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Modal Logics of Topological Relations

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Modal Logics of Topological Relations

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Abstract

The eight topological RCC8 *(or Egenhofer-Franzosa)-relations between spatial regions play a fundamental role in spatial reasoning, spatial and constraint databases, and geographical information systems. In analogy with Halpern and Shoham's modal logic of time intervals based on the Allen relations, we introduce a family of modal logics equipped with eight modal operators that are interpreted by the* RCC8*-relations. The semantics is based on region spaces induced by standard topological spaces, in particular the real plane. We investigate the expressive power and computational complexity of the logics obtained in this way. It turns our that, similar to Halpern and Shoham's logic, the expressive power is rather natural, but the computational behavior is problematic: topological modal logics are usually undecidable and often not even recursively enumerable. This even holds if we restrict ourselves to classes of finite region spaces or to substructures of region spaces induced by topological spaces. We also analyze modal logics based on the set of* RCC5*-relations, with similar results.*

1 Introduction

Reasoning about topological relations between regions in space is recognized as one of the most important and challenging research areas within Spatial Reasoning in AI and Philosophy, Spatial and Constraint Databases, and Geographical Information Systems (GISs). Research in this area can be classified according to the logical apparatus employed:

– General first-order theories of topological relations between regions are studied in AI and Philosophy [7; 29; 28], Spatial Databases [27; 33] and, from, an algebraic viewpoint, in [9; 34];

– Purely existential theories formulated as constraint satisfaction systems over jointly exhaustive and mutually disjoint sets of topological relations between regions [10; 31; 18; 33; 29]

– Modal logics of space with operators interpreted by the closure and interior operator of the underlying topological space and propositions interpreted as subsets of the topological space, see e.g., [21; 5; 1].

A similar classification can be made for Temporal Reasoning: general first-order theories [3], temporal constraint systems [2; 36; 26] and modal temporal logics like Prior's tense logics, LTL, and CTL [16; 12].

However, one of the most important and influential approaches in temporal reasoning has not yet found a fully developed analogue on the spatial reasoning research agenda: Halpern and Shoham's Modal Logic of intervals [19], in which propositions are interpreted as sets of intervals (those in which they are true) and reference to other intervals is enabled by modal operators interpreted by Allen's 13 relations between intervals. Despite its bad computational behavior (undecidable, usually not even r.e.), this framework proved extremely fruitful and influential in temporal reasoning, see e.g. [35; 4; 22].

To develop an equally powerful and useful modal language for reasoning about topological relations between regions, we first have to select a set of basic relations. In the initially mentioned research areas, there seems to be consensus that the eight RCC8-relations, which are also known as "Egenhofer-Franzosa"-relations and have been independently introduced in [29] and [11], are very natural and important—both from a theoretical and a practical viewpoint, see e.g. [27; 10]. Thus, in this paper we will consider modal logics with eight modal operators interpreted by the eight RCC8-relations, and whose formulas are interpreted as sets of regions (those in which they are true). This modal framework for reasoning about regions has been suggested in an early paper by Cohn [8] and further considered in [37]. However, it proved difficult to analyze the computational behavior of such logics and, despite several efforts, to the best of our knowledge no results have been obtained so far.

To relate this approach to previous and ongoing work on first-order theories of regions [29; 28; 27; 33], it is important to observe that the modal logic we propose is a fragment of

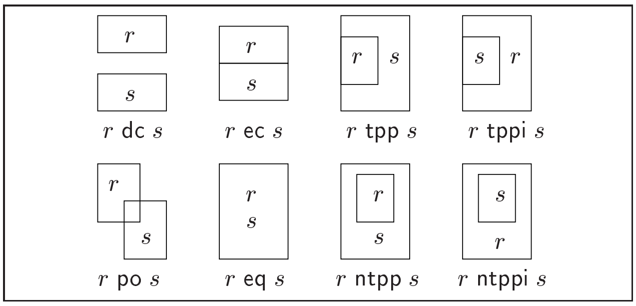

Figure 1. Examples for the RCC8 relations.

$\circ$	dc	ec	tpp	tppi	po	ntpp	ntppi
dc	$*$	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	dc	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	dc
ec	dc,ec, po,tppi, ntppi	dc,ec, po,tpp, tppi,eq	ec,po, tpp, ntpp	dc,ec	dc,ec, po,tpp, ntpp	po, tpp, ntpp	dc
tpp	dc	dc,ec	tpp,ntpp	dc,ec, po,tpp, tppi,eq	dc,ec, po,tpp, ntpp	ntpp	dc,ec, po,tppi, ntppi
tppi	dc,ec, po,tppi, ntppi	ec,po, tppi, ntppi	po,eq, tpp, tppi	tppi,ntppi	po, tppi, ntppi	po, tpp, ntpp	ntppi
po	dc,ec, po,tppi, ntppi	dc,ec, po,tppi, ntppi	po, tpp, ntpp	dc,ec, po,tppi, ntppi	$*$	po, tpp, ntpp	dc,ec, po,tppi, ntppi
ntpp	dc	dc	ntpp	dc,ec, po,tpp, ntpp	dc,ec, po,tpp, ntpp	ntpp	$*$
ntppi	dc,ec, po,tppi, ntppi	po, tppi, ntppi	po, tppi, ntppi	ntppi	po, tppi, ntppi	po, tppi, tpp,ntpp, ntppi,eq	ntppi

Figure 2. The RCC8 composition table.

first-order logic with the eight binary RCC8-relations *and infinitely many unary predicates*. More precisely, we will show that our logic has exactly the same expressive power as the two-variable fragment of this FO logic—although the latter is exponentially more succinct. Since usual first-order theories of regions admit arbitrarily many variables but *no* unary predicates, their expressive power is incomparable to the one of our modal logics. We argue that the availability of unary predicates is essential for a wide range of application areas: in contrast to describing only purely topological properties of regions, it allows to also capture other properties such as being a country (in a GIS), a ball (for a soccer-playing robot), or a protected area (in a spatial database). In our modal logic, we can then formulate constraints such as "there are no two overlapping regions that are both countries" and "every river is connected to an ocean or a lake".

The purpose of this paper is to *introduce modal logics of topological relations in a systematic way, perform an initial investigation of their expressiveness and relationships, and analyze their computational behavior*. More precisely, this paper is organized as follows: in Section 2, we introduce region spaces, which form the semantical basis for our logics. The modal language is introduced in Section 3, and a brief analysis of its expressiveness is performed. In Section 4, we identify a number of natural logics induced by different classes of region structures, and analyze their relationship. In Section 5, we then prove the undecidability of our logics. This is strengthened to a Π^1_1-hardness proof in Section 6, where it is shown that only very few of our logics are recursively enumerable. We prove undecidability of classes of finite region spaces in Section 7. Finally, in Section 8 we consider modal logics based on the non-topological, but spatial RCC5-relations and show that they, too, are usually undecidable.

2 Structures

We want to reason about models whose domains consist of regions that are related by the eight RCC8-relations dc ('disconnected'), ec ('externally connected'), tpp ('tangential proper part'), tppi ('inverse of tangential proper part'), po ('partial overlap'), eq ('equal'), ntpp ('non-tangential proper part'), and nttpi ('inverse of non-tangential proper part'). Figure 1 gives examples of the RCC8 relations in the real plane $\mathbb{R}^2$, where regions are rectangles. Different spatial ontologies give rise to different notions of regions and, therefore, different classes of models. Almost all definitions of regions provided in the literature, however, have in common that the resulting models are *region structures*

$$\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots \rangle,$$

where W is a non-empty set (of regions) and the $\mathsf{r}^{\mathfrak{R}}$ are binary relations on W that are mutually disjoint (i.e., $\mathsf{r}^{\mathfrak{R}} \cap \mathsf{q}^{\mathfrak{R}} = \emptyset$, for $\mathsf{r} \neq \mathsf{q}$), jointly exhaustive (i.e., the union of all $\mathsf{r}^{\mathfrak{R}}$ is $W \times W$), and satisfy the following:

- eq is interpreted as the identity on W, dc, ec, and po are symmetric, and tppi and ntppi are the inverse relations of ttp and ntpp, respectively;

- the rules of the RCC8 composition table (Figure 2) are satisfied in the sense that, for any entry $\mathsf{q}_1, \ldots, \mathsf{q}_k$ in row r_1 and column r_2, the first-order sentence

  $$\forall x \forall y \forall z ((\mathsf{r}_1(x,y) \wedge \mathsf{r}_2(y,z)) \rightarrow (\mathsf{q}_1(x,z) \vee \cdots \vee \mathsf{q}_k(x,z))$$

  is valid ($*$ is the disjunction over all RCC8-relations).

Denote the class of all region structures by $\mathcal{RS}$. Although of definite interest as a basic class of models representing the relation between regions in space, often more restricted definitions of region structures are considered. On the one hand, one can consider further first-order conditions on region structures, say, (fragments) of the RCC-theory [29]. Another possibility is to consider only region structures that are induced by (classes of) topological spaces. Recall that a topological space is a pair $\mathfrak{T} = (U, \mathbb{I})$, where U is a set and $\mathbb{I}$ is an *interior operator* on U, i.e., for all $s, t \subseteq U$, we have

$$\begin{array}{rclcrcl} \mathbb{I}(U) & = & U & \quad & \mathbb{I}(s) & \subseteq & s \\ \mathbb{I}(s) \cap \mathbb{I}(t) & = & \mathbb{I}(s \cap t) & \quad & \mathbb{I}\mathbb{I}(s) & = & \mathbb{I}(s). \end{array}$$

The closure $\mathbb{C}(s)$ of s is then $\mathbb{C}(s) = U - (\mathbb{I}(U - s))$. Of particular interest are n-dimensional Euclidean spaces $\mathbb{R}^n$ based on Cartesian products of the real line with the standard topology induced by the Euclidean metric.

Depending on the application domain, different definitions of regions in topological spaces have been introduced. A rather general notion identifies regions with non-empty, *regular closed* sets, i.e. non-empty subsets $s \subseteq U$ such that $\mathbb{C}\mathbb{I}(s) = s$. We write $\mathfrak{T}_{\mathsf{reg}}$ to denote the set of non-empty, regular closed subsets of the topological space $\mathfrak{T}$. Various more restrictive definitions of regions are important in the Euclidean spaces $\mathbb{R}^n$, e.g.,

- the set $\mathbb{R}^n_{\mathsf{conv}}$ of non-empty convex regular closed subsets of $\mathbb{R}^n$;
- the set $\mathbb{R}^n_{\mathsf{rect}}$ of closed hyper-rectangular subsets of $\mathbb{R}^n$, i.e., regions of the form $\prod_{i=1}^n C_i$, where $C_1, \dots, C_n$ are non-singleton closed intervals in $\mathbb{R}$.

In both cases we allow unbounded regions, in particular $\mathbb{R}^n$. However, we should note that the technical results proved in this paper also hold if we consider bounded regions, only.

Given a topological space $\mathfrak{T}$ and a set of regions $U_\mathfrak{T}$ in $\mathfrak{T}$ as introduced above, we obtain a region structure $\mathfrak{R}(\mathfrak{T}, U_\mathfrak{T}) = \langle U_\mathfrak{T}, \mathsf{dc}^\mathfrak{T}, \dots \rangle$ by putting:

$$\begin{array}{rll} (s,t) \in \mathsf{dc}^\mathfrak{T} & \text{iff} & s \cap t = \emptyset \\ (s,t) \in \mathsf{ec}^\mathfrak{T} & \text{iff} & \mathbb{I}(s) \cap \mathbb{I}(t) = \emptyset \ \wedge \ s \cap t \neq \emptyset \\ (s,t) \in \mathsf{po}^\mathfrak{T} & \text{iff} & \mathbb{I}(s) \cap \mathbb{I}(t) \neq \emptyset \ \wedge \ s \setminus t \neq \emptyset \ \wedge \\ & & t \setminus s \neq \emptyset \\ (s,t) \in \mathsf{eq}^\mathfrak{T} & \text{iff} & s = t \\ (s,t) \in \mathsf{tpp}^\mathfrak{T} & \text{iff} & s \cap \overline{t} = \emptyset \ \wedge \ s \cap \overline{\mathbb{I}(t)} \neq \emptyset \\ (s,t) \in \mathsf{ntpp}^\mathfrak{T} & \text{iff} & s \cap \overline{\mathbb{I}(t)} = \emptyset \\ (s,t) \in \mathsf{tppi}^\mathfrak{T} & \text{iff} & (t,s) \in \mathsf{tpp} \\ (s,t) \in \mathsf{ntppi}^\mathfrak{T} & \text{iff} & (t,s) \in \mathsf{ntpp} \end{array}$$

$\mathfrak{R}(\mathfrak{T}, U_\mathfrak{T})$ is called the region structure *induced* by $(\mathfrak{T}, U_\mathfrak{T})$. It is easy (but tedious) to verify that the conditions of region structures are satisfied. We set $\mathcal{TOP} = \{\mathfrak{R}(\mathfrak{T}, \mathfrak{T}_{\mathsf{reg}}) \mid \mathfrak{T}$ topological space $\}$.

3 Languages

The modal language $\mathcal{L}_{\mathsf{RCC8}}$ extends propositional logic with countably many variables $p_1, p_2, \dots$ and the Boolean connectives $\wedge$ and $\neg$ by means of the modal operators [dc], [ec], etc. (one for each RCC8 relation). A *region model* $\mathfrak{M} = \langle \mathfrak{R}, p_1^\mathfrak{M}, p_2^\mathfrak{M}, \dots \rangle$ for $\mathcal{L}_{\mathsf{RCC8}}$ consists of a region structure $\mathfrak{R} = \langle W, \mathsf{dc}^\mathfrak{R}, \dots \rangle$ and the interpretation $p_i^\mathfrak{M}$ of the variables of $\mathcal{L}_{\mathsf{RCC8}}$ as subsets of W. A formula φ is either true at $s \in W$ (written $\mathfrak{M}, s \models \varphi$) or false at s (written $\mathfrak{M}, s \not\models \varphi$), the inductive definition being as follows:

1. If φ is a prop. variable, then $\mathfrak{M}, s \models \varphi$ iff $s \in \varphi^\mathfrak{M}$.
2. $\mathfrak{M}, s \models \neg\varphi$ iff $\mathfrak{M}, s \not\models \varphi$.
3. $\mathfrak{M}, s \models \varphi_1 \wedge \varphi_2$ iff $s \models \varphi_1$ and $s \models \varphi_2$.
4. $\mathfrak{M}, s \models [\mathsf{r}]\varphi$ iff, for all $t \in W$, $(s,t) \in \mathsf{r}^\mathfrak{R}$ implies $\mathfrak{M}, t \models \varphi$.

