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Quantitative Temporal Logics: PSpace and below

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Abstract

Often, the addition of metric operators to qualitative temporal logics leads to an increase of the complexity of satisfiability by at least one exponential. In this paper, we exhibit a number of metric extensions of qualitative temporal logics of the real line that do not lead to an increase in computational complexity. The main result states that the language obtained by extending since/until logic of the real line with the operators 'sometime within n time units', n coded in binary, is PSpace-complete even without the finite variability assumption. Without qualitative temporal operators the complexity of this language turns out to depend on whether binary or unary coding of parameters is assumed: it is still PSpace-hard under binary coding but in NP under unary coding.

1 Introduction

The extension of qualitative temporal logics (TLs) with metric operators has been studied for almost fifteen years [5, 4, 7]. Of particular interest are metric extensions of temporal logics of the real line, since the resulting quantitative TLs are an important tool for the specification and verification of real-time systems [2]. Unfortunately, moving from qualitative to quantitative logics is often accompanied by an increase in computational complexity of the satisfiability problem. The most important example witnessing this effect is the PSpace-complete since/until logic of the real line [6], whose extension with a metric operator 'sometime in at least n but not more than m time units' (n and m coded in binary) becomes ExpSpace-complete if the case $n = m$ is not admitted and even undecidable if it is [1, 3, 5].

It is well known that the complexity of the metric temporal logic obtained by this extension can be reduced to PSpace again by further restricting the values of n and m, e.g., by enforcing that $n = 0$ [1]. However, in contrast to the ExpSpace-completeness and undecidability results above, this improvement has only been proven under the *finite variability assumption (FVA)* which states that no propositional variable changes its truth-value infinitely many times in any finite interval. While the FVA is a natural condition for various computer science applications, we believe that there are at least two reasons to consider also the non-FVA case: first, qualitative temporal logic originated in philosophy and mathematics to study time itself, rather than the behaviour of systems with discrete state changes as considered in computer science. If quantitative TL is used for the former purpose, the FVA is less convincing than in computer science applications. Second, even in computer science reasoning without the FVA can be fruitfully employed: assume that a formula φ of a quantitative TL describes the specification of a real-time system. Further assume that φ has been found to be unsatisfiable under FVA, indicating that the described specification is not realizable. If an additional satisfiability check *without* FVA is made revealing that dropping FVA regains satisfiability of φ, then the user obtains additional information on the source of the unrealizability of her specification: namely that it enforces an infinite number of state changes in a finite interval.

The purpose of this paper is to *investigate metric temporal logics of the real line that are at most* PSPACE*-complete*. More precisely, we prove three results. Our first and main result is that extending since/until logic of the real line with metric operators 'sometime in at most n time units', n coded in binary, is PSPACE-complete even without FVA. To show this, we propose a new method for polynomially reducing satisfiability in metric TLs whose numerical parameters are coded in binary to satisfiability in the same logic with numbers coded in unary. The essence of the reduction is to introduce new propositional variables that serve as the bits of a binary counter measuring distances. For the metric TL mentioned above, we obtain a PSpace upper bound since Hirshfeld and Rabinovich have shown that QTL, i.e., the same logic with numbers coded in unary, is PSPACE-complete without FVA [5]. We also show that our proof method can also be used for other logics such as a metric extensions of the branching time logic CTL, thus reproving the EXPTIME-completeness of metric CTL from [4].

Our second result concerns a sharpening of PSPACE lower bounds for metric temporal logics of the real line. In the current literature, such logics usually contain qualititative since/until logic as a proper fragment, and thus trivially inherit PSPACE-hardness [2, 5, 6]. We consider metric TLs with weaker qualitative operators and show that PSPACE-hardness can already be observed in the following three cases: (i) a future diamond and a future operator 'sometime in at most n time units', n coded in unary; (ii) *only* the future operator 'sometime in at most n time units', n coded in binary (i.e., no qualitative operators at all); (iii) only a metric version of the until operator for the interval $[0, 1]$.

As a third result, we explore the transition from NP to PSPACE. In particular, we show that the quantitative TL with only the metric operator 'sometime within n time units', n coded in unary, is NP-complete. This result extends the result of [9] that satisfiability of the qualitative TL with operators 'eventually in the future' and 'eventually in the past' over the real line is decidable in NP. When compared with result (ii) above, it also shows that the complexity of metric TLs without qualitative operators depends on the coding of numbers. To establish the NP upper bound, we show that satisfiability of a formula φ can be decided by first 'guessing' a system of rational linear inequalities, and checking whether this system has a solution over the real (or, equivalently, rational) numbers.

2 Preliminaries

We introduce the metric temporal language QTL of [5]. It is closely related to the language MITL of [1]. Fix a countably infinite supply $p_0, p_1, \dots$ of propositional variables. A QTL-formula is built according to the syntax rule

$$\varphi := p \ \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi\,\mathcal{S}\,\psi \mid \varphi\,\mathcal{U}\,\psi \mid \varphi\mathcal{S}^I\psi \mid \varphi\mathcal{U}^I\psi$$

with p ranging over the propositional variables and I ranging over intervals of the forms $(0,n)$, $(0,n]$, $[0,n)$, and $[0,n]$, where $n > 0$ is a natural number. The Boolean operators $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as abbreviations in the usual way. Moreover, we introduce additional future modalities as abbreviations $\Diamond_F^I\varphi = \top\,\mathcal{U}^I\,\varphi$, $\Box_F^I\varphi = \neg\Diamond_F^I\neg\varphi$, $\Diamond_F\varphi = \top\,\mathcal{U}\,\varphi$, and $\Box_F\varphi = \neg\Diamond_F\neg\varphi$.

Formulas of QTL are interpreted on the real line. Thus, a *model* $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}\rangle$ is a pair consisting of the real numbers and a valuation $\mathfrak{V}$ mapping every propositional variable p to a set $\mathfrak{V}(p) \subseteq \mathbb{R}$. The satisfaction relation '$\models$' is defined inductively as follows, where we write $w + I$ to denote the set $\{w + x \mid x \in I\}$ for each time point $w \in \mathbb{R}$ and interval I of one of the above forms; $w - I$ is defined analogously.

$\mathfrak{M}, w \models p$	iff	$w \in \mathfrak{V}(p)$
$\mathfrak{M}, w \models \top$	for	all $w \in \mathbb{R}$
$\mathfrak{M}, w \models \bot$	for	no $w \in \mathbb{R}$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi\,\mathcal{U}\,\psi$	iff	there exists $u > w$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $w < v < u$
$\mathfrak{M}, w \models \varphi\,\mathcal{S}\,\psi$	iff	there exists $u < w$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $u < v < w$
$\mathfrak{M}, w \models \varphi\,\mathcal{U}^I\,\psi$	iff	there exists $u \in w + I$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $w < v < u$
$\mathfrak{M}, w \models \varphi\,\mathcal{S}^I\,\psi$	iff	there exists $u \in w - I$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $u < v < w$.

We will also write $w \models_{\mathfrak{V}} \varphi$ for $\langle \mathbb{R}, \mathfrak{V}\rangle, w \models \varphi$. A QTL-formula φ is satisfiable if there exists a model $\mathfrak{M}$ and $w \in \mathbb{R}$ such that $\mathfrak{M}, w \models \varphi$. It is satisfiable under the *finite variability assumption (FVA)* if it is satisfiable in a model in which no propositional variable changes its truth-value infinitely many times in any finite interval.

Our presentation of QTL deviates from that of [5], where only the metric operators $\Diamond_F^{(0,1)}$ and $\Diamond_P^{(0,1)}$ are admitted. If the numerical parameters of the metric operators are coded in unary, there exists an easy polynomial translation from Hirshfeld and Rabinovich's version of QTL to ours and vice versa. However, in this paper we also consider binary coding of numbers. If we want to emphasize this fact, we shall write QTL^b instead of QTL, and likewise QTL^u will denote unary coding of numbers.

3 QTL^b is PSpace-complete without FVA

The purpose of this section is to prove that QTL^b-satisfiability without FVA is decidable in PSpace. This result is already known for QTL^u without FVA [5] and QTL^u with FVA [1]. We first show that our result indeed improves upon the existing ones by proving that QTL^b is exponentially more succinct than QTL^u.

Theorem 1. *Let ψ be a QTL-formula with numbers coded in unary that is equivalent to $\Box_F^{[0,n]}p$. Then ψ has length at least n.*

Proof. Suppose by contradiction that there exists a QTL-formula ψ with numbers coded in unary such that ψ is equivalent to $\Box_F^{[0,n]}p$, for some $n \geq 1$, and the length of ψ is strictly smaller than n. We may assume that ψ contains no other propositional letters than p: otherwise, just replace them with $\top$. Then, for $n \geq 1$, set $\mathfrak{V}_n(p) := [-n, n]$ and $\mathfrak{M}_n := \langle \mathbb{R}, \mathfrak{V}_n \rangle$. Then $\mathfrak{M}_n, 0 \models \Box_F^{[0,n]}p$. Therefore, $\mathfrak{M}_n, 0 \models \psi$. Now, it is straightforward to prove the following by induction: for every subformula χ of ψ of length $\leq k$ and all $x \geq k$ such that $n - x \geq -n + k$:

$$\mathfrak{M}, (n-k) \models \chi \;\text{ iff }\; \mathfrak{M}_n, (n-x) \models \chi.$$

Since the length of ψ is smaller than n, it follows that, in $\mathfrak{M}_n$, the points 0 and 1 satisfy the same subformulas of ψ. In particular, $\mathfrak{M}_n, 1 \models \psi$. We have derived a contradiction since $\mathfrak{M}_n, 1 \not\models \Box_F^{[0,n]}p$. ❑

We now establish the main result of this paper.

