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TERMINOLOGICAL KNOWLEDGE ACQUISITION IN PROBABILISTIC DESCRIPTION LOGIC

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ABSTRACT

For a probabilistic extension of the description logic $\mathcal{EL}^{\bot}$, we consider the task of automatic acquisition of terminological knowledge from a given probabilistic interpretation. Basically, such a probabilistic interpretation is a family of directed graphs the vertices and edges of which are labeled, and where a discrete probability measure on this graph family is present. The goal is to derive so-called concept inclusions which are expressible in the considered probabilistic description logic and which hold true in the given probabilistic interpretation. A procedure for an appropriate axiomatization of such graph families is proposed and its soundness and completeness is justified.

Keywords Data mining · Knowledge acquisition · Probabilistic description logic · Knowledge base · Probabilistic interpretation · Concept inclusion

CONTENTS

1 INTRODUCTION

Description Logics (abbrv. DLs) [2] are frequently used knowledge representation and reasoning formalisms with a strong logical foundation. In particular, these provide their users with automated inference services that can derive implicit knowledge from the explicitely represented knowledge. Decidability and computational complexity of common reasoning tasks have been widely explored for most DLs. Besides being used in various application domains, their most notable success is the fact that DLs constitute the logical underpinning of the *Web Ontology Language* (abbrv. OWL) and many of its profiles.

DLs in its standard form only allow for representing and reasoning with *crisp* knowledge without any degree of *uncertainty*. Of course, this is a serious shortcoming for use cases where it is impossible to perfectly determine the truth of a statement. For resolving this expressivity restriction, probabilistic variants of DLs [5] have been introduced. Their model-theoretic semantics is built upon so-called probabilistic interpretations, that is, families of directed graphs the vertices and edges of which are labeled and for which there exists a probability measure on this graph family.

Results of scientific experiments, e.g., in medicine, psychology, or biology, that are repeated several times can induce probabilistic interpretations in a natural way. In this document, we shall develop a suitable axiomatization technique for deducing terminological knowledge from the assertional data given in such probabilistic interpretations. More specifically, we consider a probabilistic variant $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ of the description logic $\mathcal{EL}^{\bot}$, show that reasoning in $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ is **ExpTime**-complete, and provide a method for constructing a set of rules, so-called concept inclusions, from probabilistic interpretations in a sound and complete manner.

This document also resolves an issue found by Franz Baader with the techniques described by the author in [8, Sections 5 and 6]. In particular, the concept inclusion base proposed therein in Proposition 2 is only complete with respect to those probabilistic interpretations that are also quasi-uniform with a probability ε of each world. Herein, we describe a more sophisticated axiomatization technique of not necessarily quasi-uniform probabilistic interpretations and that ensures completeness of the constructed concept inclusion base with respect to *all* probabilistic interpretations, but which, however, disallows nesting of probability restrictions. It is not hard to generalize the following results to a more expressive probabilistic description logic, for example to a probabilistic variant $\mathcal{P}_1^{>}\mathcal{M}$ of the description logic $\mathcal{M}$, for which an axiomatization technique is available [6]. That way, we can regain the same, or even a greater, expressivity as the author has tried to have tackled in [8], but without the possibility to nest probability restrictions.

2 THE PROBABILISTIC DESCRIPTION LOGIC $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$

The probabilistic description logic $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ extends the light-weight description logic $\mathcal{EL}^{\bot}$ [2] by means for expressing and reasoning with probabilities. Put simply, it is a variant of the logic Prob-$\mathcal{EL}$ introduced by Gutiérrez-Basulto, Jung, Lutz, and Schröder in [5] where nesting of probabilistic quantifiers is disallowed, only the relation symbols $>$ and $\geq$ are available for the probability restrictions, and further the bottom concept description $\bot$ is present. We introduce its syntax and semantics as follows.

2.1 SYNTAX

Fix some *signature* Σ, which is a disjoint union of a set Σ_{C} of *concept names* and a set Σ_{R} of *role names*. Then, $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ *concept descriptions* C over Σ may be constructed by means of the following inductive rules (where $A \in \Sigma_{\mathsf{C}}$, $r \in \Sigma_{\mathsf{R}}$, $\gtrdot \in \{\geq, >\}$ and $p \in [0,1] \cap \mathbb{Q}$).[1]

$$C ::= \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C \mid \mathsf{d} \gtrdot p.\, D$$
$$D ::= \bot \mid \top \mid A \mid D \sqcap D \mid \exists\, r.\, D$$

We denote the set of all $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept descriptions over Σ by $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}(\Sigma)$. An $\mathcal{EL}^{\bot}$ *concept description* is a $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept description not containing any subconcept of the form $\mathsf{d} \gtrdot p.\, C$, and we shall write $\mathcal{EL}^{\bot}(\Sigma)$ for the set of all $\mathcal{EL}^{\bot}$ concept descriptions over Σ. A *concept inclusion* (abbrv. CI) is an expression of the form $C \sqsubseteq D$, and a *concept equivalence* (abbrv. CE) is of the form $C \equiv D$, where both C and D are concept descriptions. A *terminological box* (abbrv. TBox) is a finite set of CIs and CEs. Furthermore, we also allow for so-called *wildcard concept inclusions* of the form $\mathsf{d} \gtrdot_1 p_1.\, * \sqsubseteq \mathsf{d} \gtrdot_2 p_2.\, *$ that, basically, are abbreviations for the set $\{\, \mathsf{d} \gtrdot_1 p_1.\, C \sqsubseteq \mathsf{d} \gtrdot_2 p_2.\, C \mid C \in \mathcal{EL}^{\bot}(\Sigma) \,\}$.