We use the usual abbreviations: $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$ and $\langle \mathsf{r} \rangle \varphi$ for $\neg[\mathsf{r}]\neg\varphi$.

The discussion of the expressive power of our logic starts with three simple examples. First, the useful *universal box* $\Box_u \varphi$ has the following semantics:

$$\mathfrak{M}, s \models \Box_u \varphi \text{ iff } \mathfrak{M}, t \models \varphi \text{ for all } t \in W$$

In our logic, it can obviously be expressed as $\bigwedge_{\mathsf{r} \in \mathsf{RCC8}} [\mathsf{r}]\varphi$. Second, we can express that a formula φ holds in precisely one region (is *a nominal*) by

$$\mathsf{nom}(\varphi) = \Diamond_u (\varphi \wedge \bigwedge_{\mathsf{r} \in \mathsf{RCC8} \setminus \{\mathsf{eq}\}} [\mathsf{r}]\neg\varphi),$$

where $\Diamond_u \varphi = \neg\Box_u \neg\varphi$. The definability of nominals means, in particular, that we can express RCC8-constraints [31] in our language: just observe that constraints $(x \ \mathsf{r} \ y)$, where r is an RCC8-relation, correspond to the assertion $(p_x \wedge \langle \mathsf{r} \rangle p_y) \wedge \mathsf{nom}(p_x) \wedge \mathsf{nom}(p_y)$. Another main advantage of having nominals is that we can introduce names for regions; e.g., the formulas

$$\mathsf{nom}(\textit{Elbe}), \quad \mathsf{nom}(\textit{Dresden})$$

state that "*Elbe*" (the name of a river) and "*Dresden*" each apply to exactly one region. Third, it is useful to define operators [pp] and [ppi] as abbreviations:

$$\begin{array}{rcl} [\mathsf{pp}]\varphi & = & [\mathsf{tpp}]\varphi \wedge [\mathsf{nttp}]\varphi \\ [\mathsf{ppi}]\varphi & = & [\mathsf{tppi}]\varphi \wedge [\mathsf{nttpi}]\varphi. \end{array}$$

As in the temporal case [19] and following Cohn [8], we can classify propositions φ according to whether

- they are homogeneous, i.e. they hold continuously throughout regions: $\Box_u(\varphi \rightarrow [\mathsf{pp}]\varphi)$.
- they are anti-homogeneous, i.e. they hold only in regions whose interiors are mutually disjoint:
$\Box_u(\varphi \rightarrow ([\mathsf{pp}]\neg\varphi \wedge [\mathsf{po}]\neg\varphi)$

Instances of anti-homogeneous propositions are "*river*" and "*city*", while "*occupied-by-water*" is homogeneous. The following are some example statements in our logic (neglecting for simplicity that existence of sea harbors):

$$\Box_u(\textit{harbor-city} \leftrightarrow (\textit{city} \wedge \langle \mathsf{ntppi} \rangle(\textit{harbor} \wedge \langle \mathsf{ec} \rangle \textit{river})))$$
$$\Box_u(\textit{Dresden} \rightarrow \textit{harbor-city})$$
$$\Box_u(\textit{Elbe} \rightarrow \textit{river})$$
$$\Box_u(\textit{Dresden} \rightarrow (\langle \mathsf{po} \rangle \textit{Elbe} \wedge [\mathsf{po}](\textit{river} \rightarrow \textit{Elbe})))$$
$$\Box_u(\textit{Dresden} \rightarrow [\mathsf{ppi}]\neg \textit{river})$$

From these formulas, it follows that Dresden has a harbor that is related via ec to the river Elbe.

We now relate the expressive power of the modal language $\mathcal{L}_{\mathsf{RCC8}}$ to the expressive power of first-order languages over region structures. Since spatial first-order theories are usually formulated in first-order languages equivalent to $\mathcal{FO}_{\mathsf{RCC8}}$ with eight binary relations for the RCC8 relations and *no* unary predicates [27; 28; 33; 29], we cannot reduce $\mathcal{L}_{\mathsf{RCC8}}$ to such languages. A formal proof is provided by the observation that $\mathcal{FO}_{\mathsf{RCC8}}$ is decidable over the region space consisting of rectangles in $\mathbb{R}^2$ (in fact it is reducible to the decidable first order theory of $\langle \mathbb{R}, < \rangle$), while in Section 6 we show that $\mathcal{L}_{\mathsf{RCC8}}$ is not even r.e. over that space. Thus, the proper first-order language to compare $\mathcal{L}_{\mathsf{RCC8}}$ with is the monadic extension $\mathcal{FO}^m_{\mathsf{RCC8}}$ of $\mathcal{FO}_{\mathsf{RCC8}}$ that is obtained by adding unary predicates $p_1, p_2, \ldots$. By well-known results from modal correspondence theory [15], any modal formula φ can be polynomially translated into an equivalent formula φ^* of $\mathcal{FO}^m_{\mathsf{RCC8}}$ with only two variables such that, for any region model $\mathfrak{M}$ and any region s,

$$\mathfrak{M}, s \models \varphi \text{ iff } \mathfrak{M} \models \varphi^*[s].$$

More surprisingly, the converse holds as well: this follows from recent results of [23] since the RCC8 relations are mutually exclusive and jointly exhaustive.

Theorem 1. *For every $\mathcal{FO}^m_{\mathsf{RCC8}}$-formula $\varphi(x)$ with free variable x that uses only two variables, one can effectively construct a $\mathcal{L}_{\mathsf{RCC8}}$-formula φ^* of length at most exponential in the length of $\varphi(x)$ such that, for every region model $\mathfrak{M}$ and any region s, $\mathfrak{M}, s \models \varphi^*$ iff $\mathfrak{M} \models \varphi^*[s]$.*

A proof sketch can be found in Appendix A. There, we also argue that, due to a result of Etessami, Vardi, and Wilke [13], there exist properties that can be formulated exponentially more succinct in the two-variable fragment of $\mathcal{FO}^m_{\mathsf{RCC8}}$ than in $\mathcal{L}_{\mathsf{RCC8}}$.

4 Logics

In this section, we analyze the impact of choosing different underlying classes of region structures. As discussed in Section 2, the most important such classes are induced by topological spaces.

A formula φ is *valid* in a class of regions structures $\mathcal{S}$ if it is true in all points of all models based on region structures from $\mathcal{S}$. We use $L_{\mathsf{RCC8}}(\mathcal{S})$ to denote the logic of the class $\mathcal{S}$, i.e. the set of all $\mathcal{L}_{\mathsf{RCC8}}$-formulas valid in $\mathcal{S}$. If $\mathcal{S} = \{\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})\}$ for some topological space $\mathfrak{T}$ with regions $U_{\mathfrak{T}}$, then we write $L_{\mathsf{RCC8}}(\mathfrak{T}, U_{\mathfrak{T}})$ instead of $L_{\mathsf{RCC8}}(\mathcal{S})$.

The basic logic we consider is $L_{\mathsf{RCC8}}(\mathcal{RS})$, the logic of all region structures. On arbitrary topological spaces, we investigate $L_{\mathsf{RCC8}}(\mathcal{TOP})$, the logic of all region structures induced by topological spaces in which regions are non-empty regular closed sets. On $\mathbb{R}^n$, $n \geq 1$, we investigate the family of logics

$$L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n), \text{ where } \mathbb{R}^n_{\mathsf{reg}} \supseteq U_n \supseteq \mathbb{R}^n_{\mathsf{rect}}.$$

In particular, we may have $U_n = \mathbb{R}^n_{\mathsf{conv}}$.

In many applications, it does not seem natural to enforce the presence of *all* regions with some characteristics (say, non-empty and regular closed) in every model. Instead, one could include only those regions that are "relevant" for the application. Thus, given a class $\mathcal{S}$ of region structures, we are interested in the classes $\mathsf{S}(\mathcal{S})$ of all substructures of structures in $\mathcal{S}$. Then we write $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{S})$ as abbreviation of $L_{\mathsf{RCC8}}(\mathsf{S}(\mathcal{S}))$. Going one step further, one could even postulate that the set of relevant regions is finite (but unbounded). Thus we use $\mathsf{S}_{\mathsf{fin}}(\mathcal{S})$ to denote all finite substructures of structures in $\mathcal{S}$ and write $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{S})$ for $L_{\mathsf{RCC8}}(\mathsf{S}_{\mathsf{fin}}(\mathcal{S}))$.

It is natural to ask for the relationship between the logics just introduced. We start with two examples: first, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$ (and any other logic of spaces with finitely many regions) differs from $L_{\mathsf{RCC8}}(\mathcal{RS})$, $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and the $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ since

$$[\mathsf{pp}]([\mathsf{pp}]p \rightarrow p) \rightarrow [\mathsf{pp}]p.$$

is valid in $\mathcal{S}_{\mathsf{fin}}(\mathcal{RS})$ (it states that there does not exist an infinite ascending pp-chain). Second, the logic $L_{\mathsf{RCC8}}(\mathcal{RS})$ differs from $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and the $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ since

$$\Diamond_u(p \wedge \langle \mathsf{dc} \rangle q) \rightarrow \Diamond_u(\langle \mathsf{ppi} \rangle p \wedge \langle \mathsf{ppi} \rangle q)$$

is not valid in $\mathcal{RS}$ (it states that any two disconnected regions are proper parts of a region).

These and some more relationships are summarized in Figure 3. Perhaps most interesting is the fact that $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$ and $L_{\mathsf{RCC8}}(\mathcal{RS})$ can be regarded as logics of topological spaces, and even of $\mathbb{R}^n$:

Theorem 2. *For $n > 0$:*
(i) $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$
(ii) $L_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.

$$\begin{array}{ccccc}
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{rect}}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^2, \mathbb{R}^2_{\mathsf{rect}}) & \supset & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^2, \mathbb{R}^2_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^3, \mathbb{R}^3_{\mathsf{rect}}) & \supset & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^3, \mathbb{R}^3_{\mathsf{conv}}) & & \\
\cup & & \cup & & \\
L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP}) & = & L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \\
\cup & & \cup & & \cup \\
L_{\mathsf{RCC8}}(\mathcal{RS}) & = & L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP}) & = & L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \\
\cap & & & & \\
L_{\mathsf{RCC8}}(\mathcal{TOP}) & & & & \\
\cap & & & & \\
L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) & & & &
\end{array}$$

Figure 3. Inclusions between logics.

A proof of the theorem and a justification of the inclusions in Figure 3 (in particular the fact that they are proper) can be found in Appendix B.

It is out of the scope of this paper to fully complete the picture given by Figure 3. Yet, there are a few more interesting things to be noted. For example, using the same formulas as in the corresponding finite cases, it can be shown that $L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}}) \not\subseteq L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ and $L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}}) \not\subseteq L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$. But in contrast to the finite case, the converse inclusions do not hold either because the following formula is valid in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$:

$$\Big(\bigwedge_{1 \le i \le 3} \mathsf{nom}(p_i) \wedge \bigwedge_{1 \le i < j \le 3} \Diamond_u(p_i \wedge \langle \mathsf{dc} \rangle p_j)\Big) \rightarrow \Diamond_u(\langle \mathsf{ppi} \rangle p_1 \wedge \langle \mathsf{ppi} \rangle p_2 \wedge \neg \langle \mathsf{ppi} \rangle p_3).$$

5 Undecidability

We now establish the undecidability of all logics whose region classes contain certain region spaces based on $\mathbb{R}^n$. Indeed, an even weaker (but less natural) condition is established in Appendix C.

Theorem 3. *Let $\mathfrak{R}(\mathbb{R}^n, U) \in \mathcal{S} \subseteq \mathcal{RS}$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$, for some $n > 0$. Then $L_{\mathsf{RCC8}}(\mathcal{S})$ is undecidable.*

Thus the logics $L_{\mathsf{RCC8}}(\mathcal{S})$ and $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{S})$ are undecidable, for $\mathcal{S}$ one of $\mathcal{RS}$, $\mathcal{TOP}$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$, and $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$, with $n > 0$.

The proof is by reduction of the domino problem that requires tiling of the first quadrant of the plane to the satisfiability problem.

Definition 4. Let $\mathcal{D} = (T, H, V)$ be a *domino system*, where T is a finite set of *tile types* and $H, V \subseteq T \times T$ represent the horizontal and vertical matching conditions. We say that $\mathcal{D}$ *tiles the first quadrant of the plane* iff there exists a mapping $\tau : \mathbb{N}^2 \rightarrow T$ such that, for all $(x, y) \in \mathbb{N}^2$:

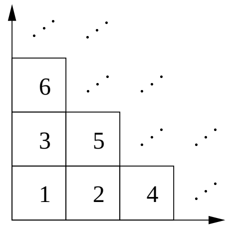

Figure 4. Enumerating tile positions.

- if $\tau(x, y) = t$ and $\tau(x + 1, y) = t'$, then $(t, t') \in H$
- if $\tau(x, y) = t$ and $\tau(x, y + 1) = t'$, then $(t, t') \in V$

Such a mapping τ is called a *solution* for $\mathcal{D}$. ◇

For reducing this domino problem to satisfiability in L_{RCC8} logics, we fix an enumeration of all the tile positions in the first quadrant of the plane as indicated in Figure 4. The function λ takes positive integers to $\mathbb{N} \times \mathbb{N}$-positions, i.e. $\lambda(1) = (0, 0)$, $\lambda(2) = (1, 0)$, $\lambda(3) = (1, 1)$, etc.