Theorem 2. *Satisfiability in QTL with numbers coded in binary is* PSpace*-complete without FVA.*

Since (qualitative) since/until logic on the real line is PSpace-hard [6], it suffices to prove the upper bound. For simplicity, we prove the upper bound for the future fragment of QTL, i.e., we omit past operators. The proofs are easily extended to the general case. Within the future fragment, we consider only the metric operators $\Diamond_F^{(0,1)}$, $\Diamond_F^{(0,1]}$, $\Diamond_F^{[0,1)}$, and $\Diamond_F^{[0,n]}$. This can be done w.l.o.g. due to the following observations:

First, satisfiability in QTL^b can be reduced to satisfiability in QTL^b without the metric operators $\psi_1 \mathcal{U}^I \psi_2$: to decide satisfiability of a QTL^b formula φ, introduce a new propositional variable p_{ψ_2} for every ψ_2 which occurs in a subformula of the form $\psi_1 \mathcal{U}^I \psi_2$ of φ. For any subformula χ of φ, we use χ^p to denote the result of replacing all outermost subformulas $\psi_1 \mathcal{U}^I \psi_2$ of χ by $\psi_1 \mathcal{U} p_{\psi_2} \wedge \Diamond_F^I p_{\psi_2}$. Set $\Box_F^+\psi = \psi \wedge \Box_F \psi$. Then φ is satisfiable iff

$$\varphi^p \wedge \Box_F^+ \Big[\bigwedge_{\psi_1 \mathcal{U}^I \psi_2 \in sub(\varphi)} (p_{\psi_2} \leftrightarrow \psi_2^p) \Big]$$

is satisfiable and the length of the latter formula is polynomial in the length of φ. Second, for any interval I of the form $(0, n)$, $(0, n]$, or $[0, n)$, $\Diamond_F^I \varphi$ is equivalent to $\Diamond_F^{(0,1)} \Diamond_F^J \varphi$, where J is obtained from I by decrementing the upper interval bound from n to $n-1$.

In the following, we reduce satisfiability of QTL^b-formulas to the satisfiability of QTL^1-formula, i.e., QTL-formulas in which all upper interval bounds have value 1. As the coding of numbers is not an issue in the latter logic, we obtain a PSPACE upper bound from the result of [5] that QTL^u satisfiability in models without FVA is decidable in PSPACE.

Let φ be a QTL-formula meeting the restrictions laid out above. Let k be the greatest number occurring as a parameter to a metric operator in φ, $n_c = \lceil \log_2(k+2) \rceil$, and $\chi_1, \ldots, \chi_\ell$ the subformulas of φ that occur as an argument to a metric operator of the form $\Diamond_F^{[0,n]}$ with $n > 1$. We reserve, for $1 \leq i \leq \ell$, fresh propositional variables x_i, y_i, and $c^i_{n_c-1}, \ldots, c^i_0$ that do not occur in φ. The sequences $c^i_{n_c-1}, \ldots, c^i_0$ of propositional variables will be used to implement binary counters, one for each χ_i. Intuitively, these counters measure the distance to the "nearest" future occurrence of the formula χ_i, rounded up to the next largest natural number. A counter value greater than or equal to $k+1$ is a special case indicating that the nearest occurrence is too far away to be of any relevance. The variables x_i and y_i will serve as markers with the following meaning: x_i holds in a point iff there is a natural number n such that χ_i^* holds at distance n, but not in between; similarly, y_i holds iff there is a natural number n such that χ_i^* does not hold at any distance up to (and including) n, but χ_i^* holds at future points that converge from the right to the future point with distance n. In the following, we call the structure imposed on the real line by the markers x_i and y_i the (one-dimensional) 'grid'.

To implement the counters, we introduce auxiliary formulas. For $1 \leq i \leq \ell$, let

- $(C_i = m)$ be a formula saying that, at the current point, the value of the i-th counter is m, for $0 \leq m < 2^{n_c}$. There are exponentially many such formulas, but we will use only polynomially many of them in the reduction.
- $(C_i \leq m)$ is a formula saying that, at the current point, the value of the i-th counter does not exceed m, for $0 \leq m < 2^{n_c}$.
- $\bigcirc\varphi := \neg(x_i \vee y_i)\,\mathcal{U}((x_i \vee y_i) \wedge \varphi)$ says that, at the next grid point, φ is satisfied.

To deal with effects of convergence, it is convenient to introduce an additional abbreviation. The formula $rc(\psi) := \big(\neg(\neg\psi\,\mathcal{U}\,\top) \wedge \neg\psi\big)$ describes convergence of ψ-points from the right against a point where ψ does not hold. We now inductively define a translation of QTL^b-formulas to QTL^1-formulas:

$$
\begin{array}{rcl}
p^* & := & p \\
(\neg\psi)^* & := & \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* & := & \psi_1^* \wedge \psi_2^* \\
(\psi_1\,\mathcal{U}\,\psi_2)^* & := & \psi_1^*\,\mathcal{U}\,\psi_2^* \\
(\Diamond_F^I \psi)^* & := & \Diamond_F^I \psi^* \\
(\Diamond_F^{[0,n]} \chi_i)^* & := & (C_i \leq n-1) \vee \big((C_i = n) \wedge \neg y_i\big)
\end{array}
$$

Here, I ranges over intervals $(0,1]$, $(0,1)$, and $[0,1)$. It remains to enforce the existence of the grid and the behavior of the counters as described above. This is done with the following auxiliary formulas, for $1 \le i \le \ell$:

$$
\begin{aligned}
\vartheta_1^i &:= (C_i = 0) \leftrightarrow \big(\chi_i^* \vee rc(\chi_i^*)\big)\\
\vartheta_2^i &:= x_i \leftrightarrow \Big[\chi_i^* \vee \big(\Box_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \wedge \Diamond_F^{(0,1]} x_i \wedge \Diamond_F \chi_i^*\big)\Big]\\
\vartheta_3^i &:= y_i \leftrightarrow \Big[rc(\chi_i^*) \vee \big(\Box_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \wedge \Diamond_F^{(0,1]} y_i \wedge \Diamond_F rc(\chi_i^*)\big)\Big]\\
\vartheta_4^i &:= \neg(C_i = 0) \wedge \Diamond_F^{(0,1]}(x_i \vee y_i) \to\\
&\qquad \Big(\bigvee_{t=0..n_c-1} \big(c_t^i \wedge \bigcirc\neg c_t^i \wedge \bigwedge_{\ell=0..t-1} (\neg c_\ell^i \wedge \bigcirc c_\ell^i) \wedge \bigwedge_{\ell=t+1..n_c-1} (c_\ell^i \leftrightarrow \bigcirc c_\ell^i) \big)\\
&\qquad \vee \bigwedge_{\ell=0..n_c-1} (c_\ell^i \wedge \bigcirc c_\ell^i) \Big)\\
\vartheta_5^i &:= \neg\Diamond_F^{[0,1)}(x_i \vee y_i) \to (C_i = 2^{n_c} - 1)
\end{aligned}
$$

Intuitively, ϑ_1^i initializes the counter, ϑ_2^i and ϑ_3^i ensure that x_i and y_i behave as described above, ϑ_4^i increments the counter when travelling to the left, and ϑ_5^i ensures that, when travelling left, the counter stays in maximal value after the last occurrence of χ_i^*. Let ϑ^i be the conjunction of ϑ_1^i to ϑ_5^i. The following finishes the reduction.

Lemma 3. *φ is satisfiable iff $\Box_F(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell) \wedge \varphi^*$ is satisfiable.*

Proof. "$\Leftarrow$": Let $\mathfrak{V}$ be a valuation and $w \in \mathbb{R}$ such that $w \models_{\mathfrak{V}} \Box_F(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell) \wedge \varphi^*$. We show, by induction, for all $v \in \mathbb{R}$ and all subformulas χ of φ:

$$v \models_{\mathfrak{V}} \chi \ \text{ iff } \ v \models_{\mathfrak{V}} \chi^* \tag{†}$$

Clearly, $w \models_{\mathfrak{V}} \varphi$ follows. The cases for propositional variables, $\neg$, $\wedge$, $\mathcal{U}$, and $\Diamond_F^I$, where I ranges over intervals $(0,1)$, $(0,1]$, and $[0,1)$, are trivial and omitted here. Consider the remaining case $\chi = \Diamond_F^{[0,n]}\chi_i$.

For the direction from right to left, suppose

$$v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^* = (C_i \le n-1) \vee \big((C_i = n) \wedge \neg y_i\big).$$

We take a time point $u \in \mathbb{R}$ and distinguish two cases:

(i) $v \models_{\mathfrak{V}} x_i \vee y_i$. Set $u = v$.

(ii) $v \not\models_{\mathfrak{V}} x_i \vee y_i$. Let $u \in v + (0,1)$ be minimal such that $u \models_{\mathfrak{V}} x_i \vee y_i$.

