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Role-Value Maps and General Concept Inclusions in the Description Logic $\mathcal{FL}_0$

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Abstract

We investigate the impact that general concept inclusions and role-value maps have on the complexity and decidability of reasoning in the Description Logic $\mathcal{FL}_0$. On the one hand, we give a more direct proof for ExpTime-hardness of subsumption w.r.t. general concept inclusions in $\mathcal{FL}_0$. On the other hand, we determine restrictions on role-value maps that ensure decidability of subsumption, but we also show undecidability for the cases where these restrictions are not satisfied.

1 Introduction

Description Logics (DLs) [5] are a well-investigated family of logic-based knowledge representation formalisms, which are descended from the knowledge representation system KL-ONE [13]. The design goal of KL-ONE was, on the one hand, to provide its users with a knowledge representation (KR) language that is equipped with a well-defined syntax and a formal, unambiguous semantics, which was not always true for early KR approaches such as semantic networks [20] and frames [18]. On the other hand, reasoning over knowledge bases written in this language was supposed to be tractable (i.e., realizable by polynomial-time inference procedures) [11]. Thus, it came as a considerable shock to the community when it was shown that the second requirement is not satisfied by the language employed by KL-ONE for two independent reasons.

On the one hand, KL-ONE provided its users with the concept constructor role-value maps (RVMs), which can be employed to link role successor sets. For example, the concept described by the RVM

$$(\mathsf{child} \circ \mathsf{friend} \sqsubseteq \mathsf{knows})$$

collects all individuals that know all the friends of their children. The general form of such an RVM is $(r_1 \circ \ldots \circ r_m \sqsubseteq s_1 \circ \ldots \circ s_n)$, where $r_1, \ldots, s_n$ are roles (i.e., binary predicates). It was shown in [21] that the presence of RVMs actually makes reasoning in KL-ONE undecidable. As a consequence, general RVMs were removed from KL-ONE-based KR languages, and are not available in any of the DLs employed by today's DL systems. One possibility for avoiding the undecidability caused by RVMs is to restrict the roles occurring in them to being functional. This approach was employed by the CLASSIC system [10], where the corresponding constructor is called the same-as constructor. However, using same-as in place of RVMs only overcomes the undecidability problem if no general concept inclusions (GCIs) are available in the terminological formalism [4]. An alternative approach for restricting RVMs with the goal of achieving decidability is to disallow role composition on the right-hand side, i.e., consider only RVMs of the form $(r_1 \circ \ldots \circ r_m \sqsubseteq s)$. For the inexpressive DL $\mathcal{EL}$, adding such restricted RVMs leaves reasoning not only decidable, but also tractable even in the presence of GCIs [2, 3]. For more expressive DLs, additional restrictions on such RVMs need to be imposed to keep reasoning decidable [14, 16].

On the other hand, KL-ONE provided its users with the concept constructors conjunction ($\sqcap$) and value restriction ($\forall r.C$). For example, using these constructors one can build the concept

$$\mathsf{Person} \sqcap \forall \mathsf{child}.\forall \mathsf{friend}.\mathsf{Nice},$$

which describes the persons all of whose children have only nice friends. It was shown in [19] that the subsumption problem in the DL $\mathcal{FL}_0$, which has only these two constructors, is coNP-hard in the presence of the simplest terminological formalism, which are so-called acyclic TBoxes. For cyclic TBoxes, the complexity increases to PSpace [1, 17], and for general TBoxes consisting of GCIs even to ExpTime [3]. Thus, w.r.t. general TBoxes, subsumption reasoning in $\mathcal{FL}_0$ is as hard as in $\mathcal{ALC}$, its closure under negation.

In the present paper, we first reconsider this ExpTime-hardness result. The original proof in [3] leads to a rather long chain of reductions, which makes it hard to understand the reasons for ExpTime-hardness and to reuse the proof ideas for other DLs. In Section 3, we provide a reduction from the problem of deciding the winner in countdown games, which was shown in [15] to be ExpTime-hard by a direct reduction from the problem of the acceptance of a word by a linearly-bounded alternating Turing machine. Then, we investigate the effect that adding RVMs has on the decidability of subsumption in $\mathcal{FL}_0$. On the one hand, we introduce two classes of RVMs that leave subsumption without GCIs decidable. On the other hand, we show that the restrictions made to achieve decidability are really needed: (i) in the presence of GCIs, even adding a single length-preserving RVM can cause undecidability; (ii) for unrestricted RVMs, undecidability even holds without GCIs. The two latter results are independent of whether global RVMs (which must hold for every element of a model) or local RVMs (which are concept constructors, as introduced above) are used.

2 The Description Logic $\mathcal{FL}_0$

In Description Logic, *concept constructors* are used to build complex *concepts* out of *concept names* (unary predicates) and *role names* (binary predicates). A particular DL is determined by the available constructors. Starting with countably infinite sets N_C and N_R of concept and role names, respectively, the set of $\mathcal{FL}_0$ concepts is inductively defined as follows:

- $\top$ (top concept) and every concept name $A \in N_C$ is an $\mathcal{FL}_0$ concept,
- if C, D are $\mathcal{FL}_0$ concepts and $r \in N_R$ is a role name, then $C \sqcap D$ (conjunction) and $\forall r.C$ (value restriction) are $\mathcal{FL}_0$ concepts.

The *semantics* of $\mathcal{FL}_0$ concepts is defined using first-order interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of a non-empty domain $\Delta^\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$ that assigns a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ to each concept name A, and a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ to each role name r. This function is extended to $\mathcal{FL}_0$ concepts as follows:

$$\top^\mathcal{I} = \Delta^\mathcal{I} \text{ and } (C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I},$$
$$(\forall r.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}\colon (x, y) \in r^\mathcal{I} \Rightarrow y \in C^\mathcal{I}\}.$$

A (general) $\mathcal{FL}_0$ *TBox* $\mathcal{T}$ is a finite set of *general concept inclusions (GCIs)*, which are expressions of the form $C \sqsubseteq D$ for $\mathcal{FL}_0$ concepts C, D. The interpretation $\mathcal{I}$ is a *model* of $\mathcal{T}$ if it satisfies all the GCIs in $\mathcal{T}$, i.e. $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for all GCIs $C \sqsubseteq D$ in $\mathcal{T}$. Given an $\mathcal{FL}_0$ TBox $\mathcal{T}$ and two $\mathcal{FL}_0$ concepts C, D, we say that C is *subsumed* by D (denoted as $C \sqsubseteq_\mathcal{T} D$) if $C^\mathcal{I} \subseteq D^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{T}$. These two concepts are *equivalent* (denoted as $C \equiv_\mathcal{T} D$) if $C \sqsubseteq_\mathcal{T} D$ and $D \sqsubseteq_\mathcal{T} C$. If the TBox is empty, we write $C \sqsubseteq D$ and $C \equiv D$ instead of $C \sqsubseteq_\emptyset D$ and $C \equiv_\emptyset D$.

For $\mathcal{FL}_0$, the subsumption problem w.r.t. general TBoxes is ExpTime-complete: the upper bound follows from the well-known ExpTime upper bound for $\mathcal{ALC}$ [7], which contains $\mathcal{FL}_0$ as a sublogic. A previous proof of the lower bound was given in [3]. In the next section, we will provide a more direct proof of this hardness result.