Our proof strategy is inspired by [25; 32]. Let $\mathcal{D} = (T, H, V)$ be a domino system. In the reduction, we use the following propositional letters:

- for each tile type $t \in T$, a letter p_t;
- propositional letters a, b, and c that are used to mark certain, important regions;
- propositional letters wall and floor that are used to identify regions corresponding to tiles with positions from the sets $\{0\} \times \mathbb{N}$ and $\mathbb{N} \times \{0\}$, respectively.

The reduction formula $\varphi_{\mathcal{D}}$ is defined as

$$a \wedge b \wedge \mathsf{wall} \wedge \mathsf{floor} \wedge [\mathsf{ntppi}]\neg a \wedge \Box_u \chi,$$

where χ is the conjunction of a number of formulas. We list these formulas together with some intuitive explanations:

1. ensure that the regions $\{s \in W \mid \mathfrak{M}, s \models a\}$ are ordered by the relation pp (i.e. the union of tpp and ntpp):
$$a \rightarrow [\mathsf{dc}]\neg a \wedge [\mathsf{ec}]\neg a \wedge [\mathsf{po}]\neg a \qquad (1)$$
2. enforce that the regions $\{s \mid \mathcal{M}, s \models a \wedge b\}$ are *discretely* ordered by ntpp. These regions will correspond to positions of the grid. In order to ensure discreteness, we use sequence of alternating $a \wedge b$ and $a \wedge \neg b$ regions as shown in Figure 5.
$$a \wedge b \rightarrow \langle \mathsf{tpp} \rangle (a \wedge \neg b) \qquad (2)$$
$$a \wedge \neg b \rightarrow \langle \mathsf{tpp} \rangle (a \wedge b) \qquad (3)$$
$$a \wedge \neg b \rightarrow [\mathsf{tpp}](a \rightarrow b) \qquad (4)$$
$$a \wedge b \rightarrow [\mathsf{tpp}](a \rightarrow \neg b) \qquad (5)$$

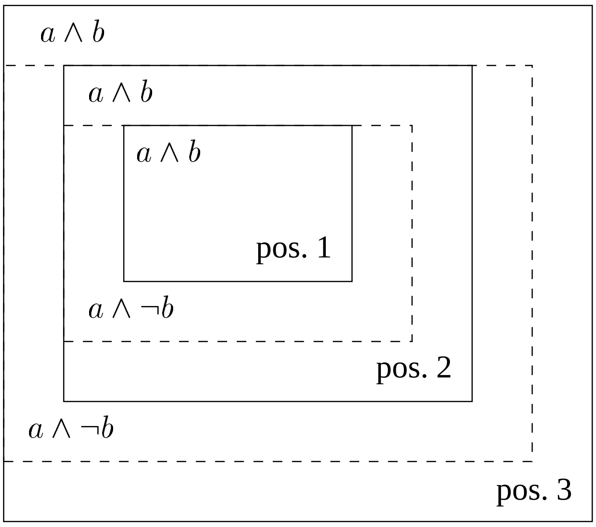

Figure 5. A discrete ordering in the plane.

If we are at an $a \wedge b$ region, we can access the region corresponding to the next grid position (w.r.t. the fixed ordering) and to the previous grid position using

$$\begin{aligned} \Diamond^{+}(\varphi) &= \langle \mathsf{tpp} \rangle (a \wedge \neg b \wedge \langle \mathsf{tpp} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^{-}(\varphi) &= \langle \mathsf{tppi} \rangle (a \wedge \neg b \wedge \langle \mathsf{tppi} \rangle (a \wedge b \wedge \varphi)). \end{aligned}$$

3. we need a way to "go right" in the grid. To this end, we introduce additional regions satisfying c as displayed in Figure 6. For example, Grid cell 2 in the figure is right of Grid cell 1, and Grid cell 4 is right of Grid cell 2.

$$a \wedge b \rightarrow \langle \mathsf{tpp} \rangle c \quad (6)$$

$$c \rightarrow \langle \mathsf{tpp} \rangle (a \wedge b) \quad (7)$$

$$c \rightarrow [\mathsf{dc}]\neg c \wedge [\mathsf{ec}]\neg c \wedge [\mathsf{po}]\neg c \wedge [\mathsf{tpp}]\neg c \wedge [\mathsf{tppi}]\neg c \quad (8)$$

We can go to the right and upper element with

$$\begin{aligned} \Diamond^{R}(\varphi) &= \langle \mathsf{tpp} \rangle (c \wedge \langle \mathsf{tpp} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^{U}(\varphi) &= \Diamond^{R}(\Diamond^{+}(\varphi)). \end{aligned}$$

Similarly, we can go to the left and down:

$$\begin{aligned} \Diamond^{L}(\varphi) &= \langle \mathsf{tppi} \rangle (c \wedge \langle \mathsf{tppi} \rangle (a \wedge b \wedge \varphi)) \\ \Diamond^{D}(\varphi) &= \Diamond^{L}(\Diamond^{-}(\varphi)). \end{aligned}$$

Considering Formulas (6) to (8), it can be checked that going to the right is a monotone and injective total function (see Appendix C).

4. axiomatize the behavior of tiles on the floor and on the wall to enforce that our "going to the right" relation

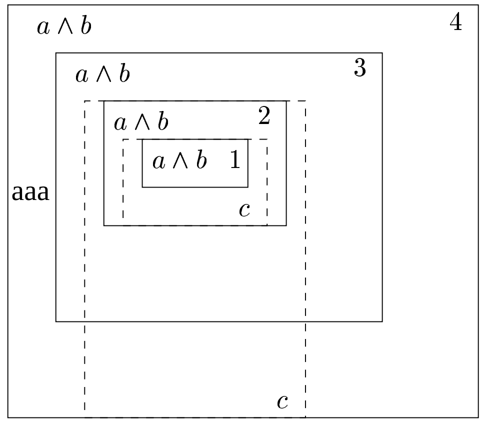

Figure 6. Two "going right" regions.

brings us to the expected position:

$$[\mathsf{ntppi}]\neg a \vee (\neg(\mathsf{floor} \wedge \mathsf{wall})) \quad (9)$$

$$\mathsf{wall} \rightarrow \Diamond^{+}\mathsf{floor} \quad (10)$$

$$\mathsf{wall} \rightarrow \Diamond^{U}(\mathsf{wall}) \quad (11)$$

$$[\mathsf{ntppi}]\neg a \vee (\mathsf{wall} \rightarrow \Diamond^{D}(\mathsf{wall})) \quad (12)$$

$$\Diamond^{R}(\neg\mathsf{wall}) \quad (13)$$

$$\neg\mathsf{wall} \rightarrow \Diamond^{L}\top \quad (14)$$

5. finally, we enforce the tiling:

$$\bigwedge_{t,t \in T} \neg(p_t \wedge p_{t'}) \quad (15)$$

$$\bigvee_{(t,t') \in H} p_t \wedge \Diamond^{R} p_{t'} \quad (16)$$

$$\bigvee_{(t,t') \in V} p_t \wedge \Diamond^{U} p_{t'} \quad (17)$$

The main strength of our reduction is that it requires only very limited prerequisites. Indeed, we will show that satisfiability of $\varphi_{\mathcal{D}}$ in *any* region model implies that $\mathcal{D}$ has a solution. Thus, to prove undecidability of some logic $L_{\mathsf{RCC8}}(\mathcal{S})$, it suffices to show that $\varphi_{\mathcal{D}}$ is satisfiable in $\mathcal{S}$ if $\mathcal{D}$ has a solution. This can be done for each region space $\mathfrak{R}(\mathbb{R}^n, U)$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$ and $n > 0$:

Lemma 5. *Let $\mathcal{D}$ be a domino system. Then:*

(i) if the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model, then the domino system $\mathcal{D}$ has a solution;

(ii) if the domino system $\mathcal{D}$ has a solution, then the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n > 0$ and each U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$.

Obviously, Theorem 3 is an immediate consequence of this lemma. A proof can be found in Appendix C, where indeed a more general variant of Lemma 5 is proved since the restriction in Point 2 is weakened.

6 Axiomatizability

In this section, we show that many of the introduced logics are Π^1_1-hard, thus *highly* undecidable and not even recursively enumerable. We start with some easy "positive" results and then prove a general "negative" result. First, we remind the reader of the following consequence of the translation of $\mathcal{L}_{\mathsf{RCC8}}$ into $\mathcal{FO}^m_{\mathsf{RCC8}}$:

Proposition 6. *If a class $\mathcal{S}$ of region structures is characterized by a finite set of axioms from $\mathcal{FO}_{\mathsf{RCC8}}$, then $L_{\mathsf{RCC8}}(\mathcal{S})$ is recursively axiomatizable.*

Recall that $\mathcal{RS}$ was defined by first-order axioms. Hence, $L_{\mathsf{RCC8}}(\mathcal{RS})$ and any $L_{\mathsf{RCC8}}(\mathcal{S})$ with $\mathcal{S}$ a first-order definable subclass of $\mathcal{S}$ are recursively enumerable. Actually, using general results on modal logics with names [17] and the fact that $\mathcal{RS}$ is axiomatized by universal first-order sentences, it is not difficult to provide a finitary axiomatization of $L_{\mathsf{RCC8}}(\mathcal{RS})$ using non-standard rules. By Theorem 2, we obtain axiomatizations for $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP})$ and every $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $n > 0$.

We now establish a non-axiomatizability result that applies to many logics $L_{\mathsf{RCC8}}(\mathcal{S})$ whose class of region spaces $\mathcal{S}$ is induced by a class of topological spaces:

Theorem 7. *The following logics are Π^1_1-hard: $L_{\mathsf{RCC8}}(\mathcal{TOP})$ and $L_{\mathsf{RCC8}}(\mathbb{R}^n, U_n)$ with $U_n \in \{\mathbb{R}^n_{\mathsf{reg}}, \mathbb{R}^n_{\mathsf{conv}}, \mathbb{R}^n_{\mathsf{rect}}\}$ and $n \geq 1$.*

To prove this result, the domino problem of Definition 4 is modified by requiring that, in solutions, a distinguished tile $t_0 \in T$ occurs infinitely often in the first column of the grid. It has been shown in [20] that this variant of the domino problem is Σ^1_1-hard. Since we reduce it to satisfiability, this yields a Π^1_1-hardness bound for validity.

As a first step toward reducing this stronger variant of the domino problem, we extend $\varphi_{\mathcal{D}}$ with the following conjunct:

$$\Box_u\Big(\langle\mathsf{ntpp}\rangle(a \wedge b \wedge \mathsf{wall} \wedge p_{t_0}) \\ \wedge [\mathsf{ntpp}]\big((a \wedge b \wedge \mathsf{wall} \wedge p_{t_0}) \rightarrow \\ \langle\mathsf{ntpp}\rangle(a \wedge b \wedge \mathsf{wall} \wedge p_{t_0})\big)\Big) \tag{18}$$

However, this is not yet sufficient: in models of $\varphi_{\mathcal{D}}$, we can have not only one discrete ordering of $a \wedge b$ regions, but rather many "stacked" such orderings (c.f. Point 5 of Claim 1 in the proof of Lemma 17). Due to this effect, the above formula does not enforce that the main ordering (there is only one for which we can ensure a proper "going to the right relation") has infinitely many occurrences of t_0.

It is thus obvious that we have to prevent stacked orderings. This is done by enforcing that there is only one "limit region", i.e. only one region approached by an infinite sequence of a-regions in the limit. We add the following formula to $\varphi_{\mathcal{D}}$:

$$\Box_u\big([\mathsf{tppi}]\langle\mathsf{po}\rangle a \rightarrow (\neg a \wedge [\mathsf{tpp}]\neg a \wedge [\mathsf{ntpp}]\neg a)\big) \tag{19}$$

Let $\varphi'_{\mathcal{D}}$ be the resulting extension of $\varphi_{\mathcal{D}}$. The classes of region spaces to which the extended reduction applies is more restricted than for the original one. We adopt the following property:

Definition 8 (Closed under infinite unions). Suppose that $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots\rangle$ is a region space. Then $\mathfrak{R}$ is called *closed under infinite unions* if $\mathfrak{R} = \mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})$ is a region space induced by a topological space $\mathfrak{T}$, and, additionally, $\mathfrak{R}$ satisfies the following property: for any sequence $r_1, r_2, \ldots \in W$ such that r_1 ntpp r_2 ntpp r_3 $\cdots$, we have $\mathbb{CI}\,(\bigcup_{i\in\omega} r_i)) \in W$. $\diamond$

We can now formulate the first part of correctness for the extended reduction. The proofs of this and the following lemma can be found in Appendix D.

Lemma 9. *Let $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}}) = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots\rangle$ be a region space that is closed under infinite unions such that all regions in $U_{\mathfrak{T}}$ are regular closed. Then the formula $\varphi'_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}$ only if the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall.*

For the second part of correctness, we again consider region spaces $\mathfrak{R}(\mathbb{R}^n, U)$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$. Note that we can *not* generalize this to a larger class of topological spaces in the same way as in the proof of Point 2 of Lemma 5 (Appendix C).

Lemma 10. *If the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall, then the formula $\varphi'_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n \geq 1$ and each U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U \subseteq \mathbb{R}^n_{\mathsf{reg}}$.*

Note that the region spaces $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$, $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$ and $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$ are closed under infinite unions. Since $\mathbb{R}^n_{\mathsf{rect}} \subseteq \mathbb{R}^n_{\mathsf{conv}} \subseteq \mathbb{R}^n_{\mathsf{reg}}$, Lemmas 9 and 10 immediately yield Theorem 7.

It is worth noting that there are a number of interesting region spaces to which this proof method does not apply. Interesting examples are the region space based on simply connected regions in $\mathbb{R}^2$ [33] and the space of polygons in $\mathbb{R}^2$ [28]. Since these spaces are not closed under infinite unions, the above proof does not show the non-axiomatizability of the induced logics. We conjecture, however, that slight modifications of the proof introduced here can be used to prove their Π^1_1-hardness as well.

7 Finite Region Spaces

We now consider logics of classes of finite region spaces. In this case, we can establish a quite general undecidability result. Moreover, undecidability of such a logic implies that it is not recursively enumerable if it is based on a class of region structures $\mathsf{S}_{\mathsf{fin}}(\mathcal{S})$ with $\mathcal{S}$ first-order definable.

