{x\gamma^i \approx v} \dots \Sigma'_n$
Since these are renamings of respective paths in (Π', γ'), obviously there is no variable that is used in one of them and has occurrences that are not used in the other. Therefore, there is a path:
$\Sigma'_1 \dots \Sigma'_{z \approx s[x\gamma^k]}\Sigma'_{x\gamma^i \approx v} \dots \Sigma'_n$ in (Π, γ).

2. Now let us consider variables that have occurrences inside $\Sigma_{``x\gamma^i" \approx v}$ in (Π', γ'). Let $y_1 \in Dom(\gamma')$ be such a variable and $y_1\gamma'$, is a renaming of an occurrence of y_1' in $\Sigma_{x\gamma^i \approx v}$ in (Π, γ).
Let $\Sigma_1 \dots \Sigma_n$ be a path in (Π', γ') starting with $y_1\gamma'$.
We have 3 cases here:
(a) If $\Sigma_1 \dots \Sigma_n$ is contained inside $\Sigma_{``x\gamma^i" \approx v}$, this path is a renaming of a path inside $\Sigma_{x\gamma^i \approx v}$ in (Π, γ), and although there may be numerous such copies in (Π', γ'), notice that all of them are shorter than $\Sigma_{x\gamma^i \approx v}$ which is no longer a path in (Π', γ') because x is solved there. So assume that the path $\Sigma_1 \dots \Sigma_n$ starts inside $\Sigma_{``x\gamma^i" \approx v}$ and has some part outside it.

i. Assume that the path $\Sigma_1 \dots \Sigma_n$ spreads beyond v. Then it has the form: $\Sigma_1 \dots \Sigma_{t\approx v[z\gamma'^l]} \Sigma_{z\gamma'^m \approx s} \dots \Sigma_n$.
   Hence, there must be also a path in (Π, γ) of the form:
   $\Sigma'_1 \dots \Sigma'_{t\approx v[z\gamma^l]} \Sigma'_{z\gamma^m \approx s} \dots \Sigma'_n$ and z must be external in both $\Sigma'_{t\approx v[z\gamma'^i]}$ and $\Sigma'_{z\gamma'^m \approx s}$. The path $\Sigma_1 \dots \Sigma_n$ is a renamed and possibly contracted version of this path.
   Notice that $\Sigma'_1 \dots \Sigma'_{t\approx v[z\gamma'^i]} \Sigma'_{z\gamma'^m \approx s} \dots \Sigma'_n$ is a subpath of a path starting with $x\gamma^i$ in (Π, γ). Namely, it is a subpath of path:
   $\Sigma'_{x\gamma'^i \approx v[z\gamma'^i]} \Sigma'_{z\gamma'^m \approx s} \dots \Sigma'_n$. Since y_1 is different than x the path $\Sigma'_1 \dots \Sigma'_{t\approx v[z\gamma'^i]} \Sigma'_{z\gamma'^m \approx s} \dots \Sigma'_n$ is strictly shorter than the path starting with $x\gamma^i$.
   Hence although there may be many renamings of the path
   $\Sigma'_1 \dots \Sigma'_{t\approx v[z\gamma'^l]} \Sigma'_{z\gamma'^m \approx s} \dots \Sigma'_n$ in (Π', γ'), they will all be shorter than a path in (Π, γ) which no longer appears in (Π', γ'), because x is solved.

ii. Assume now that the path $\Sigma_1 \dots \Sigma_n$ spreads beyond "$x\gamma^i$". Then it has the form: $\Sigma_1 \dots \Sigma_{t\approx \text{``}x\gamma^i\text{''}|_\alpha} \Sigma_{\text{``}x\gamma^k\text{''}|_\alpha \approx s} \dots \Sigma_n$.
   Then there must be a unique path in (Π, γ) of the form:
   $\Sigma'_1 \dots \Sigma'_{t\approx x\gamma^i|_\alpha} \Sigma'_{x\gamma^k|_\alpha \approx s} \dots \Sigma'_n$. The path $\Sigma_1 \dots \Sigma_n$ in (Π', γ') is a renamed and possibly contracted form of that path.