An example of a $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept description is the following.

$$\mathsf{Cat} \sqcap \mathsf{d} \geq \tfrac{1}{2}.\, \exists\, \mathsf{hasPhysicalCondition}.\, \mathsf{Alive} \sqcap \mathsf{d} \geq \tfrac{1}{2}.\, \exists\, \mathsf{hasPhysicalCondition}.\, \mathsf{Dead} \qquad (2.1)$$

[1] If we treat these two rules as the production rules of a BNF grammar, C is its start symbol.

2.2 SEMANTICS

A *probabilistic interpretation* over Σ is a tuple $\mathcal{I} := (\Delta^{\mathcal{I}}, \Omega^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbb{P}^{\mathcal{I}})$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of *objects*, called the *domain*, a non-empty, countable set $\Omega^{\mathcal{I}}$ of *worlds*, a discrete probability measure $\mathbb{P}^{\mathcal{I}}$ on $\Omega^{\mathcal{I}}$, and an *extension function* $\cdot^{\mathcal{I}}$ such that, for each world $\omega \in \Omega^{\mathcal{I}}$, any concept name $A \in \Sigma_{\mathsf{C}}$ is mapped to a subset $A^{\mathcal{I}(\omega)} \subseteq \Delta^{\mathcal{I}}$ and each role name $r \in \Sigma_{\mathsf{R}}$ is mapped to a binary relation $r^{\mathcal{I}(\omega)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Note that $\mathbb{P}^{\mathcal{I}}\colon \wp(\Omega^{\mathcal{I}}) \to [0,1]$ is a mapping which satisfies $\mathbb{P}^{\mathcal{I}}(\emptyset) = 0$ and $\mathbb{P}^{\mathcal{I}}(\Omega^{\mathcal{I}}) = 1$, and is *$\sigma$-additive*, that is, for all countable families $(\, U_n \mid n \in \mathbb{N}\,)$ of pairwise disjoint sets $U_n \subseteq \Omega^{\mathcal{I}}$ it holds true that $\mathbb{P}^{\mathcal{I}}(\bigcup\{\, U_n \mid n \in \mathbb{N}\,\}) = \sum(\, \mathbb{P}^{\mathcal{I}}(U_n) \mid n \in \mathbb{N}\,)$. In particular, we follow the assumption in [5, Section 2.6] and consider only probabilistic interpretations without any infinitely improbable worlds, i.e., without any worlds $\omega \in \Omega^{\mathcal{I}}$ such that $\mathbb{P}^{\mathcal{I}}\{\omega\} = 0$. We call a probabilistic interpretation *finitely representable* if $\Delta^{\mathcal{I}}$ is finite, $\Omega^{\mathcal{I}}$ is finite, the *active signature* $\Sigma^{\mathcal{I}} := \{\, \sigma \mid \sigma \in \Sigma$ and $\sigma^{\mathcal{I}(\omega)} \neq \emptyset$ for some $\omega \in \Omega^{\mathcal{I}}\,\}$ is finite, and if $\mathbb{P}^{\mathcal{I}}$ has only rational values. In the sequel of this document we will also utilize the notion of *interpretations*, which are the models upon which the semantics of $\mathcal{EL}^{\bot}$ is built; these are, basically, probabilistic interpretations with only one world, that is, these are tuples $\mathcal{I} := (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set of *objects*, called *domain*, and where $\cdot^{\mathcal{I}}$ is an *extension function* that maps concept names $A \in \Sigma_{\mathsf{C}}$ to subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and maps role names $r \in \Sigma_{\mathsf{R}}$ to binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

Fix some probabilistic interpretation $\mathcal{I}$. The *extension* $C^{\mathcal{I}(\omega)}$ of a $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept description C in a world ω of $\mathcal{I}$ is defined by means of the following recursive formulae.