Theorem 11. *If $\mathsf{S}_{\mathsf{fin}}(\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})) \subseteq \mathcal{S} \subseteq \mathsf{S}_{\mathsf{fin}}(\mathcal{RS})$ for some $n \geq 1$, then $L_{\mathsf{RCC8}}(\mathcal{S})$ is undecidable.*

Thus, the following logics are undecidable for each $n \geq 1$: $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$, and $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$.

To prove this result, we reduce yet another variant of the domino problem. For $k \in \mathbb{N}$, the *k-triangle* is the set $\{(i,j) \mid i + j \leq k\} \subseteq \mathbb{N}^2$. The task of the new domino problem is, given a domino system $\mathcal{D} = (T, H, V)$, to determine whether $\mathcal{D}$ tiles an arbitrary k-triangle, $k \in \mathbb{N}$, such that the position $(0,0)$ is occupied with a distinguished tile $s_0 \in T$, and some position is occupied with a distinguished tile $f_0 \in T$. It is shown in Appendix E that the existence of such a tiling is undecidable.

The reduction formula $\varphi_{\mathcal{D}}$ is defined as

$$a \wedge b \wedge \mathsf{wall} \wedge \mathsf{floor} \wedge s_0 \wedge [\mathsf{ntppi}]\neg a \wedge \Box_u \chi \\ \wedge (f_0 \vee \langle \mathsf{ntpp} \rangle (a \wedge b \wedge f_0)),$$

where χ is the conjunction of the Formulas (1), (3) to (5), and (7) to (17) of Section 5, and the following formulas:

- The first tile that has no tile to the right is on the floor:

$$\big(a \wedge b \wedge \neg \Diamond^R \top \wedge [\mathsf{ntppi}]((a \wedge b) \rightarrow \Diamond^R \top)\big) \rightarrow \mathsf{floor} \quad (20)$$

- If a tile has no tile to the right, then the next tile (if existent) also has no tile to the right:

$$(a \wedge b \wedge \neg \Diamond^R \top) \rightarrow (\neg \Diamond^+ \top \vee \Diamond^+ \neg \Diamond^R \top) \quad (21)$$

- The last tile is on the wall and we have no stacked orderings:

$$(a \wedge b \wedge \neg \Diamond^+ \top) \rightarrow (\mathsf{wall} \wedge [\mathsf{ntpp}]\neg(a \wedge b)) \quad (22)$$

The proof of the following lemma is now a variation of the proof of Lemma 5. Details are left to the reader.

Lemma 12. *Let $\mathcal{D}$ be a domino system. Then:*

(i) if the formula $\varphi_{\mathcal{D}}$ is satisfiable in a finite region model, then $\mathcal{D}$ tiles a k-triangle as required;

(ii) if $\mathcal{D}$ tiles a k-triangle, then $\varphi_{\mathcal{D}}$ is satisfiable in a region model based on a structure from $\mathsf{S}_{\mathsf{fin}}(\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}}))$, for each $n \geq 1$.

∘	dr	po	pp	ppi
dr	∗	dr,po,pp	dr,po,pp	dr
po	dr,po,ppi,	∗	po,pp	dr,po,ppi
pp	dr	dr,po,pp	pp	∗
ppi	dr,po,ppi,	po,pp	eq,po,pp,ppi	ppi

Figure 7. The RCC5 composition table.

Obviously, Theorem 11 is an immediate consequence of Lemma 12. Since $\mathcal{RS}$ is first-order definable, we can enumerate all finite region models. Thus, Theorems 11 and Theorem 2 give us the following:

Corollary 13. *The following logics are not r.e., for each $n \geq 1$: $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP})$, $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.*

8 The RCC5 set of Relations

For several applications, the RCC8 relations are weakened into a set of only 5 relations called RCC5 (or medium resolution topological relations) [18; 9]. This is done by keeping the relation eq and po but coarsening (1) the tpp and ntpp relations into a new "proper-part of" relation pp; (2) the tppi and ntppi relations into a new "has proper-part" relation ppi; and (3) the dc and ec relations into a new disjointness relation dr. The modal language $\mathcal{L}_{\mathsf{RCC5}}$ for reasoning about RCC8-style region structures $\mathfrak{R} = \langle W, \mathsf{ec}^{\mathfrak{R}}, \ldots \rangle$ thus extends propositional logic with the operators $[\mathsf{r}]$, where r ranges over the five RCC5-relations. They are interpreted by the relations $\mathsf{eq}^{\mathfrak{R}}$, $\mathsf{po}^{\mathfrak{R}}$, and

- $\mathsf{dr}^{\mathfrak{R}} = \mathsf{dc}^{\mathfrak{R}} \cup \mathsf{ec}^{\mathfrak{R}}$;
- $\mathsf{pp}^{\mathfrak{R}} = \mathsf{tpp}^{\mathfrak{R}} \cup \mathsf{nttp}^{\mathfrak{R}}$;
- $\mathsf{ppi}^{\mathfrak{R}} = \mathsf{tppi}^{\mathfrak{R}} \cup \mathsf{nttpi}^{\mathfrak{R}}$.

Given a class $\mathcal{S}$ of region structures, we denote by $L_{\mathsf{RCC5}}(\mathcal{S})$ the set of $\mathcal{L}_{\mathsf{RCC5}}$-formulas which are valid in all members of $\mathcal{S}$. The sets $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{S})$ and $L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{S})$ are defined analogously to the RCC8 case.

A number of results from our investigation of $\mathcal{L}_{\mathsf{RCC8}}$ have obvious analogues for $\mathcal{L}_{\mathsf{RCC5}}$: First, we can characterize the logics $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{TOP})$ by means of a composition table: denote by $\mathcal{RS}^5$ the class of all structures $\mathfrak{R} = \langle W, \mathsf{dr}^{\mathfrak{R}}, \mathsf{eq}^{\mathfrak{R}}, \mathsf{pp}^{\mathfrak{R}}, \mathsf{ppi}^{\mathfrak{R}}, \mathsf{po}^{\mathfrak{R}} \rangle$, where W is non-empty and the $\mathsf{r}^{\mathfrak{R}}$ are mutually exclusive and jointly exhaustive binary relations on W such that (1) eq is interpreted as the identity relation on W, (2) po and dr are symmetric, (3) pp is the inverse of ppi and (4) the rules of the RCC5-composition table (Figure 7) are valid. Second, it is possible to prove an analogue of Theorem 2, i.e. that, for $n \geq 1$, we have

(i) $L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{RS}^5) = L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathcal{TOP}) = L^{\mathsf{fin}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$

(ii) $L_{\mathsf{RCC5}}(\mathcal{RS}^5) = L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP}) = L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.

Third, on region models, $\mathcal{L}_{\mathsf{RCC5}}$ has the same expressive power as the two-variable fragment of $\mathcal{FL}^m_{\mathsf{RCC5}}$, i.e. the first-order language with the five binary RCC5-relation symbols and infinitely many unary predicates.

We now investigate the computational properties of logics based on $\mathcal{L}_{\mathsf{RCC5}}$. Analogously to the RCC8 case, the most natural logics are undecidable. Still, our RCC5 undecidability result is less powerful than the one for RCC8. More precisely, we have to restrict ourselves to region structures with certain properties: denote by $\mathcal{RS}^\exists$ the class of all region structures $\mathfrak{R} = \langle W, \mathsf{ec}^{\mathfrak{R}}, \dots \rangle$ such that, for any set $S \subseteq W$ of cardinality two or three, there exists a unique region $\mathsf{Sup}(S)$ such that

- s eq $\mathsf{Sup}(S)$ or s pp $\mathsf{Sup}(S)$ for each $s \in S$;
- for every region $t \in W$ with s pp t for each $s \in S$, we have $\mathsf{Sup}(S)$ eq t or $\mathsf{Sup}(S)$ pp t,
- for every region $t \in W$ with t dr s for each $s \in S$, we have t dr $\mathsf{Sup}(S)$.

It is easy to verify that $\mathcal{TOP} \subseteq \mathcal{RS}^\exists$ and $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{RS}^\exists$ for each $n > 0$.

Theorem 14. *Suppose $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{S} \subseteq \mathcal{RS}^\exists$, for some $n \geq 1$. Then $L_{\mathsf{RCC5}}(\mathcal{S})$ is undecidable. Thus, the following logics are undecidable, for each $n \geq 1$: $L_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.*

The proof is by reduction of the satisfiability problem for the undecidable modal logic $\mathsf{S5}^3$ (see [24] for the original proof in an algebraic setting. We use the modal notation of [14]). Due to space limitations, we refer the reader to [14] or to Appendix F for a formal definition of $\mathsf{S5}^3$, and just recall here that the domain of $\mathsf{S5}^3$ is a product $W_1 \times W_2 \times W_3$, and that there are three modal operators for referring to triples that are identical to the current one, but for one component.

With every $\mathsf{S5}^3$-formula φ, we associate a $\mathcal{L}_{\mathsf{RCC5}}$-formula

$$\Box_u \chi \wedge d \wedge \varphi^\sharp \tag{$*$}$$

such that φ is $\mathsf{S5}^3$-satisfiable iff $\Box_u \chi \wedge d \wedge \varphi^\sharp$ is satisfiable in a model from $\mathcal{S}$. In $(*)$, χ is the conjunction of the following formulas:

1. Each sets W_i of $\mathsf{S5}^3$-models is simulated by the set $\{r \in W \mid \mathfrak{M}, r \models a_i\}$. Thus, we introduce fresh variables a_i, $i = 1, 2, 3$, and state

$$a_i \to \bigwedge_{j=1,2,3} ([\mathsf{pp}]\neg a_j \wedge [\mathsf{ppi}]\neg a_j \wedge [\mathsf{po}]\neg a_j) \tag{23}$$

$$a_1 \to \neg a_2, a_2 \to \neg a_3, a_2 \to \neg a_3, \tag{24}$$

$$\bigwedge_{i=1,2,3} \Diamond_u a_i \tag{25}$$

2. the set $W_1 \times W_2 \times W_3$ is simulated by a fresh variable d, so we add

$$d \leftrightarrow (\bigwedge_{i=1,2,3} \langle\mathsf{ppi}\rangle a_i) \wedge \neg\langle\mathsf{ppi}\rangle(\bigwedge_{i=1,2,3} \langle\mathsf{ppi}\rangle a_i) \tag{26}$$

3. the sets $W_i \times W_j$, $1 \leq i < j \leq 3$ are simulated by fresh variables d_{ij}, so we add

$$d_{ij} \leftrightarrow (\bigwedge_{k=i,j} \langle\mathsf{ppi}\rangle a_k) \wedge \neg\langle\mathsf{ppi}\rangle(\bigwedge_{k=i,j} \langle\mathsf{ppi}\rangle a_k). \tag{27}$$

Now, we define $\varphi^\sharp$ inductively by

$$\begin{array}{rcl} p_i^\sharp & := & p_i \\ (\neg\varphi)^\sharp & := & d \wedge \neg\varphi^\sharp \\ (\varphi \wedge \psi)^\sharp & := & \varphi^\sharp \wedge \psi^\sharp \\ (\Diamond_1\varphi)^\sharp & := & \langle\mathsf{ppi}\rangle(d_{23} \wedge \langle\mathsf{pp}\rangle(d \wedge \varphi^\sharp)) \\ (\Diamond_2\varphi)^\sharp & := & \langle\mathsf{ppi}\rangle(d_{13} \wedge \langle\mathsf{pp}\rangle(d \wedge \varphi^\sharp)) \\ (\Diamond^3\varphi)^\sharp & := & \langle\mathsf{ppi}\rangle(d_{12} \wedge \langle\mathsf{pp}\rangle(d \wedge \varphi^\sharp)) \end{array}$$

The following Lemma immediately yields Theorem 14 and is proved in Appendix F.

Lemma 15. *Suppose $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \in \mathcal{S} \subseteq \mathcal{RS}^\exists$, for some $n \geq 1$. Then an $\mathsf{S5}^3$-formula φ is satisfiable in an $\mathsf{S5}^3$-model iff $\Box_u \chi \wedge d \wedge \varphi^\sharp$ is satisfiable in $\mathcal{S}$.*

9 Conclusion

We first compare our results with Halpern and Shoham's on interval temporal logic [19]: Theorems 3, 7, and 11 apply to logics induced by the region space $\mathfrak{R}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, which is clearly an interval structure. Interestingly, on this interval structure our results are stronger than those of Halpern and Shoham in two respects: first, we only need the RCC8 relations, which can be viewed as a "coarsening" of the Allen interval relations used by Halpern and Shoham. Second and more interestingly, by Theorem 3 we have also proved undecidability of the *substructure logic* $L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, which is a natural but much weaker variant of the full (interval temporal) logic $L_{\mathsf{RCC8}}(\mathbb{R}, \mathbb{R}_{\mathsf{conv}})$, and not captured by Halpern and Shoham's undecidability proof.

Several open questions for future research remain. Similar to the temporal case, the main challenge is to exhibit a decidable and still useful variant of the logics proposed in this paper. Perhaps the most interesting candidate is $L_{\mathsf{RCC5}}(\mathcal{RS})$, which coincides with the substructure logics

$L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L^{\mathsf{S}}_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, and to which the reduction exhibited in Section 8 does not apply. Other candidates could be obtained by modifying the set of relations, e.g. giving up some of them. It has for example been argued that dropping po still results in a useful formalism for geographic applications. Finally, it as an open problem whether $L_{\mathsf{RCC5}}(\mathcal{TOP})$ and $L_{\mathsf{RCC5}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ are recursively enumerable. Although we believe that they are r.e. (in contrast to their RCC8 counterparts), a proof is yet lacking.

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A Expressive Completeness

The proof of the following theorem is an adaption of the proof in [13], and a minor variant of the proof in [23] that is provided here for convenience. Throughout this section, we use $2\mathcal{FO}^m_{\mathsf{RCC8}}$ to denote the two-variable fragment of $\mathcal{FO}^m_{\mathsf{RCC8}}$ and assume that its two variables are called x and y.