Note that, in (ii), the required u exists: by definition of n_c, we have $n < 2^{n_c} - 1$ and thus $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^*$ implies $v \models_{\mathfrak{V}} \Diamond_F^{[0,1)}(x_i \vee y_i)$ by ϑ_5^i. Hence, there exists $u \in v + (0,1)$ such that $u \models_{\mathfrak{V}} x_i \vee y_i$. By ϑ_2^i and ϑ_3^i, there exists a minimal such u. For $m \ge 1$, let c_m denote the natural number such that $u + m \models_{\mathfrak{V}} (C_i = c_m)$. Our aim is to show that one of the following holds:

(a) $u + c_0 \models_{\mathfrak{V}} \chi_i^*$ and $u \models_{\mathfrak{V}} x_i$;

(b) $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$ and $u \models_{\mathfrak{V}} y_i$;

For suppose that this has been shown. Then we obtain $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i^*$, which can be seen by distinguishing the following four subcases, and thus get $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i$ by induction hypothesis as desired.

- Cases (i) and (a). Since $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]} \chi_i)^*$ and $v = u$, we have $c_0 \leq n$. Thus, $u + c_0 \models_{\mathfrak{V}} \chi_i^*$ yields $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i^*$.

- Cases (i) and (b). Then $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$ implies that we can find a time point $v' \in u + (c_0, c_0 + 1)$ such that $v' \models_{\mathfrak{V}} \chi_i^*$. Since $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]} \chi_i)^*$, $v = u$, and $u \models_{\mathfrak{V}} y_i$, we have $c_0 < n$. Thus, $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i^*$.

- Cases (ii) and (a). Since $v \not\models_{\mathfrak{V}} x_i \vee y_i$, ϑ_1^i to ϑ_3^i yield that $v \models_{\mathfrak{V}} \neg(C_i = 0)$. By the existence of u and by ϑ_4^i, this yields $v \models_{\mathfrak{V}} (C_i = c_0 + 1)$, and thus $c_0 < n$. Thus $u + c_0 \models_{\mathfrak{V}} \chi_i^*$ and the choice of u yield $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i^*$.

- Cases (ii) and (b). Then (b) $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$ implies that we can find a $v' \in u + (c_0, c_0 + (u - v))$ such that $v' \models_{\mathfrak{V}} \chi_i^*$. As in the third subcase, we can show that $c_0 < n$. Thus $v' \models_{\mathfrak{V}} \chi_i^*$ and the choice of u and v' yield $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i^*$.

It thus remains to show that one of (a) and (b) holds. To this end, we show by induction on m that, for $m \leq c_0$, we have

1. $u + m \models_{\mathfrak{V}} x_i \vee y_i$;

2. $c_m = c_0 - m$;

3. $v' \not\models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in [u + m, u + m + 1)$, if $m < c_0$;

First for the induction start: Point 1 holds by choice of u and Point 2 is trivial. For Point 3, assume that $m \geq c_0$. First assume that $u \models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$. This implies $c_0 = 0$ by ϑ_1^i and thus we have a contradiction. It thus remains to show that $v' \models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in (u, u+1)$. This is an immediate consequence of ϑ_2^i and ϑ_3^i together with the facts that $u \models x_i \vee y_i$ and $u \not\models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$. For the induction step, let $m < c_0$:

- Point 1. By induction, $u + m \models_{\mathfrak{V}} x_i \vee y_i$ and $u + m \not\models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$. Thus, we have $u + m + 1 \models_{\mathfrak{V}} x_1 \vee y_i$ by ϑ_2^i and ϑ_3^i;

- Point 2. By induction, we have $c_m = c_0 - m$ implying $u + m \models_{\mathfrak{V}} \neg(C_i = 0)$. Since Point 1 additionally gives us $u + m + 1 \models_{\mathfrak{V}} x_1 \vee y_i$, ϑ_4^i yields $c_m = c_{m+1} + 1$ and from Point 2 of the induction hypothesis we obtain $c_{m+1} = c_0 - (m + 1)$.

- Point 3. Assume $m + 1 < c_0$. Point 2 gives us $c_{m+1} = c_0 - (m + 1)$. We thus have $u + m + 1 \models_{\mathfrak{V}} \neg(C_i = 0)$. Thus, ϑ_1^i implies $u + m \not\models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$. It thus remains to show that $v' \models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in (u + m, u + m + 1)$. This is an immediate consequence of ϑ_2^i and ϑ_3^i together with the facts that $u \models x_i \vee y_i$ by Point 1 and $u + m \not\models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$.

In particular, we have shown that $u + c_0 \models_{\mathfrak{V}} (C_i = 0)$. Thus, $u + c_0 \models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)$ by ϑ_1^i. We have two sub-cases: first, $u + c_0 \models_{\mathfrak{V}} \chi_i^*$. By ϑ_2^i, we have $u + m \models_{\mathfrak{V}} x_i$ for all $m \leq c_0$, and thus Case (a) from above holds. The second case is $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$. Then ϑ_3^i yields $u + m \models_{\mathfrak{V}} y_i$ for all $m \leq c_0$ and Case (b) from above holds.

For the direction from left to right of (†), suppose $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i$. By the semantics, there is an $u \in w + [0, n]$ such that $u \models_{\mathfrak{V}} \chi_i$. If there is a smallest such position u, then (a) take u to be the smallest one, otherwise (b) take u to be the smallest position such that $u \models_{\mathfrak{V}} rc(\chi_i)$. The induction hypothesis yields that (a) $u \models_{\mathfrak{V}} \chi_i^*$, or (b) $u \models_{\mathfrak{V}} rc(\chi_i^*)$. Then $u \models_{\mathfrak{V}} (C_i = 0)$ by ϑ_1^i. Together with $v' \not\models_{\mathfrak{V}} \chi_i^*$ for each $v' \in (v, u)$, it follows from ϑ_2^i and ϑ_3^i that $v'' \models_{\mathfrak{V}} x_i \vee y_i$ for all v'' such that $v'' = u - j$ for some natural number $j \leq u - v$. Then ϑ_4^i yields $v' \models_{\mathfrak{V}} (C_i = j)$ for all $j \in \mathbb{N}$ with $j \leq u - v$ and each $v' \in [u - j, u - j + 1)$. Since u was chosen such that $u \in v + [0, n]$, in particular we obtain that $w \models_{\mathfrak{V}} (C_i \leq n)$. To show that $v \models_{\mathfrak{V}} (C_i \leq n - 1) \vee \big((C_i = n) \wedge \neg y_i\big)$, is thus remains to prove that $v \models_{\mathfrak{V}} (C_i = n)$ implies $v \not\models_{\mathfrak{V}} y_i$. Suppose $v \models_{\mathfrak{V}} (C_i = n)$. In Case (a), $v \not\models_{\mathfrak{V}} y_i$ by ϑ_3^i. Consider Case (b) and assume to the contrary of what is to be shown that $v \models_{\mathfrak{V}} y_i$. By ϑ_3^i, it then follows that $u - v = n$. But then, $v' \not\models_{\mathfrak{V}} \chi_i$ for all $v' \in v + [0, n]$ contradicting the assumption that $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]} \chi_i$.

"⇒": Suppose φ is satisfiable, i.e., there is a valuation $\mathfrak{V}$ and a $w \in \mathbb{R}$ such that $w \models_{\mathfrak{V}} \varphi$. For each $v \in \mathbb{R}$ and $1 \leq i \leq \ell$, let v_{χ_i} denote

- the smallest time point such that $v \leq v_{\chi_i}$ and $v \models_{\mathfrak{V}} \chi_i \vee rc(\chi_i)$ if such a time point exists;
- $v + 2^{n_c} - 1$ otherwise.

If $v_{\chi_i} \models \chi_i$, we say that v is χ_i-*exact*; if $v_{\chi_i} \models rc(\chi_i)$, we say that v is χ_i-*convergent*. We extend $\mathfrak{V}$ to the additional propositional letters x_i, y_i, and c_t^i used in φ^* as follows:

(1) $v \in \mathfrak{V}(x_i)$ iff $v_{\chi_i} - v$ is an integer and v is χ_i-exact;

(2) $u \in \mathfrak{V}(y_i)$ iff $v_{\chi_i} - v$ is an integer and v is χ_i-convergent;

(3) $u \in \mathfrak{V}(c_t^i)$ iff the t-th bit of the number $v_{\chi_i} - v$ is one or this number exceeds the value $2^{n_c} - 2$.

It is not hard to verify that $w \models_{\mathfrak{V}} \Box_F(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell)$. To show that $w \models_{\mathfrak{V}} \varphi^*$, the following can be proved by structural induction:

$$w \models_{\mathfrak{V}} \varphi \ \text{ iff } \ w \models_{\mathfrak{V}} \varphi^*$$

Details are left to the reader. ❑

4 From NP to PSpace

Qualitative since/until logic on the real line is PSpace-complete, and thus not computationally simpler than QTL^b. However, several natural fragments are only NP-complete, an important example being the qualitative TL with temporal operators 'eventually in the future' and 'eventually in the past' [9]. In this section, we explore the transition from NP to PSpace for fragments of *quantitative* logics of the real line, i.e., for QTL and it's fragments. We start with determining several weak, but still PSpace-hard fragments of QTL. Observe that two of the fragments are purely quantitative, i.e., they do not admit qualitative temporal operators at all.

Theorem 4. *Satisfiability (with and without FVA) is* PSpace*-hard for the fragments of QTL whose only temporal operators are:*

(i) $\Diamond_F$ *and* $\Diamond_F^{[0,n]}$ *with* $n > 0$ *coded in unary;*

(ii) $\Diamond_F^{[0,n]}$ *with* $n > 0$ *is coded in binary;*

(iii) $\mathcal{U}^{[0,1]}$.