Consider Example 3. After eliminating x from the goal with $x \approx a$, the new goal is $G' = \{a \approx a, z \approx ha, z \approx c\}$. This is reflected by solving x with $x \to a$ in the equational proof of $G\gamma$. We get a new equational proof of the form:

$$\begin{array}{llllll}
\text{``}x\gamma'\text{''} & & \text{``}x\gamma'\text{''} & & & \\
\downarrow & & \downarrow & & & \\
\{\, a, & hb \approx_{[<1>,b\approx a,[]]} & ha, & hb \approx_{[<1>,\,b\approx fga,\,[]]} & hfga \approx_{[\epsilon,\,hfy\approx c,\,[y\mapsto ga]]} & c\,\} \\
 & \uparrow & & \uparrow & & \\
 & z{\gamma'}^1 & & z{\gamma'}^2 & &
\end{array}$$

There is a path starting with $z{\gamma'}^1$: $\Sigma_{z\gamma'^1 \approx ha}$ in the new proof, but it is identical to the path $\Sigma_{z\gamma^1 \approx x\gamma^2} \Sigma_{x\gamma^1 \approx a}$ in the old proof.

Instead of eliminating x from the goal, we could have chosen to eliminate z with $z \to c$. After Variable Elimination, the new goal is $G' = \{x \approx a, c \approx hx, c \approx c\}$. By solving z with $z \to c$ in the equational proof of $G\gamma$, we get a new proof:

$$\begin{array}{llllll}
x{\gamma'}^1 & & & & x{\gamma'}^2 & \\
\downarrow & & & & \downarrow & \\
\{\, b \approx_{[\epsilon,b\approx a,[]]} a, & c & \approx_{[\epsilon,\,c\approx hfy',\,[y'\mapsto ga]]} hfga & \approx_{[<1>,\,fga\approx b,\,[]]} & hb, & c\,\} \\
 & \uparrow & & & & \uparrow \\
 & \text{``}z\gamma'\text{''} & & & & \text{``}z\gamma'\text{''}
\end{array}$$

There is a path starting with $x{\gamma'}^2$ in this proof: $\Sigma_{x\gamma'^2 \approx fy'\gamma'}$, but this is a renaming of a path in the original proof starting with $x\gamma^2$:
$\Sigma_{x\gamma^2 \approx z\gamma^1|_{<1>}} \Sigma_{z\gamma^2|_{<1>} \approx fy\gamma}$ in (Π, γ).

Corollary 2. *Let (Π, γ) be an equational proof, $U \subset Dom(\gamma)$ a set of unsolved variables in (Π, γ), The process of solving (Π, γ) will terminate.*

Proof. If $(\Pi, \gamma) \overset{[x \mapsto v]}{\rightarrow} (\Pi', \gamma')$, and U' a set of unsolved variables in (Π', γ'), the multiset of lengths of paths in (Π', γ') is smaller than the multiset of lengths of paths in (Π, γ).

6 Result

We prove completeness of the inference rules presented in Figure 1.

Namely, we prove that in any equational theory E, a given goal G such that $E \models G\sigma$, may be transformed by applications of rules in Figure 1 applied to equations which are not *solved*, into a *solved form* with which we can define an E-unifier more general than σ. The solved form of an equation and of a goal is defined in the following way.

Definition 15. *(solved equation and solved goal)*
Let G be a set of equations. An equation $x \approx t \in G$ is in a solved form*, if x is a variable, $x \notin Var(t)$ and $x \notin Var(G \backslash \{x \approx t\})$.*

G is in a solved form if all equations in G are in solved form.

If G is in the solved form, then we define a substitution $\theta_G = [x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$. Obviously, θ_G is the most general unifier of G.

If G is a set of goal equations, an inference performed on G with one of the rules of Figure 1 is denoted by $G \rightarrow G'$, where G' is the result of this inference. The transitive, reflexive closure of $\rightarrow$ is written as $\overset{*}{\rightarrow}$.

In order to prove completeness, we will need the measure of a goal G, of which we will show that it may be decreased by application of an inference rule if G is E-unifiable and not in solved form.