Theorem 16. *For every $\mathcal{FO}^m_{\mathsf{RCC8}}$-formula $\varphi(x)$ with free variable x that uses only two variables, one can effectively construct a $\mathcal{L}_{\mathsf{RCC8}}$-formula φ^* of length at most exponential in the length of $\varphi(x)$ such that, for every region model $\mathfrak{M}$ and any region s, $\mathfrak{M}, s \models \varphi^*$ iff $\mathfrak{M} \models \varphi^*[s]$.*

Proof. A formula ξ is called a *unary atom* if it is of the form $R_i(x,x)$, $R_i(y,y)$, $A_i(x)$, or $A_i(y)$. A $2\mathcal{FO}^m_{\mathsf{RCC8}}$-formula $\rho(x,y)$ is called a *binary atom* if it is an atom of the form $\mathsf{r}(x,y)$, $\mathsf{r}(y,x)$, or $x = y$.

Let $\varphi(x) \in 2\mathcal{FO}^m_{\mathsf{RCC8}}$. We assume $\varphi(x)$ is built using $\exists$, $\wedge$, and $\neg$ only. We inductively define two mappings $\cdot^{\sigma_x}$ and $\cdot^{\sigma_y}$ where the former one takes each $2\mathcal{FO}^m_{\mathsf{RCC8}}$-formula $\varphi(x)$ to the corresponding $\mathcal{L}_{\mathsf{RCC8}}$-formula φ^{σ_x}, and the latter does the same for $2\mathcal{FO}^m_{\mathsf{RCC8}}$-formulas $\varphi(y)$. We only give the details of $\cdot^{\sigma_x}$ since $\cdot^{\sigma_y}$ is defined analogously by switching the roles of x and y.:

– If $\varphi(x) = p_i(x)$, then put $(\varphi(x))^{\sigma_x} = p_i$.

– If $\varphi(x) = \mathsf{r}(x,x)$, then put $(\varphi(x))^{\sigma_x} = \top$ if $\mathsf{r} = \mathsf{eq}$, and $(\varphi(x))^{\sigma_x} = \bot$ otherwise.

– If $\varphi(x) = \chi_1 \wedge \chi_2$, then put $(\varphi(x))^{\sigma_x} = \chi_1^{\sigma_x} \wedge \chi_2^{\sigma_x}$.

– If $\varphi(x) = \neg\chi$, then put $(\varphi(x))^{\sigma_x} = \neg(\chi)^{\sigma_x}$.

– If $\varphi(x) = \exists y \chi(x,y)$, then $\chi(x,y)$ can be written as

$$\chi(x,y) = \gamma[\rho_1, \dots, \rho_r, \gamma_1(x), \dots, \gamma_l(x), \xi_1(y), \dots, \xi_s(y)],$$

i.e. as a Boolean combination γ of ρ_i, $\gamma_i(x)$, and $\xi_i(y)$; the ρ_i are binary atoms; the $\gamma_i(x)$ are unary atoms or of the form $\exists y \gamma_i'$; and the $\xi_i(y)$ are unary atoms or of the form $\exists x \xi_i'$. We may assume that x occurs free in $\varphi(x)$. Our first step is to move all formulas without a free variable y out of the scope of $\exists$: obviously, $\varphi(x)$ is equivalent to

$$\bigvee_{\langle w_1,\dots,w_\ell\rangle \in \{\top,\bot\}^\ell} \Big(\bigwedge_{1\le i\le \ell} (\gamma_i \leftrightarrow w_i) \wedge \\ \exists y \gamma(\rho_1, \dots, \rho_r, w_1, \dots, w_l, \xi_1, \dots, \xi_s)\Big).$$

Now we "guess" a relation r that holds between x and y, and then replace all binary atoms by either true or false. For r a topological relation and $1 \le i \le r$, let

- $\rho_i^{\mathsf{r}} = \top$ if $\rho_i = \mathsf{r}(x,y)$;
- $\rho_i^{\mathsf{r}} = \top$ if $\rho_i = \mathsf{r}(y,x)$ for $\mathsf{r} \in \{\mathsf{dc}, \mathsf{ec}, \mathsf{po}\}$;
- $\rho_i^{\mathsf{r}} = \top$ if $\rho_i = \mathsf{tpp}(y,x)$ and $\mathsf{r} = \mathsf{tppi}$ or $\rho_i = \mathsf{ntpp}(y,x)$ and $\mathsf{r} = \mathsf{ntppi}$;
- $\rho_i^{\mathsf{r}} = \top$ if ρ_i is $x = y$ and $\mathsf{r} = \mathsf{eq}$;
- $\rho_i^{\mathsf{r}} = \bot$ otherwise.

Using this notiation, our last formula is equivalent to

$$\bigvee_{\langle w_1,\dots,w_\ell\rangle \in \{\top,\bot\}^\ell} \big(\bigwedge_{1\le i\le \ell} (\gamma_i \leftrightarrow w_i) \wedge \\ \bigvee_{\mathsf{r}\in\mathsf{RCC8}} \exists y (\mathsf{r}(x,y) \wedge \\ \gamma(\rho_1^t, \dots, \rho_r^t, w_1, \dots, w_l, \xi_1, \dots, \xi_s))).$$

Now compute, recursively, $\gamma_i^{\sigma_x}$ and $\xi_i^{\sigma_y}$, and define $\varphi(x)^\sigma$ as

$$\bigvee_{\langle w_1,\dots,w_\ell\rangle \in \{\top,\bot\}^\ell} \big(\bigwedge_{1\le i\le \ell} (\gamma_i^{\sigma_x} \leftrightarrow w_i) \wedge \\ \bigvee_{\mathsf{r}\in\mathsf{RCC8}} \langle \mathsf{r}\rangle \gamma(\rho_1^t, \dots, \rho_r^t, w_1, \dots, w_l, \xi_1^{\sigma_y}, \dots, \xi_s^{\sigma_y})).$$

❑

Now for the succinctness of $2\mathcal{FO}^m_{\mathsf{RCC8}}$. In [13], Etessami, Vardi, and Wilke show that, on infinite words, the two-variable fragment of first-order logic with binary predicates "successor" and "<" as well as an infinite number of unary predicates (called $2\mathcal{FO}_{\mathsf{inf}}$ in the following) have the same expressive power as temporal logic. Here, "temporal logic" refers to the variant with operators "next", "previously", "sometime in the future", and "sometime in the past", but without "until" and "since". Etessami et al. also show that the sequence of $(\varphi_n)_{n\ge1}$ of first-order sentences with two variables defined by

$$\varphi_n := \forall x \forall y \Big(\bigwedge_{i<n} (p_i(x) \leftrightarrow p_i(y)) \to (p_n(x) \leftrightarrow p_n(y))\Big)$$

is such that the shortest temporal logic formulas equivalent to φ_n have size $2^{\Omega(n)}$. Intuitively, this formula states that any two points agreeing on $p_0, \dots, p_{n-1}$ also agree on p_n. Since the above formula does not involve the successor and "<" relations, it is not hard to prove that this result carries over to our case: the $2\mathcal{FO}^m_{\mathsf{RCC8}}$-formulas $(\varphi_n)_{n\ge1}$ above are such that the shortest $\mathcal{L}_{\mathsf{RCC8}}$ formulas equivalent to φ_n have size $2^{\Omega(n)}$.

B Logics

Theorem 2. *For $n > 0$:*
(i) $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$
(ii) $L_{\mathsf{RCC8}}(\mathcal{RS}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathcal{TOP}) = L^{\mathsf{S}}_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.

Proof. (i) follows from [6; 30], where it is proved that $\mathsf{S}_{\mathsf{fin}}(\mathcal{RS}) = \mathsf{S}_{\mathsf{fin}}(\mathcal{TOP}) = \mathsf{S}_{\mathsf{fin}}(\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}))$.

We now prove (ii) with the help of (i) and the proof of [14] Theorem 16.22. Obviously,

$$\mathcal{RS} \supseteq \mathsf{S}(\mathcal{TOP}) \supseteq \mathsf{S}(\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}))$$

Hence, by 'Löwenheim-Skolem' it is sufficient to show that every countable region space $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \ldots \rangle$ is isomorphic to some substructure of $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$. However, it is proved in [14] Theorem 16.22 that every at most countable set Γ of RCC8-constraints of the form $(x \ \mathsf{r} \ y)$, r an RCC8 relation, is satisfiable in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ provided that every finite subset of Γ is satisfiable in $\mathsf{S}(\mathcal{TOP})$. Now our claim follows immediately with the help of (i). ❑

We now provide formulas which prove the inequalities of Figure 3 which were not yet considered. Since RCC8 constraints can be expressed in $\mathcal{L}_{\mathsf{RCC8}}$, we can use constraints $(x \ \mathsf{r} \ y)$, where r is an RCC8-relation and x, y are individual variables for regions.

- $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}_{\mathsf{rect}}) \not\subseteq L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{n}, \mathbb{R}^{n}_{\mathsf{rect}})$, for all $n > 0$: let, for a set of n distinct individual variables $x_1, \ldots, x_n$,

  $$\mathsf{ec}[n] = \{(x_i \ \mathsf{ec} \ x_j) \mid 1 \leq i < j \leq n\}.$$

  Then $\mathsf{ec}[2^n]$ is satisfiable in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$, but $\mathsf{ec}[2^n + 1]$ is not satisfiable in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{rect}})$.

- $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{n}, \mathbb{R}^{n}_{\mathsf{rect}}) \not\subseteq L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{m}, \mathbb{R}^{m}_{\mathsf{conv}})$, for all $m, n \geq 2$: this follows from the observation that all $\mathsf{ec}[n]$, $n < \omega$, are satifisable in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{conv}})$, $n \geq 2$.

- $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{n}, \mathbb{R}^{n}_{\mathsf{rect}}) \not\subseteq L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathcal{RS})$, for all $n > 0$. Identical to previous case.

- $L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{3}, \mathbb{R}^{3}_{\mathsf{conv}}) \not\subseteq L^{\mathsf{fin}}_{\mathsf{RCC8}}(\mathbb{R}^{2}, \mathbb{R}^{2}_{\mathsf{conv}})$: take variables x_{ij}, $1 \leq i < j \leq 4$. Then the union of $\mathsf{ec}[4]$,

  $$\{(x_i \ \mathsf{pp} \ x_{ij}), (x_j \ \mathsf{pp} \ x_{ij}) \mid 1 \leq i < j \leq 4\}$$

  and

  $$\{(x_{ij} \ \mathsf{ec} \ x_k) \mid 1 \leq i < j \leq 4, k \in \{1,2,3,4\} - \{i,j\}\}$$

  is satisfiable in $\mathfrak{R}(\mathbb{R}^3, \mathbb{R}^3_{\mathsf{conv}})$ but not in $\mathfrak{R}(\mathbb{R}^2, \mathbb{R}^2_{\mathsf{conv}})$.

- $L_{\mathsf{RCC8}}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}}) \not\subseteq L_{\mathsf{RCC8}}(\mathcal{TOP})$, for all $n > 0$: $\langle \mathsf{ppi} \rangle \top$ is valid in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$, but not in $\mathcal{TOP}$.

C Undecidability of RCC8 logics

To ease notation, throughout the appendix we denote accessibility relations in models simply with dc, ec, etc., instead of with $\mathsf{dc}^{\mathfrak{R}}$, $\mathsf{ec}^{\mathfrak{R}}$, etc.

The purpose of this section is to prove Lemma 5. Indeed, we prove the two stated Points as independent lemmas and, as announced in Section 5, even establish a stronger variant of the second Point.

Lemma 17. *If the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model, then the domino system $\mathcal{D}$ has a solution.*

Proof. Let $\mathfrak{M} = \langle \mathfrak{R}, p_1^{\mathfrak{M}}, p_2^{\mathfrak{M}}, \ldots \rangle$ be a region model of $\varphi_{\mathcal{D}}$ with $\mathfrak{R} = \langle W, \mathsf{dc}, \mathsf{ec}, \ldots \rangle$.

Claim 1. There exists a sequence $r_1, r_2, \cdots \in W$ such that

1. $\mathfrak{M}, r_1 \models \varphi_{\mathcal{D}}$,
2. $r_1 \ \mathsf{ntpp} \ r_2 \ \mathsf{ntpp} \ r_3 \ \mathsf{ntpp} \cdots$,
3. $\mathfrak{M}, r_i \models a \wedge b$ for $i \geq 1$.
4. for each $i \geq 1$, there exists a region $s_i \in W$ such that
   (a) $r_i \ \mathsf{tpp} \ s_i$,
   (b) $\mathfrak{M}, s_i \models a \wedge \neg b$,
   (c) $s_i \ \mathsf{tpp} \ r_{i+1}$,
   (d) for each region s with $r_i \ \mathsf{tpp} \ s$ and $\mathfrak{M}, s \models a \wedge \neg b$, we have $s = s_i$, and
   (e) for each region r with $s_i \ \mathsf{tpp} \ r$ and $\mathfrak{M}, r \models a \wedge b$, we have $r = r_{i+1}$,
5. for all $r \in W$ with $\mathfrak{M}, r \models a \wedge b$, we have that $r = r_i$ for some $i \geq 1$ or $r_i \ \mathsf{ntpp} \ r$ for all $i \geq 1$.

Proof: Points 1 to 4 of this claim can be proved using a simple induction. We only do the induction start since the induction step is identical. Since $\mathfrak{M}$ is a model of $\varphi_{\mathcal{D}}$, there is a region r_1 such that $\mathfrak{M}, r_1 \models \varphi_{\mathcal{D}}$. By definition of $\varphi_{\mathcal{D}}$, Point 3 is satisfied. Due to Formulas (2) and (3), there are regions s_1 and r_2 such that $r_1 \ \mathsf{tpp} \ s_1$, $\mathfrak{M}, s_1 \models a \wedge \neg b$, $s_1 \ \mathsf{tpp} \ r_2$, and $\mathfrak{M}, r_2 \models a \wedge b$. We show that all necessary Properties are satisfied:

- Point 2. Since $r_1 \ \mathsf{tpp} \ s_1$ and $s_1 \ \mathsf{tpp} \ r_2$, we have $r_1 \ \mathsf{tpp} \ r_2$ or $r_1 \ \mathsf{ntpp} \ r_2$ according to the composition table. But then, the first possibility is ruled out by Formula (5).

