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Making Quantification Relevant Again —the Case of Defeasible $\mathcal{EL}_\bot$

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Abstract. Defeasible Description Logics (DDLs) extend Description Logics with defeasible concept inclusions. Reasoning in DDLs often employs rational or relevant closure according to the (propositional) KLM postulates. If in DDLs with quantification a defeasible subsumption relationship holds between concepts, this relationship might also hold if these concepts appear in existential restrictions. Such nested defeasible subsumption relationships were not detected by earlier reasoning algorithms—neither for rational nor relevant closure. In this report, we present a new approach for $\mathcal{EL}_\bot$ that alleviates this problem for relevant closure (the strongest form of preferential reasoning currently investigated) by the use of typicality models that extend classical canonical models by domain elements that individually satisfy any amount of consistent defeasible knowledge. We also show that a certain restriction on the domain of the typicality models in this approach yields inference results that correspond to the (weaker) more commonly known rational closure.

1 Introduction

Description Logics (DLs) are usually decidable fragments of First Order Logic. In DLs *concepts* describe groups of objects by means of other concepts (unary FOL predicates) and roles (binary relations). Such concepts can be related to other concepts as sub- and super-concepts in the TBox which is essentially a theory constraining the interpretation of the concepts. One of the main reasoning problems in DLs is to compute subsumption relationships between two given concepts, i.e., decide whether all instances of one concept must be necessarily instances of the other (w.r.t. the TBox).

While classical DLs allow only for monotonic reasoning, defeasible DLs admit a form of non-monotonic reasoning and have been intensively studied in the last years [6,7,8,4,5,9]. Most defeasible DLs allow to state relationships between concepts by defeasible concept inclusions (DCIs), which characterise typical instances of a concept and can be overwritten by more specific information that would otherwise cause an inconsistency. Often the semantics of defeasible DLs is based on a translation of propositional preferential and (the stronger) rational reasoning for conditional knowledge bases introduced by Kraus, Lehmann and Magidor (KLM) in [11] to DLs. Recent studies on DDLs investigate different semantics, such as a typicality operator under preferential model semantics in [9], a syntactic materialisation-based approach in [6,5], characterised with a different kind of preferential model semantics in [4], and extensions of rational reasoning to the inferentially stronger lexicographic and relevant closure in [5,8].

We consider here an extension of the lightweight DL $\mathcal{EL}$. In this DL complex concepts are built by conjunctions and existential restrictions, which are a form of quantification and clearly not

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expressible by propositional logic. It is well-known that the DL $\mathcal{EL}$ enjoys good computational properties: subsumption can be computed in polynomial time [3]. Despite its moderate expressivity, many applications rely on $\mathcal{EL}$, predominantly the bio-medical domain and semantic web applications using on the web ontology language and its OWL 2 EL profile. In contrast to $\mathcal{EL}$, its extension $\mathcal{EL}_\bot$ can express disjointness of concepts and thus inconsistencies. The ability to express inconsistencies renders reasoning in its defeasible variant non-trivial. We consider in this report non-monotonic subsumption in defeasible $\mathcal{EL}_\bot$ under two kinds of closures: relevant and rational closure. We develop our reasoning algorithm for the stronger relevant closure and then can treat the rational closure as a special case.

In [6] Casini et al. showed that the complexity of non-monotonic subsumption coincides with the complexity of classical reasoning of the underlying DL and devise reasoning algorithms for defeasible subsumption under rational and relevant closure. Their algorithm uses a reduction to classical reasoning and thereby allows to employ highly optimised classical DL reasoners for the reasoning task. Their reduction uses materialisation, where the idea is to encode one consistent subset of the defeasible statements in one concept which is then used in the classical subsumption query as an additional constraint for the (potential) sub-concept in the query. Essentially, the algorithms for the two types of closure differ in the subsets of DCIs from the knowledge base they use for reasoning. While rational closure utilises only a single sequence of decreasing subsets of DCIs, the stronger relevant closure admits *any* such subset during reasoning. Thus relevant reasoning is done w.r.t. a lattice of DCI sets which include more combinations of DCIs and can potentially lead to more fine-grained entailments. However, neither of the resulting algorithms in [5,6,8] is complete in the sense that not all expected subsumption relationships are inferred. The reason is, that defeasible knowledge is not propagated to concepts nested in existential restrictions and thus even un-defeated knowledge is omitted during reasoning.

The goal of this report is to devise a reduction algorithm for reasoning under relevant closure for $\mathcal{EL}_\bot$ that derives defeasible knowledge for concepts nested in existential restrictions. To this end, we introduce a kind of canonical model that is able to represent concept instances of differing typicality, i.e. instances of the same concept that satisfy different sets of DCIs. These so-called typicality models are an extension of the well-known canonical models for classical DLs of the $\mathcal{EL}$ family where the interpretation domain consists of elements representing the concepts occurring in the TBox. Now, typicality models have representatives for each pair of a concept occurring in the TBox and a set of defeasible statements. Thus, for the case of relevant closure such typicality models are built over a lattice-shaped domain according to the lattice of DCI subsets. For a simple form of these typicality models we show that it results in the same entailments as the materialisation-based approach [5] for relevant closure. We extend the simple typicality models to remedy the mentioned shortcoming regarding existential restrictions. The main idea is, to admit in this kind of model differing "amounts" of consistent defeasible information for different occurrences of the same nested concept.

The semantics of the resulting closure actually depends on which subsets of DCIs are considered. The relevant closure requires the whole powerset, i.e. the full lattice of subsets, while for the rational closure a (particular) sequence of decreasing subsets of the set of DCIs is sufficient. Therefore we present the technical results on the more general approach for relevant closure using the DCI lattice first. Then we treat the special case that uses a sequence of decreasing subsets of the set of DCIs to obtain rational reasoning in the second part of this report. Results presented here are individually published for relevant [13] and rational closure [12], respectively.

This report is structured as follows: the next section introduces the basic notions of (D)DLs and $\mathcal{EL}_\bot$. Section 3 recalls the materialisation-based approach for rational and relevant closure and investigates their shortcomings. Section 4 introduces minimal typicality models over a lattice domain and shows that the same subsumption relationships under relevant closure can be inferred as by the materialisation-based approach from [5]. In Section 5 we extend these models to maximal typicality models over a lattice domain and show that these allow to obtain the formerly omitted entailments. We present a specialisation of the approach for relevant reasoning for rational reasoning in [12] and illustrate how the more general results shown in Sections 4 and 5 apply to the weaker form of reasoning in Section 6. The report ends with conclusions and an outlook to future work.

2 Preliminaries

We introduce here the basic notions of (defeasible) DLs and their inferences. Starting from the two disjoint sets N_C of concept names and N_R of role names, complex concepts can be defined inductively. Let C and D be $\mathcal{EL}$-concepts and $r \in N_R$, then *(complex) $\mathcal{EL}$-concepts* are:

- *named concepts* A ($A \in N_C$),
- the *top-concept* $\top$,
- *conjunctions* $C \sqcap D$, and
- *existential restrictions* $\exists r.C$.

The DL $\mathcal{EL}_\bot$ extends $\mathcal{EL}$ by the *bottom-concept* $\bot$, which can be used in conjunctions and existential restrictions. We occasionally also use the concepts *negation* $\neg C$ and *disjunction* $C \sqcup D$.

The semantics of concepts is given by means of interpretations. An *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of an *interpretation domain* $\Delta^\mathcal{I}$ and a *mapping function* that assigns subsets of the domain $\Delta^\mathcal{I}$ to concept names and binary relations over $\Delta^\mathcal{I}$ to role names. The top-concept is interpreted as the whole domain ($\top^\mathcal{I} = \Delta^\mathcal{I}$) and the bottom-concept as the empty set ($\bot^\mathcal{I} = \emptyset$). The complex concepts are interpreted as follows:

- $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$,
- $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}$,
- $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, and
- $(\exists r.C)^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid \exists e.(d, e) \in r^\mathcal{I} \text{ and } e \in C^\mathcal{I}\}$.

If in an interpretation $\mathcal{I}$ $(d, e) \in r^\mathcal{I}$ holds, then e is called a *role successor* of d.

DL ontologies can state (monotonous) relationships between concepts. Let C and D be concepts. A *general concept inclusion axiom* (GCI) is of the form: $C \sqsubseteq D$. A *TBox* $\mathcal{T}$ is a finite set of GCIs. A concept C is *satisfied* by an interpretation $\mathcal{I}$ *iff* $C^\mathcal{I} \neq \emptyset$. A GCI $C \sqsubseteq D$ is *satisfied* in an interpretation $\mathcal{I}$, *iff* $C^\mathcal{I} \subseteq D^\mathcal{I}$ (written $\mathcal{I} \models C \sqsubseteq D$). An interpretation $\mathcal{I}$ is a *model of a TBox* $\mathcal{T}$, *iff* $\mathcal{I}$ satisfies all GCIs in $\mathcal{T}$ (written $\mathcal{I} \models \mathcal{T}$). Based on the notion of a model, DL reasoning problems are defined. A concept is *consistent* w.r.t. a TBox $\mathcal{T}$ *iff* some model of $\mathcal{T}$ satisfies the concept. A concept C is *subsumed by* a concept D w.r.t. a TBox $\mathcal{T}$ (written $C \sqsubseteq_\mathcal{T} D$) *iff* $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds in all models $\mathcal{I}$ of $\mathcal{T}$. Two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent, *iff* $\mathcal{I} \models \mathcal{T}_1 \iff \mathcal{I} \models \mathcal{T}_2$ holds for all interpretations $\mathcal{I}$.

As a side note, we shall use $\Longrightarrow$, $\Longleftarrow$ and $\iff$ as propositional implications (equivalence) in proofs or explanations to improve the overall readability of this report.

Several proofs in this report use product interpretations and the closure property of product models.

Definition 1. *Given two interpretations $\mathcal{I}$ and $\mathcal{J}$. The* product interpretation *of $\mathcal{I}$ and $\mathcal{J}$ is defined as $\mathcal{I} \times \mathcal{J} = (\Delta^\mathcal{I} \times \Delta^\mathcal{J}, \cdot^{\mathcal{I}\times\mathcal{J}})$, where*

– $A^{\mathcal{I}\times\mathcal{J}} = A^{\mathcal{I}} \times A^{\mathcal{J}}$ *(*$A \in N_C$*)*
– $r^{\mathcal{I}\times\mathcal{J}} = \{((a,b),(c,d)) \mid (a,c) \in r^{\mathcal{I}} \wedge (b,d) \in r^{\mathcal{J}}\}$ *(*$r \in N_R$*).*

Lemma 2. *The set of models of a given $\mathcal{EL}_\bot$ TBox $\mathcal{T}$ is closed under product.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be models of $\mathcal{T}$. As a stepping stone in this proof we show

$$C^{\mathcal{I}} \times C^{\mathcal{J}} = C^{\mathcal{I}\times\mathcal{J}} \tag{1}$$

by induction on the structure of $\mathcal{EL}_\bot$ concepts C. The base case for $C = Y$ with $Y \in \{A, \top, \bot\}$ ($A \in N_C$) is trivial by Definition 1. The induction hypothesis is $X^{\mathcal{I}} \times X^{\mathcal{J}} = X^{\mathcal{I}\times\mathcal{J}}$ for $X \in \{D, E\}$. For the case $C = D \sqcap E$, we obtain

$$\begin{aligned}(D \sqcap E)^{\mathcal{I}} \times (D \sqcap E)^{\mathcal{J}} &= (D^{\mathcal{I}} \cap E^{\mathcal{I}}) \times (D^{\mathcal{J}} \cap E^{\mathcal{J}})\\ &= (D^{\mathcal{I}} \times D^{\mathcal{J}}) \cap (E^{\mathcal{I}} \times E^{\mathcal{J}})\\ &= D^{\mathcal{I}\times\mathcal{J}} \cap E^{\mathcal{I}\times\mathcal{J}}\\ &= (D \sqcap E)^{\mathcal{I}\times\mathcal{J}}.\end{aligned}$$

For the case $C = \exists r.D$,

$$\begin{aligned}&(\exists r.D)^{\mathcal{I}} \times (\exists r.D)^{\mathcal{J}}\\ &= \{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}.(d,e) \in r^{\mathcal{I}} \wedge e \in D^{\mathcal{I}}\} \times \{d' \in \Delta^{\mathcal{J}} \mid \exists e' \in \Delta^{\mathcal{J}}.(d',e') \in r^{\mathcal{J}} \wedge e' \in D^{\mathcal{J}}\}\\ &= \{(d,d') \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid \exists e \in \Delta^{\mathcal{I}}, e' \in \Delta^{\mathcal{J}}.(d,e) \in r^{\mathcal{I}} \wedge (d',e') \in r^{\mathcal{J}} \wedge e \in D^{\mathcal{I}} \wedge e' \in D^{\mathcal{J}}\}\\ &= \{(d,d') \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid \exists (e,e') \in \Delta^{\mathcal{I}\times\mathcal{J}}.((d,d'),(e,e')) \in r^{\mathcal{I}\times\mathcal{J}} \wedge (e,e') \in D^{\mathcal{I}\times\mathcal{J}}\}\\ &= (\exists r.D)^{\mathcal{I}\times\mathcal{J}}.\end{aligned}$$

Now it is not hard to show that for any two interpretations $\mathcal{I}, \mathcal{J}$ and an $\mathcal{EL}_\bot$ TBox $\mathcal{T}$,

$$\mathcal{I} \models \mathcal{T} \wedge \mathcal{J} \models \mathcal{T} \implies \mathcal{I} \times \mathcal{J} \models \mathcal{T}.$$

For a GCI $C \sqsubseteq D \in \mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ directly implies $C^{\mathcal{I}} \times C^{\mathcal{J}} \subseteq D^{\mathcal{I}} \times D^{\mathcal{J}}$, which by (1) implies $C^{\mathcal{I}\times\mathcal{J}} \subseteq D^{\mathcal{I}\times\mathcal{J}}$, thus $\mathcal{I} \times \mathcal{J} \models \mathcal{T}$. □

We fix some notation to access parts of knowledge bases or concepts. Let X denote a concept or a TBox, then $sig(X)$ denotes the *signature* of X. We define $sig_{N_C}(X) = sig(X) \cap N_C$ and $sig_{N_R}(X) = sig(X) \cap N_R$. We also define the set $Qc(X)$ of *quantified concepts* in X as $F \in Qc(X)$ *iff* $\exists r.F$ syntactically occurs in X for some $r \in N_R$.