Without TBox, subsumption and equivalence in $\mathcal{FL}_0$ can be characterized using inclusion of formal languages. This characterization relies on transforming $\mathcal{FL}_0$ concepts into an appropriate normal form as follows. First, the semantics given to the concept constructors in $\mathcal{FL}_0$ implies that value restrictions distribute over conjunction, i.e., for all $\mathcal{FL}_0$ concepts C, D and roles r it holds that $\forall r.(C \sqcap D) \equiv \forall r.C \sqcap \forall r.D$. Using this equivalence as a rewrite rule from left to right, every

$\mathcal{FL}_0$ concept can be transformed into an equivalent one that is either $\top$ or a conjunction of concepts of the form $\forall r_1. \cdots \forall r_n.A$, where $r_1, \ldots, r_n$ are role names and A is a concept name. Such a concept can be abbreviated as $\forall w.A$, where $w = r_1 \ldots r_n$ is a word over the alphabet N_R. Note that $n = 0$ means that w is the empty word ε, and thus $\forall \varepsilon.A$ corresponds to A. Furthermore, a conjunction of the form $\forall w_1.A \sqcap \ldots \sqcap \forall w_m.A$ can be written as $\forall L.A$ where $L \subseteq N_R^*$ is the finite language $\{w_1, \ldots, w_m\}$. We use the convention that $\forall \emptyset.A$ corresponds to the top concept $\top$. Thus, any two $\mathcal{FL}_0$ concepts C, D containing only the concept names $A_1, \ldots, A_\ell$ can be represented as

$$\begin{array}{lcl} C & \equiv & \forall K_1.A_1 \sqcap \ldots \sqcap \forall K_\ell.A_\ell, \\ D & \equiv & \forall L_1.A_1 \sqcap \ldots \sqcap \forall L_\ell.A_\ell, \end{array} \qquad (1)$$

where $K_1, L_1, \ldots, K_\ell, L_\ell$ are finite languages over the alphabet of role names N_R. We call this representation the *language normal form (LNF)* of C, D.

If C, D have the LNFs shown above, then $C \sqsubseteq D$ holds iff $L_1 \subseteq K_1, \ldots, L_\ell \subseteq K_\ell$ [8]. A similar characterization of subsumption can actually also be given in the presence of a TBox, but then $K_1, \ldots, L_\ell$ are regular languages represented by automata of size exponential in the size of $\mathcal{T}$ [6].

3 ExpTime-hardness of $\mathcal{FL}_0$ with GCIs

We give a new proof of the fact that subsumption in $\mathcal{FL}_0$ w.r.t. a general TBox is ExpTime-hard. This proof is by reduction from the problem of deciding the winner in countdown games, which are two-player games for which deciding which player has a winning strategy is known to be ExpTime-complete [15].

As defined in [15], a *countdown game* is given by a weighted graph (S, T), where S is the finite set of *states* and $T \subseteq S \times \mathbb{N} \backslash \{0\} \times S$ is the finite *transition relation.* If $t = (s, d, s') \in T$, then we say that the *duration* of the transition t is d. A *configuration* of a countdown game is a pair (s, c), where $s \in S$ is a state and $c \in \mathbb{N}$. A *move* of a countdown game from a configuration (s, c) is performed in the following way: first Player 1 chooses a number d such that $0 < d \leq c$ and there is $s' \in S$ with $(s, d, s') \in T$; then Player 2 chooses a transition $(s, d, s') \in T$ of duration d; the new configuration resulting from this move is then $(s', c - d)$. There are two types of *terminal* configurations, i.e., configurations (s, c) in which no more moves are available. If $c = 0$ then the configuration (s, c) is terminal and is a *winning configuration for Player 1.* If for all transitions $(s, d, s') \in T$ from the state s we have that $d > c$, then the configuration (s, c) is terminal and it is a *winning configuration for Player 2.* The algorithmic problem of *deciding the winner* in countdown games is the following problem: given a weighted graph (S, T) and a configuration (s_0, c), where all the durations of transitions and the number c are assumed to be represented in binary, to determine whether Player 1 has a winning strategy from the configuration (s_0, c). Theorem 2 in Section 4.2 of [15] shows that this problem is ExpTime-complete by a reduction from the word problem for linearly-bounded alternating Turing machine.

Proposition 1. *Deciding whether Player 1 has a winning strategy in a countdown game (S, T) with initial configuration (s_0, c) can be reduced in polynomial time to non-subsumption in $\mathcal{FL}_0$ w.r.t. a general TBox*

Proof. Let ℓ be the maximal number of bits needed to represent c and any of the numbers occurring in T in binary. We set

$$\begin{aligned} N_C &= S \uplus \{F\} \uplus \bigcup_{i=0}^{\ell} \{\mathsf{b}_i{=}0, \mathsf{b}_i{=}1\}, \\ N_R &= \{\bar{s} \mid s \in S\} \uplus \{i \mid 0 \leq i \leq \ell\}. \end{aligned}$$

The idea is that each element of an interpretation $\mathcal{I}$ is labeled by one (or several) number(s) written in binary: $\mathsf{b}_i{=}0$ means that the ith bit of this number is equal to 0, and $\mathsf{b}_i{=}1$ that it is equal to 1. In addition, if $(x, y) \in i^{\mathcal{I}}$, the number labeling y should be the same as the one labeling x minus 2^i. Note that the size of N_C and N_R is polynomial in the size of the input since ℓ is bounded by the size of the binary representation of the largest number occurring in the input. The subsumption relationship we want to test is

$$s_0 \sqcap \hat{c} \sqsubseteq_{\mathcal{T}} F,$$

where $\hat{c}$ stands for the conjunction of all $\mathsf{b}_i{=}c_i$, where c_i is the value of the ith bit in the binary representation of c. The concept F stands for “fail,” i.e., a configuration where Player 1 has no winning strategy.

The goal is to define the TBox $\mathcal{T}$ such that any model of $\mathcal{T}$ that does not satisfy the subsumption corresponds to a winning strategy for Player 1. To do this, we use the fact that, if Player 1 has a winning strategy in configuration (s_0, c) whose first step chooses duration d, then for all $(s_0, d, s') \in T$, Player 1 must also has a winning strategy in configuration $(s', c - d)$. Thus, if $s_1, ..., s_p$ are the states such that $(s_0, d, s_i) \in T$, then we can construct inductively the structures corresponding to the winning strategies on $(s_i, c - d)$, as shown in Figure 1 (where $\mathcal{I}_i$ is the interpretation corresponding to a winning strategy in configuration $(s_i, c - d)$).

Given a duration d occurring in T, we write $\forall \tilde{d}$ as an abbreviation for $\forall i_1 i_2 ... i_k$, where $i_1, i_2, ..., i_k$ are the bits equal to 1 in the binary representation of d, written in decreasing order.[1]

The TBox $\mathcal{T}$ consists of the following GCIs:

1. $s \sqsubseteq \forall \tilde{d} \bar{s}.s'$ for all $t = (s, d, s') \in T$,

[1] The only condition needed is that the same number must always be represented in the same order, and using decreasing order is an easy way to achieve this.