- Point 4d. Suppose there is an $s \neq s_1$ with $r_1 \ \mathsf{tpp} \ s$ and $\mathfrak{M}, s \models a \wedge \neg b$. Since $r_1 \ \mathsf{tpp} \ s_1$, s_1 and s are related via one of po, tpp, and tppi by the composition table. But then, the first option is ruled out by Formula (1) and the last two by Formula (4).

- Point 4e. Analogous to the previous case.

This finishes the induction, and it thus remains to prove Point 5. Assume that there is a region r such that $\mathfrak{M}, r \models a \wedge b$, $r \neq r_i$ for all $i \geq 1$, and $r_k \ \mathsf{ntpp} \ r$ does not hold for some $k \geq 1$. Since $r_k \ \mathsf{ntpp} \ r$ does not hold and $r_k \neq r$, r_k and r are related by one of dc, ec, po, tpp, tppi, and ntppi. The first three possibilities are ruled out by Formula (1), and tpp and tppi are ruled out by Formula (5). It thus remains to treat the case $r_k \ \mathsf{ntppi} \ r$. Consider the relationship between r_1 and r. Since $r_1 \neq r$ and due to Formulas (1) and (5), there are only two possibilities for this relation;

- r_1 ntpp r. Impossible by $\varphi_{\mathcal{D}}$'s subformula $[\mathsf{ntppi}]\neg a$.

- r ntpp r_1. Then we have r_1 ntpp r ntpp r_k. Take the maximal i such that r_i ntpp r and the minimal j such that r_j ntpp r. Since $r \neq r_n$ for all $n \geq 1$, we have $j = i + 1$. By Point 4, there thus is a region s with r_i ntpp s, $\mathfrak{M}, s \models a \wedge \neg b$, and s ntpp r_j. Clearly, we have s po r which is a contradiction to Formula (1).

Claim 2. For each $i \geq 1$, there exist regions t_i and u_i such that

1. r_i tpp t_i,

2. $\mathfrak{M}, t_i \models c$,

3. for each region t with r_i tpp t and $\mathfrak{M}, t \models c$, we have $t = t_i$,

4. t_i tpp u_i,

5. $\mathcal{M}, u_i \models a \wedge b$,

6. for each region u with t tpp u and $\mathfrak{M}, r \models a \wedge b$, we have $u = u_i$.

Proof: Let $i \geq 1$. By Formula (6), there is a t_i with r_i tpp t_i and $\mathfrak{M}, t_i \models c$. Let us show that t_i satisfies Property 3. To this end, let $t \neq t_i$ such that r_i tpp t and $\mathfrak{M}, t \models c$. Then t and t_i are related via one of po, tpp, and tppi. But then, all these options are ruled out by Formula (8).

Now for Points 4 to 6. By Formula (7), there is an r such that t_i tpp r and $\mathfrak{M}, r \models a \wedge b$. Point 6 can now be be proved analogously to Point 3, using Formulas (1) and (4) instead of Formula (8). This finishes the proof of Claim 2.

Before proceeding, let us introduce some notation.

- for $i, j > 0$, we write $i \Rightarrow j$ if the tile position $\lambda(j)$ can be reached from $\lambda(i)$ by going one step to the right. Similarly, we define a relation $i \Uparrow j$ for going one step up;

- for $i, j > 0$ we write $r_i \rightarrow r_j$ if $u_i = r_j$. Similarly, we write $r_i \uparrow r_j$ if $r_i \rightarrow r_{j-1}$.

Clearly, the "$\rightarrow$" and "$\uparrow$" relations are partial functions by Claims 1 and 2. The following claim establishes some other important properties of "$\rightarrow$": first, it may only move ahead in the sequence $r_1, r_2, \ldots$, but never back. And second, it is monotone and injective.

Claim 3. Let $i, j \geq 1$. Then the following holds:

1. if $r_i \rightarrow r_j$, then $i < j$;

2. if $i < j$, $r_i \rightarrow r_k$, and $r_j \rightarrow r_\ell$, then $k < \ell$.

Proof: First for Point 1. Suppose $r_i \rightarrow r_j$ and $i = j$. Then $u_i = r_i$ and, by Claim 2, r_i tpp t_i tpp r_i, which is clearly impossible: the composition table then yields that r_i is related to itself via tpp or ntpp, in contrast to the fact that r_i eq r_i, and the relations are mutually disjoint. Now suppose $r_i \rightarrow r_j$ and $i > j$. Then $u_i = r_j$ and Claim 2 yields r_i tpp t_i tpp r_j. Since $i > j$, Claim 1 gives us r_j ntpp r_i: a contradiction.

Now for Point 2. Assume $r_i \rightarrow r_k$, $r_j \rightarrow r_\ell$, and $k = \ell$. This means that $u_i = u_j = r_k$. By Claim 2, we have s_i tpp u_i and s_j tpp u_j. Thus, $s_i = s_j$ or s_i and s_j are related by one of po, tpp, and tppi. The last three possibilities are ruled out by Formula (8). Thus we get $s_i = s_j$. This, however, is a contradiction to the facts that $i < j$, and, by Claims 1 and 2, r_i ntpp r_j, r_i tpp s_i, and r_j tpp s_j.

Now assume $r_i \rightarrow r_k$, $r_j \rightarrow r_\ell$, and $k > \ell$. By Claim 1, we have r_i ntpp r_j. By Claim 2, we have r_i tpp t_i and r_j tpp t_j. It is easily verified that t_i and t_j are thus related by one of ec, po, tpp, and ntpp. All possibilities but ntpp are ruled out by Formula (8), and hence t_i ntpp t_j. We now make another derivation for the relationship between t_i and t_j, and, in this way, obtain a contradiction. Since $r_i \rightarrow r_k$ and $r_j \rightarrow r_\ell$, we have $u_i = r_k$ and $u_j = r_\ell$. By Claim 2, we thus get t_i tpp r_k and t_j tpp r_ℓ. By Claim 1 and since $k > \ell$, we have r_ℓ ntpp r_k. Thus, we obtain that t_i and t_j are related by one of ntppi, tppi, and po. This is a contradiction to the previously derived t_i ntpp t_j, thus finishing the proof of Claim 3.

The following lemma establishes the core part of the proof: the fact that the "$\rightarrow$" relation "coincides" with the "$\Rightarrow$" relation. More precisely, this follows from Point 3 of the following claim. For technical reasons, we simultaneously prove some other, technical properties. The remainder closely follows the lines of Marx and Reynolds [25].

Claim 4. Let $i \geq 1$ and $i \Rightarrow j$. Then the following holds:

1. if $\lambda(j)$ is on the floor, then $\mathfrak{M}, r_j \models \mathsf{floor}$;

2. $\mathfrak{M}, r_j \not\models \mathsf{wall}$;

3. $r_i \rightarrow r_j$ and $r_i \uparrow r_{j+1}$.

4. if $\lambda(j+1)$ is on the wall, then $\mathfrak{M}, r_{j+1} \models \mathsf{wall}$

Proof: All subclaims are proved simultaneously by induction on i. First for the induction start. Then we have $i = 1$ and $j = 2$.

1. Clearly, $\lambda(2)$ is on the floor. Since $\mathfrak{M}, r_1 \models \varphi_{\mathcal{D}}$, we have $\mathfrak{M}, r_1 \models \mathsf{wall}$. Thus Formula (10) yields $\mathfrak{M}, r_2 \models \mathsf{floor}$.

2. We have $1 \Rightarrow 2$. Point 1 gives us $\mathfrak{M}, r_2 \models \mathsf{floor}$. Since r_1 ntpp r_2, we also have $\mathfrak{M}, r_2 \not\models [\mathsf{ntppi}]\neg a$. Thus, Formula (9) yields $\mathfrak{M}, r_2 \not\models \mathsf{wall}$.

3. By Point 2, we have $\mathfrak{M}, r_2 \not\models \mathsf{wall}$. By Formula (14), there are regions $r, s \in W$ such that $\mathfrak{M}, r \models a \wedge b$, r tpp s, $\mathfrak{M}, s \models c$, and s tpp r_2. By Point 5 of Claim 1, we have either $r = r_i$ for some $i \geq 1$ or r_i ntpp r for all $i \geq 1$. In the first case, we have $r_i \rightarrow r_2$. Claim 3.1 yields $i = 1$ and we are done. In the second case, we have r_2 ntpp r: contradiction to r tpp s and s tpp r_2.

4. Since $\lambda(3)$ is on the wall, we have to show that $\mathfrak{M}, r_3 \models \mathsf{wall}$. By Point 3, we have $r_1 \uparrow r_3$. Thus, Formula (11) yields the desired result.

Now for the induction step.

1. Suppose that $\lambda(j)$ is on the floor. Since obviously $j > 1$, $\lambda(j-1)$ is on the wall. Since $i > 1$, there is a k with $i - 1 \Rightarrow k$. It is readily checked that $j - 1 = k + 1$. Thus, IH (Point 4) yields $\mathfrak{M}, r_{j-1} \models \mathsf{wall}$ and we can use Formula (10) to conclude that $\mathfrak{M}, r_j \models \mathsf{floor}$ as required.

2. First assume that $\lambda(j)$ is on the floor. Since $j > 1$, we have $\mathfrak{M}, r_j \not\models [\mathsf{ntppi}]\neg a$. Thus, Point 1 and Formula (9) yield $\mathfrak{M}, r_j \not\models \mathsf{wall}$ as required.

   Now assume that $\lambda(j)$ is not on the floor. Suppose, to the contrary of what is to be shown, that $\mathfrak{M}, r_j \models \mathsf{wall}$. Since $j > 1$, we have $\mathfrak{M}, r_j \not\models [\mathsf{ntppi}]\neg a$. Thus, by Formula (12) we obtain $\mathfrak{M}, r_j \models \Diamond^D\mathsf{wall}$. Since j is not on the floor, $i \Rightarrow j$ implies $i - 1 \Rightarrow j - 1$. Thus, the IH (Point 3) yields $r_{i-1} \uparrow r_j$. Hence, we can use $\mathfrak{M}, r_j \models \Diamond^D\mathsf{wall}$ to derive $\mathfrak{M}, r_{i-1} \models \mathsf{wall}$. By IH (Point 2), we cannot have $m \Rightarrow i - 1$ for any m. Thus, $\lambda(i-1)$ is on the wall implying that $\lambda(i)$ is on the floor. We have established a contradiction since, with $i \Rightarrow j$, this yields that j is on the floor.

3. We first show $r_i \rightarrow r_j$. By Point 2, we have $\mathfrak{M}, r_j \not\models \mathsf{wall}$. Let us show that we have $r_k \rightarrow r_j$ for some $k < j$. By Formula (14), there are regions $r, s \in W$ such that $\mathfrak{M}, r \models a \wedge b$, r tpp s, $\mathfrak{M}, s \models c$, and s tpp r_j. By Point 5 of Claim 1, we have either $r = r_k$ for some $k \geq 1$ or r_n ntpp r for all $n \geq 1$. In the first case, Claim 3.1 yields $k < j$ and we are done. In the second case, we have r_j ntpp r: contradiction to r tpp s and s tpp r_j.

   Next, we show that $k = i$. To this end, assume that $k \neq i$. We distinguish two cases:

   - $k < i$. Let ℓ be such that $k \Rightarrow \ell$. By IH (Point 3), we have $r_k \rightarrow r_\ell$. Due to functionality of "$\rightarrow$" (Claim 2) and since $r_k \rightarrow r_j$, we have $\ell = j$. Due to the injectivity of "$\Rightarrow$", we get $k = i$, which is a contradiction.
   - $i < k$. We first show that $r_i \rightarrow r_\ell$ for some $\ell > i$. By Claim 2, there are regions t and r with r_i tpp t, $\mathfrak{M}, t \models c$, t tpp r, and $\mathfrak{M}, u_i \models a \wedge b$. By Point 5 of Claim 1, we have either $r = r_\ell$ for some $\ell \geq 1$ or r_n ntpp r for all $n \geq 1$. In the first case, Claim 3.1 yields $\ell > i$. Now for the second case. Since $r_k \rightarrow r_j$, there is a $t' \in W$ with r_k tpp t', $\mathfrak{M}, c \models a \wedge b$, and t' tpp r_j. Since $i < j$, we have r_i ntpp r_j. To sum up:
     - t' tpp r_j,
     - r_j ntpp r,
     - r_i ntpp r_j,
     - r_i tpp t,
     - t tpp r.

     It is straightforward to verify that this implies that t' and t are related by one of dc, po, and ec. This is a contradiction since $\mathfrak{M}, t \models c$, $\mathfrak{M}, t' \models c$, and by Formula (8).

     Thus $r_i \rightarrow r_\ell$ for some $\ell > i$. We make a case distinction as follows:
     - $\ell < j$. There are two subcases: the tile position $\lambda(\ell)$ may or may not be on the wall.
       First assume that it is not. Then there is an $h < \ell$ with $h \Rightarrow \ell$. By definition of the "$\Rightarrow$" function, $i \Rightarrow j$, $h \Rightarrow \ell$, and $\ell < j$ this implies $h < i$. Thus we can use IH (Point 3) to conclude $r_h \rightarrow r_\ell$, a contradiction to the injectivity of "$\rightarrow$" (Claim 3.2) and the facts that $r_i \rightarrow r_\ell$ and $h < i$.
       Now assume that ℓ is on the wall. Since $1 < i < \ell$, there is a h such that $h \uparrow \ell$ and $h \rightarrow \ell - 1$. Thus, IH (Point 4) yields $\mathfrak{M}, \ell \models \mathsf{Wall}$. But then, $r_i \rightarrow r_\ell$ and Formula (13) yield a contradiction.
     - $\ell = j$. Then $r_i \rightarrow r_j$ and $r_k \rightarrow r_j$, which is a contradiction to the injectivity of "$\rightarrow$" (Claim 3.2) since $i \neq k$.
     - $\ell > j$. Contradiction to Claim 3.2.