Definition 16. *(measure for an equational proof)*
Let (Π, γ) be an equational proof and $U \subset Dom(\gamma)$ be a set of unsolved variables in (Π, γ). The measure $M(\Pi, \gamma)$ is a multiset of the lengths of paths starting with occurrences of variables in U.

Definition 17. *(measure of a goal)*
Let E be an equational theory, and G, an unsolved part of a goal G', such that there is a ground substitution γ, for which $E \models G'\gamma$ and hence there is an equational proof (Π', γ') of $G'\gamma$ and its subproof, (Π, γ), which is a proof of $G\gamma$, and all variables in $Var(G)$ are unsolved in (Π', γ').

The measure of G' with respect to (Π', γ') is a 4-tuple (m, n, o, p), where $m = M(\Pi, \gamma)$, n is the length of Π, o is the size of terms in $G\gamma$, p is the number of equations in G, of the form $t \approx x$, where x is a variable and t is not a variable.

Measures for different goals are compared with respect to lexicographic order.

Decomposition

$$\frac{\{f(s_1,\cdots,s_n) \approx f(t_1,\cdots,t_n)\} \cup G}{\{s_1 \approx t_1,\cdots,s_n \approx t_n\} \cup G}$$

where $f(s_1,\cdots,s_n) \approx f(t_1,\cdots,t_n)$ is selected in the goal.

Mutate

$$\frac{\{u \approx f(v_1,\cdots,v_n)\} \cup G}{\{u \approx s, t_1 \approx v_1,\cdots,t_n \approx v_n\} \cup G}$$

where $u \approx f(v_1,\cdots,v_n)$ is selected in the goal, and $s \approx f(t_1,\cdots,t_n) \in E$.[a]

Variable Mutate

$$\frac{\{u \approx f(v_1,\cdots,v_n)\} \cup G}{\{u \approx s, x \approx f(v_1,\cdots,v_n)\} \cup G}$$

where $s \approx x \in E$, x is a variable, and $u \approx f(v_1,\cdots,v_n)$ is selected in the goal.

Variable Decomposition (for cycle)

$$\frac{\{x \approx f(t_1,\cdots,t_n)\} \cup G}{\{x \approx f(x_1,\cdots,x_n)\} \cup (\{x_1 \approx t_1,\cdots,x_n \approx t_n\} \cup G)[x \mapsto f(x_1,\cdots,x_n)]}$$

where x is a variable, $x \approx f(t_1,\cdots,t_n)$ is selected in the goal,
$x \in Var(f(t_1,\cdots,t_n))$.

Variable Elimination

$$\frac{\{x \approx v\} \cup G}{\{x \approx v\} \cup G[x \mapsto v]}$$

where $x \notin Var(v)$

Orient

$$\frac{\{t \approx x\} \cup G}{\{x \approx t\} \cup G}$$

where x is a variable.
and t is not a variable.

Trivial

$$\frac{\{x \approx x\} \cup G}{G}$$

where $x \approx x$ is selected in the goal.

[a] We assume that E is closed under symmetry.

Fig. 1. E-Unification with eager Variable Elimination

Theorem 1. *Let E be a set of equations, such that $E \models G\gamma$ for some ground substitution γ. Then there is H a set of equations in the solved form, such that $G \xrightarrow{*} H$ and $\theta_H[Var(G)] \leq_E \gamma$.*

Proof. If G is already in the solved form, then $\theta_G \leq_E \gamma$.

If G is not in solved form, then there is an unsolved part of G, G', such that $u \approx v \in G'$, if $u \approx v$ is not in solved form. Assume that $u \approx v$ was selected for an inference. If $E \models G\gamma$, there must be an equational proof (Π, γ) of $G'\gamma$. We will call it an **actual proof** of $G\gamma$. There must be a subproof in (Π, γ), of $u\gamma \approx v\gamma$ and $u\gamma$, $v\gamma$ are the extreme terms in this subproof. We can also assume that all solved variables in G are solved in (Π, γ) and all unsolved variables in G are unsolved in (Π, γ). Hence there is a graph G_Π for all unsolved variables in (Π, γ). As we have seen, there are sometimes choices in constructing G_Π. The choices reflect the selection function, but in any case, we can always choose such G_Π that if $x \approx v$ is selected for an inference, $x\gamma$ is a maximal node in G_Π.