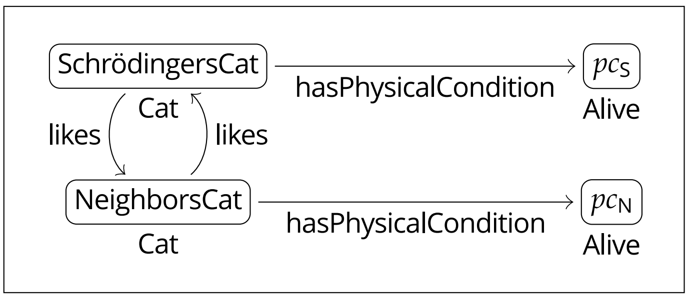
$$\begin{aligned}
\bot^{\mathcal{I}(\omega)} &:= \emptyset \\
\top^{\mathcal{I}(\omega)} &:= \Delta^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}(\omega)} &:= C^{\mathcal{I}(\omega)} \cap D^{\mathcal{I}(\omega)} \\
(\exists\, r.\, C)^{\mathcal{I}(\omega)} &:= \{\, \delta \mid \delta \in \Delta^{\mathcal{I}},\ (\delta,\epsilon) \in r^{\mathcal{I}(\omega)},\ \text{and } \epsilon \in C^{\mathcal{I}(\omega)} \text{ for some } \epsilon \in \Delta^{\mathcal{I}}\,\} \\
(\mathsf{d} \geqslant p.\, C)^{\mathcal{I}(\omega)} &:= \{\, \delta \mid \delta \in \Delta^{\mathcal{I}} \text{ and } \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \geqslant p\,\}
\end{aligned}$$

Please note that we use the abbreviation $\{\delta \in C^{\mathcal{I}}\} := \{\, \omega \mid \omega \in \Omega^{\mathcal{I}}$ and $\delta \in C^{\mathcal{I}(\omega)}\,\}$. All but the last formula can be used similarly to recursively define the *extension* $C^{\mathcal{I}}$ of an $\mathcal{EL}^{\bot}$ concept description C in an interpretation $\mathcal{I}$.

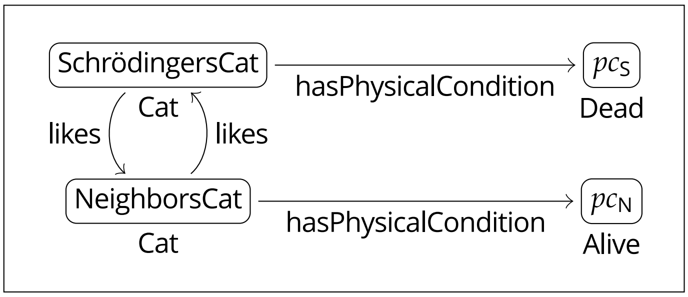
A toy example of a probabilistic interpretation is the following $\mathcal{I}_{\mathsf{Cat}}$. As one quickly verifies, only the object SchrödingersCat belongs to the extension of the concept description from Equation (2.1).

$\mathcal{I}_{\mathsf{Cat}}(\omega_1)$:

$\mathbb{P}^{\mathcal{I}_{\mathsf{Cat}}}\{\omega_1\} := \frac{1}{2}$



$\mathcal{I}_{\mathsf{Cat}}(\omega_2)$:
$\mathbb{P}^{\mathcal{I}_{\mathsf{Cat}}}\{\omega_2\} := \frac{1}{2}$



A concept inclusion $C \sqsubseteq D$ or a concept equivalence $C \equiv D$ is *valid* in a probabilistic interpretation $\mathcal{I}$ if $C^{\mathcal{I}(\omega)} \subseteq D^{\mathcal{I}(\omega)}$ or $C^{\mathcal{I}(\omega)} = D^{\mathcal{I}(\omega)}$, respectively, is satisfied for all worlds $\omega \in \Omega^{\mathcal{I}}$, and we shall then write $\mathcal{I} \models C \sqsubseteq D$ or $\mathcal{I} \models C \equiv D$, respectively. A wildcard CI $\mathsf{d} \gtrdot_1 p_1.* \sqsubseteq \mathsf{d} \gtrdot_2 p_2.*$ is *valid* in $\mathcal{I}$, written $\mathcal{I} \models \mathsf{d} \gtrdot_1 p_1.* \sqsubseteq \mathsf{d} \gtrdot_2 p_2.*$, if, for each $\mathcal{EL}^\bot$ concept description C, the CI $\mathsf{d} \gtrdot_1 p_1.C \sqsubseteq \mathsf{d} \gtrdot_2 p_2.C$ is valid in $\mathcal{I}$. Furthermore, $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, denoted as $\mathcal{I} \models \mathcal{T}$, if each concept inclusion in $\mathcal{T}$ is valid in $\mathcal{I}$. A TBox $\mathcal{T}$ *entails* a concept inclusion $C \sqsubseteq D$, symbolized by $\mathcal{T} \models C \sqsubseteq D$, if $C \sqsubseteq D$ is valid in every model of $\mathcal{T}$. In the sequel of this document, we may also use the denotation $C \leq_{\mathcal{Y}} D$ instead of $\mathcal{Y} \models C \leq D$ where $\mathcal{Y}$ is either an interpretation or a terminological box and $\leq$ is a suitable relation symbol, e.g., one of $\sqsubseteq$, $\equiv$, $\sqsupseteq$, and we may analogously write $C \not\leq_{\mathcal{Y}} D$ for $\mathcal{Y} \not\models C \leq D$.