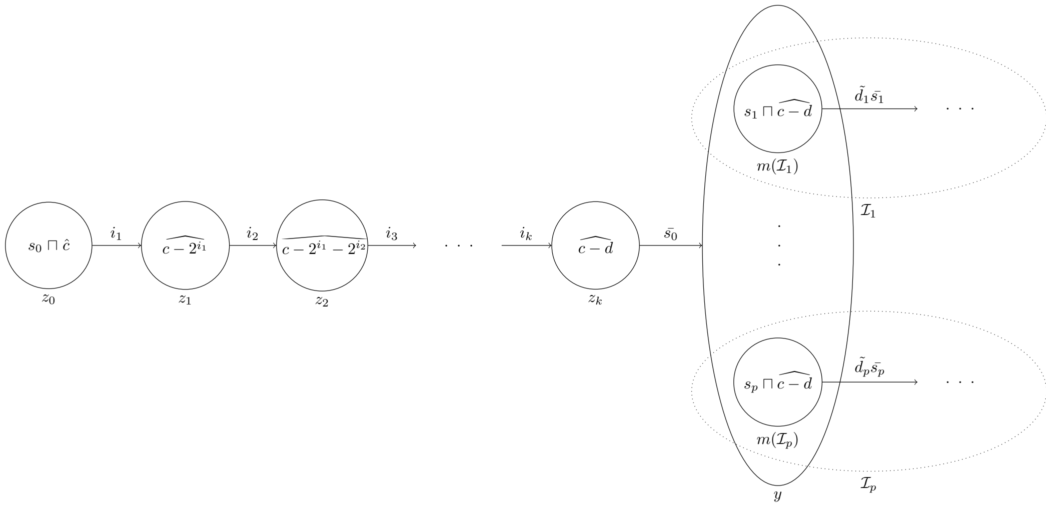

Figure 1: The interpretation corresponding to a winning strategy of Player 1.

2. $s \sqcap \mathsf{b}_i{=}1 \sqcap \bigsqcap_{d \in E_s} \forall \tilde{d}\bar{s}.F \sqsubseteq F$ for all $s \in S$ and all $i, 1 \leq i \leq \ell$, where $E_s = \{d \mid \exists s' \text{ s.t. } (s, d, s') \in T\}$,

3. $\mathsf{b}_i{=}x \sqsubseteq \forall k.\mathsf{b}_i{=}x$ for all $0 \leq i < k \leq \ell$ and $x \in \{0, 1\}$,

4. $\mathsf{b}_i{=}1 \sqcap \mathsf{b}_j{=}x \sqsubseteq \forall k.\mathsf{b}_j{=}x$ for all $k \leq i < j$ and $x \in \{0, 1\}$,

5. $\bigsqcap_{j=k}^{i-1} \mathsf{b}_j{=}0 \sqcap \mathsf{b}_i{=}1 \sqsubseteq \forall k.(\bigsqcap_{j=k}^{i-1} \mathsf{b}_j{=}1 \sqcap \mathsf{b}_i{=}0)$ for all $k \leq i$,

6. $\mathsf{b}_i{=}x \sqsubseteq \forall \bar{s}.\mathsf{b}_i{=}x$ for all $s \in S$, $1 \leq i \leq \ell$, and $x \in \{0, 1\}$,

7. $\bigsqcap_{j=i}^{\ell} \mathsf{b}_j{=}0 \sqsubseteq \forall i.F$ for all $i, 1 \leq i \leq \ell$,

8. $F \sqsubseteq \forall i.F$ for all $i, 1 \leq i \leq \ell$,

9. $F \sqsubseteq \forall \bar{s}.F$ for all $s \in S$.

The intuition underlying these GCIs is the following:

- The GCIs in 1. say that, if we choose the duration d, then we must consider every state accessible this way. The GCIs in 2. reflect the fact that, if at least one of the configurations in which we could end up this way does not have a winning strategy, then choosing d does not yield a winning strategy either, unless the number of the configuration is already 0.

- The GCIs in 3., 4., 5., and 6. are there to ensure that subtraction is performed properly.

- The GCIs in 7. say that, if we choose a duration greater than the number in the current configuration (which is checked by verifying that the subtraction would return a negative number), then this leads to a failure for Player 1. The GCIs in 8. and 9. propagate this information forward to the next element corresponding to a configuration (so that it can then be propagated backwards using the GCIs in 2.).

Let us now prove the correctness of the reduction in a formal way, i.e., we show that $s_0 \sqcap \hat{c} \not\sqsubseteq_{\mathcal{T}} F$ iff Player 1 has a winning strategy.

"$\Rightarrow$" We show by *induction on* c that, for all states $s \in S$, if there exists a model $\mathcal{I}$ of $\mathcal{T}$ that contains an element x such that $x \in (s \sqcap \hat{c})^{\mathcal{I}} \backslash F^{\mathcal{I}}$, then Player 1 has a winning strategy for the configuration (s, c).

If $c = 0$, then doing nothing is already a winning strategy, and we are done.

Otherwise, x belongs to at least one of the concepts $\mathsf{b}_i{=}1$ in $\mathcal{I}$. Since x also belongs to s, but does not belong to F, it necessarily does not belong to one of the concepts $\forall \tilde{d}\bar{s}.F$ for $d \in E_s$, due to the GCIs in 2. Let us fix a duration d for which this is the case. We show that $c - d \geq 0$ and that the element $y \in (\tilde{d}\bar{s})^{\mathcal{I}}(x)$ that does not belong to F indeed belongs to $s' \sqcap \widehat{(c-d)}$, for all s' such that $(s, d, s') \in T$. Once this is shown, we can apply the induction hypothesis to y to get a winning strategy for each $(s', c - d)$ accessible from (s, c) by choosing d. Consequently, choosing d and then applying these winning strategies yields a winning strategy for (s, c).

The fact that y belongs to s' directly follows from the GCIs in 1.

Let us now show that $(z_1, z_2) \in i^{\mathcal{I}}$ and $z_1 \in \hat{n}^{\mathcal{I}}$ imply that $z_2 \in \widehat{(n - 2^i)}^{\mathcal{I}}$ if $n \geq 2^i$, and $z_2 \in F^{\mathcal{I}}$ otherwise.

- *Case* $n \geq 2^i$. First note that the $i - 1$ lowest bits of $n - 2^i$ are the same as the ones of n. This is ensured by the GCIs in 3. Then, the other bits are changed according to the rules of binary subtraction, which are reflected by the GCIs in 4. and 5. The idea is that, if the jth bit is the lowest bit equal to 1 and such that $j \geq i$, then the GCIs in 5. take care of the bits between the ith one and the jth one, and the GCIs in 4. take care of the bits above the jth one.

- *Case* $n < 2^i$. Then we have that all bits above the ith one (included) have value 0, and thus z_1 belongs to $\prod_{j=i}^{\ell} \mathsf{b}_j{=}0$. Consequently, the GCIs in 7. yield that $z_2 \in F^{\mathcal{I}}$.