Proof. The proof is only sketched here, details are easily filled in. First for Point (i) of Theorem 4, we reduce satisfiability in qualitative TL on the natural numbers with the only temporal operators $\bigcirc$ and $\Diamond_F$, where $\Diamond_F$ is not strict, i.e., $\Diamond_F\varphi$ is equivalent to the QTL formula $\varphi \vee \Diamond_F\varphi$. This logic is known to be PSpace-hard [8]. Let φ be a formula of this logic, and a a propositional variable that does not occur in φ. The main idea of the reduction is to construct a discrete model on the real line by alternating intervals making a true and intervals making $\neg a$ true, with the former representing the time points of discrete time. This structure is enforced such that the length of the a-intervals is from the interval $[2, 3)$, the length of the $\neg a$-intervals is from $[7, 8)$, and the length of an a-interval together with the subsequent $\neg a$ interval is from $(9, 10)$. This is done by the formula $\vartheta = \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3$:

$$\begin{aligned}
\vartheta_1 &= \Box_F^{[0,2]} a, \\
\vartheta_2 &= \Box_F(a \to \Diamond_F^{[0,3]}\Box_F^{[0,7]}\neg a), \\
\vartheta_3 &= \Box_F(a \to \Diamond_F^{[0,10]}\Box_F^{[0,2]} a).
\end{aligned}$$

Inductively define a translation $(\cdot)$ as follows:

$$\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(\bigcirc\psi)^* &:= \Diamond_F^{[0,3]}\big(\Box_F^{[0,7]}\neg a \wedge \Diamond_F^{[0,8]}(\psi^* \wedge a)\big) \\
(\Diamond_F\psi)^* &:= \Diamond_F(\psi^* \wedge a)
\end{aligned}$$

Additionally, a formula ϑ' is needed to take care of uniformity, i.e., to make sure that the same propositional variables hold in all points of an interval that makes a true:

$$\vartheta' = \Box_F \bigwedge_{p \text{ used in } \varphi} \Big(\big(p \wedge a \to \Box_F^{[0,3]}(a \to p)\big) \wedge \big(\neg p \wedge a \to \Box_F^{[0,3]}(a \to \neg p)\big)\Big).$$

Then φ is satisfiable iff $\varphi^* \wedge \vartheta \wedge \vartheta' \wedge a$ is.

For Point (ii) of Theorem 4, we reduce the word problem of a deterministic Turing machine M that solves a PSPACE-hard problem and for which there exists a polynomial p such that M's space consumption on input $w \in \Sigma^*$ is bounded by $p(w)$ and M's time consumption on w is bounded by $2^{p(w)}$. The general idea is to use a sequence of a intervals and $\neg a$ intervals as in the previous relation, with each a interval representing one configuration of the Turing machine computation. Let $w = a_0 \cdots a_{n-1}$ be an input to M. In the reduction, we use the following propositional variables:

- all states q of M are used as propositional variables;
- to describe the state inscription, we fix a variable s^i for each alphabet symbol s and each tape position $i \in \{0, \ldots, p(w)\}$;
- the head position is denoted using propositional variables $h_0, \ldots, h_{p(n)}$.

We first state that the head position, tape inscription, and state are uniquely described:

$$\chi_1 := \bigvee_{1 \le i \le p(n)} \big(h_i \wedge \bigwedge_{\substack{1 \le j \le p(n) \\ j \ne i}} \neg h_j\big) \wedge \bigwedge_{1 \le i \le p(n)} \bigvee_{s \in \Sigma} \big(s^i \wedge \bigwedge_{\substack{t \in \Sigma \\ s \ne t}} \neg t^i\big) \wedge \bigvee_{q \in Q} \big(q \wedge \bigwedge_{\substack{q' \in Q \\ q' \ne q}} \neg q'\big).$$

We also need to formalize the transition relation Δ of M, which we assume to be given as a set of quintuples (s, q, s', d, q') with $d \in \{L, R\}$:

$$\begin{aligned}\chi_2 := \bigwedge_{(s,q,t,d,q') \in \Delta} \bigwedge_{i \le p(n)} \Big((q \wedge h_i) \to \big(\bigcirc(t^i \wedge q' \wedge \big((d = \mathrm{L}) \to \bigcirc h_{i-1}\big) \wedge \\ \big((d = \mathrm{R}) \to \bigcirc h_{i+1}\big)\ \big)\Big)\end{aligned}$$

$$\bigwedge_{i \le p(n)} \big(\neg h_i \to \bigwedge_{s \in \Sigma} (s_i \to \bigcirc s_i)\big)$$

where $\bigcirc \varphi$ abbreviates the formula $(\bigcirc \varphi)^*$ as introduced in the proof of (i). It remains to describe the initial configuration. Recall that the input is $w = a_0 \cdots a_{n-1}$, let q_0 the initial state and $\not b$ denote the blank symbol.

$$\chi_3 := h_0 \wedge q_0 \wedge a_0^0 \wedge \cdots \wedge a_{n-1}^{n-1} \wedge \bigwedge_{n \le i \le p(n)} \not b^i$$

Take ϑ and ϑ' from (i) with φ in ϑ' denoting $\chi_1 \wedge \chi_2 \wedge \chi_3$, replace $\Box_F$ by $\Box_F^{[0, 2^{p(n)}]}$, and denote the result by σ and σ', respectively. It is readily checked that M accepts w iff the following formula is satisfiable, where F denotes the set of final states of M:

$$\chi_3 \wedge \Box_F^{[0, 2^{p(n)}]}(\chi_1 \wedge \chi_2) \wedge \Diamond_F^{[0, 2^{p(n)}]} \bigvee_{q \in F} q.$$

For Point (iii) of Theorem 4, we reduce satisfiability in $\mathrm{QTL}_{\mathcal{U}}$, the QTL-fragment with only temporal operator $\mathcal{U}$, which is known to be PSPACE-hard without FVA [6].

The idea of the reduction is to embed the whole real line into the interval $(0,1)$: given a formula φ of $\mathrm{QTL}_{\mathcal{U}}$, fix a fresh propositional variable a that does not occur in φ. Define a translation $(\cdot)^*$ that recursively replaces every subformula of the form $\varphi\,\mathcal{U}\,\psi$ with $\varphi\,\mathcal{U}^{[0,1]}(a \wedge \psi)$. Then φ is satisfiable iff $\varphi^* \wedge a \wedge \big(a\,\mathcal{U}^{[0,1]}(\Box_F^{[0,1]}\neg a)\big)$ is. For the FVA case, we note that that the PSPACE-hardness proof for $\mathrm{QTL}_{\mathcal{U}}$ does not depend on variables changing their value an infinite number of times in (i) any finite interval, and (ii) in any infinite interval. By (i), $\mathrm{QTL}_{\mathcal{U}}$ is PSPACE-hard also with FVA, and by (ii) we can use the same reduction as in the non-FVA case. ❑

We now exhibit a purely quantitative temporal logic of the real line for which satisfiability is NP-complete: the fragment of QTL with only the quantitative diamond and numbers coded in unary, with and without FVA. This logic may appear rather weak since it does not allow to make statements about *all* time points. Still, it is useful for reasoning about the behaviour of systems up to a previously fixed time point. Note that our NP-completeness result shows that Points (i) and (ii) of Theorem 4 are optimal in the following sense: in Point (i) we cannot drop $\Diamond_F$, and in Point (ii) we cannot switch to unary coding.

Theorem 5. *In the fragment of QTL with temporal operators $\Diamond_F^I$ and $\Diamond_P^I$, I of the form $(0,n)$, $[0,n)$, $[0,n]$, or $(0,n]$, and $n > 0$ coded in unary, satisfiability is decidable in* NP*, both, with and without FVA.*

The lower bound is immediate from propositional logic and thus we only have to prove the upper bound. Since numbers are coded in unary, we may restrict our attention to temporal operators whose upper interval bound is 1. In the proof, we only consider the temporal operator $\Diamond_F^{[0,1]}$. An extension to past operators and open intervals is straightforward.

Let φ be a formula whose satisfiability is to be decided. We introduce some convenient abbreviations: m_φ denotes the nesting depth of operators $\Diamond_F^I$ in φ (henceforth *diamond depth*), $n_\varphi = 2|\varphi|^3 + |\varphi|^2$, and $r_\varphi = |\varphi| \times n_\varphi$. Denote by $cl(\varphi)$ the closure of the set of subformulas of φ under single negation. A *type t for φ* is a subset of $cl(\varphi)$ such that (i) $\neg\psi \in t$ iff $\psi \notin t$ for each $\neg\psi \in cl(\varphi)$, and (ii) $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$ for each $\psi_1 \wedge \psi_2 \in cl(\varphi)$. For a model $\langle \mathbb{R}, \mathfrak{V}\rangle$ and $w \in \mathbb{R}$, set

$$\begin{aligned} t(w) &= \{\psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \psi\}, \\ t^<(w) &= \{\Diamond_F^I\psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \Diamond_F^I\psi\}. \end{aligned}$$

Notice that $t(w)$ is a type for φ. First, we devise an algorithm for satisfiability without FVA. To begin with, we show that satisfiability of φ implies satisfiability of φ in a 'homogeneous' model. In particular, in such models the number of realized types is polynomial in the length of φ.