For the proof, we have to consider all possible forms of an unsolved goal equation $u \approx v$ selected for an inference. These forms are analyzed in the following cases.

1. Assume that neither u nor v is a variable.
   Let $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$ be a subproof in (Π, γ) of $u\gamma \approx v\gamma$, such that $u\gamma$ and $v\gamma$ are extreme terms in $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$.
   Assume also that there is no step at the root in $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$. Hence u and v must have the same root symbols.
   Then if we apply **Decomposition** to this equation, we get equations $s_1 \approx t_1, \ldots, s_n \approx t_n$, such that there is a subproof in (Π, γ) for each $s_i\gamma \approx t_i\gamma$, $i \in \{1, \ldots, n\}$, and if $u\gamma$, $v\gamma$ were the extreme terms in $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$, each of $s_i\gamma$, $t_i\gamma$ are extreme terms in their respective subproofs. Hence $E \models \{s_1\gamma \approx t_1\gamma, \ldots, s_n\gamma \approx t_n\gamma\}$. The sum of the lengths of the subproofs is equal to the length of the original subproof of $u\gamma \approx v\gamma$, but $\Sigma_{i=1}^{n}|s_i\gamma| + |t_i\gamma| < |u\gamma| + |v\gamma|$. Let (m, n, o, p) be the measure of the goal before Decomposition and (m', n', o', p') after Decomposition. $m' = m$, $n' = n$ and $o' < o$.
2. Assume that neither u nor v is a variable.
   Let $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$ be a subproof in (Π, γ) of $u\gamma \approx v\gamma$, such that $u\gamma$ and $v\gamma$ are extreme terms in $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$.
   Assume also that there is a step at the root in $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$.
   $(\Sigma_{u\gamma \approx v\gamma}, \gamma)$ has the form: $u\gamma \approx \cdots \approx w_i \approx_{[\epsilon, s_i \approx t_i, \gamma]} w_{i+1} \approx \cdots \approx v\gamma$. Let us choose i in such a way, that this is the rightmost root step in this proof and assume that t_i is not a variable.
   Then there is no root step between w_{i+1} and $v\gamma$. Since the i'th step is at the root position, $s_i\gamma = w_i$ and $t_i\gamma = w_{i+1}$. Since there is no root step between $t_i\gamma$ and $v\gamma$, both these terms must have the same root symbol and thus we can at once decompose them, obtaining possible empty set of equations: $t_1 \approx v_1, \ldots, t_n \approx v_n$, such that for each $k \in \{1, \ldots, n\}$, $t_n\gamma \approx v_n\gamma$ has a subproof in (Π, γ), and moreover each $t_i\gamma$, $v_i\gamma$ are extreme subterms in their respective subproofs. In this case **Mutate** is applicable, and we see that $E \models \{u\gamma \approx s_i\gamma, t_1\gamma \approx v_1\gamma, \ldots, t_n\gamma \approx v_n\gamma\}$.

Let (m, n, o, p) be the measure of the goal before Mutate and (m', n', o', p') after Mutate. $m' = m$ and $n' < n$.