   The second part of Point 3, i.e. $r_i \uparrow r_{j+1}$, is now an immediate consequence of the fact that $r_i \rightarrow r_j$.

4. Suppose that $\lambda(j+1)$ is on the wall. Then $\lambda(i)$ is also on the wall. Since additionally $i > 1$, there is a k such that $k \Uparrow i$ and $k \Rightarrow i - 1$. By IH (Point 4), the latter yields $\mathfrak{M}, r_i \models \mathsf{wall}$. Since Point 3 yields $r_i \uparrow r_{j+1}$, Formula (11) yields $\mathfrak{M}, r_{j+1} \models \mathsf{wall}$.

This finishes the proof of Claim 4. By definition of "$\Rightarrow$", "$\Uparrow$", "$\rightarrow$", and "$\uparrow$", Point 3 of this claim yields the following:

$$i \Rightarrow j \text{ implies } r_i \rightarrow r_j \quad \text{and} \quad i \Uparrow j \text{ implies } r_i \uparrow r_j. \quad (*)$$

Using this property, we can finally define the solution of $\mathcal{D}$: set $\tau(i,j)$ to the unique $t \in T$ such that $\mathfrak{M}, r_n \models p_t$, where $\lambda_n = (i,j)$. This is well-defined due to Formulas (15) and (16). Thus, it remains to check the matching conditions:

- Let $(i,j) \in \mathbb{N}^2$, $\lambda_n = (i,j)$, and $\lambda_m = (i+1,j)$. Then $n \Rightarrow m$. By ($*$), this yields $r_n \to r_m$. By Formula (16), there are $(t,t') \in H$ such that $\mathfrak{M}, r_n \models p_t$ and $\mathfrak{M}, r_m \models p_{t'}$. Since this implies $\tau(i,j) = t$ and $\tau(i+1,j) = t'$, the horizontal matching condition is satisfied.
- The vertical matching condition can be verified analogously using Formula (17).

❏

Now for the second Point of Lemma 5. We start with identifying a property of region spaces ensuring that, if the domino system $\mathcal{D}$ is satisfiable, then $\varphi_{\mathcal{D}}$ is satisfiable in all region spaces having this property. Thus, our proof will not be restricted to the topological spaces $\mathfrak{R}(\mathbb{R}^n, U)$ with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$.

Definition 18 (Domino ready). Let $\mathfrak{R} = \langle W, \mathsf{dc}^{\mathfrak{R}}, \mathsf{ec}^{\mathfrak{R}}, \ldots \rangle$ be a region space. Then $\mathfrak{R}$ is called *domino ready* if it satisfies the following property: the set W contains sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ such that, for $i,j \geq 1$, we have

1. x_i tpp x_{i+1};
2. x_i ntpp x_j if $j > i+1$;
3. x_{2i-1} tpp y_i;
4. y_i tpp x_{2j-1} iff the grid position $\lambda(j)$ can be reached from $\lambda(i)$ by going one step to the right and $2j-1 \leq k$;
5. y_i ntpp y_j if $j > i$.

◇

Lemma 19. *Let $\mathfrak{R} = \langle W, \mathsf{dc}, \mathsf{ec}, \ldots \rangle$ be a region space that is domino ready. If the domino system $\mathcal{D}$ has a solution, then the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}$.*

Proof. Let $\mathfrak{R}$ be a region space satisfying the condition from the lemma, $\mathcal{D} = (T, H, V)$ a domino system, and τ a solution of $\mathcal{D}$. We introduce new names for regions listed in the condition of Lemma 19 that are closer to the names used in the proof of Lemma 17:

- $r_i := x_{2i-1}$ for $i \geq 1$;
- $s_i := x_{2i}$ for $i \geq 1$;
- $t_i := y_i$.

Now define a region model $\mathfrak{M}$ based on $\mathfrak{R}$ by interpreting the propositional letters as follows:

- $a^{\mathfrak{M}} = \{r_i, s_i \mid i \geq 1\}$;
- $b^{\mathfrak{M}} = \{r_i \mid i \geq 1\}$;
- $c^{\mathfrak{M}} = \{t_i \mid i \geq 1\}$;
- $\mathsf{wall}^{\mathfrak{M}} = \{r_i \mid \lambda(i) \text{ is on the wall}\}$;
- $\mathsf{floor}^{\mathfrak{M}} = \{r_i \mid \lambda(i) \text{ is on the floor}\}$;
- $p_t^{\mathfrak{M}} = \{r_i \mid \tau(\lambda(i)) = t\}$.

It is now easy to verify that χ is satisfied by every region of $\mathfrak{M}$, and that $\mathfrak{M}, r_1 \models \varphi_{\mathcal{D}}$. ❏

For establishing the second point of Lemma 5, it obviously remains to show that the region spaces $\mathfrak{R}(\mathbb{R}^n, U)$, with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$, are domino ready.

Lemma 20. *If the domino system $\mathcal{D}$ has a solution, then the formula $\varphi_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n > 0$ and U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$.*

Proof. By Lemma 19, it suffices to show that each topological space $\mathfrak{R}(\mathbb{R}^n, U)$ with $n > 0$ and $\mathbb{R}^n_{\mathsf{rect}} \subseteq U$ is domino ready. We start with $n = 1$. Thus, we must exhibit the existence of two sequences of convex, closed intervals $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ satisfying Properties 1 to 5 from Definition 18: for $i \geq 1$, set

- $x_i := [-j, j]$ if $i = 2j-1$;
- $x_i := [-j, j-1]$ if $i = 2j$;
- $y_i := [-i, j]$ if $\lambda(j)$ is the grid position reached from $\lambda(i)$ by going a single step to the right.

It is readily checked that these sequences of intervals are as required. To find sequences for $n > 1$, just use the n-dimensional products of these intervals. ❏

Note that we can also prove this lemma if we use only *bounded* rectangles of $\mathbb{R}^n$ as regions: the construction from Lemma 20 can be easily modified such that the sequence of $a \wedge b$-rectangles converges against a finite rectangle, rather than against $\mathbb{R}^n$.

D Π^1_1-hardness of $\mathsf{RCC8}$-logics

Lemma 9. *Let $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}}) = \langle W, \mathsf{dc}, \mathsf{ec}, \ldots \rangle$ be a region space that is closed under infinite unions such that all regions in $U_{\mathfrak{T}}$ are regular closed. Then the formula $\varphi'_{\mathcal{D}}$ is satisfiable in a region model based on $\mathfrak{R}$ only if the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall*

Proof. Let $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}}) = \langle W, \mathsf{dc}, \mathsf{ec}, \ldots \rangle$ be a region space as in the lemma, $\mathfrak{M} = \langle \mathfrak{R}, p_1^{\mathfrak{M}}, p_2^{\mathfrak{M}}, \ldots \rangle$ a region model based on $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})$, and $w \in W$ such that $\mathfrak{M}, w \models \varphi'_{\mathcal{D}}$. We may establish Claims 1 to 4 as in the proof of Lemma 17, and we will use the same terminology in what follows. We first strengthen Point 5 of Claim 1 as follows:

Claim 1'. There exists a sequence $r_1, r_2, \cdots \in W$ such that

1. $\mathfrak{M}, r_1 \models \varphi_{\mathcal{D}}$,
2. r_1 ntpp r_2 ntpp r_3 ntpp $\cdots$,
3. $\mathfrak{M}, r_i \models a \wedge b$ for $i \geq 1$.
4. for each $i \geq 1$, there exists a region $s_i \in W$ such that
   (a) r_i tpp s_i,
   (b) $\mathfrak{M}, s_i \models a \wedge \neg b$,
   (c) s_i tpp r_{i+1},
   (d) for each region s with r_i tpp s and $\mathfrak{M}, s \models a \wedge \neg b$, we have $s = s_i$, and
   (e) for each region r with s_i tpp r and $\mathfrak{M}, r \models a \wedge b$, we have $r = r_{i+1}$,

5'. for all $r \in W$ with $\mathfrak{M}, r \models a \wedge b$, we have $r = r_i$ for some $i \geq 1$.

Proof of Point 5': since $\mathfrak{R}(\mathfrak{T}, U_{\mathfrak{T}})$ is closed under infinite unions, we have $t = \mathbb{CI}\,(\bigcup_{i \in \omega} r_i) \in W$. We first show that

$$t \models [\mathsf{tppi}]\langle \mathsf{po} \rangle a \qquad (*)$$

To this end, suppose t tppi q. Then we have the following:

1. $q - r_i \neq \emptyset$ for all $i > 0$.

   Since t tppi q, there exists $x \in q$ such that $x \notin \mathbb{I}(t)$. Suppose $x \in r_i$, for some r_i. Since r_i ntpp r_{i+1}, this yields $x \in \mathbb{I}(r_{i+1})$. Therefore $x \in \mathbb{I}(t)$ and we have a contradiction.

2. There exists $n > 0$ such that $i \geq n$ implies $r_i - q \neq \emptyset$.

   Suppose $r_i \subseteq q$, for all $i > 0$. Then $s = \bigcup_{i \in \omega} r_i \subseteq q$. Since $q \in U_{\mathfrak{T}}$, we have $q = \mathbb{CI}(q)$. Thus $t = \mathbb{CI}(s) \subseteq q$, and we have a contradiction to t tppi q.

3. There exists $m > 0$ such that $i \geq m$ implies $\mathbb{I}(r_j) \cap \mathbb{I}(q) \neq \emptyset$.

   Since $q = \mathbb{CI}(q)$, we have $\mathbb{I}(q) \neq \emptyset$. Take any $x \in \mathbb{I}(q)$. Since $t = \mathbb{CI}\,(\bigcup_{i \in \omega} r_i)$ and t tppi q, this yields $x \in \bigcup_{i \in \omega} r_i$. Thus there is a j with $x \in r_j$. Then $x \in \mathbb{I}(r_{j+1})$. Set $m := j + 1$. Since r_m ntpp r_i for all $i > m$, we have $x \in \mathbb{I}(q) \cap \mathbb{I}(r_{j+1})$ for all $i \geq m$.

Take $k = \max\{n, m\}$. Using the above Points 1 to 3, it is easily verified that q po r_k, thus finishing the proof of $(*)$.

Now we can establish Point 5'. By Point 5 of the original Claim 1, for all $r \in W$ with $\mathfrak{M}, r \models a \wedge b$, we have that $r = r_i$ for some $i \geq 1$ or r_i ntpp r for all $i \geq 1$. It thus suffices to show that the latter alternative yields a contradiction. Thus assume r_i ntpp r for all $i \geq 1$. Since r_1 ntpp r_2 ntpp $\cdots$ and $t = \mathbb{CI}\,(\bigcup_{i \in \omega} r_i)$, it is not hard to verify that this yields $r = t$, t tpp r, or t ntpp r. By $(*)$, t satisfies $[\mathsf{tppi}]\langle \mathsf{po} \rangle a$. By Formula (19), t thus also satisfies $\neg a \wedge [\mathsf{tpp}]\neg a \wedge [\mathsf{ntpp}]\neg a$: contradiction since $\mathfrak{M}, r \models a$.

Finally, we can define a solution of $\mathcal{D}$ as in the proof of Lemma 17. By Point 5' of Claim 1' and Formula (18), this solution is such that the tile t_0 occurs infinitely often on the wall. ❑

Lemma 10. *If the domino system $\mathcal{D}$ has a solution with t_0 occurring infinitely often on the wall, then the formula $\varphi'_{\mathcal{D}}$ is satisfiable in region models based on $\mathfrak{R}(\mathbb{R}^n, U)$, for each $n \geq 1$ and U with $\mathbb{R}^n_{\mathsf{rect}} \subseteq U \subseteq \mathbb{R}^n_{\mathsf{reg}}$.*

Proof. Let τ be a solution of $\mathcal{D}$ with t_0 appearing infinitely often on the wall. It was shown in the proof of Lemma 20 that the region spaces we are considering are domino ready. Thus we can use τ to construct a model $\mathfrak{M}$ based on the region space $\mathfrak{R}(\mathbb{R}^n, U)$ exactly as in the proof of Lemma 19. It suffices to show that $\mathfrak{M}$ satisfies, additionally, Formulas (18) and (19). This is easy for Formula (18) since τ has been chosen such that t_0 appears infinitely often. Thus, let us concentrate on Formula (19).

Let $r_1, r_2, \ldots$ be the regions from the construction of $\mathfrak{M}$ in the proof of Lemma 19. If

$$t = \mathbb{CI}\,(\bigcup_{i \in \omega} r_i) = \mathbb{R}^n \in W,$$

then t satisfies $\neg a \wedge [\mathsf{tpp}]\neg a \wedge [\mathsf{ntpp}]\neg a$ since, clearly, t is not related via eq, tpp, and ntpp to any of the r_i. To show that Formula (19) holds, it thus suffices to prove that, for all $s \in W$, $\mathfrak{M}, s \models [\mathsf{tppi}]\langle \mathsf{po} \rangle a$ implies $s = t$.

Hence fix an $s \in W$ and assume that $s \neq t$ and $\mathfrak{M}, s \models [\mathsf{tppi}]\langle \mathsf{po} \rangle a$. We distinguish two cases:

- t and s are related by one of dc, ec, po, tpp, and ntpp. Then we find a region x such that s tppi x and t dc x. Since r_i ntpp t for all $i > 0$, we thus have r_i dc x for all $i > 0$. Since only the r_i regions satisfy a, we obtain $\mathfrak{M}, x \not\models \langle \mathsf{po} \rangle a$ in contradiction to $\mathfrak{M}, s \models [\mathsf{tppi}]\langle \mathsf{po} \rangle a$.
- t and s are related by one of tppi and ntppi. Analogous to Points 2 and 3 in the proof of $(*)$ in the proof of Lemma 9, we can prove that
  1. There exists $n > 0$ such that $i \geq n$ implies $r_i - s \neq \emptyset$.