Let $\tilde{d} = i_1 \dots i_k$. Let us suppose that $c < d$. Then there exists j such that $c - (\sum_{\mu=1}^{j} 2^{i_\mu}) \geq 0$ and $c - (\sum_{\mu=1}^{j+1} 2^{i_\mu}) < 0$. By what we have just shown in the first item above, we know that any element of $(i_1 \dots i_j)^\mathcal{I}(x)$ belongs to $(\widehat{c - (\sum_{\mu=1}^{j} 2^{i_\mu})})$. By the second item, we can then deduce that any element of $(i_1 \dots i_{j+1})^\mathcal{I}(x)$ is in $F^\mathcal{I}$. By the GCIs in 8. and 9., the concept F is then propagated to y, which contradicts the fact that y does not belong to F. Thus, we have $c \geq d$. Knowing this, and using again the first item shown above as well as the GCIs in 6, we can also deduce that $y \in \widehat{(c-d)}^\mathcal{I}$. The finishes the proof of the only-if-direction.

"$\Leftarrow$" Assume that Player 1 has a winning strategy. First note that winning strategies can be represented as trees as follows. The *tree $T_{s,c}$ of the winning strategy in configuration* (s, c) is defined inductively: if $c > 0$ and d is the number chosen by Player 1 in this configuration, we set $T_{s,c} = (d, \{T_{s',c-d} \mid (s, d, s') \in T\})$; for $c = 0$ we set $T_{s,0} = (0, \emptyset)$.

We construct a counterexample $\mathcal{I}_{s,c}$ that satisfies the TBox but not the subsumption $s_0 \sqcap \hat{c} \sqsubseteq F$ by induction on c. During the construction, we will mark a certain element $m(\mathcal{I}_{s,c})$ of the interpretation $\mathcal{I}_{s,c}$, which is the one violating the subsumption relation. There will be three important invariants during the construction, which will obviously be verified by construction:

1. $m(\mathcal{I}_{s,c})$ belongs only to the concept names required by $s \sqcap \hat{c}$, and no other concept names;
2. $F^{\mathcal{I}_{s,c}} = \emptyset$;
3. If there exists $x \in (\tilde{d}\bar{s}')^{\mathcal{I}_{s,c}}(m(\mathcal{I}_{s,c}))$ for some duration d and some state s', then $s' = s$.

First, we consider the base case where $c = 0$:

- *Construction*: We just take the interpretation consisting of a single element, which belongs to the concept names s and b_i=0 for all i, and to no other concept names. Then $m(\mathcal{I}_{s,0})$ is this one element.
- *Correctness of the construction*: We only have one element, which indeed satisfy the GCIs in 1.–9. In fact, the value restrictions on the right-hand sides are trivially satisfied since this element does not have role successors. Moreover, there is no i such that this element belongs to b_i=1, so the left-hand sides of the GCIs in 2. are not satisfied. Finally, the element $m(\mathcal{I}_{s,0})$ does not satisfy the subsumption $s_0 \sqcap \hat{0} \sqsubseteq F$ since it obviously belongs to the left-hand side, but not to the right-hand side.

In the step case, we have $c > 0$:

- *Construction*: We know that the tree $T_{s,c}$ of the winning strategy is of the form $T_{s,c} = (d, E)$, where E contains the trees for the winning strategies for the configurations $(s', c - d)$ where s' ranges over the states satisfying $(s, d, s') \in T$. The induction hypothesis applied to the configurations $(s', c - d)$ (where $c - d < c$ since $d > 0$) yields interpretations $\mathcal{I}_1, ..., \mathcal{I}_p$ corresponding to the elements of E. Let $i_1, \ldots, i_k$ be the digits of d equal to 1. We create a new interpretation $\mathcal{I}$ that looks like the one depicted in Figure 1 as follows. i The idea is that we will take the union of all $\mathcal{I}_p$, merge their marked elements into a new element y, and add new elements $z_0, \ldots, z_k$ (where z_0 is the marked element of the new interpretation) such that $y \in (\tilde{d}\bar{s})^{\mathcal{I}}(z_0)$. More formally: set $\Gamma_\nu = \Delta^{\mathcal{I}_\nu} \backslash \{m(\mathcal{I}_\nu)\}$ and define
  - $\Delta^{\mathcal{I}} = \biguplus_{\nu=1}^{p} \Gamma_\nu \uplus \{z_0, z_1, ...z_k, y\}$,
  - $m(\mathcal{I}) = z_0$,
  - for $0 \leq j \leq k - 1$, add (z_j, z_{j+1}) to $(i_{j+1})^{\mathcal{I}}$,
  - add (z_k, y) to $\bar{s}^{\mathcal{I}}$,
  - for all elements γ in $\biguplus_{\nu=1}^{p} \Gamma_\nu$, for all roles r, add (y, γ) to $r^{\mathcal{I}}$ iff there exists ν such that $(m(\mathcal{I}_\nu), \gamma) \in r^{\mathcal{I}_\nu}$,
  - for all pairs (α, β) in $\big(\biguplus_{\nu=1}^{p} \Gamma_\nu\big)^2$, for all roles r, add (α, β) to $r^{\mathcal{I}}$ iff $(\alpha, \beta) \in \biguplus_{\nu=1}^{n} r^{\mathcal{I}_\nu}$,
  - add z_0 to $s^{\mathcal{I}}$,
  - for all $0 \leq j \leq k$, add z_j to the concept names corresponding occurring in the conjunction $\widehat{c - (\sum_{j=1}^{i} 2^{i_j})}$,
  - for all concept names A, add y to $A^{\mathcal{I}}$ if there exists ν such that $m(\mathcal{I}_\nu) \in A^{\mathcal{I}_\nu}$,
  - for all elements γ in $\biguplus_{\nu=1}^{p} \Gamma_\nu$, for all concept names A, add γ to $A^{\mathcal{I}}$ iff $\gamma \in \biguplus_{\nu=1}^{n} A^{\mathcal{I}_p}$.
- *Correctness of the construction*:
  - First, note that, by the induction hypothesis, the GCIs in the TBox are satisfied by all elements of $\mathcal{I}$ belonging to $\biguplus_{\nu=1}^{p} \Gamma_\nu$. This is the case since they belong to the same concept names and are linked (going forward using a role) to the same other elements as they are in their respective interpretations $\mathcal{I}_\nu$.

- Next, consider the element y obtained by "merging" the elements $m(\mathcal{I}_\nu)$:
  - *The GCIs in 1.* Let $z \in (\tilde{d'}\bar{s'})^{\mathcal{I}}(y)$ for some d' and s'. Then there exists ν such that $z \in (\tilde{d'}\bar{s'})^{\mathcal{I}_\nu}(m(\mathcal{I}_\nu))$. By Invariant 3, we know that this yields $m(\mathcal{I}_\nu) \in (s')^{\mathcal{I}_\nu}$. By the induction hypothesis, we thus obtain $z \in (s'')^{\mathcal{I}}$, for all s'' such that $(s', d, s'') \in T$. Consequently, the right-hand side of the GCI is verified.
  - *The GCIs in 2.* By Invariant 1, two cases are possible. The first one is that each $m(\mathcal{I}_\nu)$ does not belong to any of the concepts b_i=1 (in the case where $d = c$). But then y does not belong to them either, and thus y does not satisfy the left-hand side of the GCI.