Lemma 6. *Let φ be satisfiable without FVA. Then there is a sequence $x_0, \ldots, x_{n_\varphi}$ in $\mathbb{R}$ such that $0 = x_0 < x_1 < \cdots < x_{n_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that $\langle \mathbb{R}, \mathfrak{V}\rangle, 0 \models \varphi$ and*

- *$|\{t(w) \mid 0 \le w \le m_\varphi\}| \le r_\varphi$;*

- *for every n with $0 \leq n < n_\varphi$ and each type t for φ, the set $\{w \in \mathbb{R} \mid x_n < w < x_{n+1}$ and $w \models_{\mathfrak{V}} t\}$ is either empty or dense in the interval (x_n, x_{n+1}).*

Proof. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with $\mathfrak{M}, 0 \models \varphi$. By the semantics, we clearly have the following:

(∗) for any $\Diamond_F^I \psi \in sub(\varphi)$, the set $\{w \in \mathbb{R} \mid 0 \leq w \leq m_\varphi$ and $w \models_{\mathfrak{V}'} \Diamond_F^I \psi\}$ is a union of intervals of length at least 1 and at most two intervals of length smaller than 1.

The two possibly shorter intervals are the one starting at 0 and the one ending at m_φ. Using (∗), we can show that there is a sequence $y_0, \ldots, y_k$ in $\mathbb{R}$ for some $k \leq 2|\varphi|^2 + |\varphi|$ such that

- $0 = y_0 < \cdots < y_k = m_\varphi$ and
- $t^<(w) = t^<(w')$ whenever $y_i < w < w' < y_{i+1}$ for any $i < k$.

To see this, take a formula $\Diamond_F^I \psi \in sub(\varphi)$. The *toggle points* for $\Diamond_F^I \psi$ in the interval $[0, m_\varphi]$ are those points x such that either (i) there is a $y > x$ such that the truth value of $\Diamond_F^I \psi$ at x is different from the truth value of $\Diamond_F^I \psi$ at all points z with $x < z < y$ or (ii) there is a $y < x$ such that the truth value of $\Diamond_F^I \psi$ at x is different from the truth value of $\Diamond_F^I \psi$ at all points z with $y < z < x$. By (∗), there are at most $2 \cdot m_\varphi + 1 < 2 \cdot |\varphi| + 1$ toggle points for each formula $\Diamond_F^I \psi$, and thus at most $2|\varphi|^2 + |\varphi|$ toggle points altogether. These points form the required sequence $y_0, \ldots, y_k$.

We convert this sequence into the desired sequence $x_0, \ldots, x_{n_\varphi}$ by arranging the elements of the set

$$\{y_0, \ldots, y_k\} \cup \bigcup_{\substack{i<k \\ 1 \leq j < m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\}$$

in ascending order according to '$<$', possibly introducing (arbitrary) intermediate points to obtain a sequence of length $n_\varphi + 1$.

To obtain a valuation $\mathfrak{V}$ as required by the lemma, fix a set T_i of types for each $i < n_\varphi$ as follows: for each $\Diamond_F^I \psi \in sub(\varphi)$, choose a $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ if such a w exists. Then, T_i is the set of types $t(w)$ of all points w chosen in this way. Clearly $|T_i| \leq |\varphi|$. For each $i < n_\varphi$, take a collection $(X_t^i)_{t \in T_i}$, of subsets of (x_i, x_{i+1}) which form a partitioning of (x_i, x_{i+1}) such that each X_t^i is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) := (\mathfrak{V}'(p) \cap \{x_0, \ldots, x_{n_\varphi}\}) \cup \bigcup_{i<n_\varphi, t \in T_i} \{X_t^i \mid p \in t\}.$$

Let t_i, $i \leq n_\varphi$, be the type $\{\psi \in sub(\varphi) \mid x_i \models_{\mathfrak{V}'} \psi\}$ for φ realized in point x_i of the original model $\mathfrak{M}$. To show that $\mathfrak{V}$ is as required, it is sufficient to show that, for each $k \leq m_\varphi$, each $\psi \in sub(\varphi)$ with diamond depth bounded by k, and each $w \in [0, m_\varphi - k]$, we have

$$
\begin{aligned}
w \models_{\mathfrak{V}} \psi \quad \text{iff} \quad & \text{there is an } i \leq n_\varphi \text{ such that} \\
& \text{(a) } w = x_i \text{ and } \psi \in t_i, \text{ or} \\
& \text{(b) } w \in X_t^i \text{ and } \psi \in t \text{ for some } t \in T_i.
\end{aligned}
$$

Proof. Let k, ψ, and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\Diamond_F^{[0,1]}$.

"$\Rightarrow$": Suppose $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$. Then there is a $w' \in w + [0,1]$ such that $w' \models_{\mathfrak{V}} \psi$ by the semantics. Distinguish four cases:

- $w = x_i$ for some $i < n_\varphi$ and $w' = x_j$ for some $j \geq i$. The induction hypothesis in (a) yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. Since $x_j - x_i \leq 1$, it follows by the semantics that $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Hence $\Diamond_F^{[0,1]}\psi \in t_i$.

- $w = x_i$ for some $i < n_\varphi$ and $w' \in X_t^j$ for some $j \geq i$ and $t \in T_j$. The induction hypothesis in (b) yields $\psi \in t$. Then, by definition of T_j, there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. Note that there is an i' with $i < i' \leq n_\varphi$ such that $x_{i'} = x_i + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' - x_i > 1$ contradicting $w' \in w + [0,1]$. This implies that $w'' - w < 1$ and $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ by the semantics. Hence $\Diamond_F^{[0,1]}\psi \in t_i$.

- $w \in X_t^i$ for some $i < n_\varphi$ and $t \in T_i$, and $w' = x_j$ for some $j > i$. By (a), the induction hypothesis yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. Since $x_j - w \leq 1$, it follows by the semantics that $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. But then by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $w'' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for any $t' \in T_i$. Hence $\Diamond_F^{[0,1]}\psi \in t$.

- $w \in X_t^i$ for some $i < n_\varphi$ and $t \in T_i$, and $w' \in X_{t'}^j$ for some $j \geq i$ and $t' \in T_j$. The induction hypothesis in (b) yields $\psi \in t'$. Then by definition of T_j, there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. Note that there is an i' with $i < i' \leq n_\varphi$ such that $x_{i'} = x_{i+1} + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' - w > 1$ contradicting $w' \in w + [0,1]$. Thus, there is a $v \in (x_i, x_{i+1})$ such that $w'' - v \leq 1$. It follows by the semantics that $v \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. But then by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $v' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $v' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t''$ for any $t'' \in T_i$. Hence $\Diamond_F^{[0,1]}\psi \in t$.

"$\Leftarrow$": Let $i \leq n_\varphi$ such that

(a) $w = x_i$ and $\Diamond_F^{[0,1]}\psi \in t_i$. Then $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. By the semantics, there is a $w' \in x_i + [0,1]$ such that $w' \models_{\mathfrak{V}'} \psi$. Distinguish two cases:

  - $w' = x_j$ for some $j \geq i$. Then $\psi \in t_j$. The induction hypothesis in (a) yields $w' \models_{\mathfrak{V}} \psi$. Since $w' - x_i \leq 1$, it follows by the semantics that $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
  - $w' \in (x_j, x_{j+1})$ for some $j \geq i$. By definition of T_j, there is a $t \in T_j$ such that $\psi \in t$. The induction hypothesis in (b) yields $w'' \models_{\mathfrak{V}} \psi$ for any $w'' \in X_t^j$. Since X_t^j is dense in (x_j, x_{j+1}), there is such a w'' such that $w'' \leq w'$. Then $w'' - x_i \leq 1$. Hence, $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$ by the semantics.

(b) $w \in X_t^i$ and $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$. By definition of T_i, there is a $w' \in (x_i, x_{i+1})$ such that $w' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for any $w'' \in (x_i, x_{i+1})$. In particular, $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then $v \models_{\mathfrak{V}'} \psi$ for some $v \in w + [0,1]$ by the semantics. Distinguish two cases:

- $v = x_j$ for some $j > i$. Then $\psi \in t_j$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{V}} \psi$. Since $x_j - w \leq 1$, it follows by the semantics that $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_j, x_{j+1})$ for some $j \geq i$. By definition of T_j, there is a $t \in T_j$ such that $\psi \in t$. The induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for any $v' \in X_t^j$. Take a $v' \in X_t^j$ such that $v' \leq v$ if $j > i$, and $v' \geq v$ otherwise. Such a v' exists since X_t^j is dense in (x_j, x_{j+1}). Then, $v' - w \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

❏

Lemma 6 suggests the following idea for deciding in non-deterministic polynomial time whether a formula φ is satisfiable: guess a (polynomially bounded) set of types for φ to be realized in a homogeneous model, a sequence $v_0, \ldots, v_{n_\varphi}$ of variables, and construct a system of linear inequalities whose solution in $\mathbb{R}$ determines a sequence of points $x_0, \ldots, x_{n_\varphi}$ from which we can build a homogeneous model realizing the guessed types. More precisely, to decide the satisfiability of φ, we non-deterministically choose

- a set T of types for φ such that $|T| \leq r_\varphi$;
- a type $t_i \in T$ such that $\varphi \in t_0$, for every $i \leq n_\varphi$;
- a non-empty set of types $T_i \subseteq T$, for every $i < n_\varphi$.