3. Assume that u and v are the same as in the previous case, but now t_i is a variable.
   In this case we don't want to "decompose" variable t_i, but we see that the rule **Variable Mutate** gives us two equations such that: $E \models u\gamma \approx s_i\gamma$ and $E \models t_i\gamma \approx v\gamma$.
   Both $u\gamma \approx s_i\gamma$ and $t_i\gamma \approx v\gamma$ have subproofs in (Π, γ) and the terms of these equations are extreme terms in their respective subproofs.
   Let (m, n, o, p) be the measure of the goal before Variable Mutate and (m', n', o', p') after Variable Mutate. $m' = m$ and $n' < n$.
4. Assume that u is a variable x, v is not a variable and $x \in Var(v)$.
   Let $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$ be a subproof in (Π, γ) of $x\gamma \approx v[x]\gamma$, such that $x\gamma$ and $v[x]\gamma$ are extreme terms in $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$. Hence $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$ is a subproof associated with $x\gamma$.
   Since x has an occurrence in $v[x]\gamma$, the subproof $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$ must have length greater than 0 (an equation of the type $a \approx a$ would have proof of the length 0).
   Again, we look at the subproof $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$. If there is a step at the root in the subproof, the right rule to apply is **Mutate** or **Variable Mutate**, depending on the form of equation from E used in the step at the root. Hence the analysis is exactly the same as in 2 or 3.
5. Assume that u is a variable x, v is not a variable and $x \in Var(v)$.
   Like in the previous case, we argue that an appropriate subproof $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$ in (Π, γ) has length greater than 0. But this time assume that there is no step at the root in $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$.
   It means that $x\gamma$ and $v[x]\gamma$ must have the same symbol at their roots (and all the steps in the subproof $(\Sigma_{x\gamma\approx v[x]\gamma}, \gamma)$ are performed under the root). The subproof has the form: $x\gamma = f(u_1, \ldots, u_n)\gamma \approx \cdots \approx f(v_1, \cdots, v_n)\gamma = v\gamma$.
   The right rule to apply to $x \approx v[x]$ is **Variable Decomposition** , we get an equation $x \approx f(x_1, \ldots, x_n)$ in the conclusion. Since γ is general extension of itself, $x_1, \ldots, x_n$ are fresh variables, that are already in $Dom(\gamma)$. $\gamma_{x_i} = [x_i \mapsto u_i]$, for $1 \leq i \leq n$. $E \models x\gamma \approx f(x_1, \ldots, x_n)\gamma$ and $E \models x_i\gamma \approx v_i\gamma$. We notice also that since $x\gamma = f(x_1, \ldots, x_n)\gamma$, $G[x \mapsto f(x_1, \ldots, x_n)]\gamma = G\gamma$. Therefore $E \models G[x \mapsto f(x_1, \ldots, x_n)]\gamma$. And since $x\gamma = f(x_1, \ldots, x_n)\gamma$, $E \models x_i\gamma \approx v_i[x \mapsto f(x_1, \ldots, x_n)]\gamma$.
   For each equation in $(\{x_1 \approx v_1, \ldots, x_n \approx v_n\} \cup G)[x \mapsto f(x_1, \ldots, x_n)])\gamma$ there is a subproof in (Π, γ).
   In the consequence of application of Variable Decomposition, x gets solved. Indeed, $f(x_1, \ldots, x_n)\gamma = ass(x\gamma)$ and we state that $(\Pi, \gamma) \overset{[x\mapsto f(x_1,\ldots,x_n)]}{\rightarrow} (\Pi', \gamma')$, where $\gamma' = \gamma$, $\Pi' = \Pi$ and x is solved in (Π', γ').
   The actual equational proof for the goal will be changed to (Π', γ'). By Corollary 2, $M(\Pi', \gamma') < M(\Pi, \gamma)$.
   Let (m, n, o, p) be the measure of the goal before Variable Decomposition and (m', n', o', p') after Variable Decomposition. $m' < m$.

6. Assume that v is a variable and u is not a variable. Then **Orient** applies. Obviously, Orient preserves the set of E-unifiers for $u \approx v$. Let (m, n, o, p) be the measure of the goal before Orient and (m', n', o', p') after Orient. $m' \leq m, n' \leq n, o' \leq o$ and $p' < p$.
7. Assume that $x \approx v$ was selected for an inference and $x \notin Var(v)$. Then $E \models x\gamma \approx v\gamma$ and there is a subproof $(\Sigma_{x\gamma \approx v\gamma}, \gamma)$ in the proof (Π, γ) such that $x\gamma$ and $v\gamma$ are the extreme terms of $(\Sigma_{x\gamma \approx v\gamma}, \gamma)$. If x is unsolved in the goal G, x is also unsolved in (Π, γ).