    The second case is that there is an i such that each $m(\mathcal{I}_\nu)$ belongs to the concept b_i=1. Since these elements must satisfy the GCIs from 2. in $\mathcal{I}_\nu$, but do not belong to the right hand-side F in $\mathcal{I}_\nu$, this means that, for all ν, if $m(\mathcal{I}_\nu) \in (s')^{\mathcal{I}_\nu}$, then there exists d', s'' such that $(s', d', s'') \in T$ and $(\tilde{d'}\bar{s'})^{\mathcal{I}_p}(m(\mathcal{I}_p)) \setminus F^{\mathcal{I}_p} \neq \emptyset$. Consequently, for all s' such that $y \in (s')^{\mathcal{I}}$, we have $(\tilde{d'}\bar{s'})^{\mathcal{I}}(y) \setminus F^{\mathcal{I}} \neq \emptyset$. This shows that y does not belong to the left-hand side of the GCI.
  - *The GCIs in 3., 4., 5., 6. and 7.* We know that the elements $m(\mathcal{I}_\nu)$ only satisfy concept names corresponding to their states s_ν and to $c - d$, by Invariant 1. The value $c - d$ is the same for all of them, so if one of them satisfies one of the left-hand side of these GCIs, all of them satisfy it. By the induction hypothesis, all of them thus also satisfy the right-hand side. Consequently, y also satisfy it.
  - *The GCIs in 8. and 9.* The element y cannot satisfy the left-hand side of any of them, due to Invariant 2.
- The elements $z_1, ..., z_k$ do not belong to F and any of the state-concepts s, which implies that they trivially satisfy the GCIs in 1., 2., 8., and 9. The GCIs in 3., 4., 5., and 7. only reflect the way subtraction of a power of 2 works, and are thus also satisfied. The GCIs in 6. are also obviously satisfied since z_k is the only z_j such that the right-hand side does not quantify over an empty set, and the only element of this set is y, which is indeed "labeled" by $c - d$, the same number as the "label" of z_k.
- The element z_0 satisfies the GCIs in 1. by the definition of E_s and Invariant 1. The GCIs in 2. have their left-hand side falsified by z_0 since y is in $(\tilde{d}\bar{s})^{\mathcal{I}}$ and does not satisfy F. The GCIs in 3., 4., 5., 6., and 7. are satisfied for the same reason as for the other z_j. Finally, the GCIs in 8. and 6. are satisfied because their left-hand side is not verified due to Invariant 2.

This finishes our proof of the correctness of the reduction, and thus of the proposition. □

Given the ExpTime-hardness result for deciding the winner in countdown games shown in [15], this proposition yields the following hardness result for $\mathcal{FL}_0$.

Theorem 1. *Subsumption in $\mathcal{FL}_0$ w.r.t. general TBoxes is ExpTime-hard.*

4 Decidable role-value maps in $\mathcal{FL}_0$

Role-value maps actually come in two variants [7]: local RVMs are concept constructors whereas global RVMs are axioms that constrain the interpretation of roles. To be more precise,

- a *local role-value map* is a concept constructor with the syntax $(r_1 \circ \ldots \circ r_m \sqsubseteq s_1 \circ \ldots \circ s_n)$ where $r_1, \ldots, s_n$ are role names. To define its semantics, let
$$(t_1 \circ \ldots \circ t_k)^{\mathcal{I}}(d) = \{e \mid (d, e) \in t_1^{\mathcal{I}} \circ \ldots \circ t_k^{\mathcal{I}}\},$$
for role names $t_1, \ldots, t_k$, where "$\circ$" on the right-hand side is composition of binary relations. Then, $(r_1 \circ \ldots \circ r_m \sqsubseteq s_1 \circ \ldots \circ s_n)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid (r_1 \circ \ldots \circ r_m)^{\mathcal{I}}(d) \subseteq (s_1 \circ \ldots \circ s_n)^{\mathcal{I}}(d)\}$.
- a *global role-value maps* has the same syntax as a local one, but is viewed to be an axiom. An interpretation $\mathcal{I}$ is a model of this axiom if $(r_1 \circ \ldots \circ r_m)^{\mathcal{I}}(d) \subseteq (s_1 \circ \ldots \circ s_n)^{\mathcal{I}}(d)$ holds for all $d \in \Delta^{\mathcal{I}}$.

In the presence of GCIs, local RVMs can express global ones since the global RVM $(r_1 \circ \ldots \circ r_m \sqsubseteq s_1 \circ \ldots \circ s_n)$ has the same models as the GCI $\top \sqsubseteq (r_1 \circ \ldots \circ r_m \sqsubseteq s_1 \circ \ldots \circ s_n)$. However, in the present section we consider only global RVMs without GCIs.

To simplify notation, we write $t_1 \ldots t_k$ in place of $t_1 \circ \ldots \circ t_k$, and again view this expression as a word over the alphabet of role names. Thus, a set $\mathcal{T}$ of global RVMs can be written as $\mathcal{T} = \{u_1 \sqsubseteq v_1, ..., u_k \sqsubseteq v_k\}$ where $u_1, \ldots, v_k \in N_R^*$. Such a set induces the following string-rewriting relation [9] between words over N_R:

$$v \rightarrow_{\mathcal{T}} u \quad \text{iff} \quad \begin{array}{l}\text{there are } x, y \in N_R^* \text{ and } 1 \leq i \leq n \\ \text{such that } v = xv_iy \text{ and } u = xu_iy.\end{array}$$

As usual, we denote the reflexive, transitive closure of $\rightarrow_{\mathcal{T}}$ as $\xrightarrow{*}_{\mathcal{T}}$. More formally, we define
$$\xrightarrow{*}_{\mathcal{T}} = \bigcup_{n \geq 0} \xrightarrow{n}_{\mathcal{T}},$$
where $\xrightarrow{0}_{\mathcal{T}} = \{v, v) \mid v \in N_R^*\}$ and $\xrightarrow{n+1}_{\mathcal{T}} = \xrightarrow{n}_{\mathcal{T}} \circ \rightarrow_{\mathcal{T}}$.

Given a formal language L over N_R, i.e., a subset of N_R^*, we now define the languages

$$L^{\downarrow\mathcal{T}} = \{x \in N_R^* \mid \exists y \in L \text{ with } y \xrightarrow{*}_{\mathcal{T}} x\},$$
$$L^{\uparrow\mathcal{T}} = \{x \in N_R^* \mid \exists y \in L \text{ with } x \xrightarrow{*}_{\mathcal{T}} y\},$$

which can be used to characterize subsumption w.r.t. $\mathcal{T}$ as follows.

Theorem 2. *Let $\mathcal{T}$ be a finite set of global RVMs, and C, D be $\mathcal{FL}_0$ concepts with LNFs as in* (1)*. Then the following are equivalent:*

1. *$C \sqsubseteq_{\mathcal{T}} D$, i.e., $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models of $\mathcal{T}$;*
2. *$L_i \subseteq K_i^{\downarrow\mathcal{T}}$ for all $i, 1 \leq i \leq \ell$;*
3. *$\{w\}^{\uparrow\mathcal{T}} \cap K_i \neq \emptyset$ for all $i, 1 \leq i \leq \ell$ and $w \in L_i$.*

Regarding the proof of this theorem, first note that 2. and 3. are easily seen to be equivalent. In fact, given a word $w \in L_i$, we have $w \in K_i^{\downarrow\mathcal{T}}$ iff $\exists y \in K_i$ with $y \xrightarrow{*}_{\mathcal{T}} w$ iff $\exists y \in K_i$ with $y \in \{w\}^{\uparrow\mathcal{T}}$ iff $\{w\}^{\uparrow\mathcal{T}} \cap K_i \neq \emptyset$.