Intuitively, the type t_i is to be realized at point x_i, and the types in T_i are those types realized in the interval (x_i, x_{i+1}). Then, we take variables $v_0, \ldots, v_{n_\varphi}$ and check whether the system of inequalities given in Figure 1 has a solution in $\mathbb{R}$. The Inequalities 2 to 9 are only added if $i < n_\varphi$. To understand the inequalities (in particular 4 and 5), note that the point x_i described by variable v_i is not intended to realize the *whole* type t_i, but only those elements of t_i whose diamond depth is at most $\lfloor m_\varphi - x_i \rfloor$. Similarly, points from (x_i, x_{i+1}) described by a type $t \in T_i$ realize only elements of t whose diamond depth is at most $\lfloor m_\varphi - x_i \rfloor$; cf. the structural induction in the proof of Lemma 6.

The algorithm returns 'φ is satisfiable' if there is a solution to this system of inequalities, and 'φ is not satisfiable' otherwise. By considering the contrapositive, it is easily seen that φ is unsatisfiable if the algorithm answers 'no': if φ has a model, then by Lemma 6 it also has a homogeneous model, and this model suggests a choice of types such that the corresponding system of inequalities is satisfiable. Conversely, if the algorithm returns 'yes', we can construct a homogeneous model:

Lemma 7. *If the algorithm returns 'φ is satisfiable', then φ is satisfiable.*

Proof. Suppose there are types t_i, $i \leq n_\varphi$, and sets of types T_i, $i < n_\varphi$, such that there is a solution $x_0, \ldots, x_{n_\varphi}$ for the corresponding system of inequalities. For $i < n_\varphi$, take a partitioning $(X_t^i)_{t \in T_i}$ of (x_i, x_{i+1}) such that each X_t^i is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by putting, for every propositional variable p,

$$\mathfrak{V}(p) := \bigcup_{i \leq n_\varphi} \Big(\{x_i \mid p \in t_i\} \cup \bigcup_{i < n_\varphi, t \in T_i} \{X_t^i \mid p \in t\} \Big).$$

(1) $0 = v_0 < v_1 < \cdots < v_{n_\varphi} = m_\varphi$

(2) $v_j - v_i > 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t_j$

(3) $v_j - v_i \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t$ for some $t \in T_j$

(4) $m_\varphi - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$, but there is no $j \geq i$ such that $\psi \in t_j$ or $\psi \in t$ for a $t \in T_j$

(5) $m_\varphi - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, there is no $j > i$ such that $\psi \in t_j$, and there is no $j \geq i$ such that $\psi \in t'$ for some $t' \in T_j$

(6) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t_j$ and, for every j' with $i \leq j' < j$, $\psi \notin t$ for any $t \in T_{j'}$

(7) $v_j - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t$ for some $t \in T_j$ and there is no j' with $i \leq j' \leq j$ such that $\psi \in t_{j'}$

(8) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, $\psi \notin t'$ for any $t' \in T_i$, and $j > i$ is minimal such that $\psi \in t_j$ or $\psi \in t'$ for some $t' \in T_j$

(9) $v_j - v_{i+1} \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, and ($j \geq i$ and $\psi \in t'$ for some $t' \in T_j$) or ($j > i$ and $\psi \in t_j$)

Figure 1: The system of inequalities.

It is now straightforward to prove that, for all $k \leq m_\varphi$, all $\psi \in sub(\varphi)$ with diamond depth bounded by k, and all $w \in [0, m_\varphi - k]$, we have

$$w \models_{\mathfrak{V}} \psi \quad \text{iff} \quad \text{there is an } i \leq n_\varphi \text{ such that}$$
$$\text{(a) } w = x_i \text{ and } \psi \in t_i, \text{ or}$$
$$\text{(b) } w \in X_t^i \text{ and } \psi \in t \text{ for some } t \in T_i.$$

It is an immediate consequence that $0 \models_{\mathfrak{V}} \varphi$. ❏

We discuss the proof for satisfiability under FVA. Again, the first step is to show that if φ is satisfiable under FVA, then it is satisfiable in a homogeneous model (this time with FVA) in which only polynomially many types are realized:

Lemma 8. *Suppose φ is satisfiable with FVA. Then there exists a sequence $z_0, \ldots, z_{r_\varphi}$ in $\mathbb{R}$ such that $0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that $\langle \mathbb{R}, \mathfrak{V} \rangle, 0 \models \varphi$ and*

- *$|\{t(w) \mid 0 \leq w \leq m_\varphi\}| \leq r_\varphi$;*
- *for all n with $0 \leq n < r_\varphi$, all $\psi \in sub(\varphi)$, and all $z_n < w < w' < z_{n+1}$, $w \models_{\mathfrak{V}} \psi$ iff $w' \models_{\mathfrak{V}} \psi$.*

Proof. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with fva satisfying φ in 0. First, construct a sequence $0 = y_0 < y_1 < \cdots < y_k = m_\varphi$, $k \leq 2|\varphi|^2 + |\varphi|$, as in Lemma 6. Then the sequence $x_0, \ldots, x_{n_\varphi}$ is obtained by arranging the elements of the set

$$\{y_0, \ldots, y_k\} \cup \bigcup_{\substack{i<k \\ 1\leq j<m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\} \cup \bigcup_{\substack{i\leq k \\ 1\leq j<m_\varphi}} \{y_i - j \mid y_i - j > 0\}$$

in ascending order according to $<$ (where we possibly have to add new x_i to obtain a sequence of length $n_\varphi + 1$). Let

$$\sigma = \min\{x_{i+1} - x_i \mid 0 \leq i < n_\varphi\},$$

and set, for $i < n_\varphi$, $\sigma_i = \frac{1}{|\varphi|^{i+1}} \times \sigma$. The sequence

$$0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$$

is obtained by adding to the sequence $x_0, \ldots, x_{n_\varphi}$ the points

$$y_i^j = x_i + \frac{j}{|\varphi|} \times \sigma_i,$$

for all $i < n_\varphi$ and $j \leq |\varphi|$. For $i < n_\varphi$, denote by t^{-i} the type t which is realized in some interval of the form (x_i, y). Note that such an interval exists since we are in a model with FVA. Also, denote by t^{+i} the type which is realized in some interval of the form (y, x_{i+1}). Now, for $i < n_\varphi$, take for any $\Diamond_F^I \psi \in sub(\varphi)$ such that there exists $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ such a type $t(w)$ and denote the collection of selected types plus the types t^{-i} and t^{+i} by T_i. Notice that $|T_i| \leq |\varphi|$. Let $t_0^i, \ldots, t_{|\varphi|-1}^i$ be an ordering of the types in T_i such that $t_0^i = t^{-i}$ (if T_i has cardinality $< |\varphi|$, then take some t from T_i more than once in this ordering.) Define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) = \{x_i \mid i \leq n_\varphi, x_i \models_{\mathfrak{V}'} p\} \cup \bigcup_{i<n_\varphi, j<|\varphi|} \{(y_i^j, y_i^{j+1}] \mid p \in t_j^i\} \cup \bigcup_{i<n_\varphi} \{(y_i^{|\varphi|}, x_{i+1}) \mid p \in t^{+i}\}.$$

We show that $\mathfrak{V}$ is as required. To this end, it is sufficient to show by induction that, for each $k \leq m_\varphi$, every $\psi \in sub(\varphi)$ in which the number of nestings of $\Diamond_F^{[0,1]}$ does not exceed k, and all $w \in [0, m_\varphi - k]$:

$w \models_{\mathfrak{V}} \psi \Leftrightarrow$ there is an $i \leq n_\varphi$ such that

(a) $w = x_i$ and $x_i \models_{\mathfrak{V}'} \psi$, or

(b) $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\psi \in t_\ell^i$ for some $\ell < |\varphi|$, or

(c) $w \in (y_i^{|\varphi|}, x_{i+1})$ and $\psi \in t^{+i}$.

Proof. Let k, ψ, and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\Diamond_F^{[0,1]}$.

"$\Rightarrow$": Suppose $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$. Then there is a $w' \in w + [0,1]$ such that $w' \models_{\mathfrak{V}} \psi$. Distinguish four cases:

- $w = x_i$ for some $i < n_\varphi$ and $w' = x_j$ for some $j \geq i$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}'} \psi$. Since $x_j - x_i \leq 1$, it follows by the semantics that $x_i \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$.

- $w = x_i$ for some $i < n_\varphi$ and $w' \in (x_j, x_{j+1})$ for some $j \geq i$. If $w' \in (y_j^\ell, y_j^{\ell+1}]$ for some $\ell < |\varphi|$, then the induction hypothesis in (b) yields $\psi \in t_\ell^j$. Otherwise, i.e., if $w' \in (y_j^{|\varphi|}, x_{j+1})$, $\psi \in t^{+j}$ by the induction hypothesis in (c). Since t_ℓ^j, $t^{+j} \in T_j$, it follows by definition of T_j that there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{Y}'} \psi$. Note that there is an i' with $i < i' \leq n_\varphi$ such that $x_{i'} = x_i + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' - w > 1$ contradicting $w' \in w + [0, 1]$. Hence $w'' - w < 1$. Then it follows by the semantics that $w \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$.

- $w \in (x_i, x_{i+1})$ for some $i < n_\varphi$, and $w' = x_j$ for some $j > i$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}'} \psi$. Since $x_j - w \leq 1$, $w \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$ by the semantics. Then it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$ for all $w'' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for any $t' \in T_i$. Hence, $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ if $w \in (y_i^\ell, y_i^{\ell+1}]$ for some $\ell < |\varphi|$, and $\Diamond_F^{[0,1]}\psi \in t^{+i}$ if $w \in (y_i^{|\varphi|}, x_{i+1})$.