In extensions of $\mathcal{EL}$ that are in the Horn fragment of DLs, canonical models are widely used for reasoning [3]. For an $\mathcal{EL}_\bot$-TBox $\mathcal{T}$, the *canonical model* $\mathcal{I}_\mathcal{T} = (\Delta, \cdot^{\mathcal{I}_\mathcal{T}})$ of $\mathcal{T}$ with $\Delta = \{d_F \mid F \in Qc(\mathcal{T})\}$ has the mapping function satisfying the conditions:

– $d_F \in A^{\mathcal{I}_\mathcal{T}}$ *iff* $F \sqsubseteq_\mathcal{T} A$ and
– $(d_F, d_G) \in r^{\mathcal{I}_\mathcal{T}}$ *iff* $F \sqsubseteq_\mathcal{T} \exists r.G$.

Once the canonical model is computed, subsumption relationships between concepts can be directly read-off from it [3,1].

In defeasible DLs it can be stated that a concept is subsumed by another concept as long as there is no contradicting information. A *defeasible concept inclusion* (DCI) is of the form $C \mathrel{\sqsubseteq\!\!\!\sim} D$ and states that C *usually* entails D. A *DBox* $\mathcal{D}$ is a finite set of DCIs. A *defeasible knowledge base*

(DKB) $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ consists of a TBox $\mathcal{T}$ and a DBox $\mathcal{D}$. The definitions for $sig(X)$, $sig_{N_C}(X)$, $sig_{N_R}(X)$ and $Qc(X)$ extend to DBoxes or DKBs in the obvious way. A *materialisation* of a DBox $\mathcal{D}$ is the concept $\overline{\mathcal{D}} = \bigsqcap_{E \mathrel{\sqsubseteq\!\!\!\sim} F \in \mathcal{D}} (\neg E \sqcup F)$.

The semantics of DBoxes differ from the ones for TBoxes, since DCIs need not hold at each element in the model whereas GCIs do. The satisfaction of DCIs for $d \in \Delta^\mathcal{I}$ is captured by $\mathcal{I}, d \models \mathcal{D}$ *iff* $\forall G \mathrel{\sqsubseteq\!\!\!\sim} H \in \mathcal{D}. d \in G^\mathcal{I} \Longrightarrow d \in H^\mathcal{I}$. Usually, the semantics of DBoxes is given by means of ranked/ordered interpretations—called preferential model semantics [4,9]. Instead of using these, we define a new kind of model for DKBs (in Sect. 4) that extends canonical models for $\mathcal{EL}_\bot$. The main idea is to use several copies of the representatives, such as d_F, for each existentially quantified concept, where each copy satisfies a different set of DCIs from the powerset of the DBox.

We want to develop a decision procedure for (defeasible) subsumption relationships between concepts, say C and D, w.r.t. a given DKB $\mathcal{K}$ under relevant closure. For the remainder of the report we make two simplifying assumptions for the sake of ease of presentation. We assume w.l.o.g. that

1. concepts C and D appear syntactically in $Qc(\mathcal{T})$ which can be achieved by adding $\exists r.E \sqsubseteq \top$ with $E \in \{C, D\}$ to $\mathcal{T}$ and
2. all quantified concepts in $\mathcal{K}$ are consistent i.e., $\forall F \in Qc(\mathcal{K}). F \not\sqsubseteq_\mathcal{T} \bot$ and thus $\bot \notin Qc(\mathcal{K})$.

To motivate our approach for reasoning under relevant closure in defeasible $\mathcal{EL}_\bot$, we discuss first earlier approaches for this task and identify their main shortcoming.

3 Minimal Relevant Closure by Materialisation

We recall the reduction algorithms for reasoning by Casini et al. from [5]. Rational entailment in [5] uses materialisation of DCIs to decide defeasible subsumptions $C \mathrel{\sqsubseteq\!\!\!\sim} D$ w.r.t. a given DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. Since C might be inconsistent w.r.t. the materialisation of the entire DBox $\mathcal{D}$, the algorithm needs to determine a subset $\mathcal{D}' \subseteq \mathcal{D}$ whose materialisation is consistent with C and $\mathcal{T}$ in order to decide whether $\overline{\mathcal{D}'} \sqcap C \sqsubseteq_\mathcal{T} D$ holds. To obtain $\mathcal{D}'$, $\mathcal{D}$ is iteratively reduced to that subset containing all DCIs whose left-hand sides are inconsistent in conjunction with the materialisation of the current DBox:

$$\mathcal{E}(\mathcal{D}) = \{C \mathrel{\sqsubseteq\!\!\!\sim} D \in \mathcal{D} \mid \mathcal{T} \models \overline{\mathcal{D}} \sqcap C \sqsubseteq \bot\}.$$

They define $\mathcal{E}^1(\mathcal{D}) = \mathcal{E}(\mathcal{D})$ and $\mathcal{E}^j(\mathcal{D}) = \mathcal{E}(\mathcal{E}^{j-1}(\mathcal{D}))$ (for $j > 1$). Using $\mathcal{E}()$, the DCIs in $\mathcal{D}$ can be ranked according to their level of exceptionality, i.e., $r_\mathcal{K}(G \mathrel{\sqsubseteq\!\!\!\sim} H) = i - 1$, for the smallest i s.t. $G \mathrel{\sqsubseteq\!\!\!\sim} H \notin \mathcal{E}^i(\mathcal{D})$, or $r_\mathcal{K}(G \mathrel{\sqsubseteq\!\!\!\sim} H) = \infty$ if no such i exists. A DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ is *well-separated* if no DCI in $\mathcal{D}$ has an infinite rank of exceptionality [4]. We assume w.l.o.g. that all DKBs are well-separated: any DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ can be transformed into a well-separated DKB $\mathcal{K}'$ by testing a quadratic number of subsumptions in the size of $\mathcal{D}$:

$$\mathcal{K}' = \big(\mathcal{T} \cup \{C \sqsubseteq \bot \mid r_\mathcal{K}(C \mathrel{\sqsubseteq\!\!\!\sim} D) = \infty\}, \mathcal{D} \setminus \{C \mathrel{\sqsubseteq\!\!\!\sim} D \mid r_\mathcal{K}(C \mathrel{\sqsubseteq\!\!\!\sim} D) = \infty\}\big).$$

Based on the level of exceptionality $r_\mathcal{K}()$, the algorithm from [5] partitions the DBox $\mathcal{D}$ into $(E_0, E_1, \ldots, E_n)$ where $E_i = \{G \mathrel{\sqsubseteq\!\!\!\sim} H \in \mathcal{D} \mid r_\mathcal{K}(G \mathrel{\sqsubseteq\!\!\!\sim} H) = i\}$, i.e. $\mathcal{D} = \bigcup_{i=0}^n E_i$. To find the maximal (w.r.t. cardinality) subset $\mathcal{D}'$ of $\mathcal{D}$, whose materialisation is consistent with C and $\mathcal{T}$ the procedure starts with $\mathcal{D}' = \mathcal{D}$. If $\overline{\mathcal{D}'} \sqcap C \sqsubseteq_\mathcal{T} \bot$, then E_i is removed from $\mathcal{D}'$ for the smallest not yet used i. This test and removal is done iteratively until a subset of $\mathcal{D}$ is reached whose materialisation is consistent with C and $\mathcal{T}$.

While rational closure treats inconsistencies with the granularity of the partitions E_i, relevant closure uses a more fine-grained treatment. To illustrate this, let $G \mathrel{\tilde{\sqsubseteq}} H \in E_0$ and assume that C is only consistent with $\mathcal{D} \setminus E_0$ (or its subsets). In this situation

$$(\neg G \sqcup H) \sqcap \overline{\mathcal{D} \setminus E_0} \sqcap C \sqsubseteq_{\mathcal{T}} \bot$$

need not hold, since the inconsistency may be due to other DCIs in E_0. Still $G \mathrel{\tilde{\sqsubseteq}} H$ is never used for reasoning about C. This effect is called *inheritance blocking*, as it might be possible to include $G \mathrel{\tilde{\sqsubseteq}} H$ for reasoning about C, but other DCIs induce some inconsistency and so block the inheritance of property $G \mathrel{\tilde{\sqsubseteq}} H$ for C. Relevant closure disregards only DCIs that are *relevant* for the inconsistency of C, thereby averting inheritance blocking. General relevant closure and two specific constructions (basic and minimal relevant closure) are introduced in [5] in terms of justification.

Definition 3. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{J} \subseteq \mathcal{D}$, and C a concept. $\mathcal{J}$ is a C-*justification *w.r.t. $\mathcal{K}$ iff $\overline{\mathcal{J}} \sqcap C \sqsubseteq_{\mathcal{T}} \bot$ and $\overline{\mathcal{J}'} \sqcap C \not\sqsubseteq_{\mathcal{T}} \bot$ for all $\mathcal{J}' \subset \mathcal{J}$.*

Let *justifications*$(\mathcal{K}, C) = (\mathcal{J}_1, \ldots, \mathcal{J}_m)$ be the function that returns all C-justifications w.r.t. $\mathcal{K}$ that are of minimal set cardinality. This set can be computed in exponential time [10].

To present a simplified (but equivalent) version of the algorithm from [5] for computing entailment of defeasible subsumptions under minimal relevant semantics, we need to define the $\mathcal{D}' \subseteq \mathcal{D}$ that is consistent with C and ultimately used for deciding $\overline{\mathcal{D}'} \sqcap C \sqsubseteq_{\mathcal{T}} D$. Let *partition*$(\mathcal{D}) = (E_0, \ldots, E_n)$ be a function that computes the above defined partitioning of DBoxes and let $\mathcal{J} \subseteq \mathcal{D}$. Then *min*(*partition*$(\mathcal{D}), \mathcal{J})$ returns E_i for the *smallest* i, s.t. $\mathcal{J} \cap E_i \neq \emptyset$. Given $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and the subsumption query $C \mathrel{\tilde{\sqsubseteq}} D$, we define the rank-minimal part of all C-justifications w.r.t. $\mathcal{K}$ as $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$ for $(\mathcal{J}_1, \ldots, \mathcal{J}_m) =$ *justifications*$(\mathcal{K}, C)$, where $\mathcal{M}_i = \mathcal{J}_i \cap$ *min*(*partition*$(\mathcal{D}), \mathcal{J}_i)$, for $1 \leq i \leq m$. In order to obtain a subset of $\mathcal{D}$ that is consistent with C, at least one statement from every justification has to be removed from $\mathcal{D}$. By a preference of exceptionality rank[1], the removed statements shall be the rank-minimal[2] parts of all justifications, i.e. $\mathcal{D}' = \mathcal{D} \setminus (\bigcup_{i=1}^{m} \mathcal{M}_i)$. We denote non-monotonic entailments obtained by minimal relevant closure and materialisation as $\models_m^{rel}$ and define it as

$$\mathcal{K} \models_m^{rel} C \mathrel{\tilde{\sqsubseteq}} D \text{ iff } \overline{\mathcal{D}'} \sqcap C \sqsubseteq_{\mathcal{T}} D$$

for $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and $\mathcal{D}'$ as defined above.

The following example illustrates the problem of inheritance blocking caused by rational closure, but not by minimal relevant closure.

Example 4. Let $\mathcal{K}_{ex1} = (\mathcal{T}_{ex1}, \mathcal{D}_{ex1})$ with:

$$\begin{aligned}
\mathcal{T}_{ex1} &= \{Boss \sqsubseteq Worker,\ Boss \sqcap \exists superior.Worker \sqsubseteq \bot\},\\
\mathcal{D}_{ex1} &= \{Worker \mathrel{\tilde{\sqsubseteq}} \exists superior.Boss,\ Worker \mathrel{\tilde{\sqsubseteq}} Productive,\\
&\qquad Boss \mathrel{\tilde{\sqsubseteq}} Responsible\}, \text{ and}\\
partition(\mathcal{D}_{ex1}) &= \big(E_0 = \{Worker \mathrel{\tilde{\sqsubseteq}} \exists superior.Boss,\ Worker \mathrel{\tilde{\sqsubseteq}} Productive\},\\
&\qquad E_1 = \{Boss \mathrel{\tilde{\sqsubseteq}} Responsible\}\big).
\end{aligned}$$

Rational closure detects the inconsistency $\overline{\mathcal{D}_{ex1}} \sqcap Boss \sqsubseteq_{\mathcal{T}_{ex1}} \bot$, but $\overline{\mathcal{D}_{ex1} \setminus E_0} \sqcap Boss \not\sqsubseteq_{\mathcal{T}_{ex1}} \bot$ holds. Thus $Boss \mathrel{\tilde{\sqsubseteq}} Worker \sqcap Responsible$ is entailed from $\mathcal{K}_{ex1}$, while $Boss \mathrel{\tilde{\sqsubseteq}} Productive$ is

[1] One could consider more refined preferences such as a quantitative ranking of DCIs, for instance.

[2] Removing only the rank-minimal parts characterises minimal relevant closure, for a slightly different technique for the basic relevant closure see [5].

not, even though the DCI $Worker \sqsubsetsim Productive$ does not cause the inconsistency of $Boss$. For minimal relevant closure, $\mathcal{J}_1 = \{Worker \sqsubsetsim \exists superior.Boss\}$ is the only $Boss$-justification w.r.t. $\mathcal{K}_{ex1}$. Therefore, the largest consistent DBox subset of $\mathcal{D}_{ex1}$ for reasoning about the concept $Boss$ is $\mathcal{D}' = \{Worker \sqsubsetsim Productive, Boss \sqsubsetsim Responsible\}$ and providing the consequence $\overline{\mathcal{D}'} \sqcap Boss \sqsubseteq_{\mathcal{T}_{ex1}} Productive$.

Example 4 also illustrates the issue caused by employing materialisation that is addressed in this report. Materialising the DCI $Worker \sqsubsetsim Productive$ to $\neg Worker \sqcup Productive$ in conjunction with $\exists superior.Worker$ yields a concept that is not subsumed by $\exists superior.Productive$. The defeasible information is unjustly disregarded when reasoning about quantified concepts yielding uniformly non-typical role successors. Hence, in Example 4, both rational and relevant closure (based on materialisation) are oblivious to the conclusion $Worker \sqsubsetsim \exists superior.Responsible$.

4 Typicality Models for Propositional Relevant Entailment

In order to achieve relevant entailment also for quantified concepts, DCIs need to hold for concepts in (nested) existential restrictions. A naive idea to extend the materialisation approach is to enrich all concepts in existential restrictions with materialisations of the DBox. However, for Example 4, enriching the concept $\exists superior.Boss$ with $Worker \sqsubsetsim \exists superior.Boss$ to $\exists superior.(Boss \sqcap (\neg Worker \sqcup \exists superior.Boss))$ leads to infinitely many such enriching steps (due to $Boss \sqsubseteq Worker$). Instead, our approach is to extend the canonical models for the classical members of the $\mathcal{EL}$-family to DDLs. Our new kind of models captures varying numbers of DCIs from a DKB to be satisfied by role successors. Their interpretation domain essentially consist of copies of the domain of a classical canonical model for each set of DCIs. How many such copies are introduced for the domain of a typicality model, or equivalently, how many sets of DCIs are considered in the model can determine the semantics and thus the strength of the resulting reasoning. For instance, in order to capture and satisfy any subset of DCIs from the DBox, an exponential number of copies (in the size of the DBox) of the classical domain is required. The shape of the domain containing these copies can be viewed as a *lattice* over the subsets of the DBox. To develop the semantics for reasoning under nested relevant entailment and an appropriate reasoning procedure we proceed in two steps:

1. We introduce minimal typicality models over a lattice domain where all domain elements have non-typical role successors only, i.e., no role successor needs to satisfy any DCI. We show that these minimal typicality models yield exactly the same subsumption relationships as the materialisation-based relevant entailment in [5].
2. We extend minimal typicality models to maximal typicality models, where each role successor required by $\mathcal{K}$ is chosen such that it satisfies a subset of DCIs from $\mathcal{D}$ that is of maximal cardinality while not causing an inconsistency. We define subsumption under nested relevant entailment based on maximal typicality models over a lattice domain and show that these models then yield more subsumption relationships than the materialisation-based relevant entailment.