Our proof of 2. $\Rightarrow$ 1. uses the following proposition, which is an easy consequence of the semantics of global RVMs and value restrictions, and our definition of $\rightarrow_{\mathcal{T}}$.

Proposition 2. *If $x \xrightarrow{*}_{\mathcal{T}} y$, then $\forall x.A \sqsubseteq_{\mathcal{T}} \forall y.A$ holds for all $A \in N_C$.*

Proof. We show by induction on n that $x \xrightarrow{n}_{\mathcal{T}} y$ implies $\forall x.A \sqsubseteq_{\mathcal{T}} \forall y.A$ for all $A \in N_C$.

- The case $n = 0$ is trivial since then $x = y$, and we obviously have $\forall x.A \sqsubseteq_{\mathcal{T}} \forall x.A$.
- Let $n \geq 0$ and assume that $x \xrightarrow{n+1}_{\mathcal{T}} y$. Then there exists z such that $x \xrightarrow{n}_{\mathcal{T}} z$ and $z \rightarrow_{\mathcal{T}} y$. By the induction hypothesis, $\forall x.A \sqsubseteq_{\mathcal{T}} \forall z.A$ holds for all $A \in N_C$.

  Since $z \rightarrow_{\mathcal{T}} y$, there exist words u, v and an index i such that $z = uv_iv$ and $y = uu_iv$. Let $\mathcal{I}$ a model of $\mathcal{T}$ and $A \in N_C$, and assume that $\mathcal{I}$ does not satisfy the subsumption $\forall z.A \sqsubseteq \forall y.A$, i.e., that there is an element d such that $d \in (\forall z.A)^{\mathcal{I}}$, but $d \notin (\forall y.A)^{\mathcal{I}}$. Then there are elements e, f, g such that $(d, e) \in u^{\mathcal{I}}$, $(e, f) \in u_i^{\mathcal{I}}$, $(f, g) \in v^{\mathcal{I}}$, and $g \notin A^{\mathcal{I}}$. Since $\mathcal{I}$ must verify the role-value map $u_i \sqsubseteq v_i$, we obtain $(e, f) \in v_i^{\mathcal{I}}$, and thus. $(d, g) \in z^{\mathcal{I}}$. This contradicts our assumption that $d \in (\forall z.A)^{\mathcal{I}}$.

  Consequently, we have shown $\forall z.A \sqsubseteq_{\mathcal{T}} \forall y.A$, which together with $\forall x.A \sqsubseteq_{\mathcal{T}} \forall z.A$ yields $\forall x.A \sqsubseteq_{\mathcal{T}} \forall y.A$.

Since $x \xrightarrow{*}_{\mathcal{T}} y$ implies that there is an n such that $x \xrightarrow{n}_{\mathcal{T}} y$, this completes the proof of the proposition. □

This proposition yields that $C \sqsubseteq_\mathcal{T} \forall w.A_i$ holds for all $w \in K_i^{\downarrow\mathcal{T}}$ and all $i, 1 \leq i \leq \ell$. If $L_i \subseteq K_i^{\downarrow\mathcal{T}}$, then this implies that $C \sqsubseteq_\mathcal{T} \forall L_i.A_i$ for all $i, 1 \leq i \leq \ell$, and thus we have $C \sqsubseteq_\mathcal{T} D$. This completes the proof of 2. $\Rightarrow$ 1. of Theorem 2.

We show 1. $\Rightarrow$ 2. by contraposition. Thus, assume that there is an i and a word $w = t_1 \ldots t_p$ such that $w \in L_i \setminus K_i^{\downarrow\mathcal{T}}$. We use w and i to build a counterexample to the subsumption $C \sqsubseteq_\mathcal{T} D$, i.e., a model $\mathcal{I}_{w,i}$ of $\mathcal{T}$ in which $C^\mathcal{I} \not\subseteq D^\mathcal{I}$. To build $\mathcal{I}_{w,i}$, we construct an increasing sequence of models $\mathcal{I}_q$ for $q \geq 0$ by induction on q, and then define $\mathcal{I}_{w,i}$ as the union of all the $\mathcal{I}_q$.

- For $\mathcal{I}_0$, we start with a sequence of individuals $d_0, \ldots, d_p$ and connect them with the roles in w, i.e., we set $(d_0, d_1) \in t_1^{\mathcal{I}_0}, \ldots, (d_{p-1}, d_p) \in t_p^{\mathcal{I}_0}$. Additionally, we set $\Delta^{\mathcal{I}_0} = \{d_0, \ldots, d_p\}$ and define $A_i^{\mathcal{I}_0} = \Delta_{\mathcal{I}_0} \setminus \{d_p\}$, and $B^{\mathcal{I}_0} = \Delta_{\mathcal{I}_0}$ for all $B \in N_C \setminus \{A_i\}$.

- For $q \geq 0$, assume that $\mathcal{I}_q$ is already defined. We construct $\mathcal{I}_{q+1}$ by extending $\mathcal{I}_q$ with additional individuals in order to add the role paths required by the RVMs in $\mathcal{T}$. We say that a pair of individuals (d, e) *violates* the RVM $r_1 \ldots r_m \sqsubseteq s_1 \ldots s_n$ in $\mathcal{I}_q$ if $e \in (r_1 \circ \ldots \circ r_m)^{\mathcal{I}_q}(d)$, but $e \notin (s_1 \circ \ldots \circ s_n)^{\mathcal{I}_q}(d)$. Then, to get $\mathcal{I}_{q+1}$, we consider all RVMs $r_1 \ldots r_m \sqsubseteq s_1 \ldots s_n \in \mathcal{T}$ and all pairs (d, e) that violate this RVM in $\mathcal{I}_q$, and for each of them add new individuals $f_1, \ldots, f_{n-1}$ to the domain, and connect them via the roles $s_1, \ldots, s_n$ as follows: $(d, f_1) \in s_1^{\mathcal{I}_{q+1}}, (f_1, f_2) \in s_2^{\mathcal{I}_{q+1}}, \ldots, (f_{n-1}, e) \in s_n^{\mathcal{I}_{q+1}}$. The new individuals introduced this way are made to belong to all concepts $B \in N_C$.

It is easy to see that the interpretations $\mathcal{I}_q$ for $q \geq 0$ are finite. This is clearly the case for $q = 0$. In addition, if we already know that $\mathcal{I}_q$ is finite, then there can only be finitely many pairs violating a GCI from $\mathcal{T}$ in $\mathcal{I}_q$, and to remove a violation only finitely many new individuals are introduced. Since $\mathcal{T}$ is also finite, this implies that $\mathcal{I}_{q+1}$ is finite. However, in general this process of removing violations needs to be iterated infinitely, and the resulting interpretation $\mathcal{I}_{w,i}$ is the limit of this infinite process obtained as the infinite union of the interpretations $\mathcal{I}_q$. The interpretation $\mathcal{I}_{w,i}$ may thus be infinite, but it satisfies the following important properties.