2.3 COMPLEXITY

Reasoning in $\mathcal{P}_1^{>}\mathcal{EL}^\bot$, more specifically deciding subsumption with respect to a TBox, is **ExpTime**-complete, and, consequently, more expensive than reasoning in its non-probabilistic sibling $\mathcal{EL}^\bot$, for which the similar decision problem is well-known to be **P**-complete, that is, reasoning in $\mathcal{EL}^\bot$ is tractable.

Propositio 1 In $\mathcal{P}_1^{>}\mathcal{EL}^\bot$, the problem of deciding whether a terminological box entails a concept inclusion is **ExpTime**-complete.

Approbatio Containment in **ExpTime** follows from [5, Theorem 3] and the fact that $\mathcal{P}_1^{>}\mathcal{EL}^\bot$ is a sublogic of Prob-$\mathcal{ALC}$. **ExpTime**-hardness is a consequence of [5, Theorem 13 and Sections A.1, A.2, and A.3], where **ExpTime**-hardness of the logics Prob-$\mathcal{EL}^{\sim p}$ using non-nested probability restrictions for $\sim \in \{\geq, >\}$, that is, of sublogics of $\mathcal{P}_1^{>}\mathcal{EL}^\bot$, is demonstrated. □

2.4 CONCEPT INCLUSION BASES IN $\mathcal{EL}^\bot$

In the next section, we will use techniques for axiomatizing concept inclusions in $\mathcal{EL}^\bot$ as developed by Baader and Distel in [1, 4] for greatest fixed-point semantics, and as adjusted by Borchmann, Distel, and the author in [3] for the role-depth-bounded case. A brief introduction is as follows. A *concept inclusion base* for an interpretation $\mathcal{I}$ is a TBox $\mathcal{T}$ such that, for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{I} \models C \sqsubseteq D$ if, and only if, $\mathcal{T} \models C \sqsubseteq D$. For each finite interpretation $\mathcal{I}$ with finite active signature, there is a *canonical base* $\mathsf{Can}(\mathcal{I})$ with respect to greatest fixed-point semantics, which has minimal cardinality among all concept inclusion bases for $\mathcal{I}$, cf. [4, Corollary 5.13 and Theorem 5.18], and similarly there is a minimal *canonical base* $\mathsf{Can}(\mathcal{I}, d)$ with respect to an upper bound $d \in \mathbb{N}$ on the role depths, cf. [3, Theorem 4.32]. The construction of both canonical bases is built upon the notion of a *model-based most specific concept description*, which, for an interpretation $\mathcal{I}$ and a subset $X \subseteq \Delta^{\mathcal{I}}$, is a concept description C such that $X \subseteq C^{\mathcal{I}}$ and, for each concept description D, it holds true that $X \subseteq D^{\mathcal{I}}$ implies $\emptyset \models C \sqsubseteq D$. These exist either if greatest fixed-point semantics is applied (in order to be able to express cycles present in $\mathcal{I}$) or if the role depth of C is bounded by some $d \in \mathbb{N}$, and these are then denoted as $X^{\mathcal{I}}$ or $X^{\mathcal{I}_d}$, respectively. This mapping $\cdot^{\mathcal{I}} \colon \wp(\Delta^{\mathcal{I}}) \to \mathcal{EL}^{\bot}(\Sigma)$ is the adjoint of the extension function $\cdot^{\mathcal{I}} \colon \mathcal{EL}^{\bot}(\Sigma) \to \wp(\Delta^{\mathcal{I}})$, and the pair of both constitutes a *Galois connection*, cf. [4, Lemma 4.1] and [3, Lemmas 4.3 and 4.4], respectively.

As a variant of these two approaches, the author presented in [9] a method for constructing canonical bases relative to an existing terminological box. If $\mathcal{I}$ is an interpretation and $\mathcal{B}$ is a terminological box such that $\mathcal{I} \models \mathcal{B}$, then a *concept inclusion base* for $\mathcal{I}$ *relative* to $\mathcal{B}$ is a terminological box $\mathcal{T}$ such that, for each concept inclusion $C \sqsubseteq D$, it holds true that $\mathcal{I} \models C \sqsubseteq D$ if, and only if, $\mathcal{T} \cup \mathcal{B} \models C \sqsubseteq D$. The appropriate *canonical base* is denoted by $\mathsf{Can}(\mathcal{I}, \mathcal{B})$, cf. [9, Theorem 1].