To devise an algorithm that computes the same entailments as materialisation-based relevant entailment, we use the same subsets of the DBox based on justifications as Casini et al. in [5] (and as discussed in Section 3). To decide the entailment of $C \sqsubsetsim D$ w.r.t. $\mathcal{K} = (\mathcal{T}, \mathcal{D})$, the subset $\mathcal{D}'$ of $\mathcal{D}$ is constructed by removing rank-minimal parts of all justifications relevant for the inconsistency of C. Since we need to distinguish the subsets obtained from C-justifications for different concepts C, we denote from now on, $\mathcal{D}' = \mathcal{D} \setminus (\bigcup_{i=1}^{m} \mathcal{M}_i)$ as $\mathcal{D}_C$ (e.g. for $\mathcal{D}_X \subseteq \mathcal{D}$, use X-justifications w.r.t. $\mathcal{K}$).

In order to infer all the undefeated facts for a concept F, the representative domain element of F in a model needs to satisfy the largest subset of $\mathcal{D}$ that is still satisfiable together with the TBox. If minimal relevant closure is used, this subset is obviously $\mathcal{D}_F$. Now, in $\mathcal{EL}_\bot$-concepts a syntactical sub-concept F can occur in multiple existential restrictions, causing several role successors in a model. These in turn might be able to satisfy any subset of DCIs of $\mathcal{D}$ "up to" $\mathcal{D}_F$ each. To be able to capture elements satisfying any such set of DCIs, typicality interpretations (potentially) need a representative domain element for each subset of the given DBox $\mathcal{D}$. To this end we introduce typicality domains, where each domain element is associated with a concept and a set of DCIs from $\mathcal{D}$.

Definition 5. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The domain Δ is a typicality domain over $\mathcal{K}$ if the domain elements are of the form $d_F^{\mathcal{U}}$, where*

- *$F \in Qc(\mathcal{K})$,*
- *$\mathcal{U} \subseteq \mathcal{D}$, and*
- *$\{d_F^{\emptyset} \mid F \in Qc(\mathcal{K})\} \subseteq \Delta$.*

The set of represented subsets of $\mathcal{D}$ *in Δ is $\Gamma(\Delta) = \{\mathcal{U} \subseteq \mathcal{D} \mid \exists d_F^{\mathcal{U}} \in \Delta\}$. The* shape of a typicality domain *is that of a*

- sequence*, if $\Gamma(\Delta)$ is totally ordered by $\subseteq$,*
- lattice*, if no further restrictions are imposed on $\Gamma(\Delta)$.*

Using the notion of differently shaped typicality domains, we can characterise different kinds of *typicality models*, depending on the strength of reasoning we want to obtain. For the main part of this report we consider relevant reasoning and therefore a lattice-shaped typicality domain (or "lattice domain" for short).

Definition 6. *An interpretation over a typicality domain is called a* typicality interpretation*.*

Typicality interpretations are characterised by the elements of its domain being associated with a set of DCIs and a concept, e.g. $d_F^{\emptyset}$ or $d_C^{\mathcal{U}}$ for $\mathcal{U} \subseteq \mathcal{D}$. Such an association is only possible because a typicality domain is directly linked with a defeasible knowledge base, hence we use: Δ over the DKB $\mathcal{K}$. As opposed to the classical case, our typicality interpretations are therefore also directly associated with such a DKB. In particular, it does not make sense to check whether a typicality interpretation $\mathcal{I}$ with the typicality domain $\Delta^{\mathcal{I}}$ over the DKB $\mathcal{K}$ satisfies a different DKB $\mathcal{K}' \neq \mathcal{K}$ as the following definition reveals. Observe that a typicality domain always contains representatives for each concept (occurring in existential restrictions) that are associated with the empty set of DCIs. If in this report the shape of the underlying typicality domain is not explicitly specified in the claims or proofs, the result applies to all shapes. Typicality interpretations over a lattice domain are the basis for our relevant reasoning semantics and we define under which conditions a DKB is satisfied in such interpretations.

Definition 7 (model of $\mathcal{K}$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. A typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a model of $\mathcal{K}$ (written $\mathcal{I} \models \mathcal{K}$) iff*

1. *$\mathcal{I} \models \mathcal{T}$ and*
2. *$\mathcal{I}, d_F^{\mathcal{U}} \models \mathcal{U}$ for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{I}}$.*

This definition ensures that a model of a DKB satisfies the TBox at each element and it satisfies a subset $\mathcal{U}$ of the DBox at all elements associated with this subset. As stated before, typicality models extend classical canonical models. As such, we want to characterise subsumption entailment by a typicality model in a similar way as for canonical models, where, in order to determine $C \sqsubsetsim_{\mathcal{T}} D$, the representative domain element d_C is checked for containment in $D^{\mathcal{I}_\mathcal{T}}$. The main difference is, of course, that typicality models do not have one representative of a concept C but several representatives, satisfying different sets of defeasible statements. Hence, the resulting semantics of defeasible subsumption do not only depend on the shape of the underlying domain, but also on the representative element that is selected for extracting information about subsumptions. Intuitively, this element should satisfy as many DCIs as possible. However, for a lattice domain, there might be several domain elements satisfying sets of DCIs with the same cardinality such that no representative satisfying a set of DCIs with higher cardinality is consistent (and therefore exists in the interpretation). Nevertheless, for lattice shaped typicality domains, we shall still select the unique representative $d_C^{\mathcal{D}_C}$, satisfying the consistent set of DCIs $\mathcal{D}_C$ as constructed in [5] for minimal relevant closure. The reason for doing so is our aim to recreate the same entailment as in [5], before extending this result to consider role successors satisfying as many DCIs as possible.

Definition 8. *Let $\mathcal{I}$ be a typicality interpretation over a lattice domain. Then $\mathcal{I}$* satisfies a defeasible subsumption *$C \sqsubsetsim D$ (written $\mathcal{I} \models C \sqsubsetsim D$) iff $d_C^{\mathcal{D}_C} \in D^{\mathcal{I}}$.*

In order to reduce this reasoning problem to a classical one, we construct a model for $\mathcal{K}$ by means of a TBox. We use auxiliary concept names from the set $N_C^{aux} \subseteq N_C \setminus sig(\mathcal{K})$ to introduce representatives for all $F \in Qc(\mathcal{K})$ for each subset of the given DBox.

Definition 9 (extended TBox). *Given concept F and DBox $\mathcal{D}$, we use $F_\mathcal{D} \in N_C^{aux}$ to define the* extended TBox *of F w.r.t. $\mathcal{D}$:*

$$\mathcal{T}_\mathcal{D}(F) \;=\; \mathcal{T} \cup \{F_\mathcal{D} \sqsubseteq F\} \cup \{F_\mathcal{D} \sqcap G \sqsubseteq H \mid G \sqsubsetsim H \in \mathcal{D}\}. \tag{2}$$

In this definition $\{F_\mathcal{D} \sqsubseteq F\}$ ensures that all constraints on F hold for the auxiliary concept as well. The last set of GCIs in Eq. (2) ensures that every element in $F_\mathcal{D}^\mathcal{I}$ (for $\mathcal{I} \models \mathcal{T}_\mathcal{D}(F)$) satisfies the DCIs in $\mathcal{D}$.

To simplify upcoming proofs, we introduce the notion of witness models and show some of their properties. The idea of a witness model is that it instantiates each named concept by a given element.

Definition 10 (witness model). *Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation, $C_\mathcal{D} \in N_C^{aux}$, and let $o \in \Delta^\mathcal{I}$. The interpretation $\mathcal{I}(C_\mathcal{D}, o) = (\Delta^\mathcal{I}, \cdot^{\mathcal{I}(C_\mathcal{D},o)})$ with*

- $C_\mathcal{D}^{\mathcal{I}(C_\mathcal{D},o)} = \{o\}$,
- $\forall A \in N_C \setminus N_C^{aux}. A^{\mathcal{I}(C_\mathcal{D},o)} = A^\mathcal{I}$, *and*
- $r^{\mathcal{I}(C_\mathcal{D},o)} = r^\mathcal{I}$ *for all* $r \in N_R$.

is the witness model *of $\mathcal{I}$ with o for $C_\mathcal{D}$. The element o is the* witness element *of $\mathcal{I}(C_\mathcal{D}, o)$.*

The following proposition characterises the relation between witness models and their basis (i.e. for $\mathcal{I}(C_{\mathcal{D}_i}, o)$, the basis is $\mathcal{I}$) with regard to their property of being a model for a TBox $\mathcal{T}$ and extended TBoxes $\mathcal{T}_\emptyset(C)$ and $\mathcal{T}_{\mathcal{D}_i}(C)$.

Proposition 11. *For an $\mathcal{EL}_\bot$ TBox $\mathcal{T}$ with $sig(\mathcal{T}) \cap N_C^{aux} = \emptyset$, DBox $\mathcal{D}$ with $sig(\mathcal{D}) \cap N_C^{aux} = \emptyset$, a concept C with $sig(C) \cap N_C^{aux} = \emptyset$, a (typicality) interpretation $\mathcal{I}$ and the concept name $C_\mathcal{D} \in N_C^{aux}$, the following holds*

1. $o \in C^{\mathcal{I}} \wedge \mathcal{I} \models \mathcal{T} \implies \mathcal{I}(C_\emptyset, o) \models \mathcal{T}_\emptyset(C)$
2. $o \in C_{\mathcal{D}}^{\mathcal{I}} \wedge \mathcal{I} \models \mathcal{T}_{\mathcal{D}}(C) \implies \mathcal{I}(C_{\mathcal{D}}, o) \models \mathcal{T}_{\mathcal{D}}(C)$

Proof. In either case $\mathcal{I} \models \mathcal{T}$ obviously implies $\mathcal{I}(C_{\mathcal{D}}, o) \models \mathcal{T}$ since left- and right-hand sides of GCIs in $\mathcal{T}$ clearly have the same extensions under both interpretations.

Claim 1 is easy to see, since $\mathcal{T}_\emptyset(C) = \mathcal{T} \cup \{C_\emptyset \sqsubseteq C\}$ and $o \in C^{\mathcal{I}}$ implies $C_\emptyset^{\mathcal{I}(C_\emptyset,o)} \subseteq C^{\mathcal{I}(C_\emptyset,o)}$, hence $\mathcal{I}(C_\emptyset, o) \models \mathcal{T}_\emptyset(C)$.

For 2, we only need to show that all GCIs $G \sqsubseteq H \in \mathcal{T}_{\mathcal{D}}(C) \setminus \mathcal{T}$ are satisfied by $\mathcal{I}(C_{\mathcal{D}}, o)$. Since $C_{\mathcal{D}} \notin sig(H)$, $H^{\mathcal{I}(C_{\mathcal{D}},o)} = H^{\mathcal{I}}$ and $C_{\mathcal{D}}^{\mathcal{I}(C_{\mathcal{D}},o)} = \{o\} \subseteq C_{\mathcal{D}}^{\mathcal{I}}$ implies $G^{\mathcal{I}(C_{\mathcal{D}},o)} \subseteq G^{\mathcal{I}}$ in $\mathcal{EL}_\bot$.

Using the notion of witness models we can show that the auxiliary concept $F_\emptyset$ introduced in the extended TBox $\mathcal{T}_\emptyset(F)$ and the concept F from $\mathcal{T}$ have the same subsumers.

Proposition 12. *Let $\mathcal{T}$ be a TBox and F, G be concepts with $sig(G) \cap N_C^{aux} = \emptyset$. Then $F \sqsubseteq_{\mathcal{T}} G$ iff $F_\emptyset \sqsubseteq_{\mathcal{T}_\emptyset(F)} G$.*

Proof. Note that $\mathcal{T}_\emptyset(F) = \mathcal{T} \cup \{F_\emptyset \sqsubseteq F\}$.

The direction $F \sqsubseteq_{\mathcal{T}} G \implies F_\emptyset \sqsubseteq_{\mathcal{T}_\emptyset(F)} G$ follows by monotonicity of $\mathcal{EL}_\bot$ and transitivity of subsumption, since $F_\emptyset \sqsubseteq F \in \mathcal{T}_\emptyset(F)$. We show the other direction by contraposition. Assume $F \not\sqsubseteq_{\mathcal{T}} G$, i.e. there exists a model $\mathcal{I}$ of $\mathcal{T}$ s.t. $F^{\mathcal{I}} \not\subseteq G^{\mathcal{I}}$. Let $d \in F^{\mathcal{I}}$ and $d \notin G^{\mathcal{I}}$. By 1 of Proposition 11 it is clear, that $\mathcal{I}(F_\emptyset, d) \models \mathcal{T}_\emptyset(F)$ with $d \in F_\emptyset^{\mathcal{I}(F_\emptyset,d)} \setminus G^{\mathcal{I}(F_\emptyset,d)}$. □

To use typicality interpretations for reasoning under materialisation-based relevant entailment, the DCIs from $\mathcal{U} \subseteq \mathcal{D}$ need to be satisfied at the elements $d_F^{\mathcal{U}}$ representing $F \in Qc(\mathcal{K})$, but not (necessarily) for the role successors of these elements. In fact, it suffices to construct a typicality interpretation of minimally typical role successors, i.e. to use only the TBox for reasoning about role successors induced by existential restrictions. Such interpretations can be defined for general typicality domains.