Proposition 3. *The interpretation $\mathcal{I}_{w,i}$ satisfies all the RVMs in $\mathcal{T}$, and for all words u we have that $(d_0, d_p) \in u^{\mathcal{I}_{w,i}}$ implies $u \xrightarrow{*}_\mathcal{T} w$.*

Proof. Let $r_1 \ldots r_m \sqsubseteq s_1 \ldots s_n$ be a role-value map in $\mathcal{T}$ and assume that $e \in (r_1 \circ \ldots \circ r_m)^{\mathcal{I}_{w,i}}(d)$. Since $\mathcal{I}_{w,i}$ is the union of the interpretations $\mathcal{I}_q$, there is a $q \geq 0$ such that $e \in (r_1 \circ \ldots \circ r_m)^{\mathcal{I}_q}(d)$. If the pair (d, e) does not violate the above RVM in $\mathcal{I}_q$, then we have $e \in (s_1 \circ \ldots \circ s_n)^{\mathcal{I}_q}(d)$. Otherwise we have $e \in (s_1 \circ \ldots \circ s_n)^{\mathcal{I}_{q+1}}(d)$. In both cases we obtain $e \in (s_1 \circ \ldots \circ s_n)^{\mathcal{I}_{w,i}}(d)$, which shows that $\mathcal{I}_{w,i}$ satisfies the RVM $r_1 \ldots r_m \sqsubseteq s_1 \ldots s_n$.

Regarding the second property stated in the proposition, first note that $(d_0, d_p) \in u^{\mathcal{I}_{w,i}}$ implies that there is a $q \geq 0$ such that $(d_0, d_p) \in u^{\mathcal{I}_q}$. Thus, it is sufficient to show that the property holds for all interpretations $\mathcal{I}_q$, which we can show by induction. It is clearly satisfied for $q = 0$ since the only path from d_0 to d_p in $\mathcal{I}_0$ is labeled with w.

When going from $\mathcal{I}_q$ to $\mathcal{I}_{q+1}$, we make several steps, where each one removes a violation of an RVM in $\mathcal{I}_q$. Clearly, it is sufficient to show that each such step preserves the property. Thus, assume that $\mathcal{I}$ is an interpretation containing the individuals d_0 and d_p satisfying that $(d_0, d_p) \in u^{\mathcal{I}}$ implies $u \xrightarrow{*}_{\mathcal{T}} w$. We can assume that $\mathcal{I}$ does not have cyclic paths since the interpretations $\mathcal{I}_q$ satisfy this property. Now, assume that there are an RVM $r_1 \ldots r_m \sqsubseteq s_1 \ldots s_n$ in $\mathcal{T}$ and individuals d, e in $\Delta^{\mathcal{I}}$ such that $e \in (r_1 \circ \ldots \circ r_m)^{\mathcal{I}}(d)$. Let $\mathcal{I}'$ be the interpretation obtained from $\mathcal{I}$ by adding the new individuals $f_1, \ldots, f_{n-1}$ and the role relationships $(d, f_1) \in s_1^{\mathcal{I}'}, (f_1, f_2) \in s_2^{\mathcal{I}'}, \ldots, (f_{n-1}, e) \in s_n^{\mathcal{I}'}$. Now, let $(d_0, d_p) \in u^{\mathcal{I}'}$. If this path does not use any of the new individuals, then $(d_0, d_p) \in u^{\mathcal{I}}$, and thus $u \xrightarrow{*}_{\mathcal{T}} w$. Otherwise, there are x, y such that $u = xs_1 \ldots s_n y$, $(d_0, d) \in x^{\mathcal{I}}$, and $(e, d_p) \in y^{\mathcal{I}}$. Then $e \in (r_1 \circ \ldots \circ r_m)^{\mathcal{I}}(d)$ yields $(d_0, d_p) \in (xr_1 \ldots r_m y)^{\mathcal{I}}$, and thus we know that $xr_1 \ldots r_m y \xrightarrow{*}_{\mathcal{T}} w$. Since in addition $u = xs_1 \ldots s_n y \rightarrow_{\mathcal{T}} xr_1 \ldots r_m y$ holds, we have $u \xrightarrow{*}_{\mathcal{T}} w$ as required. □

Since $\mathcal{I}_{w,i}$ is a model of $\mathcal{T}$, it is sufficient to show that $d_0 \in C^{\mathcal{I}_{w,i}} \setminus D^{\mathcal{I}_{w,i}}$. First, suppose that $d_0 \in C^{\mathcal{I}_{w,i}}$ does not hold. By our definition of the interpretation of concept names in $\mathcal{I}_{w,i}$, this can only be the case if there is a word $u \in K_i$ such that $(d_0, d_p) \in u^{\mathcal{I}_{w,i}}$. The above proposition yields $u \xrightarrow{*}_{\mathcal{T}} w$, and thus $w \in K_i^{\downarrow \mathcal{T}}$, contradicting our choice of w. Consequently, we must have $d_0 \in C^{\mathcal{I}_{w,i}}$. Finally, we have $d_0 \notin D^{\mathcal{I}_{w,i}}$ since $w \in L_i$, $(d_0, d_p) \in w^{\mathcal{I}_{w,i}}$, and $d_p \notin A_i^{\mathcal{I}_{w,i}}$. This completes the proof of Theorem 2.

In order to derive decidability results for subsumption w.r.t. RVMs in $\mathcal{FL}_0$ from this theorem, we need to find restrictions under which the condition 2. or 3. is decidable. We say that the finite set of RVMs $\mathcal{T}$ is *downward (upward) admissible* if for every finite language L we can effectively compute a representation of $L^{\downarrow \mathcal{T}}$ ($L^{\uparrow \mathcal{T}}$) for which the word problem is decidable. For example, if all RVMS $u_i \sqsubseteq v_i$ in $\mathcal{T}$ satisfy $|u_i| \leq |v_i|$, then $L^{\downarrow \mathcal{T}}$ is also finite (and thus trivially has a decidable word problem) and can effectively be computed. Thus, such a set of RVMs is downward admissible. Symmetrically $|u_i| \geq |v_i|$ for all RVMS $u_i \sqsubseteq v_i$ in $\mathcal{T}$ implies that $\mathcal{T}$ is upward admissible. More generally, one can also have downward (upward) admissible sets of RVMs where the languages $L^{\downarrow \mathcal{T}}$ ($L^{\uparrow \mathcal{T}}$) are not necessarily finite, but one can compute a finite automaton or a pushdown automaton accepting them. We say that $\mathcal{T}$ is *admissible* if it is downward admissible or upward admissible.

Corollary 1. *If $\mathcal{T}$ is a finite, admissible set of RVMs, then the subsumption relation $\sqsubseteq_{\mathcal{T}}$ is decidable.*

Proof. If $\mathcal{T}$ is downward admissible, then we can use condition 2 to decide subsumption: to test whether $L_i \subseteq K_i^{\downarrow\mathcal{T}}$, we must decide for each of the finitely many words $u \in L_i$ whether $u \in K_i^{\downarrow\mathcal{T}}$, which is possible since the word problem for $K_i^{\downarrow\mathcal{T}}$ is decidable.

If $\mathcal{T}$ is upward admissible, then we can use condition 3: to check whether $\{w\}^{\uparrow\mathcal{T}} \cap K_i \neq \emptyset$ it is sufficient to decide, for the finitely many words $u \in K_i$ whether $u \in \{w\}^{\uparrow\mathcal{T}}$. □

5 Undecidable role-value maps in $\mathcal{FL}_0$

The decidability results proved in the previous section depend, on the one hand, on the absence of GCIs. On the other hand, they require the string-rewriting system induced by the role-value maps to be well-behaved (see the definition of *admissible* above).