3 AXIOMATIZATION OF CONCEPT INCLUSIONS IN $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$

In this section, we shall develop an effective method for axiomatizing $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept inclusions which are valid in a given finitely representable probabilistic interpretation. After defining the appropriate notion of a *concept inclusion base*, we show how this problem can be tackled using the aforementioned existing results on computing concept inclusion bases in $\mathcal{EL}^{\bot}$. More specifically, we devise an extension of the given signature by finitely many probability restrictions $\mathsf{d} > p. C$ that are treated as additional concept names, and we define a so-called *probabilistic scaling* $\mathcal{I}_{\mathsf{d}}$ of the input probabilistic interpretation $\mathcal{I}$ which is a (single-world) interpretation that suitably interprets these new concept names and, furthermore, such that there is a correspondence between CIs valid in $\mathcal{I}$ and CIs valid in $\mathcal{I}_{\mathsf{d}}$. This correspondence makes it possible to utilize the above mentioned techniques for axiomatizing CIs in $\mathcal{EL}^{\bot}$.

Definitio 2 A *concept inclusion base* for a probabilistic interpretation $\mathcal{I}$ is a terminological box $\mathcal{T}$ which is *sound* for $\mathcal{I}$, that is, $\mathcal{T} \models C \sqsubseteq D$ implies $\mathcal{I} \models C \sqsubseteq D$ for each concept inclusion $C \sqsubseteq D$,[1] and which is *complete* for $\mathcal{I}$, that is, $\mathcal{I} \models C \sqsubseteq D$ only if $\mathcal{T} \models C \sqsubseteq D$ for any concept inclusion $C \sqsubseteq D$.

3.1 THE ALMOST CERTAIN SCALING

A first important step is to significantly reduce the possibilities of concept descriptions occuring as a filler in the probability restrictions, that is, of fillers C in expressions $\mathsf{d} > p. C$. As it turns out, it suffices to consider only those fillers that are model-based most specific concept descriptions of some suitable *scaling* of the given probabilistic interpretation $\mathcal{I}$.

Definitio 3 Let $\mathcal{I}$ be a probabilistic interpretation $\mathcal{I}$ over some signature Σ. Then, its *almost certain scaling* is defined as the interpretation $\mathcal{I}_{\times}$ over Σ with the following components.

$$\Delta^{\mathcal{I}_\times} := \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}$$

$$\cdot^{\mathcal{I}_\times} : \begin{cases} A \mapsto \{\, (\delta, \omega) \mid \delta \in A^{\mathcal{I}(\omega)} \,\} & \text{for each } A \in \Sigma_{\mathsf{C}} \\ r \mapsto \{\, ((\delta, \omega), (\epsilon, \omega)) \mid (\delta, \epsilon) \in r^{\mathcal{I}(\omega)} \,\} & \text{for each } r \in \Sigma_{\mathsf{R}} \end{cases}$$

[1] Of course, soundness is equivalent to $\mathcal{I} \models \mathcal{T}$.

Lemma 4 Consider a probabilistic interpretation $\mathcal{I}$ and a concept description $\mathsf{d} \mathrel{\geqslant} p.C$. Then, the concept equivalence $\mathsf{d} \mathrel{\geqslant} p.C \equiv \mathsf{d} \mathrel{\geqslant} p.C^{\mathcal{I}_\times\mathcal{I}_\times}$ is valid in $\mathcal{I}$.

Approbatio Using structural induction on C, it can be proven that $C^{\mathcal{I}(\omega)} \times \{\omega\} = C^{\mathcal{I}_\times} \cap (\Delta^{\mathcal{I}} \times \{\omega\})$ is satisfied for each world $\omega \in \Omega^{\mathcal{I}}$, cf. [10, Lemma 16]. It follows that $C^{\mathcal{I}(\omega)} = \pi_1(C^{\mathcal{I}_\times} \cap (\Delta^{\mathcal{I}} \times \{\omega\}))$ (where π_1 projects pairs to their first components). By applying well-known properties of Galois connections we obtain that $C^{\mathcal{I}(\omega)} = C^{\mathcal{I}_\times\mathcal{I}_\times\mathcal{I}(\omega)}$, and so $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} = \mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}_\times\mathcal{I}_\times\mathcal{I}}\}$ holds true. $\square$

3.2 FINITELY MANY PROBABILITY BOUNDS

As next step, we restrict the probability bounds p occuring in probability restrictions $\mathsf{d} \mathrel{\geqslant} p.C$. Apparently, it is sufficient to consider only those values p that can occur when evaluating the extension of $\mathcal{P}_1^{\geqslant}\mathcal{EL}^{\bot}$ concept descriptions in $\mathcal{I}$, which, obviously, are the values $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\}$ for any $\delta \in \Delta^{\mathcal{I}}$ and any $C \in \mathcal{EL}^{\bot}(\Sigma)$. Denote the set of all these probability values as $P(\mathcal{I})$. Of course, we have that $\{0,1\} \subseteq P(\mathcal{I})$. If $\mathcal{I}$ is finitely representable, then $P(\mathcal{I})$ is finite too, it holds true that $P(\mathcal{I}) \subseteq \mathbb{Q}$, and the following equation is satisfied, which can be demonstrated using arguments from the proof of Lemma 4.