Definition 13. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{U} \subseteq \mathcal{D}$. A* minimal typicality model *$\mathcal{I}_{\mathcal{K}}$ of $\mathcal{K}$ consists of a typicality domain $\Delta^{\mathcal{I}_{\mathcal{K}}}$ with the property*

$$d_F^{\mathcal{U}} \in \Delta^{\mathcal{I}_{\mathcal{K}}} \iff F_{\mathcal{U}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \bot \qquad (*)$$

and an interpretation mapping that satisfies the following conditions for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{I}_{\mathcal{K}}}$:

- *$d_F^{\mathcal{U}} \in A^{\mathcal{I}_{\mathcal{K}}}$ iff $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} A$, for $A \in sig_{N_C}(\mathcal{K})$ and*
- *$(d_F^{\mathcal{U}}, d_G^{\emptyset}) \in r^{\mathcal{I}_{\mathcal{K}}}$ iff $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} \exists r.G$, for $r \in sig_{N_R}(\mathcal{K})$.*

For a minimal typicality model $\mathcal{I}_{\mathcal{K}}$ we can show that $d_C^{\mathcal{D}_C} \in D^{\mathcal{I}_{\mathcal{K}}}$ (Definition 8) is equivalent to reasoning with the extended TBox, i.e. deciding $C_{\mathcal{D}_C} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_C}(C)} D$ as follows.

Proposition 14. *For a given well–separated DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and a minimal typicality model $\mathcal{I}_{\mathcal{K}}$ over a typicality domain $\Delta^{\mathcal{I}_{\mathcal{K}}}$, the following holds for all $d_F^{\mathcal{U}} \in \Delta^{\mathcal{I}_{\mathcal{K}}}$:*

1. $d_F^{\mathcal{U}} \in F_{\mathcal{U}}^{\mathcal{I}_{\mathcal{K}}}$
2. $d_F^{\mathcal{U}} \in G^{\mathcal{I}_{\mathcal{K}}}$ *iff* $F_{\mathcal{U}} \sqsubseteq_{\mathcal{T}_{\mathcal{U}}(F)} G$

Proof. Claim 1 is trivial since $F_\mathcal{U} \in N_C$ and by Property $(*)$ in Definition 13, $d_F^\mathcal{U} \in \Delta^{\mathcal{I}_\mathcal{K}} \iff F_\mathcal{U} \not\sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} \bot$.

We show 2 by induction on the structure of G. The base case, where $G = A$ ($A \in N_C$) follows by Definition 13 as $F_\mathcal{U} \in N_C$. The cases for $G = \top$ and $G = \bot$ are both trivial. Assume the property holds for two concepts D and E, the case of the induction step where $G = D \sqcap E$ follows quickly from the semantics of intersection and the induction hypothesis. It remains to show the induction step for $G = \exists r.E$ under the hypothesis $d_X^{\mathcal{U}'} \in E^{\mathcal{I}_\mathcal{K}} \iff X_{\mathcal{U}'} \sqsubseteq_{\mathcal{T}_{\mathcal{U}'}(X)} E$ for any $d_X^{\mathcal{U}'} \in \Delta^{\mathcal{I}_\mathcal{K}}$. $d_F^\mathcal{U} \in G^{\mathcal{I}_\mathcal{K}}$ implies $\exists d_X^\emptyset \in \Delta^{\mathcal{I}_\mathcal{K}}.(d_F^\mathcal{U}, d_X^\emptyset) \in r^{\mathcal{I}_\mathcal{K}} \wedge d_X^\emptyset \in E^{\mathcal{I}_\mathcal{K}}$. By Definition 13 this implies $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} \exists r.X$. By IH, $d_X^\emptyset \in E^{\mathcal{I}_\mathcal{K}} \iff X_\emptyset \sqsubseteq_{\mathcal{T}_\emptyset(X)} E$ and thus, by Proposition 12 $X \sqsubseteq_\mathcal{T} E$ for a well–separated $\mathcal{K}$. Thus $\mathcal{T} \subseteq \mathcal{T}_\mathcal{U}(F)$ implies $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} \exists r.E$. For the other direction, let $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} \exists r.E$, 1 directly implies that $d_F^\mathcal{U} \in F_\mathcal{U}^{\mathcal{I}_\mathcal{K}}$ and thus $d_F^\mathcal{U} \in (\exists r.E)^{\mathcal{I}_\mathcal{K}} = G^{\mathcal{I}_\mathcal{K}}$. □

Proposition 14 is the main ingredient for showing that a minimal typicality model $\mathcal{I}_\mathcal{K}$ from Definition 13 is in fact a model of the given DKB $\mathcal{K}$ according to Definition 7.

Lemma 15. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. Then, a minimal typicality model $\mathcal{I}_\mathcal{K}$ over a typicality domain $\Delta^{\mathcal{I}_\mathcal{K}}$ over $\mathcal{K}$ is a model of $\mathcal{K}$.*

Proof. We need to show that 1 and 2 of Definition 7 hold for $\mathcal{I}_\mathcal{K}$.

1. For all GCIs $G \sqsubseteq H \in \mathcal{T}$ and any $d_F^\mathcal{U} \in \Delta^{\mathcal{I}_\mathcal{K}}$, $d_F^\mathcal{U} \in G^{\mathcal{I}_\mathcal{K}}$ *iff* $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} G$ by Proposition 14 and $\mathcal{T} \subseteq \mathcal{T}_\mathcal{U}(F)$ implies $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} H$, which again by Proposition 14 holds *iff* $d_F^\mathcal{U} \in H^{\mathcal{I}_\mathcal{K}}$.
2. For 2 of Definition 7 we can use a similar argument. For all $d_F^\mathcal{U} \in \Delta^{\mathcal{I}_\mathcal{K}}$ and $G \mathrel{\underset{\sim}{\sqsubset}} H \in \mathcal{U}$ we need to show $d_F^\mathcal{U} \in G^{\mathcal{I}_\mathcal{K}} \implies d_F^\mathcal{U} \in H^{\mathcal{I}_\mathcal{K}}$. $d_F^\mathcal{U} \in G^{\mathcal{I}_\mathcal{K}}$ is equivalent to $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} G$ due to Proposition 14, which implies $F_\mathcal{U} \equiv_{\mathcal{T}_\mathcal{U}(F)} F_\mathcal{U} \sqcap G$. $G \mathrel{\underset{\sim}{\sqsubset}} H \in \mathcal{U}$ implies $F_\mathcal{U} \sqcap G \sqsubseteq H \in \mathcal{T}_\mathcal{U}(F)$, thus $F_\mathcal{U} \sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} H$ which is again equivalent to $d_F^\mathcal{U} \in H^{\mathcal{I}_\mathcal{K}}$ by Proposition 14. □

Using this result and Prop. 12, it is not hard to show that a minimal typicality model, restricted to elements regarding the empty set of DCIs, yields the classical canonical model for the $\mathcal{EL}_\bot$ TBox $\mathcal{T}$.

In order to recreate relevant (or rational) reasoning by materialisation using typicality interpretations, we need to define our entailment semantics. We can achieve this with different results by fixing the typicality domain of a minimal typicality model to a specific structure.

Definition 16. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. A* minimal relevant typicality model $\mathfrak{L}_\mathcal{K}$ *is a minimal typicality model over the lattice domain $\Delta^{\mathfrak{L}_\mathcal{K}} = \{d_F^\mathcal{U} \mid F \in Qc(\mathcal{K}), \mathcal{U} \subseteq \mathcal{D}, F_\mathcal{U} \not\sqsubseteq_{\mathcal{T}_\mathcal{U}(F)} \bot\}$.*

$\mathfrak{L}_\mathcal{K}$ is well-defined, as $\Delta^{\mathfrak{L}_\mathcal{K}}$ is clearly a lattice shaped typicality domain, due to the initial assumption $F \not\sqsubseteq_\mathcal{T} \bot$ (for all $F \in Qc(\mathcal{K})$), which implies that all $d_F^\emptyset$ exist in $\Delta^{\mathfrak{L}_\mathcal{K}}$ by Prop. 12. Also, $\Delta^{\mathfrak{L}_\mathcal{K}}$ obviously satisfies property $(*)$ of Definition 13. Therefore Proposition 14 and Lemma 15 apply to $\mathfrak{L}_\mathcal{K}$ and we can deduce that $\mathfrak{L}_\mathcal{K}$ satsfies $\mathcal{K}$ according to Definition 7. The minimal relevant typicality model need not use the complete lattice domain of $2^{|\mathcal{D}|} * |Qc(\mathcal{K})|$ elements due to inconsistent combinations of the represented concept $F, \mathcal{U}$ and $\mathcal{T}$. We use $\mathfrak{L}_\mathcal{K}$ to characterise relevant entailment.

Example 17 (Minimal typicality model). Consider again the DKB $\mathcal{K}_{ex1}$ from Example 4 with the consistent subsets of the DBox $\mathcal{D}_{Worker} = \mathcal{D}_{ex1}$, and $\mathcal{D}_{Boss} = \{Worker \mathrel{\underset{\sim}{\sqsubset}} Productive,\ Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible\}$ w.r.t. $Worker$ and $Boss$, respectively. The subset-lattice of $\mathcal{D}_{ex1}$ and $\mathfrak{L}_{\mathcal{K}_{ex1}}$ are illustrated in Figure 1 using obvious abbreviations and omitting labels for clarity. Note, that

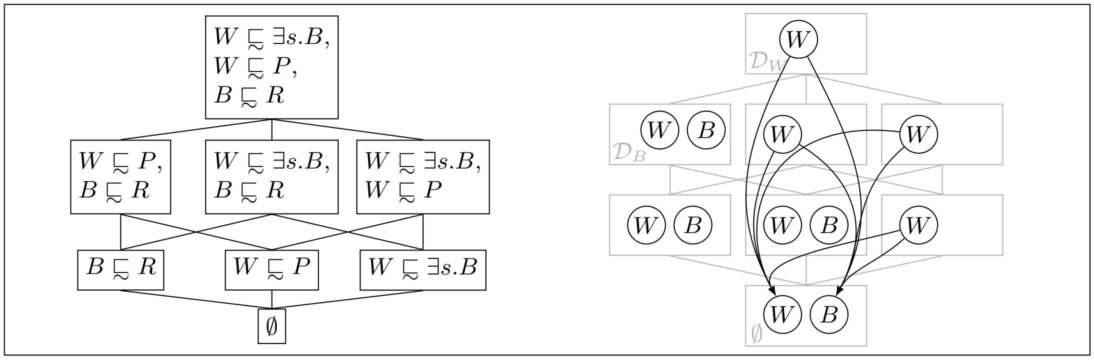

Fig. 1: a Subset lattice of $\mathcal{D}_{ex1}$ and **b** $\mathfrak{L}_{\mathcal{K}_{ex1}}$

the domain elements are grouped in *grey boxes* according to the subset-lattice indicating which DBox subsets are satisfied by which domain elements.

According to Definition 8, $\mathfrak{L}_{\mathcal{K}_{ex1}} \models Worker \mathrel{\underset{\sim}{\sqsubset}} \exists superior.Boss$, as well as $\mathfrak{L}_{\mathcal{K}_{ex1}} \models Boss \mathrel{\underset{\sim}{\sqsubset}} Responsible \sqcap Productive$, because $d_{Worker}^{\mathcal{D}_{Worker}}$ and $d_{Boss}^{\mathcal{D}_{Boss}}$ satisfy $\mathcal{D}_{Worker}$ and $\mathcal{D}_{Boss}$, respectively.

We want to characterise different entailment relations based on different kinds of typicality models for a given DKB $\mathcal{K}$ which vary in the defeasible information admitted for required role successors. We use the minimal relevant typicality model over a lattice domain to characterise relevant entailment of propositional nature $\models_p^{rel}$.

Definition 18. *Let $\mathcal{K}$ be a DKB. $\mathcal{K}$* propositionally entails *a defeasible subsumption relationship $C \mathrel{\underset{\sim}{\sqsubset}} D$ under relevant closure (written $\mathcal{K} \models_p^{rel} C \mathrel{\underset{\sim}{\sqsubset}} D$) iff $\mathfrak{L}_{\mathcal{K}} \models C \mathrel{\underset{\sim}{\sqsubset}} D$.*

This form of entailment is called *propositional* since all role successors are uniformly non-typical and since DCIs are neglected for quantified concepts. Next, we investigate the relationship between $\models_m^{rel}$ (Sec. 3) and $\models_p^{rel}$. Our approach to decide propositional entailments based on the extended TBox for a concept F, coincides with enriching F with the materialisation of the given DBox. Note that the following is a very general result, detached from the notion of typicality models.

Lemma 19. *Let $\mathcal{T}$ be a TBox $\mathcal{T}$, $\mathcal{D}$ a DBox, and C, D be concepts, with $sig(X) \cap N_C^{aux} = \emptyset$ (for $X \in \{\mathcal{T}, \mathcal{D}, C, D\}$). Then $\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} D$ iff $C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} D$.*

Proof. We show this lemma by induction on the size of $\mathcal{D}$. The base case for $\mathcal{D} = \emptyset$ is already shown in Proposition 12.

Assume the following hypothesis holds for any C and DBox with $|\mathcal{D}| = k$ ($k \geq 0$).

$$\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} D \textit{ iff } C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} D \tag{IH}$$

For the induction step, we need to show that

$$\overline{\mathcal{D}'} \sqcap C \sqsubseteq_{\mathcal{T}} D \textit{ iff } C_{\mathcal{D}'} \sqsubseteq_{\mathcal{T}_{\mathcal{D}'}(C)} D \tag{IS}$$

holds for the DBox $\mathcal{D}'$ extending $\mathcal{D}$ by one defeasible statement, i.e. $|\mathcal{D}'| = k+1$. For $sig(H) \cap N_C^{aux} = \emptyset$, let $\mathcal{D}' = \mathcal{D} \cup \{G \mathrel{\underset{\sim}{\sqsubset}} H\}$ and therefore $\mathcal{T}_{\mathcal{D}'}(C)$ is equivalent to $\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}$ up to the renaming of $C_{\mathcal{D}'}$, hence we show

$$(\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} D \textit{ iff } C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}} D. \tag{IS'}$$

We consider two cases for the influence of the newly introduced DCI $G \mathrel{\underset{\sim}{\sqsubset}} H$. In both cases we are able to reduce the left- and right-hand side of the "*iff*" in (IS') to the induction hypothesis respectively. We denote the left-hand side of (IH), (IS) and (IS') as the materialisation side and the right-hand side as the extension side.

Case 1: $\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} G$
By (IH) this case is equivalent to $C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} G$. For the materialisation side,

$$\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} G \implies (\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} H$$

and thus

$$\begin{aligned}(\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C &\equiv_{\mathcal{T}} (\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H \\ &\equiv_{\mathcal{T}} (\neg G \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H) \sqcup (H \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H).\end{aligned}$$

Due to the condition for this case ($\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} G$), $\neg G \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H$ is equivalent to $\bot$, which means

$$\begin{aligned}(\neg G \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H) \sqcup (H \sqcap \overline{\mathcal{D}} \sqcap C \sqcap H) &\equiv_{\mathcal{T}} \bot \sqcup \overline{\mathcal{D}} \sqcap C \sqcap H \\ &\equiv_{\mathcal{T}} \overline{\mathcal{D}} \sqcap C \sqcap H.\end{aligned}$$

On the extension side, $C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} G \implies C_{\mathcal{D}} \equiv_{\mathcal{T}_{\mathcal{D}}(C)} C_{\mathcal{D}} \sqcap G$. Therefore $\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}$ is equivalent to $\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqsubseteq H\}$, which is equivalent to $\mathcal{T}_{\mathcal{D}}(C \sqcap H)$ (containing $(C \sqcap H)_{\mathcal{D}} \sqsubseteq C \sqcap H$) up to the renaming of $(C \sqcap H)_{\mathcal{D}} \in N_C^{aux}$.