   Hence we know that $v\gamma = ass(x)$ and $(\Pi, \gamma) \overset{[x \mapsto v\gamma]}{\to} (\Pi', \gamma')$.
   $M(\Pi, \gamma) > M(\Pi', \gamma')$.
   The right rule to apply is therefore **Variable Elimination**.
   Since $E \models G\gamma$, also $E \models G\gamma'$ and (Π', γ') is the proof of $G\gamma'$. We change the actual equational proof to (Π', γ') and take it as the basis of completeness argument of further inferences. Since $x\gamma' = v\gamma'$, $E \models G[x \mapsto v]\gamma'$.
   Let (m, n, o, p) be the measure of the goal before Variable Elimination and (m', n', o', p') after Variable Elimination.
   $m' < m$ after Variable Elimination. Notice also that after Variable Elimination, for each $u' \approx v'$ in G' there is a subproof in (Π', γ') such that $u'\gamma'$ and $v'\gamma'$ are the extreme terms in this subproof. If $u' \approx v'[x]$ was in G', then after Variable Elimination, $u' \approx v'[v]$ in G' and obviously (because of extension) there is a subproof $(\Sigma_{u'\gamma' \approx v'[v]\gamma'}, \gamma')$ in (Π', γ').
8. Assume that u and v are occurrences of the same variable x. Since the proof of $x\gamma \approx x\gamma$ has length 0, we can get rid of this equation in the goal by applying **Trivial**.
   Let (m, n, o, p) be the measure of the goal before Trivial and (m', n', o', p') after Trivial. $m' = m$, $n' = n$ and $o' < o$.

7 Conclusion

E-unification procedures are inherently non-deterministic, because there are usually many ways to apply inferences to goal equations and many possibilities of solving a goal. It means that a search space for a solution may be very extensive. Any restrictions of this non-determinism that we may justify are therefore welcome as restrictions of this search space. Eager Variable Elimination means that the rule should be applied whenever an equation $x \approx v$ is selected and x does not appear in v. In this case, we would not try to apply other rules to this equation. On the other hand, we may see that the ground equational proof of an instance of a goal, may be made longer by Variable Elimination. This means that we will have to do more Mutate inferences in order to reach solution. One can think about some *memoization* techniques to detect and reduce such possible overhead.

We think that the proof of completeness of eager Variable Elimination opens some possibilities of finding new classes of equational theories defined syntactically, for which E-unification problem may be proved solvable and tractable.

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A Normalization of equational proofs

If $\Pi[i]|_{\alpha_n}, \Pi[i+1]|_{\alpha_n}, \dots, \Pi[i+k]|_{\alpha_n}$ is a subproof sequence in Π, there may be a step in Π between two consecutive terms $\Pi[i+n]$ and $\Pi[i+n+1]$ at the position α_{i+n} such that $\alpha_{i+n} \geq \alpha_n$ or $\alpha_{i+n} = \alpha_n$, then we can write this step in the subproof as:

$\Pi[i+n]|_{\alpha_n} \approx_{[\alpha', s_{i+n} \approx t_{i+n}, \gamma]} \Pi[i+1]|_{\alpha_n}$, where $\alpha_n \alpha' = \alpha_{i+n}$.

But it may also be that α_{i+n} is a parallel position to α_n, and then $\Pi[i+n]|_{\alpha_n} == \Pi[i+n+1]|_{\alpha_n}$.

Definition 18. *(fake step)*
If $\Pi[i]|_{\alpha_n}, \Pi[i+1]|_{\alpha_n}, \dots, \Pi[i+k]|_{\alpha_n}$ is a subproof sequence in an equational proof (Π, γ), a step $\Pi[i+n]|_{\alpha_n} \approx_{[\alpha', s_{i+n} \approx t_{i+n}, \gamma]} \Pi[i+1]|_{\alpha_n}$ in the subproof, where $\alpha_n \alpha' = \alpha_{i+n}$ and $\alpha' || \alpha_n$ is called a **fake step** *in the subproof and is written as $\Pi[i+n]|_{\alpha_n} = \Pi[i+n+1]|_{\alpha_n}$.*

Note that $=$ is thus overloaded with a second meaning. Until now $s = t$ meant only that s and t were the same ground objects. Here it means that there is no step taken in the proof between these ground subterms.

Accordingly, if for some variable x, $x\gamma = w_i$ and $w_i = w_{i+1}$ in a subproof, also $x\gamma = w_{i+1}$, because there is no real step between w_i and w_{i+1}.