$$P(\mathcal{I}) = \{\, \mathbb{P}^{\mathcal{I}}\{\delta \in X^{\mathcal{I}_\times\mathcal{I}}\} \mid \delta \in \Delta^{\mathcal{I}} \text{ and } X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}} \,\}$$

For each $p \in [0,1)$, we define $(p)^+_{\mathcal{I}}$ as the next value in $P(\mathcal{I})$ above p, that is, we set

$$(p)^+_{\mathcal{I}} := \bigwedge\{\, q \mid q \in P(\mathcal{I}) \text{ and } q > p \,\}.$$

If the considered probabilistic interpretation $\mathcal{I}$ is clear from the context, then we may also write p^+ instead of $(p)^+_{\mathcal{I}}$. To prevent a loss of information due to only considering probabilities in $P(\mathcal{I})$, we shall use the wildcard concept inclusions $\mathsf{d} > p.\ast \sqsubseteq \mathsf{d} \geq p^+.\ast$ for $p \in P(\mathcal{I}) \setminus \{1\}$.

3.3 THE PROBABILISTIC SCALING

Having found a finite number of representatives for probability bounds as well as a finite number of fillers to be used in probability restrictions, we now show that we can treat these finitely many concept descriptions as concept names of a signature Γ extending Σ in a way such that a concept inclusion is valid in $\mathcal{I}$ if, and only if, the concept inclusion projected onto this extended signature Γ is valid in a suitable *scaling* of $\mathcal{I}$ that interprets Γ.

Definitio 5 Assume that $\mathcal{I}$ is a probabilistic interpretation over a signature Σ. Then, the signature Γ is defined as follows.

$$\Gamma_{\mathsf{C}} := \Sigma_{\mathsf{C}} \cup \{\, \mathsf{d} \geq p.X^{\mathcal{I}_\times} \mid p \in P(\mathcal{I}) \setminus \{0\},\ X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}, \text{ and } \bot \not\equiv_\emptyset X^{\mathcal{I}_\times} \not\equiv_\emptyset \top \,\}$$
$$\Gamma_{\mathsf{R}} := \Sigma_{\mathsf{R}}$$

The *probabilistic scaling* of $\mathcal{I}$ is defined as the interpretation $\mathcal{I}_{\mathsf{d}}$ over Γ that has the following components.

$$\Delta^{\mathcal{I}_{\mathsf{d}}} := \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}$$
$$\cdot^{\mathcal{I}_{\mathsf{d}}} \colon \begin{cases} A \mapsto \{\, (\delta,\omega) \mid \delta \in A^{\mathcal{I}(\omega)} \,\} & \text{for each } A \in \Gamma_{\mathsf{C}} \\ r \mapsto \{\, ((\delta,\omega),(\epsilon,\omega)) \mid (\delta,\epsilon) \in r^{\mathcal{I}(\omega)} \,\} & \text{for each } r \in \Gamma_{\mathsf{R}} \end{cases}$$

Note that $\mathcal{I}_{\mathsf{d}}$ extends $\mathcal{I}_{\times}$ by also interpreting the new concept names in $\Gamma_{\mathsf{C}} \setminus \Sigma_{\mathsf{C}}$, that is, the restriction $\mathcal{I}_{\mathsf{d}}\restriction_{\Sigma}$ equals $\mathcal{I}_{\times}$.

Definitio 6 The *projection* $\pi_{\mathcal{I}}(C)$ of a $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept description C with respect to some probabilistic interpretation $\mathcal{I}$ is obtained from C by replacing each subconcept of the form $\mathsf{d} \gtrdot p.\, D$ with suitable elements from $\Gamma_{\mathsf{C}} \setminus \Sigma_{\mathsf{C}}$, and, more specifically, we recursively define it as follows.

$$\begin{aligned}
\pi_{\mathcal{I}}(A) &:= A && \text{if } A \in \Sigma_{\mathsf{C}} \cup \{\bot, \top\} \\
\pi_{\mathcal{I}}(C \sqcap D) &:= \pi_{\mathcal{I}}(C) \sqcap \pi_{\mathcal{I}}(D) \\
\pi_{\mathcal{I}}(\exists\, r.\, C) &:= \exists\, r.\, \pi_{\mathcal{I}}(C) \\
\pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, C) &:= \begin{cases}
\bot & \text{if } \gtrdot p = > 1 \\
\top & \text{otherwise if } \gtrdot p = \geq 0 \\
\bot & \text{otherwise if } C^{\mathcal{I}_\times \mathcal{I}_\times} \equiv_\emptyset \bot \\
\top & \text{otherwise if } C^{\mathcal{I}_\times \mathcal{I}_\times} \equiv_\emptyset \top \\
\mathsf{d} \geq p.\, C^{\mathcal{I}_\times \mathcal{I}_\times} & \text{otherwise if } \gtrdot = \geq \text{ and } p \in P(\mathcal{I}) \\
\mathsf{d} \geq p^{+}.\, C^{\mathcal{I}_\times \mathcal{I}_\times} & \text{otherwise}
\end{cases}
\end{aligned}$$

Lemma 7 A $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$ is valid in some probabilistic interpretation $\mathcal{I}$ if, and only if, the projected CI $\pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ is valid in $\mathcal{I}_{\mathsf{d}}$.