Combining both sides yields $\overline{\mathcal{D}} \sqcap (C \sqcap H) \sqsubseteq_{\mathcal{T}} D$ *iff* $(C \sqcap H)_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C \sqcap H)} D$, which holds by (IH).

Case 2: $\overline{\mathcal{D}} \sqcap C \not\sqsubseteq_{\mathcal{T}} G$
By (IH) this case is equivalent to $C_{\mathcal{D}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} G$. Again, we investigate $\mathcal{D}' = \mathcal{D} \cup \{G \mathrel{\underset{\sim}{\sqsubset}} H\}$. For the extension side, $\mathcal{T}_{\mathcal{D}'}(C)$ is equivalent to $\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}$ up to the renaming of $C_{\mathcal{D}'}$. Thus we need to show that:

$$C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} D \iff C_{\mathcal{D}} \sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}} D.$$

" $\implies$ " is again trivial. We show " $\impliedby$ " by contraposition and assume that $C_{\mathcal{D}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} D$, i.e. there exists an interpretation $\mathcal{I} \models \mathcal{T}_{\mathcal{D}}(C)$ s.t. there is a $d \in C_{\mathcal{D}}^{\mathcal{I}}$ with $d \notin D^{\mathcal{I}}$ and another interpretation $\mathcal{J} \models \mathcal{T}_{\mathcal{D}}(C)$, which, due to the condition for this case ($C_{\mathcal{D}} \not\sqsubseteq_{\mathcal{T}_{\mathcal{D}}(C)} G$) satisfies $e \in C_{\mathcal{D}}^{\mathcal{J}} \setminus G^{\mathcal{J}}$ for some $e \in \Delta^{\mathcal{J}}$. By Lemma 2, $\mathcal{I} \times \mathcal{J} \models \mathcal{T}_{\mathcal{D}}(C)$ and $(d, e) \in C_{\mathcal{D}}^{\mathcal{I} \times \mathcal{J}} \setminus (D^{\mathcal{I} \times \mathcal{J}} \cup G^{\mathcal{I} \times \mathcal{J}})$ by Definition 1. Let $\mathcal{I}'$ denote the witness model $(\mathcal{I} \times \mathcal{J})(C_{\mathcal{D}}, (d, e))$ with the witness element (d, e) for $C_{\mathcal{D}}$. Claim 2 of Proposition 11 implies $\mathcal{I}' \models \mathcal{T}_{\mathcal{D}}(C)$. Furthermore, $(C_{\mathcal{D}} \sqcap G)^{\mathcal{I}'} = \emptyset$ and thus $\mathcal{I}' \models \mathcal{T}_{\mathcal{D}}(C) \cup \{C_{\mathcal{D}} \sqcap G \sqsubseteq H\}$ with $(d, e) \in C_{\mathcal{D}}^{\mathcal{I}'} \setminus D^{\mathcal{I}'}$.

For the materialisation side, we need to show a similar statement, where $\overline{\mathcal{D}} \sqcap C \not\sqsubseteq_{\mathcal{T}} G$ implies

$$\overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} D \iff (\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C \sqsubseteq_{\mathcal{T}} D.$$

" $\implies$ " holds obviously. For the other direction we proceed by contraposition. Assume $\overline{\mathcal{D}} \sqcap C \not\sqsubseteq_{\mathcal{T}} D$ holds, i.e. there exists an interpretation $\mathcal{I} \models \mathcal{T}$ with $d \in (\overline{\mathcal{D}} \sqcap C)^{\mathcal{I}} \setminus D^{\mathcal{I}}$ and by the condition of this case ($\overline{\mathcal{D}} \sqcap C \not\sqsubseteq_{\mathcal{T}} G$) a $\mathcal{J} \models \mathcal{T}$ with $e \in (\overline{\mathcal{D}} \sqcap C)^{\mathcal{J}} \setminus G^{\mathcal{J}}$. The product of $\mathcal{I}$ and $\mathcal{J}$ satisfies $\mathcal{T}$ by Lemma 2 and has $(d, e) \in (\overline{\mathcal{D}} \sqcap C)^{\mathcal{I} \times \mathcal{J}} \setminus (G^{\mathcal{I} \times \mathcal{J}} \cup D^{\mathcal{I} \times \mathcal{J}})$, i.e. $(d, e) \in (\neg G)^{\mathcal{I} \times \mathcal{J}}$. Therefore $(d, e) \in ((\neg G \sqcup H) \sqcap \overline{\mathcal{D}} \sqcap C)^{\mathcal{I} \times \mathcal{J}} \setminus D^{\mathcal{I} \times \mathcal{J}}$.

To sum up, both subsumptions in (IS') were shown to be equivalent to the subsumptions in (IH) under two covering case assumptions. This concludes the induction proof. □

Although the entailment relations $\models_m^{rel}$ as introduced in [5] and $\models_p^{rel}$ are defined in different ways and are based on distinct semantics, they yield the same consequences (for subsumption) w.r.t. DKBs.

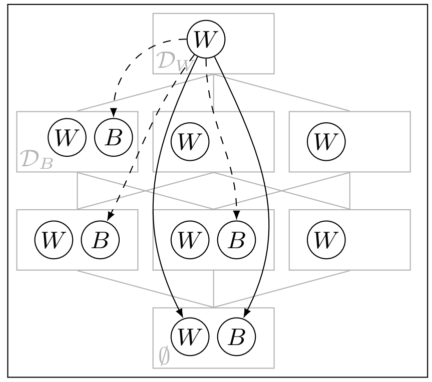

Fig. 2: Increasing typicality of role successors

Theorem 20. *$\mathcal{K} \models_p^{rel} C \sqsubsetapprox D$ iff $\mathcal{K} \models_m^{rel} C \sqsubsetapprox D$.*

Proof. $\mathcal{K} \models_p^{rel} C \sqsubsetapprox D$ is defined as $\mathfrak{L}_\mathcal{K} \models C \sqsubsetapprox D$, i.e. $d_C^{\mathcal{D}_C} \in D^{\mathfrak{L}_\mathcal{K}}$ which is equivalent to $C_{\mathcal{D}_C} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_C}(C)} D$ by Proposition 14. This subsumption in turn is equivalent to deciding $\overline{\mathcal{D}_C} \sqcap C \sqsubseteq_\mathcal{T} D$ by Lemma 19, which is precisely the definition of $\mathcal{K} \models_m^{rel} C \sqsubsetapprox D$ (from Section 3). □

In addition, this result shows that entailments based on minimal relevant typicality models also bear the shortcomings for defeasible reasoning regarding nested existential restrictions—a nuisance which we want to alleviate next.

5 Maximal Typicality Models for Relevant Entailment

We illustrate by continuing on Example 17 how defeasible information is disregarded for nested existential restrictions and our proposed countermeasure.

Example 21. Consider again the DKB $\mathcal{K}_{ex1}$ from Example 17, with $\mathfrak{L}_{\mathcal{K}_{ex1}}$ (as depicted in Figure 1). No defeasible information is used for reasoning over the *superior* successors of the element $d_{Worker}^{\mathcal{D}_{Worker}}$ and thus $\mathfrak{L}_\mathcal{K} \not\models Worker \sqsubsetapprox \exists superior.Responsible$. However, the defeasible statement $Boss \sqsubsetapprox Responsible$ remains *undefeated* for $d_{Boss}^{\emptyset}$.

Instead of satisfying $Boss \sqsubsetapprox Responsible$ at the element $d_{Boss}^{\emptyset}$, we can "upgrade" the existing *superior* relationship to another representative of *Boss*, that satisfies the DCI. For instance, we could upgrade from $(d_{Worker}^{\mathcal{D}_{Worker}}, d_{Boss}^{\emptyset})$ to $(d_{Worker}^{\mathcal{D}_{Worker}}, d_{Boss}^{\{Boss \sqsubsetapprox Responsible\}})$—as illustrated in Figure 2 by the *dashed arrows*.[3] Our method upgrades typicality of role successors as much as possible, i.e., it picks representatives of the same concept that satisfy more and more DCIs as long as it does not result in inconsistencies. Here, this method even yields the conclusion $Worker \sqsubsetapprox \exists superior.Productive$.

Upgrading the typicality of a role successor depends on the information present in the model. Different orders of such upgrade steps can yield different models of increased typicality. In order to handle sets of models over the same typicality domain Δ over the same DKB $\mathcal{K}$, we need the notions of intersection and inclusion of models.

[3] Note that Figure 2 depicts only an excerpt of $\mathfrak{L}_{\mathcal{K}_{ex1}}$ for comprehensibility.

Definition 22. *For two interpretations $\mathcal{I}, \mathcal{J}$ over the same domain Δ we define*

- $\mathcal{I} \cap \mathcal{J} = (\Delta, \cdot^{\mathcal{I}\cap\mathcal{J}})$ *with*
  - $A^{\mathcal{I}\cap\mathcal{J}} = A^{\mathcal{I}} \cap A^{\mathcal{J}}$ *(for $A \in N_C$) and*
  - $r^{\mathcal{I}\cap\mathcal{J}} = r^{\mathcal{I}} \cap r^{\mathcal{J}}$ *(for $r \in N_R$)*
- $\mathcal{I} \subseteq \mathcal{J}$ *iff* $\forall A \in N_C . A^{\mathcal{I}} \subseteq A^{\mathcal{J}}$ *and* $\forall r \in N_R . r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$.

Using this definition we can formalise the notion of "upgrading the typicality" of a typicality interpretation, i.e. introduce copies of role edges that point to elements that represent the same concept, but satisfy more defeasible statements.

Definition 23. *Let $\mathcal{I}$ be a typicality interpretation for $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. The* set of more typical role edges *for a given role r is defined as*

$$TR_{\mathcal{I}}(r) = \{(d_G^{\mathcal{X}}, d_H^{\mathcal{U}'}) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus r^{\mathcal{I}} \mid \exists \mathcal{U} \subseteq \mathcal{D}.(d_G^{\mathcal{X}}, d_H^{\mathcal{U}}) \in r^{\mathcal{I}} \wedge \mathcal{U} \subset \mathcal{U}' \subseteq \mathcal{D}_H\}.$$

Let $\mathcal{I}$ and $\mathcal{J}$ be typicality interpretations. $\mathcal{J}$ is a typicality extension *of $\mathcal{I}$ iff*

1. $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$,
2. $A^{\mathcal{J}} = A^{\mathcal{I}}$ *(for $A \in N_C$),*
3. $r^{\mathcal{J}} = r^{\mathcal{I}} \cup R$, *where* $R \subseteq TR_{\mathcal{I}}(r)$ *(for $r \in sig_{N_R}(\mathcal{K})$), and*
4. $\exists r \in sig_{N_R}(\mathcal{K}).\ r^{\mathcal{I}} \subset r^{\mathcal{J}}$.

The set of all typicality extensions *of a typicality interpretation $\mathcal{I}$ is $typ(\mathcal{I})$.*

Observe that a typicality interpretation cannot be a typicality extensions of itself. With typicality extensions at hand we can transform typicality interpretations into a set of more typical interpretations. Unfortunately, this operation does not preserve the property of being a typicality model. Let us demonstrate this by Example 21, let $\mathcal{K}_{ex2} = (\mathcal{T}_{ex2}, \mathcal{D}_{ex1})$, and $\mathcal{T}_{ex2} = \mathcal{T}_{ex1} \cup \{\exists superior.Responsible \sqsubseteq \exists coworker.Worker\}$. Since the minimal relevant typicality model $\mathfrak{L}_{\mathcal{K}_{ex2}}$ coincides with the minimal relevant typicality model $\mathfrak{L}_{\mathcal{K}_{ex1}}$, Figure 2 depicts a typicality extension of $\mathfrak{L}_{\mathcal{K}_{ex2}}$ according to Definition 23. However, the extension in Figure 2 is no longer a model of $\mathcal{T}_{ex2}$, as the newly introduced GCI

$$\exists superior.Responsible \sqsubseteq \exists coworker.Worker$$

is no longer satisfied for $d_{Worker}^{\mathcal{D}}$. It can be extended to a model by introducing a *coworker* successor for $d_{Worker}^{\mathcal{D}}$ that belongs to $Worker$. In order not to introduce unwanted inconsistencies, the successor in this new relationship needs to be picked such that it only contains the information *strictly* required by $\mathcal{K}$, i.e. $d_{Worker}^{\emptyset}$ is picked. In general, the neccessary role-successors are drawn from the least typical domain elements, those where no DCIs need to hold. We formalise the particular *model completions* that we are interested in.

Definition 24. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and Δ a typicality domain over $\mathcal{K}$. A typicality interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ is a* model completion *of a typicality interpretation $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ iff*

1. $\mathcal{J} \subseteq \mathcal{I}$,
2. $\mathcal{I} \models \mathcal{K}$, *and*
3. $\forall E \in Qc(\mathcal{K}).d_F^{\mathcal{U}} \in (\exists r.E)^{\mathcal{I}} \implies (d_F^{\mathcal{U}}, d_E^{\emptyset}) \in r^{\mathcal{I}}$ *(for any $F \in Qc(\mathcal{K})$ and $\mathcal{U} \subseteq \mathcal{D}$).*

The set of all model completions *of $\mathcal{J}$ is denoted as $mc(\mathcal{J})$.*

Note that $\mathcal{K}$ is an important parameter to compute $mc(\mathcal{J})$ and $\mathcal{K}$ gives rise to the underlying typicality domain of $\mathcal{J}$. Additionally, $mc(\mathcal{J})$ is finite due to the unique domain of model completions of $\mathcal{J}$ ($\Delta^{\mathcal{J}}$) and the fact that only concept and role names in $sig(\mathcal{K})$ need to be considered.

An interpretation that is a model completion to itself is called a *safe model* and obviously satisfies the properties of Definition 24. So, for any typicality interpretation $\mathcal{J}$, all interpretations in $mc(\mathcal{J})$ are safe models. The following proposition shows that it is no restriction to consider model extensions of $\mathcal{I}$ that belong to $mc(\mathcal{I})$, because if $mc(\mathcal{I}) = \emptyset$ then $\mathcal{I}$ cannot be completed into any model.