In a normalized proof such fake steps in a subproof will be possible only at the beginning and at the end of a subproof. Hence subproof in a normalized proof will have always the following form: $w_1 = \dots = w_1 \approx w_2 \approx \dots \approx w_k = \dots = w_k$.

Accordingly I will call a subproof which can be written in the form: $w_1 = \dots = w_1 \approx w_2 \approx \dots \approx w_k = \dots = w_k$, a subproof in a normal form, and the part of it which can be written in the form: $w_1 \approx w_2 \approx \dots \approx w_k$, a proper form of the subproof.

Definition 19. *(proof in a normal form)*
An equational proof (Π, γ) is in normal form, if all its subproofs are in normal forms.

Lemma 2. *Each equational proof may be normalized.*

Proof. If (Π, γ) has no proper subproofs, then (Π, γ) is in normal form.

Let (Π, γ) be an equational proof not in normal form. We will construct a new proof (Π', γ) which is in a normal form in a recursive way, such that (Π', γ) differs from (Π, γ) only in the order of steps and is the proof of the same ground equation.

We should identify all subproofs in (Π, γ) and their proper forms. For induction assume that all smaller proofs than (Π, γ) may be normalized. As the measure for a proof, let us assume the pair (n, M), where n is length of a proof (number of steps) and M is multiset of sizes of its terms.

Each proper subproof in (Π, γ) is either shorter than (Π, γ) or has smaller terms. Hence for induction argument we can assume that each proper subproof of (Π, γ) is in normal form and thus its proper form can be easily identified.

In the course of the following construction, we will move step by step through the proof sequence Π, remembering the set A of recent parallel positions at which the steps have been taken. At the beginning A is an empty set.

1. First step. Take the first step in the proof (Π, γ): $\Pi[1] \approx_{[\alpha_1, s_1 \approx t_1, \rho_1]} \Pi[2]$.
   (a) If $\alpha_1 = \epsilon$, write it as the first step of the proof sequence Π'. There is a subproof (Π'', γ) in (Π, γ) starting with $\Pi[2]$ at position ϵ. The subproof is shorter than (Π, γ), hence it has normal form. Embed normal form of (Π'', γ) into $\Pi[1]$ and stop.
   (b) If $\alpha_1 > \epsilon$, then there is a part of (Π, γ) at the depth α_1 and a subproof starting with $\Pi[1]|_{\alpha_1}$ (all terms in this subproof are subterms at position α_1 of some consecutive terms in Π).
   Embed the normalized, proper part of the subproof into $\Pi[1]$. Put α_1 as the first element of A and go to the next step in Π.
2. Next step. Assume that we are done with $i-1$'th consecutive step in Π. Now we consider next step: $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \rho_i]} \Pi[i+1]$.
   (a) If $\alpha_i \geq \beta$ for any $\beta \in A$ then the step belongs to the subproof already embedded into proof sequence Π', hence go to the next step in Π.
   (b) If $\alpha_i < \beta$, for any $\beta \in A$ (step above the last subproof steps), then $\Pi[i]$ must be the last term of the part containing the previous subproof. $\Pi[i]_{\alpha_i}$ is the first term in a subproof Π'' of Π, which is in the part of Π for the depth α_i and is composed of the terms of Π at the depth α_i.
   Hence embed normalized, proper form of Π'' into $\Pi[i]$, attaching it into Π'. Replace each β in A, such that $\alpha_i < \beta$ with α_i in A and go to the next step of Π.
   (c) If $\alpha_i || \beta$, for each $\beta \in A$ then there is a subproof sequence Π''' starting with $\Pi[i]$ at the depth α_i. Let $\Pi'[l]$ be the last term in the proof sequence Π' constructed up to now. From the construction of Π', we know that $\Pi[i]_{\alpha_i} == \Pi'[l]_{\alpha_i}$ (steps at a parallel position could not change this subterm). Embed normalized, proper version of $\Pi''s'$ into $\Pi'[l]_{\alpha_i}$, attaching it to Π', add α_i as the next element to A and go to the next step of Π.