Approbatio We start with showing that $C \sqsubseteq D$ is valid in $\mathcal{I}$ if, and only if, $\pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ is valid in $\mathcal{I}$. We do this by proving that C and the projection $\pi_{\mathcal{I}}(C)$ have the same extension in any world of $\mathcal{I}$, and analogously for D. In particular, we proceed with structural induction on C, and according to the definition of a projection, the only non-trivial case considers probabilistic restrictions occuring in C. For that purpose, we denote by $\pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, E)$ the concept description that is obtained from the projection $\pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, E)$ by replacing $E^{\mathcal{I}_\times \mathcal{I}_\times}$ with E. It is readily verified that then $\mathsf{d} \gtrdot p.\, E$ and $\pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, E)$ have the same extension in each world of $\mathcal{I}$. An application of Lemma 4 now yields that, in every world of $\mathcal{I}$, also the extensions of $\pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, E)$ and $\pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\, E)$ are the same.

Eventually, the equivalence of $\mathcal{I} \models \pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ and $\mathcal{I}_{\mathsf{d}} \models \pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ then follows from the very definition of the probabilistic scaling $\mathcal{I}_{\mathsf{d}}$ and the fact that the projections $\pi_{\mathcal{I}}(C)$ and $\pi_{\mathcal{I}}(D)$ can be interpreted as $\mathcal{EL}^{\bot}$ concept descriptions over Γ. □

Lemma 8 Assume that $\mathcal{T}$ is a $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ terminological box and that $C \sqsubseteq D$ is an $\mathcal{EL}^{\bot}$ concept inclusion. Then, $\mathcal{T} \models C \sqsubseteq D$ implies $\mathcal{T} \models \mathsf{d} \gtrdot p.\, C \sqsubseteq \mathsf{d} \gtrdot p.\, D$ for any $\gtrdot \in \{\geq, >\}$ and any $p \in [0,1] \cap \mathbb{Q}$.

Approbatio Fix some model $\mathcal{I}$ of $\mathcal{T}$ and let $\mathbb{P}^{\mathcal{I}}\{\delta \in C^{\mathcal{I}}\} \gtrdot p$ for an object $\delta \in \Delta^{\mathcal{I}}$. From $\mathcal{T} \models C \sqsubseteq D$ we infer that, for each world $\omega \in \Omega^{\mathcal{I}}$, it holds true that $\delta \in C^{\mathcal{I}(\omega)}$ implies $\delta \in D^{\mathcal{I}(\omega)}$. Consequently, we have that $\{\delta \in C^{\mathcal{I}}\} \subseteq \{\delta \in D^{\mathcal{I}}\}$ and, thus, $\mathbb{P}^{\mathcal{I}}\{\delta \in D^{\mathcal{I}}\} \gtrdot p$ due to the monotonicity of the probability measure $\mathbb{P}^{\mathcal{I}}$. □

3.4 A CONCEPT INCLUSION BASE

As final step, we show that each concept inclusion base of the probabilistic scaling $\mathcal{I}_{\mathsf{d}}$ induces a concept inclusion base of $\mathcal{I}$. While soundness is easily verified, completeness follows from the fact that $C \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(C) \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(D) \sqsubseteq_{\emptyset} D$ holds true for every valid CI $C \sqsubseteq D$ of $\mathcal{I}$.

Theorema 9 Fix some finitely representable probabilistic interpretation $\mathcal{I}$. If $\mathcal{T}_{\mathsf{d}}$ is a concept inclusion base for the probabilistic scaling $\mathcal{I}_{\mathsf{d}}$ (with respect to the set $\mathcal{B}$ of all tautological $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept inclusions used as background knowledge), then the following terminological box $\mathcal{T}$ is a concept inclusion base for $\mathcal{I}$.

$$\mathcal{T} := \mathcal{T}_{\mathsf{d}} \cup \{\, \mathsf{d} > p.* \sqsubseteq \mathsf{d} \geq p^{+}.* \mid p \in P(\mathcal{I}) \setminus \{1\} \,\}$$

Approbatio Soundness is apparently satisfied. We proceed with showing completeness; thus, fix some $\mathcal{P}_1^{>}\mathcal{EL}^{\bot}$ concept inclusion $C \sqsubseteq D$ which is valid in $\mathcal{I}$. We shall demonstrate the validity of the following subsumptions.

$$C \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(C) \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(D) \sqsubseteq_{\emptyset} D$$

Lemma 7 immediately yields that $\pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ is valid in the probabilistic scaling $\mathcal{I}_{\mathsf{d}}$. Since $\mathcal{T}_{\mathsf{d}}$ is complete for $\mathcal{I}_{\mathsf{d}}$ relative to $\mathcal{B}$, it follows that $\mathcal{T}_{\mathsf{d}} \cup \mathcal{B}$ entails $\pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ with respect to non-probabilistic entailment, and, thus, $\mathcal{T}$ entails $\pi_{\mathcal{I}}(C) \sqsubseteq \pi_{\mathcal{I}}(D)$ with respect to probabilistic entailment.