Proposition 25. *For a DKB $\mathcal{K}$, a typicality domain Δ over $\mathcal{K}$, and a typicality interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$,*

$$\exists \mathcal{J}.(\mathcal{I} \subseteq \mathcal{J} \wedge \mathcal{J} \models \mathcal{K}) \implies mc(\mathcal{I}) \neq \emptyset.$$

Proof. Since the premise of this implication requires $\mathcal{J}$ to satisfy Condition 1 and 2 of Definition 24, it suffices to show that we can extend $\mathcal{J}$ to also satisfy Condition 3 without violating the other two. Given $\mathcal{J}$, create $\mathcal{J}' = (\Delta, \cdot^{\mathcal{J}'})$ with

- $A^{\mathcal{J}'} = A^{\mathcal{J}}$ ($A \in sig_{N_C}(\mathcal{K})$) and
- $r^{\mathcal{J}'} = r^{\mathcal{J}} \cup \{(d_F^{\mathcal{U}}, d_D^{\emptyset}) \mid d_F^{\mathcal{U}} \in (\exists r.D)^{\mathcal{J}}, D \in Qc(\mathcal{K})\}$ ($r \in sig_{N_R}(\mathcal{K})$).

We show that the extension of all concepts C with $Qc(C) \subseteq Qc(\mathcal{K})$ under $\mathcal{J}'$ is equivalent to their extension under $\mathcal{J}$, i.e. $C^{\mathcal{J}} = C^{\mathcal{J}'}$ by induction on concepts C. $C = \top$ and $C = \bot$ are obvious. The case $C = A$ for $A \in sig_{N_C}(\mathcal{K})$ follows from the definition of $\mathcal{J}'$. The case of $C = D \sqcap E$ is also fairly easy to see.

For the case $C = \exists r.D$, $(\exists r.D)^{\mathcal{J}} \subseteq (\exists r.D)^{\mathcal{J}'}$ is easy to see since $\mathcal{J}'$ clearly extends $\mathcal{J}$ without removing anything that holds in $\mathcal{J}$. We show $(\exists r.D)^{\mathcal{J}'} \subseteq (\exists r.D)^{\mathcal{J}}$. For any $d_F^{\mathcal{U}} \in (\exists r.D)^{\mathcal{J}'}$ it follows that there exists some $d_G^{\mathcal{U}'} \in \Delta$ such that $(d_F^{\mathcal{U}}, d_G^{\mathcal{U}'}) \in r^{\mathcal{J}'}$ and $d_G^{\mathcal{U}'} \in D^{\mathcal{J}'}$. Either $(d_F^{\mathcal{U}}, d_G^{\mathcal{U}'}) \in r^{\mathcal{J}}$, in which case the induction hypothesis implies $d_F^{\mathcal{U}} \in (\exists r.D)^{\mathcal{J}}$, or $G = D$ and $\mathcal{U}' = \emptyset$, which is the case if $d_F^{\mathcal{U}} \in (\exists r.D)^{\mathcal{J}}$ by the definition of $\mathcal{J}'$.

Using the just shown equivalence of the extensions, it is easy to see that every left- and right-hand side of DCIs and GCIs in $\mathcal{K}$ is extended to the same set of domain elements under $\mathcal{J}$ and $\mathcal{J}'$, hence $\mathcal{J} \models \mathcal{K} \implies \mathcal{J}' \models \mathcal{K}$. Clearly $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{J}'$ and by the definition of $\mathcal{J}'$, Condition 3 of Definition 24 is also satisfied for $\mathcal{J}'$. Therefore, $\mathcal{J}' \in mc(\mathcal{I})$. □

Note that if $\mathcal{J} \models \mathcal{K}$, then $\mathcal{J}$ does not necessarily belong to $mc(\mathcal{J})$, however by the construction in the proof of Proposition 25, at least $\mathcal{J}' \in mc(\mathcal{J})$.

Since model completions introduce minimal typical role successors they may necessitate further typicality extensions. So, typicality extensions and model completions need to be applied alternatingly until a maximum is reached. Maximality for typicality extensions is characterised in the following way: a typicality interpretation

$$\mathcal{I} \text{ is } \textit{typicality extensible iff } \exists \mathcal{J} \in typ(\mathcal{I}).mc(\mathcal{J}) \neq \emptyset.$$

Intuitively, a typicality interpretation is typicality extensible if it admits to some typicality extension that is, or can be completed to a safe model. Therefore, a typicality interpretation is *maximal iff* it is not typicality extensible. To formalise the process of increasing typicality and completing to a model until reaching maximal typicality, we introduce some notation and an upgrade operator. Given a typicality domain Δ over the DKB $\mathcal{K}$, define the *set of all safe models* over a typicality domain Δ over the DKB $\mathcal{K}$ as

$$P(\Delta) = \{\mathcal{J} \mid \mathcal{J} = (\Delta, \cdot^{\mathcal{J}}) \wedge \mathcal{J} \in mc(\mathcal{J})\}.$$

Definition 26. *The typicality upgrade operator* $T : 2^{P(\Delta)} \rightarrow 2^{P(\Delta)}$ *is defined for* $S \subseteq P(\Delta)$ *as:*

1. $T(S) = S \setminus \{\mathcal{I}\} \cup \bigcup_{\mathcal{J} \in typ(\mathcal{I})} mc(\mathcal{J})$, *if* $\mathcal{I} \in S$ *is typicality extensible,*
2. $T(S) = S$, *otherwise.*

For a given set of model completions $S \subseteq P(\Delta)$, *the* fixpoint of T *is* $T_m(S)$ *if* $T_m(S) = T_{m+1}(S)$ *with*

1. $T_0(S) = S$ *and*
2. $T_i(S) = T(T_{i-1}(S))$ $(i > 0)$.

The set of maximal typicality extensions *of the typicality models in* S *is* $typ^{max}(S) = T_m(S)$.

Observe that the operator T replaces a model by another model (case 1), thus T does not change the cardinality of the set and $|typ^{max}(S)| = |S|$ holds.

It is clear that neither typicality extensions (Definition 23) nor model completions (Definition 24) supply a unique typicality extension or model completion in every case. Thus the typicality upgrade operator is defined over sets of typicality interpretations. Applying the operator T to $\{\mathcal{I}\}$ easily leads to an exponential number of typicality interpretations in the size of the domain $\Delta^{\mathcal{I}}$. Nevertheless the fixpoint of T will always be reached.

Proposition 27. *For a finite set of model completions* $S \subseteq P(\Delta)$, $typ^{max}(S)$

1. *is finite, and*
2. *can be computed in finite time.*

Proof. We prove Claim 1 by showing that the typicality operator T can never produce models outside of $P(\Delta)$ and that $P(\Delta)$ must be finite. It follows that the fixpoint of T, $typ^{max}(S)$, can also not exceed $P(\Delta)$ (if it exists). Since models of $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ only need to satisfy axioms contained in $\mathcal{K}$, it is enough to consider interpretations using only concept and role names occurring in $sig(\mathcal{K})$, which is finite. Any typicality domain Δ over a DKB $\mathcal{K}$ is finite due to finite sets $Qc(\mathcal{K})$ and $\mathcal{D}$. Therefore, the set $P(\Delta)$ of all safe models over Δ is finite w.r.t. $sig(\mathcal{K})$. Since $S \subseteq P(\Delta)$ and the first case in the definition of $T(S)$ only adds safe models for each removed element of S, $T(S) \subseteq P(\Delta)$ as well, contradicting that the fixpoint of T can be infinite (if it exists), thus proving Claim 1.

To show that the fixpoint of T always exists, we prove Claim 2 with a standard technique for showing termination, called well-founded set orders (cf. [2] pp. 21–25). Using a particular well-founded set-order we show that one application of T on S produces a a set of safe models from $P(\Delta)$ that is strictly smaller than S w.r.t. to the set-order. Well-foundedness of this order then implies that continuous applications of T must eventually result in a fixpoint. The set order we use measures typicality interpretations over a typicality domain Δ (of a DKB $\mathcal{K}$) by the number of “unused” role connections (w.r.t. Δ and $sig(\mathcal{K})$). While the cardinality of $T(S)$ might be larger than the cardinality of S, the added interpretations have less “unused” role connections than the removed interpretation. Let $\rho = |\Delta \times \Delta| * |sig_{N_R}(\mathcal{K})|$ be the total number of distinct role edges possible over the domain Δ and let $|\mathcal{I}|_{N_R} = \sum_{r \in (sig_{N_R}(\mathcal{K}))} |r^{\mathcal{I}}|$ be the number of role edges used in interpretations $\mathcal{I} \in P(\Delta)$. Note that for all $\mathcal{I} \in P(\Delta)$, $|\mathcal{I}|_{N_R} \leq \rho$ holds. For a given typicality domain Δ let

$$\sigma : P(\Delta) \to \mathbb{N} \text{ with } \sigma(\mathcal{I}) = \rho - |\mathcal{I}|_{N_R}$$

be the function that provides the number of unused role relationships for a safe model.

We define the relation $\succ \subseteq P(\Delta) \times P(\Delta)$ as an ordering over interpretations in $P(\Delta)$ according to the number of unused role relationships, that is $\mathcal{I}_1 \succ \mathcal{I}_2$ *iff* $\sigma(\mathcal{I}_1) > \sigma(\mathcal{I}_2)$. Since $|\mathcal{I}|_{N_R}$ has ρ as an upper bound, it is easy to see that $(P(\Delta), \succ)$ is well-founded, i.e. there cannot be a chain $\mathcal{I}_1 \succ \mathcal{I}_2 \succ \ldots$ that is infinitely descending.

Let $\succ_{set}$ be the extension of $\succ$ to sets over $P(\Delta)$, by [2] $\succ_{set}$ is well-founded if $\succ$ is well-founded. Note that for any $S \subseteq P(\Delta)$, $T(S)$ is well-defined as in the definition of the typicality operator (Definition 26) either Condition (1) holds for some $\mathcal{I} \in S$ or Condition (2) holds. Obviously, as soon as Condition (2) holds, $T(S) = S$, resulting in a fixpoint of T. Hence, we only need to show that for every S with $S \neq T(S)$, $S \succ_{set} T(S)$ holds, i.e. for every interpretation $\mathcal{I} \in T(S) \setminus S$ there is some interpretation $\mathcal{I}' \in S \setminus T(S)$ such that $\mathcal{I}' \succ \mathcal{I}$. Observe that the following holds for any typicality interpretation $\mathcal{I}$ and thus also for $\mathcal{I} \in P(\Delta)$

$$\begin{aligned} &\mathcal{J} \in \mathit{typ}(\mathcal{I}) \implies \sigma(\mathcal{J}) < \sigma(\mathcal{I}) \text{ and thus} \\ &\forall \mathcal{J} \in \mathit{typ}(\mathcal{I}).\mathcal{J}' \in \mathit{mc}(\mathcal{J}) \implies \sigma(\mathcal{J}') < \sigma(\mathcal{I}). \end{aligned} \tag{3}$$

Let now $\mathcal{I} \in S$ be the chosen interpretation in Condition (1) of Definition 26, then $\mathcal{I} \notin T(S)$ and for all $\mathcal{J}' \in \bigcup_{\mathcal{J} \in typ(\mathcal{I})} \mathit{mc}(\mathcal{J}) \implies \mathcal{I} \succ \mathcal{J}'$ by Observation (3). This shows that $S \succ_{set} T(S)$ and together with the well-foundedness of the relation $\succ_{set}$ this implies that Condition (1) from Definition 26 can only hold a finite number of times for repeated applications of T to one input set $S \subseteq P(\Delta)$. Thus $\mathit{typ}^{max}(S)$ can be computed in finite time. □

Remark 28. Obviously all interpretations in $\mathit{typ}^{max}(S)$ are safe models (c.f. Definition 24), regardless of the chosen $S \subseteq P(\Delta)$.

The following example illustrates that this typicality extension can quickly lead to multiple different maximal typicality interpretations, starting from a single interpretation.

Example 29. We extend the DKB from Example 17 to DKB $\mathcal{K}_{ex3} = (\mathcal{T}_{ex3}, \mathcal{D}_{ex1})$ with the TBox

$$\mathcal{T}_{ex3} = \mathcal{T}_{ex1} \cup \{\exists \mathit{superior}.\exists \mathit{superior}.\mathit{Responsible} \sqsubseteq \bot\}.$$

Let the role edge $(d^{\mathcal{D}}_{Worker}, d^{\emptyset}_{Worker}) \in \mathit{superior}^{\mathfrak{L}_{\mathcal{K}_{ex3}}}$ be upgraded to $(d^{\mathcal{D}}_{Worker}, d^{\mathcal{D}}_{Worker})$ and likewise $(d^{\mathcal{D}}_{Worker}, d^{\emptyset}_{Boss}) \in \mathit{superior}^{\mathfrak{L}_{\mathcal{K}_{ex3}}}$ to $(d^{\mathcal{D}}_{Worker}, d^{\mathcal{D}_{Boss}}_{Boss})$. If both of these upgrades exist in the same typicality extension $\mathcal{J}$, it does not admit to a model completion, as an inconsistency would be caused by $d^{\mathcal{D}}_{Worker} \in (\exists \mathit{superior}.\exists \mathit{superior}.\mathit{Responsible})^{\mathcal{J}}$. The typicality upgrade $(d^{\mathcal{D}}_{Worker}, d^{\{Worker \sqsubseteq Productive\}}_{Boss})$, however, is "allowed" to occur in a typicality extension, leading to the entailment of $Worker \,\underset{\sim}{\sqsubset}\, \exists \mathit{superior}.(\mathit{Boss} \sqcap \mathit{Productive})$. This shows that inheritance blocking can be remedied even for quantified concepts when upgrading typicality of successors in a lattice domain.

It is clear that, given an arbitrary typicality model $\mathcal{I}$ with a typicality domain $\Delta^{\mathcal{I}}$ over a DKB $\mathcal{K}$, the above described process leads to a variety of maximal typicality models in $\mathit{typ}^{max}(\mathcal{I})$. Recall the overall structure of our approach, where we want to

1. replicate materialisation-based relevant entailment by means of typicality interpretations (Theorem 20) and

2. directly extend this result to obtain nested relevant entailment.

To this end we are using maximal typicality models of the unique minimal typicality model $\mathfrak{L}_\mathcal{K}$ to define our inference semantics. There are several options to obtain inferences from a set of models. Since in classical DL reasoning entailment considers all models, we pick semantics closely related to *cautious* reasoning here. To this end we build a single model that is canonical in the sense that it is the biggest model (w.r.t. $\subseteq$) contained in *all* maximal typicality models obtained from the minimal relevant typicality model $\mathfrak{L}_\mathcal{K}$ (Definition 16). Since the domain of $\mathfrak{L}_\mathcal{K}$ is large enough to support relevance based inferences, this strategy provides nested relevant entailment.

Definition 30. *Let $\mathcal{K}$ be a DKB. The* relevant canonical model *is* $\mathfrak{RC}_\mathcal{K} = \bigcap_{\mathcal{I}\in typ^{max}(\{\mathfrak{L}_\mathcal{K}\})} \mathcal{I}$.