- $w \in (x_i, x_{i+1})$ for some $i < n_\varphi$, and $w' \in (x_j, x_{j+1})$ for some $j \geq i$. If $w' \in (y_j^\ell, y_j^{\ell+1}]$ for some $\ell < |\varphi|$, then the induction hypothesis in (b) yields $\psi \in t_\ell^j$. Otherwise, i.e., if $w' \in (y_j^{|\varphi|}, x_{j+1})$, $\psi \in t^{+j}$ by the induction hypothesis in (c). Since t_ℓ^j, $t^{+j} \in T_j$, it follows by definition of T_j that there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{Y}'} \psi$. Note that there is an $i' > i + 1$ such that $x_{i'} = x_{i+1} + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' - w > 1$ contradicting $w' \in w + [0, 1]$. Thus there is a $v \in (x_i, x_{i+1})$ such that $w'' - v \leq 1$. By the semantics, $v \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$. Then it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $v' \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$ for all $v' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for any $t' \in T_i$. Hence $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ if $w \in (y_i^\ell, y_i^{\ell+1}]$ for some $\ell < |\varphi|$, and $\Diamond_F^{[0,1]}\psi \in t^{+i}$ if $w \in (y_i^{|\varphi|}, x_{i+1})$.

"$\Leftarrow$": Let $i \leq n_\varphi$ such that

(a) $w = x_i$ and $x_i \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$. By the semantics, there is a $w' \in x_i + [0, 1]$ such that $w' \models_{\mathfrak{Y}'} \psi$. Distinguish two cases:

  - $w' = x_j$ for some $j \geq i$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}} \psi$. Since $x_j - x_i \leq 1$, it follows by the semantics that $x_i \models_{\mathfrak{Y}} \Diamond_F^{[0,1]}\psi$.
  - $w' \in (x_j, x_{j+1})$ for some $j \geq i$. By definition of T_j, there is an $\ell < |\varphi|$ such that $t_\ell^j \in T_j$ and $\psi \in t_\ell^j$. Then the induction hypothesis in (b) yields $w'' \models_{\mathfrak{Y}} \psi$ for all $w'' \in (y_j^\ell, y_j^{\ell+1}]$. Fix such a w''. Note that there is an $j' > j$ such that $x_{j'} = x_j + 1$. But then $x_{j+1} \leq x_{j'}$; otherwise $x_j \geq x_{j'}$ and thus $w' - x_i > 1$ contradicting $w' \in x_i + [0, 1]$. Therefore, $w'' - x_i < 1$. Hence, $x_i \models_{\mathfrak{Y}} \Diamond_F^{[0,1]}\psi$.

(b) $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ for some $\ell < |\varphi|$. By definition of T_i, there is a $w' \in (x_i, x_{i+1})$ such that $w' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for any $w'' \in (x_i, x_{i+1})$. In particular, $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then $v \models_{\mathfrak{V}'} \psi$ for some $v \in w + [0,1]$ by the semantics. Distinguish three cases:

- $v = x_j$ for some $j > i$. The induction hypothesis in (a) yields $v \models_{\mathfrak{V}} \psi$. Since $v - w \leq 1$, it follows by the semantics that $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_i, x_{i+1})$. By definition of T_i, there is a $t \in T_i$ such that $\psi \in t$. Distinguish two subcases: First, suppose that $\psi \in t_{\ell'}^i$ for some $\ell' \geq \ell$, or $\psi \in t^{+i}$. The induction hypothesis in (b) or (c) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^{\ell'}, y_j^{\ell'+1}]$, or all $v' \in (y_i^{|\varphi|}, x_{i+1})$, respectively. Then there is such a v' such that $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
  Second, suppose there is no $\ell' \geq \ell$ such that $\psi \in t_{\ell'}^i$, and $\psi \notin t^{+i}$. Note that this implies $\ell > 0$. Since $\psi \notin t^{+i}$, there is an interval of the form (y, x_{i+1}) such that $y' \not\models_{\mathfrak{V}'} \psi$ for all $y' \in (y, x_{i+1})$. Take such a y'. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$, it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $y' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then there is a $v' \in y' + [0,1]$ such that $v' \models_{\mathfrak{V}'} \psi$ and $v' \geq x_{i+1}$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $v' \in (x_j, x_{j+1})$ where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction. Therefore $j = i'$, i.e., $x_j = x_i + 1$. By definition of T_j, there is an $\ell' < |\varphi|$ such that $t_{\ell'}^j \in T_j$ and $\psi \in t_{\ell'}^j$. Then the induction hypothesis in (b) yields $v'' \models_{\mathfrak{V}} \psi$ for all $v'' \in (y_j^{\ell'}, y_j^{\ell'+1}]$. Take such a v''. Since $\ell > 0$ and $\sigma_j \leq \frac{\sigma_i}{|\varphi|}$ by definition of σ_j, it holds that $y_i^\ell + 1 \geq x_j + \sigma_j$. Then $y_j^{\ell'+1} - y_i^\ell < 1$ and thus $v'' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_j, x_{j+1})$ for some $j > i$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. By definition of $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction.
  Distinguish three subcases:
  * $\ell = 0$ and $w' \models_{\mathfrak{V}'} \psi$ for some w' with $x_i < w' \leq x_j$. Then it is easy to see that there is a $v'' \geq w$ such that $v'' \models_{\mathfrak{V}} \psi$ and $v'' - w \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
  * $\ell = 0$ and $w' \not\models_{\mathfrak{V}'} \psi$ for all w' with $x_i < w' \leq x_j$. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$, it follows by definition of $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all w'' with $x_i < w'' < w$. Take such a w''. Then there is a $v'' \in w'' + [0,1]$ such that $v'' \models_{\mathfrak{V}'} \psi$ and $v'' > x_j$. This implies that $\psi \in t^{-j} = t_0^j$. Then the induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^0, y_j^1]$. Clearly, there is such a v' such that $w - v' \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

* $1 \leq \ell < |\varphi|$. By definition of T_j, there is an $\ell' < |\varphi|$ such that $t^j_{\ell'} \in T_j$ and $\psi \in t^j_{\ell'}$. The induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y^{\ell'}_j, y^{\ell'+1}_j]$. Take such a v'. Since $\sigma_j \leq \frac{\sigma_i}{|\varphi|}$ by definition of σ_j, it holds that $y^{\ell}_i + 1 \geq x_j + \sigma_j$ and thus $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond^{[0,1]}_F \psi$.

(c) $w \in (y^{|\varphi|}_i, x_{i+1})$ and $\psi \in t^{+i}$. This case is similar to (b) and left to the reader.

❏

Using Lemma 8, one can now modify the decision procedure for satisfiability without FVA to obtain a decision procedure running in nondeterministic polynomial time for satisfiability with FVA. The crucial step is to determine a set of rational linear inequalities which represent the truth conditions in models of the form decsribed in Lemma 8. We leave this exercise to the reader.

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A ExpTime-completeness of RTCTL Reproved

We demonstrate the generality of the reduction technique proposed in Section 3 by reproving the result of Emerson et al. that RTCTL, i.e., branching-time logic CTL extended with metric operators, is in ExpTime [4]. A similar (but simpler) reduction can be used to show that the corresponding extension of linear-time logic LTL is in PSpace. For the sake of completeness, we first introduce the syntax and semantics of RTCTL.

Definition 9 (RTCTL Syntax). Let $p_0, p_1, \ldots$ be a countably infinite set of propositional variables. *RTCTL* formulas are built according to the syntax rule

$$\varphi := p_i \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid E\bigcirc\varphi \mid E(\psi\,\mathcal{U}\,\varphi) \mid A(\psi\,\mathcal{U}\,\varphi) \mid E(\psi\mathcal{U}^{\leq k}\varphi) \mid A(\psi\mathcal{U}^{\leq k}\varphi)$$

where k denotes a natural number that is coded in binary. A *CTL formula* is an RTCTL formulas that does not use the metric version of the until operator. $\Diamond$

The abbreviations $\rightarrow$, and $\leftrightarrow$ are defined as usual. Moreover, we abbreviate $A\bigcirc\varphi = \neg E\bigcirc\neg\varphi$ and $A\Box\varphi = \neg E(\top\,\mathcal{U}\,\neg\varphi)$.

A *model* $\mathfrak{M} = \langle S, R, \mathfrak{V}\rangle$ is a triple consisting of the set of states S, a binary relation $R \subseteq S \times S$, and a valuation $\mathfrak{V}$ mapping every propositional variable p to a subset $\mathfrak{V}(p)$ of S. W.l.o.g., we assume that the graph of $\mathfrak{M}$ is a tree since any model can be unwound into a tree. Moreover, we assume that for every state, there is an R-successor. Given a state $w \in S$, a *w-fullpath* is an infinite sequence $u_0u_1 \cdots \in S^\omega$ of states such that $u_0 = w$ and $(u_i, u_{i+1}) \in R$ for all positions $i \geq 0$.