We use the operator $\pi'_{\mathcal{I}}$ from the proof of Lemma 7 again. Using structural induction on D, it is apparent that $\pi'_{\mathcal{I}}(D) \sqsubseteq_{\emptyset} D$ holds true. Since the probability restriction constructor (more specifically, the mapping $E \mapsto \mathsf{d} \gtrdot p.\,E$) is monotone and $E^{\mathcal{I}_\times \mathcal{I}_\times} \sqsubseteq_{\emptyset} E$ holds true for each $E \in \mathcal{EL}^{\bot}(\Sigma)$, we further obtain that $\pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E) \sqsubseteq_{\emptyset} \pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E)$, and then structural induction yields $\pi_{\mathcal{I}}(D) \sqsubseteq_{\emptyset} \pi'_{\mathcal{I}}(D)$.

It remains to show that $C \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(C)$. For each concept description $E \in \mathcal{EL}^{\bot}(\Sigma)$, the CI $E \sqsubseteq E^{\mathcal{I}_\times \mathcal{I}_\times}$ is trivially valid in $\mathcal{I}_\times$, and since the restriction $\mathcal{I}_{\mathsf{d}}\restriction_{\Sigma}$ equals $\mathcal{I}_\times$, we conclude that $\mathcal{I}_{\mathsf{d}} \models E \sqsubseteq E^{\mathcal{I}_\times \mathcal{I}_\times}$. Completeness of $\mathcal{T}_{\mathsf{d}}$ for $\mathcal{I}_{\mathsf{d}}$ relative to $\mathcal{B}$ together with the fact that E does not contain any probability restrictions (i.e., subconcepts of the form $\mathsf{d} \gtrdot q.\,F$) yields that $\mathcal{T}_{\mathsf{d}} \models E \sqsubseteq E^{\mathcal{I}_\times \mathcal{I}_\times}$ with respect to non-probabilistic entailment, and so $\mathcal{T}$ entails $E \sqsubseteq E^{\mathcal{I}_\times \mathcal{I}_\times}$ with respect to probabilistic entailment. According to Lemma 8 then $\mathcal{T}$ entails $\mathsf{d} \gtrdot p.\,E \sqsubseteq \mathsf{d} \gtrdot p.\,E^{\mathcal{I}_\times \mathcal{I}_\times}$. It is readily verified that the set of wildcard CIs in $\mathcal{T}$ entails $\mathsf{d} \gtrdot p.\,E^{\mathcal{I}_\times \mathcal{I}_\times} \sqsubseteq \pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E^{\mathcal{I}_\times \mathcal{I}_\times})$ and, furthermore, that $\pi'_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E^{\mathcal{I}_\times \mathcal{I}_\times}) \equiv_{\emptyset} \pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E)$. Using the condensed result $\mathsf{d} \gtrdot p.\,E \sqsubseteq_{\mathcal{T}} \pi_{\mathcal{I}}(\mathsf{d} \gtrdot p.\,E)$ within a structural induction on C then shows that $C \sqsubseteq \pi_{\mathcal{I}}(C)$ is entailed by $\mathcal{T}$. □

Note that, according to the proof of Theorema 9, we can expand the above TBox $\mathcal{T}$ to a finite TBox that does not contain wildcard CIs and is still a CI base for $\mathcal{I}$ by replacing each wildcard CI $\mathsf{d} > p.* \sqsubseteq \mathsf{d} \geq q.*$ with the CIs $\mathsf{d} > p.\,X^{\mathcal{I}_\times} \sqsubseteq \mathsf{d} \geq q.\,X^{\mathcal{I}_\times}$ where $X \subseteq \Delta^{\mathcal{I}} \times \Omega^{\mathcal{I}}$ such that $\bot \not\equiv_{\emptyset} X^{\mathcal{I}_\times} \not\equiv_{\emptyset} \top$. The same hint applies to the following canonical base.

Corollarium 10 Let $\mathcal{I}$ be a finitely representable probabilistic interpretation, and let $\mathcal{B}$ denote the set of all $\mathcal{EL}^{\bot}$ concept inclusions over Γ that are tautological with respect to probabilistic entailment, i.e., are valid in every probabilistic interpretation. Then, the *canonical base* for $\mathcal{I}$ that is defined as

$$\mathsf{Can}(\mathcal{I}) := \mathsf{Can}(\mathcal{I}_{\mathsf{d}}, \mathcal{B}) \cup \{\, \mathsf{d} > p.* \sqsubseteq \mathsf{d} \geq p^{+}.* \mid p \in P(\mathcal{I}) \setminus \{1\} \,\}$$

is a concept inclusion base for $\mathcal{I}$, and it can be computed effectively.

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