Note that the intersection over all maximal typicality models is well-defined as $typ^{max}(\{\mathfrak{L}_\mathcal{K}\})$ is finite and since $\mathfrak{L}_\mathcal{K}$ is a safe model (i.e. $\mathfrak{L}_\mathcal{K} \in mc(\mathfrak{L}_\mathcal{K})$) it is not empty, as shown in the following. First, we show that the intersection of any set of safe models for a DKB $\mathcal{K}$ remains a safe model for $\mathcal{K}$.

Proposition 31. *Let $\mathcal{K}$ be a DKB and let $\mathcal{M} \subseteq P(\Delta)$ such that $\forall \mathcal{J} \in \mathcal{M}.\mathcal{J} \models \mathcal{K}$, as well as $\mathcal{I}^* = \bigcap_{\mathcal{J}\in\mathcal{M}} \mathcal{J}$. It holds that*

1. $C^{\mathcal{I}^*} = \bigcap\limits_{\mathcal{J}\in\mathcal{M}} C^\mathcal{J}$ *for $Qc(C) \subseteq Qc(\mathcal{K})$, and*
2. $\mathcal{I}^* \models \mathcal{K}$

Proof. We prove Claim 1 by induction on the structure of C, where the base case follows immediately from the definition of intersections of interpretations (Def. 22). For $C = D \sqcap E$:

$$\begin{aligned}(D \sqcap E)^{\mathcal{I}^*} &\overset{\text{Def.}}{=} D^{\mathcal{I}^*} \cap E^{\mathcal{I}^*}\\ &\overset{\text{IH}}{=} \bigcap_{\mathcal{J}\in\mathcal{M}} D^\mathcal{J} \cap \bigcap_{\mathcal{J}\in\mathcal{M}} E^\mathcal{J}\\ &\overset{\text{Def.}}{=} \bigcap_{\mathcal{J}\in\mathcal{M}} (D \sqcap E)^\mathcal{J}\end{aligned}$$

For $C = \exists r.D$:

$$\begin{aligned}(\exists r.D)^{\mathcal{I}^*} &\overset{\text{Def.}}{=} \{d \in \Delta \mid \exists e \in \Delta.(d,e) \in r^{\mathcal{I}^*} \wedge e \in D^{\mathcal{I}^*}\} && (4)\\ &\overset{\text{Def.}}{=} \{d \in \Delta \mid \exists e \in \Delta. \bigwedge_{\mathcal{J}\in\mathcal{M}} (d,e) \in r^\mathcal{J} \wedge e \in D^{\mathcal{I}^*}\} && (5)\\ &\overset{\text{IH}}{=} \{d \in \Delta \mid \exists e \in \Delta. \bigwedge_{\mathcal{J}\in\mathcal{M}} (d,e) \in r^\mathcal{J} \wedge e \in \bigcap_{\mathcal{J}\in\mathcal{M}} D^\mathcal{J}\} && (6)\\ &= \bigcap_{\mathcal{J}\in\mathcal{M}} \{d \in \Delta \mid \exists e \in \Delta.(d,e) \in r^\mathcal{J} \wedge e \in D^\mathcal{J}\} && (7)\\ &\overset{\text{Def.}}{=} \bigcap_{\mathcal{J}\in\mathcal{M}} (\exists r.D)^\mathcal{J} && (8)\end{aligned}$$

While the inclusion "$\subseteq$" from (6) to (7) is easy to see, we inspect the "$\supseteq$" inclusion more closely. Element d in $\{d \in \Delta \mid \exists e \in \Delta.(d,e) \in r^\mathcal{J} \wedge e \in D^\mathcal{J}\}$ for any $\mathcal{J} \in \mathcal{M}$ implies $d \in (\exists r.D)^\mathcal{J}$, which implies $(d, d^\emptyset_D) \in r^\mathcal{J}$ by 3. of Definition 24. Therefore and by Proposition 14, the set $\{d \in \Delta \mid \exists e \in \Delta.\ (d,e) \in r^\mathcal{J} \wedge e \in D^\mathcal{J}\}$ is equivalent to $\{d \in \Delta \mid (d, d^\emptyset_D) \in r^\mathcal{J} \wedge d^\emptyset_D \in D^\mathcal{J}\}$, making it easy to see that the inclusion holds.

Claim 2 is easily proven using Claim 1. Since all $\mathcal{J} \in \mathcal{M}$ are models of $\mathcal{K}$, it holds for every GCI $C \sqsubseteq D \in \mathcal{T}$ that $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Therefore, $\bigcap_{\mathcal{J}\in\mathcal{M}} C^{\mathcal{J}} \subseteq \bigcap_{\mathcal{J}\in\mathcal{M}} D^{\mathcal{J}}$ holds and Claim 1 implies that $\mathcal{I}^* \models \mathcal{T}$. Condition 2 in the Definition of models (Def 7) is equivalent to

$$G \sqsubsetapprox H \in \mathcal{U} \implies G^{\mathcal{J}} \cap \{d_F^{\mathcal{X}} \in \Delta^{\mathcal{J}} \mid \mathcal{X} = \mathcal{U}\} \subseteq H^{\mathcal{J}} \cap \{d_F^{\mathcal{X}} \in \Delta^{\mathcal{J}} \mid \mathcal{X} = \mathcal{U}\} \text{ for } \mathcal{U} \subseteq \mathcal{D}. \qquad (*)$$

and $(*)$ holds for all $\mathcal{J} \in \mathcal{M}$. This way, it is not hard to see that, as before, Claim 1 implies $\mathcal{I}^*, d_F^{\mathcal{U}} \models \mathcal{U}$ for all $F \in Qc(\mathcal{K})$ and $\mathcal{U} \subseteq \mathcal{D}$.

Since all $\mathcal{J} \in \mathcal{M}$ satisfy Property 3 in the definition of model completions (Definition 24), it is easy to see that Claim 1 implies that $\mathcal{I}^*$ also satisfies Property 3 of Definition 24 and is therefore a safe model. □

Lemma 32. *The relevant canonical model $\mathfrak{RE}_{\mathcal{K}}$ is a model of the DKB $\mathcal{K}$.*

Proof. Follows immediately from $\mathfrak{RE}_{\mathcal{K}}$ being well-defined and Proposition 31. □

The main reason for Lemma 32 to hold is that the intersection of models of any set $S \subseteq P(\Delta)$ yields another model already contained in $P(\Delta)$. This result is ensured by Condition 3 in the definition of model completions. The relevant canonical model $\mathfrak{RE}_{\mathcal{K}}$ is used to define (and later on decide) nested relevant entailment of the form $C \sqsubsetapprox_{\mathcal{K}} D$, which requires to propagate DCIs to concepts occurring in existential restrictions. We capture this stronger and quantifier-aware relevant entailment.

Definition 33. *Let $\mathcal{K}$ be a DKB. A defeasible subsumption relationship $C \sqsubsetapprox D$ holds under* nested relevant entailment *(written $\mathcal{K} \models_q^{rel} C \sqsubsetapprox D$) iff $\mathfrak{RE}_{\mathcal{K}} \models C \sqsubsetapprox D$.*

We are ready to state our main result: nested relevant entailment allows for strictly more inferences than the materialisation-based relevant entailment from [5] to compute the relevant closure.

Theorem 34. *For two $\mathcal{EL}_\bot$ concepts C, D and an $\mathcal{EL}_\bot$ DKB $\mathcal{K}$ the following holds:*

1. $\mathcal{K} \models_m^{rel} C \sqsubsetapprox D \implies \mathcal{K} \models_q^{rel} C \sqsubsetapprox D$, *and*
2. $\mathcal{K} \models_m^{rel} C \sqsubsetapprox D \;\not\Longleftarrow\; \mathcal{K} \models_q^{rel} C \sqsubsetapprox D$

Proof. Claim 1 follows from the fact that the minimal typicality model $\mathfrak{L}_{\mathcal{K}}$ is included (according to Definition 22) in all maximal typicality models of $\mathfrak{L}_{\mathcal{K}}$, i.e. $\mathcal{J} \in typ^{max}(\{\mathfrak{L}_{\mathcal{K}}\}) \implies \mathfrak{L}_{\mathcal{K}} \subseteq \mathcal{J}$ and thus $\mathfrak{L}_{\mathcal{K}} \subseteq \mathfrak{RE}_{\mathcal{K}}$ and Claim 2 can be shown using Example 17 as a counter-example. In preparation to do so, let s denote the role *superior*, and W, B, R denote the concepts *Worker*, *Boss* and *Responsible* respectively, also let $\mathcal{K} = \mathcal{K}_{ex1}$, $\mathcal{T} = \mathcal{T}_{ex1}$ and $\mathcal{D} = \mathcal{D}_{ex1}$ for brevity and recall that $\mathcal{D}_W = \mathcal{D}$. It needs to be verified, that

$$\forall \mathcal{J} \in typ^{max}(\{\mathfrak{L}_{\mathcal{K}}\}).(d_W^{\mathcal{D}}, d_B^{\mathcal{D}_B}) \in s^{\mathcal{J}} \overset{(i)}{\Longrightarrow} (d_W^{\mathcal{D}}, d_B^{\mathcal{D}_B}) \in s^{\mathfrak{RE}_{\mathcal{K}}} \overset{(ii)}{\Longrightarrow} \mathfrak{RE}_{\mathcal{K}} \models W \sqsubsetapprox \exists s.R.$$

Implication (i) follows from the definition of the relevant canonical model (Definition 30) and implication (ii) follows from the definition of when a typicality interpretation satisfies a defeasible subsumption relationship (Definition 8) and the fact that $d_B^{\mathcal{D}_B} \in R^{\mathfrak{L}_{\mathcal{K}}}$.

In order to show that $\forall \mathcal{J} \in typ^{max}(\{\mathfrak{L}_{\mathcal{K}}\}).(d_W^{\mathcal{D}}, d_B^{\mathcal{D}_B}) \in s^{\mathcal{J}}$ holds , we proceed by contradiction and assume that $\exists \mathcal{I} \in typ^{max}(\{\mathfrak{L}_{\mathcal{K}}\}).(d_W^{\mathcal{D}}, d_B^{\mathcal{D}_B}) \notin s^{\mathcal{I}}$, then an interpretation $\mathcal{I}'$, coinciding with $\mathcal{I}$ in everything but $s^{\mathcal{I}'}$, where $s^{\mathcal{I}'} = s^{\mathcal{I}} \cup \{(d_W^{\mathcal{D}}, d_B^{\mathcal{D}_B})\}$ is clearly in $typ(\mathcal{I})$ where $X^{\mathcal{I}} = X^{\mathcal{I}'}$ for every left- and right-hand side X of inclusion statements in $\mathcal{T}$ and $\mathcal{D}$, i.e. $\mathcal{I}' \models \mathcal{K}$, hence $\mathcal{I}' \in mc(\mathcal{I}')$, i.e. $mc(\mathcal{I}') \neq \emptyset$. Therefore case (i) of Definition 26 applies to $\mathcal{I}$, contradicting that $\mathcal{I} \in typ^{max}(\{\mathfrak{L}_{\mathcal{K}}\})$. □

Our presented approach for deciding defeasible subsumption relationships under (nested) relevant entailment mends a deficit in the materialisation approach and computes strictly more entailments than materialisation-based relevant entailment. Since relevant closure is stronger than rational closure, the reduction presented here preserves all merits for rational closure from [12] and eliminates remaining problems such as inheritance blocking.

In this part of the report we showed technical results that alleviate the deficit of relevant reasoning in defeasible description logics. Next we describe how these results can be utilized to devise an analogous approach for the weaker nested rational entailment. The main difference is, as we shall see, that for rational reasoning it suffices to consider a sequence of decreasing DBox subsets to gain consistency for domain elements, whereas (nested) relevant entailment needs to consider the whole lattice of subsets of the DBox. The use of a sequence of decreasing DBox subsets in the domain of the typicality interpretations promises a smaller domain size and thus lower complexity for rational reasoning.

6 Rational Reasoning with Typicality Models

Originally, we introduced typicality models for reasoning in defeasible DLs to allow defeasible information to be propagated to quantified concepts based on *rational* closure semantics. We lifted these results [12] which considered a specific sequence of DBox subsets as in the original approach [6] by Casini et al., to consider *any* subset of defeasible information from the DBox. This more general setting allows for a lattice-shaped domain as the basis for typicality models in order to obtain relevant reasoning, as shown in Sections 4 and 5. The domain of a typicality model containing only representatives for a fixed set of DBox subsets is included in the full lattice shaped domain from Definition 5.

For illustration consider again Example 4 and recall that $\mathcal{K}_{ex1} = (\mathcal{T}_{ex1}, \mathcal{D}_{ex1})$ with the *partition* of $\mathcal{D}_{ex1}$ into E_0 and E_1. Rational closure by materialisation considers the sequence of DBox subsets $\mathcal{D}_0 = \mathcal{D}_{ex1}$, $\mathcal{D}_1 = \mathcal{D}_0 \backslash E_0$ and $\mathcal{D}_2 = \mathcal{D}_1 \backslash E_1$. Given the query $C \sqsubsetsim D$, the materialisation-based procedure finds the smallest i, s.t. $\overline{\mathcal{D}_i} \sqcap C \not\sqsubseteq_\mathcal{T} \bot$ and return the answer to $\overline{\mathcal{D}_i} \sqcap C \sqsubseteq_\mathcal{T} D$. Now, in the typicality model approach this means that one representative for every quantified concept in $Qc(\mathcal{K})$ needs to (potentially) exist for each DBox subset in the sequence of DBox subsets considered by rational closure. Therefore, the resulting typicality domain is a sequence and called a *sequence domain*. This is illustrated in Figure 3, where **a** depicts the minimal typicality model $\mathfrak{L}_{\mathcal{K}_{ex1}}$, highlighting the subdomain parts w.r.t. $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2$ drawn with thick borders. This subdomain is extracted and rearranged in part **b** of Figure 3 (the neat structure of the sequence domain allows for labelling the domain elements again in the figure).

Since the results from Sections 4 and 5 consider any subset of defeasible statements from the DBox, they still apply in principle when restricting the typicality domain to the sequence shape. Nevertheless, we consider the special case of rational closure in detail and discuss its differences to the general approach in the following.

First of all, we use the same sequence of DBox subsets as constructed in [6]. The algorithm from [5] was described in Section 3, including the partition of $\mathcal{D} = (E_0, E_1, \ldots, E_{n-1})$. To formally capture the sequence of DBox subsets we use a partition function *sequence*. We define $\mathit{sequence}(\mathcal{D}) = (\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n)$, with $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_{i+1} = \mathcal{D}_i \setminus E_i$ for $0 \leq i < n$. Obviously, $\mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \ldots \supseteq \mathcal{D}_n$ and for well-separated DKBs $\mathcal{D}_n = \emptyset$ holds.