2. There exists $m > 0$ such that $i \geq m$ implies $\mathbb{I}(r_j) \cap \mathbb{I}(s) \neq \emptyset$.

Thus, there is a $k = \max\{n, m, 1\} > 1$ and a relation $\mathsf{r} \in \{\mathsf{po}, \mathsf{tppi}, \mathsf{ntppi}\}$ such that $s \ \mathsf{r} \ r_k$. Then we find a region x such that $s \ \mathsf{tppi} \ x$, $r_{k-1} \ \mathsf{dc} \ x$, and $x \ \mathsf{ntpp} \ r_k$. Since only the r_i regions satisfy a, we obtain $\mathfrak{M}, x \not\models \langle \mathsf{po} \rangle a$ in contradiction to $\mathfrak{M}, s \models [\mathsf{tppi}]\langle \mathsf{po} \rangle a$.

❏

E The Domino Problem for k-triangles

Recall that, for $k \in \mathbb{N}$, the *k-triangle* is the set

$$\{(i,j) \mid i + j \leq k\} \subseteq \mathbb{N}^2.$$

We are going to prove that the following domino problem is undecidable: given a domino system $\mathcal{D} = (T, H, V)$, determine whether $\mathcal{D}$ tiles an arbitrary k-triangle, $k \in \mathbb{N}$, such that the position $(0,0)$ is occupied by a distinguished tile $s_0 \in T$ and some position is occupied by a distinguished tile $f_0 \in T$.

The proof is via a reduction of the halting problem for Turing machines started on the empty tape. The basic idea of the proof is to represent a run of the Turing machine as a sequence of columns, each of which represents a configuration.

Let $\mathfrak{A}$ be a single-tape right-infinite Turing machine with state space Q, initial state q_0, halt state q_f, tape alphabet Σ ($b \in \Sigma$ stands for blank), and transition relation $\Delta \subseteq Q \times \Sigma \times Q \times \Sigma \times \{L, R\}$. W.l.o.g., we assume that Turing machines have the following properties:

- the initial state q_0 is only used at the beginning of computations, but not later;
- the TM comes to a stop only if it reaches q_f;
- if the TM holds, its last step is to the right;
- if the TM holds, then it labels the halting position with a special symbol $\# \in \Sigma$ before;
- the blank symbol is never written.

It is easily checked that every TM can be modified to satisfy these requirements. The configurations of $\mathfrak{A}$ will be represented by finite words of one of the forms

1. $\$xb^m$,
2. $\$a_0 \cdots a_k x y a'_0 \cdots a'_\ell b^m$,
3. $\$a_0 \cdots a_k y x a'_0 \cdots a'_\ell b^m$,

where

- "\$" marks the left end of the tape,
- $m > 0$,
- all a_i and a'_i are in Σ,
- $x \in A := Q \times \Sigma \times \{L, R\}$ represents the active tape cell, the current state, and the direction into which the TM has moved to reach the current position, and
- $y \in A^\dagger := \{(q, \sigma, M)^\dagger \mid (q, \sigma, M) \in A\}$ represents the previously active tape cell, the previous state, and the direction to which $\mathfrak{A}$ moved to reach the current position

Note that the only difference between elements of A and elements of $A^\dagger$ is that the latter are marked with a $\dagger$. Intuitively, the elements of A describe the current head position while the elements of $A^\dagger$ describe the previous one. Also note that, for technical reasons, the information whether the last step was to the left or to the right is stored twice in each column: both in the x cell and in the y cell. Configurations of Form 1 do not comprise the description of a previous state and thus represent the initial configuration.

Given a Turing machine $\mathfrak{A}$, we define a domino system $\mathcal{D}_\mathfrak{A} = (T, H, V, s_0, f_0)$ as follows:

- $T := \Sigma \cup A \cup A^\dagger \cup \{\$\}$;
- $s_0 := \langle q_0, b, L \rangle$;
- $f_0 := \langle q_f, \# \rangle$;
- $$\begin{aligned} V := {} & \{(\sigma, \sigma') \in \Sigma^2 \mid \sigma = b \text{ implies } \sigma' = b\} \cup \\ & \{(\sigma, \langle q, \sigma', L \rangle), (\langle q, \sigma', R \rangle, \sigma) \mid \\ & \qquad \sigma, \sigma' \in \Sigma, q \in Q\} \cup \\ & \{(\langle q, \sigma', L \rangle^\dagger, \sigma), (\sigma, \langle q, \sigma', R \rangle^\dagger) \mid \\ & \qquad \sigma, \sigma' \in \Sigma, q \in Q\} \cup \\ & \{(\langle q, \sigma, L \rangle, \langle q', \sigma', L \rangle^\dagger), (\langle q', \sigma', R \rangle^\dagger, \langle q, \sigma, R \rangle) \mid \\ & \qquad \sigma, \sigma' \in \Sigma, q, q' \in Q\} \cup \end{aligned}$$
- $$\begin{aligned} H := {} & \{(q_f, \sharp)\} \cup \\ & \{(\sigma, \sigma) \mid \sigma \in \Sigma\} \cup \\ & \{(\langle q, \sigma, M \rangle, \langle q', \sigma', M' \rangle^\dagger) \mid \\ & \qquad (q, \sigma, q', \sigma', M') \in \Delta, M \in \{L, R\}\} \cup \\ & \{(\sigma, \langle q, \sigma', M \rangle), (\langle q, \sigma, M \rangle^\dagger, \langle q', \sigma', M' \rangle) \mid \\ & \qquad \sigma, \sigma' \in \Sigma, q, q' \in Q, M, M' \in \{L, R\}\} \cup \\ & \{(\langle q, \sigma, M \rangle^\dagger, \sigma) \mid q \in Q, \sigma \in \Sigma, M \in \{L, R\}\} \end{aligned}$$

It is now a routine task to prove that $\mathfrak{A}$ halts on the empty tape iff the domino system $\mathcal{D}_\mathfrak{A}$ tiles some k-triangle with s_0 at position $(0,0)$ and f_0 used at some position: such a tiling immediately yields a terminating run of $\mathcal{A}$ while a run of $\mathcal{A}$ induces the tiling of a finite rectangle such that s_0 is at position $(0,0)$ and f_0 occurs somewhere. This rectangle can then be extended to an enclosing triangle by padding with the blank symbol on the top and with symbols from Σ to the right (such that every row has a constant tiling beyond the halting column—for this we need the first component of H).

F Undecidability of RCC5

The aim of this section is to prove Lemma 15. For the sake of completeness, we start with introducing the modal logic $\mathsf{S5}^3$. The language $\mathcal{L}_3$ is the extension of propositional logic by means of unary modal operators $\Diamond_1$, $\Diamond_2$ and $\Diamond_3$. $\mathcal{L}_3$ is interpreted in $\mathsf{S5}^3$-models

$$\mathfrak{W} = \langle W_1 \times W_2 \times W_3, p_1^{\mathfrak{W}}, p_2^{\mathfrak{W}}, \ldots \rangle$$

where the W_i are non-empty and $p_i^{\mathfrak{W}} \subseteq W_1 \times W_2 \times W_3$. The truth-relation $\models$ between pairs $(\mathfrak{W}, (w_1, w_2, w_3))$ with $w_i \in W_i$, and $\mathcal{L}_3$-formulas φ is defined inductively as follows:

- $\mathfrak{W}, (w_1, w_2, w_3) \models p_i$ iff $(w_1, w_2, w_3) \in p_i^{\mathfrak{W}}$;
- $\mathfrak{W}, (w_1, w_2, w_3) \models \neg\varphi$ iff $\mathfrak{W}, (w_1, w_2, w_3) \not\models \varphi$;
- $\mathfrak{W}, (w_1, w_2, w_3) \models \varphi_1 \wedge \varphi_2$ iff $\mathfrak{W}, (w_1, w_2, w_3) \models \varphi_1$ and $\mathfrak{W}, (w_1, w_2, w_3) \models \varphi_2$;
- $\mathfrak{W}, (w_1, w_2, w_3) \models \Diamond_1\varphi$ iff there exists $w_1' \in W_1$ such that $\mathfrak{W}, (w_1', w_2, w_3) \models \varphi$;
- $\mathfrak{W}, (w_1, w_2, w_3) \models \Diamond_2\varphi$ iff there exists $w_2' \in W_2$ such that $\mathfrak{W}, (w_1, w_2', w_3) \models \varphi$;
- $\mathfrak{W}, (w_1, w_2, w_3) \models \Diamond_3\varphi$ iff there exists $w_3' \in W_3$ such that $\mathfrak{W}, (w_1, w_2, w_3') \models \varphi$.

Now $\varphi \in \mathcal{L}_3$ is called $\mathsf{S5}^3$-*satisfiable* if there exists a $\mathsf{S5}^3$-model $\mathfrak{W}$ and a point (w_1, w_2, w_3) such that $\mathfrak{W}, (w_1, w_2, w_3) \models \varphi$.

The two directions of Lemma 15 are proved separately.

Lemma 21. *Suppose $\Box_u\chi \wedge d \wedge \varphi^\sharp$ is satisfiable in some $\mathfrak{R} \in \mathcal{RS}^\exists$. Then φ is $\mathsf{S5}^3$-satisfiable.*

Proof. Suppose

$$\mathfrak{M} = \langle \mathfrak{R}, a_1^{\mathfrak{M}}, a_2^{\mathfrak{M}}, a_3^{\mathfrak{M}}, d^{\mathfrak{M}}, d_{12}^{\mathfrak{M}}, \ldots, p_1^{\mathfrak{M}}, \ldots \rangle$$

satisfies $\Box_u\chi \wedge d \wedge \varphi^\sharp$, where $\mathfrak{R} = \langle W, \mathsf{dr}^{\mathfrak{R}}, \ldots \rangle \in \mathcal{RS}^\exists$. Define

$$\mathfrak{W} = \langle W_1 \times W_2 \times W_3, p_1^{\mathfrak{M}}, \ldots \rangle$$

by

- $W_i = a_i^{\mathfrak{M}}$, for $i = 1, 2, 3$;
- for all $(w_1, w_2, w_3) \in W_1 \times W_2 \times W_3$ and $i < \omega$, $(w_1, w_2, w_3) \in p_i^{\mathfrak{W}}$ iff $\mathsf{Sup}(\{w_1, w_2, w_3\}) \in p_i^{\mathfrak{M}}$.

By Formula (25), the W_i are non-empty. Now, the function $f : W_1 \times W_2 \times W_3 \to d^{\mathfrak{M}}$, defined by putting

$$f(w_1, w_2, w_3) = \mathsf{Sup}\{w_1, w_2, w_3\},$$

is a well-defined bijection:

- f is well-defined (i.e., $\mathsf{Sup}\{w_1, w_2, w_3\} \in d^{\mathfrak{M}}$), by Formula (26);
- f is injective since, by Formulas (23) and (24), we have w_1 dr w_2 for distinct $w_1, w_2 \in W_1 \cup W_2 \cup W_3$. Therefore w dr $\mathsf{Sup}\{w_1, w_2, w_3\}$ for every $w \in W_1 \cup W_2 \cup W_3$ different for w_1, w_2, w_3;
- By Formula (26), f is surjective.

Using Formula (27) one can show in the same way that $f_{ij} : W_i \times W_j \to d_{ij}^{\mathfrak{M}}$, $1 \leq i < j \leq 3$, defined by

$$f(w_i, w_j) = \mathsf{Sup}\{w_i, w_j\},$$

are well-defined bijections. Moreover, for all $(w_1, w_2, w_3) \in W_1 \times W_2 \times W_3$ and $u \in W_i$, $v \in W_j$, $1 \leq i < j \leq 3$, we obtain $\mathsf{Sup}\{u, v\}$ pp $\mathsf{Sup}\{w_1, w_2, w_3\}$ iff $u = w_i$ and $v = w_j$.

Now it is straightforward to show by induction for all subformulas ψ of φ and all $(w_1, w_2, w_3) \in W_1 \times W_2 \times W_3$:

$$\mathfrak{W}, (w_1, w_2, w_3) \models \psi \Leftrightarrow \mathfrak{M}, f(w_1, w_2, w_2) \models \psi^\sharp.$$

Take $(w_1, w_2, w_3) \in W_1 \times W_2 \times W_3$ such that $f(w_1, w_2, w_3) \models \varphi^\sharp$. Then $(w_1, w_2, w_3) \models \varphi$. ❏

Lemma 22. *Suppose φ is satisfiable in an $\mathsf{S5}^3$-model. Then $\Box_u\chi \wedge d \wedge \varphi^\sharp$ is satisfiable in $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$.*

Proof. Clearly, if φ is satisfiable, then it is satisfiable in a countable model

$$\mathfrak{W} = \langle W_1 \times W_2 \times W_3, p_1^{\mathfrak{W}}, \ldots \rangle$$

in which the W_i are mutually disjoint. Now let $n > 0$ and define a model $\mathfrak{M}$ for $\Box_u\chi \wedge d \wedge \varphi^\sharp$ on $\mathfrak{R}(\mathbb{R}^n, \mathbb{R}^n_{\mathsf{reg}})$ as follows. Let $f : W_1 \cup W_2 \cup W_3 \to \mathbb{R}^n_{\mathsf{reg}}$ be an injective mapping and set

- $a_i^{\mathfrak{M}} = \{f(w) \mid w \in W_i\}$, for $i = 1, 2, 3$;
- $d^{\mathfrak{M}} = \{f(w_1) \cup f(w_2) \cup f(w_3) \mid (w_1, w_2, w_3) \in W_1 \times W_2 \times W_3\}$;
- $d_{ij}^{\mathfrak{M}} = \{f(w_i) \cup f(w_j) \mid (w_i, w_j) \in W_i \times W_j\}$, for $1 \leq i < j \leq 3$;

- $p_i^{\mathfrak{M}} = \{f(w_1) \cup f(w_2) \cup f(w_3) \mid (w_1, w_2, w_3) \models p_i\}$ for $i < \omega$.

It is straightforward to prove that χ is true in every point of $\mathfrak{M}$. One can easily prove by induction, for every subformula ψ of φ and every $(w_1, w_2, w_3) \in W_1 \times W_2 \times W_3$, $\mathfrak{W}, (w_1, w_2, w_3) \models \psi$ iff $\mathfrak{M}, f(w_1) \cup f(w_2) \cup f(w_3) \models \psi^\sharp$. Hence $\Box_u \chi \wedge d \wedge \varphi^\sharp$ is satisfied in $\mathfrak{M}$. ❑