Definition 10 (RTCTL Semantics). Let $\mathfrak{M} = \langle S, R, \mathfrak{V}\rangle$ be a model. Define the truth-relation "$\models$" of RTCTL inductively as follows: for all states $w \in S$,

- $\mathfrak{M}, w \models \top$ and $\mathfrak{M}, w \not\models \bot$;
- $\mathfrak{M}, w \models p$ iff $w \in \mathfrak{V}(p)$ for all propositional variables p;
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$;
- $\mathfrak{M}, w \models \psi \wedge \varphi$ iff $\mathfrak{M}, w \models \psi$ and $\mathfrak{M}, w \models \varphi$;
- $\mathfrak{M}, w \models E\bigcirc\varphi$ iff there exists an R-successor v of w such that $\mathfrak{M}, v \models \varphi$;
- $\mathfrak{M}, w \models E(\psi\,\mathcal{U}\,\varphi)$ iff there exists a w-fullpath $u_0u_1\cdots$ and a position $i \geq 0$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions j with $0 \leq j < i$;
- $\mathfrak{M}, w \models A(\psi\,\mathcal{U}\,\varphi)$ iff for all w-fullpaths $u_0u_1\cdots$, there is a position $i \geq 0$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions j with $0 \leq j < i$.
- $\mathfrak{M}, w \models E(\psi\mathcal{U}^{\leq k}\varphi)$ iff there exists a w-fullpath $u_0u_1\cdots$ and a position i with $0 \leq i \leq k$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions j with $0 \leq j < i$;
- $\mathfrak{M}, w \models A(\psi\mathcal{U}^{\leq k}\varphi)$ iff for all w-fullpaths $u_0u_1\cdots$, there is a position i with $0 \leq i \leq k$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions j with $0 \leq j < i$.

$\Diamond$

Our aim is to prove the following result:

Theorem 11. *Satisfiability in RTCTL is* ExpTime*-complete.*

The lower bound is an immediate consequence of the fact that CTL is a fragment of RTCTL, and the former is ExpTime-hard. We prove a matching upper bound by a polynomial reduction to satisfiability CTL, which is known to be in ExpTime.

The reduction is similar to the reduction presented in Section 3. In particular, the main idea is to replace subformulas $E(\psi\mathcal{U}^{\leq k}\varphi)$ and $A(\psi\mathcal{U}^{\leq k}\varphi)$ with a binary counter that is implemented using propositional variables to represent the bits. However, there are also two significant differences: first, RTCTL is interpreted in discrete models, and thus it is not necessary to construct a 'grid' using variables x_i and y_i to measure the distance 'exactly one' as in the QTL reduction. Second, RTCTL models are not linear, and therefore we cannot simply increment the value of a distance-measuring counter when going to a predecessor state. Instead, we have to increment the least value or greatest counter value of successor nodes, depending on whether we are simulating a formula $E(\psi\mathcal{U}^{\leq k}\varphi)$ or $A(\psi\mathcal{U}^{\leq k}\varphi)$. For identifying the least and greatest counter value among the successors, we use a marking scheme based on additional propositional variables. Before we describe this marking in detail, let us fix some formalities.

Let φ be a RTCTL formula whose satisfiability is to be decided. As an upper bound for the number of counter bits needed, let $n_c = \lceil \log_2 k \rceil$ where k is one plus the largest natural number occurring as a parameter to an until operator in φ. For simplicity, we assume w.l.o.g. that φ contains at least one subformula of the form $E(\psi\mathcal{U}^{\leq k}\varphi)$ and at least one subformula of the form $A(\psi\mathcal{U}^{\leq k}\varphi)$. Now, let $\chi_0, \ldots, \chi_{\ell'}$ be an enumeration of all subformulas of φ of the form $E(\psi\mathcal{U}^{\leq k}\varphi')$, and let $\chi_{\ell'+1}, \ldots, \chi_\ell$ be an enumeration of all subformulas of φ of the form $A(\psi\mathcal{U}^{\leq k}\varphi')$. If $\chi_i = Q(\psi\mathcal{U}^{\leq k}\varphi')$ for some $i \leq \ell$, we use ψ_i to denote ψ and φ_i to denote φ'. For the reduction, we use the following propositional variables:

- the bits of the i-th counter, $i \leq \ell$ are represented using propositional variables c_j, with $j < n_c$;
- to mark the bits of the i-th counter, $i \leq \ell$, we use propositional variables m_j, with $j < n_c$.

Intuitively, the marking scheme for finding the greates counter value among the successors can be understood as follows: start marking bits of the counters in successor nodes by proceeding from the highest ($n_c - 1$-st) to the lowest (0-th), using the following two rules to mark a bit number i of a successor s' of s:

1. if, in s', all bits higher than i are marked and all successors of s whose $i+1$-st bit are marked agree on the value of the i-th bit, then mark the i-th bit of s';
2. if, in s', all bits higher than i are marked and the successors of s whose $i+1$-st bit are marked do not agree on the value of the i-th bit, then mark the i-th bit of s' iff it is one.

The result of this marking is that only those successors of s have *all* marking bits set whose counter value is highest among all the successors of s. A corresponding marking scheme for finding the lowest value is obtained by changing the last part of the second rule to "iff it is zero". The marking of the i-th counter, $i \leq \ell$, can be implemented using the following formula, where $(i \leq \ell')$ abbreviates $\top$ if $i \leq \ell'$ and $\bot$ otherwise:[1]

$$\begin{aligned}
\vartheta_1^i \;:=\; \bigwedge_{t<n_c} \; & (A\bigcirc c_t^i \vee A\bigcirc\neg c_t^i) \rightarrow A\bigcirc(m_t^i \leftrightarrow \bigwedge_{i<j<n_c} m_j^i) \wedge \\
& (E\bigcirc c_t^i \wedge E\bigcirc\neg c_t^i \wedge (i \leq \ell')) \rightarrow A\bigcirc(m_t^i \leftrightarrow (\neg c_t^i \wedge \bigwedge_{i<j<n_c} m_j^i)) \\
& (E\bigcirc c_t^i \wedge E\bigcirc\neg c_t^i \wedge (i > \ell')) \rightarrow A\bigcirc(m_t^i \leftrightarrow (c_t^i \wedge \bigwedge_{i<j<n_c} m_j^i))
\end{aligned}$$

We now inductively define a translation $(\cdot)^*$ of subformulas of φ to CTL formulas, where the formula $(C_i \leq n)$ is defined as in Section 3:

$$\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(E\bigcirc\psi)^* &:= E\bigcirc\psi^* \\
E(\psi_1 \,\mathcal{U}\, \psi_2)^* &:= E(\psi_1^* \,\mathcal{U}\, \psi_2^*) \\
A(\psi_1 \,\mathcal{U}\, \psi_2)^* &:= A(\psi_1^* \,\mathcal{U}\, \psi_2^*) \\
E(\psi_1 \mathcal{U}^{\leq k} \psi_2)^* &:= (C_i \leq k) \text{ if } \chi_i = E(\psi_1 \mathcal{U}^{\leq k} \psi_2) \\
A(\psi_1 \mathcal{U}^{\leq k} \psi_2)^* &:= (C_i \leq k) \text{ if } \chi_i = A(\psi_1 \mathcal{U}^{\leq k} \psi_2)
\end{aligned}$$

It remains to properly update the counters, which is done by the following formulas, for $i \leq \ell$, where the formulas $(C_i \leq n)$ and $(C_i = n)$ are defined as in Section 3:

$$\begin{aligned}
\vartheta_2^i &:= (C_i = 0) \leftrightarrow \varphi_i^* \\
\lambda &:= \neg\psi_i^* \vee \\
&\quad\; ((i \leq \ell') \wedge A\bigcirc(C_i = 2^{n_c} - 1)) \vee \\
&\quad\; ((i > \ell') \wedge E\bigcirc(C_i = 2^{n_c} - 1)) \\
\vartheta_3^i &:= (\neg\varphi_i^* \wedge \lambda) \rightarrow (C_i = 2^{n_c} - 1) \wedge \\
&\quad\; (\neg\varphi_i^* \wedge \neg\lambda) \rightarrow \Big(\bigvee_{t=0..n_c-1} \big(c_t^i \wedge E\bigcirc(m_t^i \wedge \neg c_t^i) \wedge \bigwedge_{\ell=0..t-1} (\neg c_\ell^i \wedge E\bigcirc(m_\ell^i \wedge c_\ell^i)) \wedge \\
&\qquad\qquad \bigwedge_{\ell=t+1..n_c-1} (c_\ell^i \leftrightarrow E\bigcirc(m_\ell^i \wedge c_\ell^i)) \big) \Big)
\end{aligned}$$

Intuitively, ϑ_2^i initializes the counter and ϑ_3^i, and ϑ_3^i ensures that the counter when is incremented correctly when travelling to a predecessor state. Similar to the QTL reduction, the counter value $C_i = 2^{n_c} - 1$ is used to express that, on all (for χ_i being

[1]Recall that $\chi_0, \ldots, \chi_{\ell'}$ are existentially path-quantified while $\chi_{\ell'+1}, \ldots, \chi_\ell$ are universally quantified.

existentially path-quantified) resp. some (universal path quantification) path, the formula φ_i is too far to be of any relevance, or that ψ_i does not hold on some point on the way to the next φ_i occurrence.

Let ϑ^i be the conjunction of ϑ^i_1 to ϑ^i_3. It is left to the reader to prove the following lemma, which finishes the reduction:

Lemma 12. *φ is satisfiable iff $\varphi^* \wedge \bigwedge_{i \leq \ell} A\Box(\vartheta^i_1 \wedge \vartheta^i_2 \wedge \vartheta^i_3)$ is satisfiable.*