The definition of a typicality domain (Definition 5) already covers the case of the sequence domain: the set $\Gamma(\Delta)$, collecting the represented subsets of the given DBox, has to be totally ordered by $\subseteq$, thus inducing the sequence shape. The conditions for a typicality interpretation satisfying a DKB (Definition 7) are independent of the shape of the underlying typicality domain. Thus, entailment of individual defeasible subsumption relationships (Definition 8) is "read

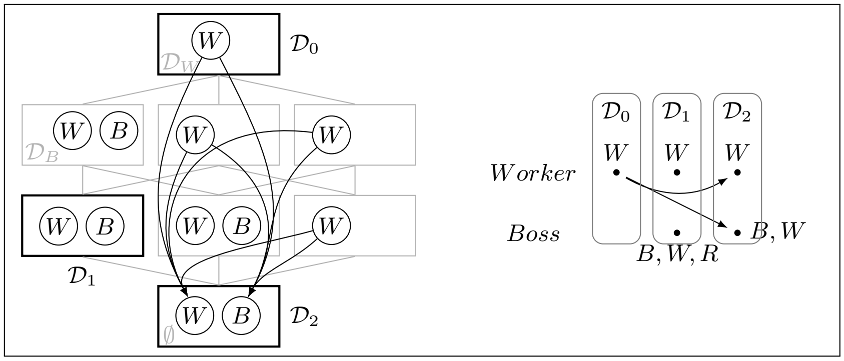

Fig. 3: a Lattice domain (left) with highlighted subdomain that is extracted as sequence domain **b** (right).

off" from an interpretation as one would do from a canonical model. The most typical domain element of those representing the same concept in a typicality interpretation over a sequence domain is the element satisfying as many DCIs as possible.

Definition 35. *Let* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *be a typicality interpretation over a sequence domain for DKB* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$*, with sequence*$(\mathcal{D}) = (\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n)$*. Then* $\mathcal{I}$ *satisfies a defeasible subsumption* $C \sqsubseteq\!\!\sim D$ *(written* $\mathcal{I} \models C \sqsubseteq\!\!\sim D$*) iff* $d_C^{\mathcal{D}_i} \in D^{\mathcal{I}}$ *for the smallest* $i \in \{0, \ldots, n\}$ *s.t.* $d_C^{\mathcal{D}_i} \in \Delta^{\mathcal{I}}$*.*

The definition of the extended TBox of F w.r.t. $\mathcal{D}$, i.e., $\mathcal{T}_{\mathcal{D}_i}(F)$ (Definition 9) as well as the Propositions 11 and 12 are independent of the shape of a typicality interpretation and therefore hold for sequence domains as well. Analogous to the case of relevant reasoning, we can make use of minimal typicality models to characterise our rational semantics by fixing the underlying domain.

Definition 36. *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB and sequence*$(\mathcal{D}) = (\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n)$*. The* minimal rational typicality model $\mathfrak{S}_{\mathcal{K}}$ *is a minimal typicality model with the sequence domain* $\Delta^{\mathfrak{S}_{\mathcal{K}}} = \{d_F^{\mathcal{D}_i} \mid F \in Qc(\mathcal{K}), F_{\mathcal{D}_i} \not\sqsubseteq_{\mathcal{T}_{\mathcal{D}_i}(F)} \bot, 0 \leq i \leq n\}$*.*

Again, $\Delta^{\mathfrak{S}_{\mathcal{K}}}$ is clearly a typicality domain and it satisfies Property $(*)$, which essentially admits only representatives of concepts not causing an inconsistency in the domain, of minimal typicality models (Definition 13). In addition Propositions 11 and 12 hold for $\mathfrak{S}_{\mathcal{K}}$. Therefore $\mathfrak{S}_{\mathcal{K}}$ is well-defined and satisfies $\mathcal{K}$ by Lemma 15. We review Example 17 in the context of rational closure: the illustration of $\mathfrak{S}_{\mathcal{K}_{ex1}}$ is part **b** in Figure 3. Deciding entailments according to Definition 35, means to check containment of the most typical (in part **b** of Figure 3 the left-most) domain element, representing the left-hand side of a subsumption query.

Model $\mathfrak{S}_{\mathcal{K}_{ex1}}$ gives evidence for subsumption relationships such as $Worker \sqsubseteq\!\!\sim \exists superior.Boss$ and $Boss \sqsubseteq\!\!\sim Responsible$. However, neither $Boss \sqsubseteq\!\!\sim Productive$ (inheritance blocking), nor $Worker \sqsubseteq\!\!\sim \exists superior.Responsible$ (neglecting quantified concepts), nor the combination of both $Worker \sqsubseteq\!\!\sim \exists superior.Productive$ are entailed. The advantage of considering the weaker rational reasoning lies in the reduced size of the typicality model, which is quadratic as opposed to the exponential lattice domain and thus leads potentially to lower computational complexity.

Next we recreate rational reasoning by materialisation by characterising propositional entailment (under rational semantics) of a DKB $\mathcal{K}$. The following definition and theorem are analogous to the definition of propositional entailment of a defeasible subsumption relationship under relevant closure (Definition 18) and Theorem 20. Still, the former are specific to relevant semantics and need to be slightly adjusted.

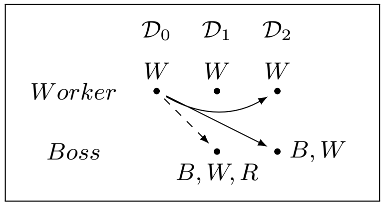

Fig. 4: Upgraded role successor edge in $\mathfrak{S}_{\mathcal{K}_{ex1}}$.

Definition 37. *Let $\mathcal{K}$ be a DKB. $\mathcal{K}$ propositionally entails a defeasible subsumption relationship $C \sqsubsim D$ with rational semantics (written $\mathcal{K} \models_p^{rat} C \sqsubsim D$) iff $\mathfrak{S}_\mathcal{K} \models C \sqsubsim D$.*

Since Lemma 19 is independent of typicality interpretations, it can be used it to prove the analogous version for rational semantics of main Theorem 20, which shows equivalence to the materialisation-based approach.

Theorem 38. $\mathcal{K} \models_p^{rat} C \sqsubsim D$ *iff* $\mathcal{K} \models_m^{rat} C \sqsubsim D$

Proof. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. In order to decide $\mathcal{K} \models_m^{rat} C \sqsubsim D$, the algorithm in [6] picks the smallest $j \in \{0, \ldots, n\}$ for $sequence(\mathcal{D}) = (\mathcal{D}_0, \ldots, \mathcal{D}_n)$ s.t. $\overline{\mathcal{D}_j} \sqcap C \not\sqsubseteq_\mathcal{T} \bot$ and returns the result for $\overline{\mathcal{D}_j} \sqcap C \sqsubseteq_\mathcal{T} D$. Definition 35 also selects the smallest $i \in \{0, \ldots, n\}$ s.t. $d_C^{\mathcal{D}_i} \in \Delta^{\mathfrak{S}_\mathcal{K}}$. We need to show that $\mathcal{D}_i = \mathcal{D}_j$, because Lemmas 14 and 19 then immediately imply equivalence of both entailment semantics. Since both semantics use the same $sequence(\mathcal{D})$, $\mathcal{D}_i = \mathcal{D}_j$ *iff* $i = j$. Assume to the contrary that $i < j$, thus $\mathcal{D}_j$ being selected by the materialisation algorithm implies $\overline{\mathcal{D}_i} \sqcap C \sqsubseteq_\mathcal{T} \bot$ which is equivalent to $C_{\mathcal{D}_i} \sqsubseteq_{\mathcal{T}_{\mathcal{D}_i}(C)} \bot$ by Lemma 19. Thus from Definition 36 it follows that $d_C^{\mathcal{D}_i} \notin \Delta^{\mathfrak{S}_\mathcal{K}}$, contradicting that $\mathcal{D}_i$ was chosen in Definition 35. For $i > j$ the same argument holds in reverse, when assuming $d_C^{\mathcal{D}_i} \in \Delta^{\mathfrak{S}_\mathcal{K}}$ was selected in Definition 35, then $d_C^{\mathcal{D}_j} \notin \Delta^{\mathfrak{S}_\mathcal{K}}$, which implies by Definition 36 and Lemma 19 that $\overline{\mathcal{D}_j} \sqcap C \sqsubseteq_\mathcal{T} \bot$ holds, contradicting that $\mathcal{D}_j$ was chosen by the materialisation algorithm. □

The next step in alleviating the shortcoming of neglecting quantification in rational defeasible subsumption is to upgrade role successor typicality of the minimal rational typicality model. The procedure for upgrading is analogous to the one described in Section 5. Most of the results in Section 5 are independent of the shape of the underlying typicality domain and therefore apply to the present case in the same way. We recall Example 21. Using $\mathfrak{S}_{\mathcal{K}_{ex1}}$, we cannot obtain conclusions such as $Worker \sqsubsim \exists superior.Responsible$, even though $Worker \sqsubsim \exists superior.Boss$ is entailed and $Boss \sqsubsim Responsible$ remains undefeated for the successor elements in general. An upgrade of $(d_W^{\mathcal{D}_0}, d_B^{\emptyset}) \in s^{\mathfrak{S}_{\mathcal{K}_{ex1}}}$ to $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1})$ with s yields the desired entailment in the resulting upgraded interpretation according to Definition 35, as illustrated in Figure 4.

The technique for upgrading works exactly as in the relevant case: typicality extensions (Definition 23), model completions (Definition 24) and the fixpoint operator T (Definition 26) are entirely independent of the shape of the underlying typicality domain. Thus, the results following these notions (Propositions 25, 27, and 31) remain valid for upgrading the minimal rational typicality model as well. We define the rational canonical model that is used to decide nested rational entailment in an analogous way to the relevant canonical models (Definition 30).

Definition 39. *The rational canonical model is* $\mathfrak{RA}_\mathcal{K} = \bigcap_{\mathcal{I} \in typ^{max}(\{\mathfrak{S}_\mathcal{K}\})} \mathcal{I}$.

Naturally, we have an analogous result to Lemma 32 for $\mathfrak{RA}_\mathcal{K}$, i.e. $\mathfrak{RA}_\mathcal{K}$ is a model of $\mathcal{K}$ by Proposition 31. Nested rational entailment is characterised as follows.

Definition 40. *Let $\mathcal{K}$ be a DKB. A defeasible subsumption relationship $C \mathrel{\tilde{\sqsubseteq}} D$ holds under* nested rational entailment *(written $\mathcal{K} \models_q^{rat} C \mathrel{\tilde{\sqsubseteq}} D$) iff $\mathfrak{RA}_\mathcal{K} \models C \mathrel{\tilde{\sqsubseteq}} D$.*

Finally, it remains to show that reasoning with the rational canonical models yields strictly more entailments than materialisation-based rational entailment.

Theorem 41. *For two $\mathcal{EL}_\bot$ concepts C, D and an $\mathcal{EL}_\bot$ DKB $\mathcal{K}$ the following holds*

1. $\mathcal{K} \models_m^{rat} C \mathrel{\tilde{\sqsubseteq}} D \implies \mathcal{K} \models_q^{rat} C \mathrel{\tilde{\sqsubseteq}} D$, *and*
2. $\mathcal{K} \models_m^{rat} C \mathrel{\tilde{\sqsubseteq}} D \not\Longleftarrow \mathcal{K} \models_q^{rat} C \mathrel{\tilde{\sqsubseteq}} D$.

Proof. The proof works analogous to the proof of Theorem 34. Claim 1 simply follows from Theorem 20 and the fact that $\mathfrak{S}_\mathcal{K} \subseteq \mathfrak{RA}_\mathcal{K}$. Claim 2 can be shown with $\mathcal{K}_{ex1}$ from Example 4. Assume $\mathcal{I} \in typ^{max}(\mathfrak{S}_{\mathcal{K}_{ex1}})$ with $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1}) \notin s^\mathcal{I}$. It is easy to see that the interpretation $\mathcal{J}$ with $\Delta^\mathcal{J} = \Delta^\mathcal{I}$, $A^\mathcal{J} = A^\mathcal{I}$ and $s^\mathcal{J} = s^\mathcal{I} \cup \{(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1})\}$ is a safe model of $\mathcal{K}_{ex1}$, contradicting the maximality of $\mathcal{I}$. Therefore all maximal typicality models $\mathcal{J} \in typ^{max}(\mathfrak{S}_{\mathcal{K}_{ex1}})$ need to satisfy $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1}) \in s^\mathcal{J}$, hence $(d_W^{\mathcal{D}_0}, d_B^{\mathcal{D}_1}) \in s^{\mathfrak{RA}_{\mathcal{K}_{ex1}}}$. This means $\mathfrak{RA}_{\mathcal{K}_{ex1}} \models Worker \mathrel{\tilde{\sqsubseteq}} \exists superior.Responsible$, i.e. $\mathcal{K}_{ex1} \models_q^{rat} Worker \mathrel{\tilde{\sqsubseteq}} \exists superior.Responsible$, however, as covered before, $\mathcal{K}_{ex1} \not\models_m^{rat} Worker \mathrel{\tilde{\sqsubseteq}} \exists superior.Responsible$. □

This shows that reasoning based on minimal rational typicality models allow for reasoning involving concepts nested existential restrictions and thus improves earlier approaches for deciding subsumption relationships under rational semantics.

7 Conclusions and Future Work

In this report we have extended a new approach for reasoning in DDLs to characterise entailment under relevant closure (for deciding subsumption) in the DDL $\mathcal{EL}_\bot$. The new approach is motivated by the observation that earlier reasoning procedures for this problem do not treat existential restrictions in an adequate way. The key idea is to extend canonical models such that for each concept from the DKB, several copies representing different amounts of defeasible concept inclusions are introduced. The new approach supports the propagation of defeasible information to concepts nested in existential restrictions. In principle, our new approach can be extended to more expressive settings, e.g. to more expressive DLs or to ABox reasoning. While rational closure needs to consider only one sequence of increasing subsets of the DBox [12], relevant closure needs (potentially) *all* subsets of the given defeasible information—forming a lattice. In minimal relevant typical models (over a lattice domain) the role successors are "a-typical" in the sense that they satisfy only the GCIs from the TBox. Such models can be computed by a reduction to classical TBox reasoning. We showed that the obtained entailments coincide with the ones obtained by earlier materialisation-based algorithms. We extended these models to maximally typical models, which have role successors of "maximal typicality". Entailment over these models propagates defeasible information to role successors and thus allows for more entailments. The presented approach was shown to be more general than the analogous approach for rational reasoning in having less restrictions on the shape of the underlying typicality domain.

There are several paths for future work. Besides the extensions to more expressive DLs, an extension to ABox reasoning, i.e., reasoning about data, would be an interesting topic to investigate. Furthermore, a completion-like algorithm as for classical $\mathcal{EL}$ [3,1] would be desirable to effectively compute these models. The current definition of typicality extensions and model completions leaves plenty of room for developing practical algorithms worth implementing.

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