First, we show that, even without GCIs, RVMs can cause undecidability in $\mathcal{FL}_0$.

Theorem 3. *There exists a fixed finite set of global role-value maps* $\mathcal{T}$ *such that subsumption of* $\mathcal{FL}_0$ *concepts w.r.t.* $\mathcal{T}$ *is undecidable.*

Proof. We prove this theorem by reduction from the word problem for string-rewriting systems. As shown in [9] (Theorem 2.5.9), there is a fixed finite string-rewriting system R such that its word problem (i.e., given two words u, v, decide whether $u \overset{*}{\leftrightarrow}_R v$ holds or not) is undecidable. Here $\overset{*}{\leftrightarrow}_R$ denotes the reflexive, transitive, and *symmetric* closure of the rewrite relation

$$\rightarrow_R = \{(xu_iy, xv_iy) \quad | \quad (u_i, v_i) \in R \text{ and } x, y \in \Sigma^*\},$$

where Σ is the finite alphabet over which the strings in R are built.

Let $R = \{(u_i, v_i) \mid 1 \leq i \leq n\}$ be such a string-rewriting system over the alphabet Σ. We set $N_R = \Sigma$ and define the set of RVMs corresponding to R as

$$\mathcal{T}_R = \{u_1 \sqsubseteq v_1, ..., u_n \sqsubseteq v_n, v_1 \sqsubseteq u_1, ..., v_n \sqsubseteq u_n\}.$$

It is easy to see that the relations $\overset{*}{\leftrightarrow}_R$ and $\overset{*}{\rightarrow}_{\mathcal{T}_R}$ coincide. Now, assume that, given words u, v over Σ, we want to test whether $u \overset{*}{\leftrightarrow}_R v$ holds. We claim that this is the case iff $\forall u.A \sqsubseteq_{\mathcal{T}_R} \forall v.A$ holds. In fact, by Theorem 2 we know that $\forall u.A \sqsubseteq_{\mathcal{T}_R} \forall v.A$ holds iff $\{v\} \subseteq \{u\}^{\downarrow\mathcal{T}_R}$. The latter is obviously equivalent to $u \overset{*}{\rightarrow}_{\mathcal{T}_R} v$, which in turn is equivalent to $u \overset{*}{\leftrightarrow}_R v$. □

Using a trick originally introduced in [21], we can easily transfer this undecidability result from global RVMs to local ones.

Lemma 1. *Let $\mathcal{T}$ be a set of role-value maps, $\mathcal{U}$ an $\mathcal{FL}_0$ TBox, and C, D two $\mathcal{FL}_0$ concepts. In addition, let Σ be the set of all role names occurring in C, D, $\mathcal{U}$, and $\mathcal{T}$, and let s be a new role name not contained in Σ. We define the concept E as follows:*

$$
\begin{aligned}
E \;=\; & \textstyle\bigsqcap_{u\sqsubseteq v\in\mathcal{T}}(u \sqsubseteq v) \sqcap \forall s.\,\big(\bigsqcap_{u\sqsubseteq v\in\mathcal{T}}(u \sqsubseteq v)\big) \sqcap \\
& \textstyle\bigsqcap_{r\in\Sigma}(r \sqsubseteq s) \sqcap \bigsqcap_{r\in\Sigma}(sr \sqsubseteq s).
\end{aligned}
$$

Then, we have $C \sqsubseteq_{\mathcal{T}\cup\mathcal{U}} D$ iff $C \sqcap E \sqsubseteq_{\mathcal{U}} D$.

Proof. "$\Rightarrow$" Let us suppose that we have a counterexample to $C \sqcap E \sqsubseteq_{\mathcal{U}} D$, i.e. an interpretation $\mathcal{I}$ that satisfies $\mathcal{U}$ and an element $x \in \Delta^{\mathcal{I}}$ such that $x \in (C \sqcap E)^{\mathcal{I}} \setminus D^{\mathcal{I}}$. We show for all elements y of $\Delta^{\mathcal{I}}$ that the following holds: if there exists a non-empty word w such that $y \in w^{\mathcal{I}}(x)$, then $y \in s^{\mathcal{I}}(x)$. The proof is by induction on $|w|$.

- If $|w| = 1$, then $w = r$ for a role name r. Since x belongs to E, it satisfies $r \sqsubseteq s$ for all $r \in N_R$. Thus $y \in w^{\mathcal{I}}(x)$ implies $y \in s^{\mathcal{I}}(x)$ as required.
- If $|w| > 1$, then $w = w'r$ with r a role name and w' a non-empty word. Then, there exists z such that $z \in w'^{\mathcal{I}}(x)$ and $y \in r^{\mathcal{I}}(z)$. By induction hypothesis, we have $z \in s^{\mathcal{I}}(x)$, and thus thus $y \in (sr)^{\mathcal{I}}(x)$. Since $x \in E^{\mathcal{I}}$, we also have that $x \in (sr \sqsubseteq s)^{\mathcal{I}}$, which yields $y \in s^{\mathcal{I}}(x)$.

As an immediate consequence of this property and the definition of E, we know that x and all elements reachable from x satisfy the global RVMs in $\mathcal{T}$.

Now, consider the interpretation $\mathcal{I}'$ obtained from $\mathcal{I}$ by removing all elements not reachable from x. It is easy to see that $\mathcal{I}'$ still satisfies $\mathcal{U}$, and in which x still belongs to $C \sqcap E$, but not to D. Since $\mathcal{I}'$ contains only x and elements reachable from x, it satisfies the RVMs in $\mathcal{T}$. Consequently, it is a counterexample to the subsumption $C \sqsubseteq_{\mathcal{T}\cup\mathcal{U}} D$.

"$\Leftarrow$" Let us suppose that we have a counterexample to $C \sqsubseteq_{\mathcal{T}\cup\mathcal{U}} D$, i.e., an interpretation $\mathcal{I}$ that satisfies $\mathcal{T}\cup\mathcal{U}$ and an element $x \in \Delta^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. We consider the interpretation $\mathcal{I}'$ that is the same as $\mathcal{I}$, except that it interprets s as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Since s was chosen to be a new role, it does not appear in $\mathcal{U}$ or C, D. Thus, $\mathcal{I}'$ is still a model of $\mathcal{U}$ and x still belongs to C, but not to D. In addition, x belongs to the concepts in the first line of our definition of E since $\mathcal{I}$, and thus also $\mathcal{I}'$, satisfies the RVMs in $\mathcal{T}$ globally. The second line of our definition of E is satisfied by x since $s^{\mathcal{I}'}$ connects every pair of individuals. Consequently, $\mathcal{I}'$ is a counterexample to the subsumption $C \sqcap E \sqsubseteq_{\mathcal{U}} D$. □

The following corollary then follows directly from this lemma (for the case $\mathcal{U} = \emptyset$) and Theorem 3.

Corollary 2. *Subsumption in $\mathcal{FL}_0$ extended with local role-value maps is undecidable even without a TBox.*

Next, we show that, in the presence of GCIs, undecidability can also be caused by RVMs that satisfy the admissibility condition introduced in the previous section. In fact, we will see that a single global RVM of the form $tr \sqsubseteq rt$ is sufficient to obtain undecidability. Since this RVM is length-preserving, it is both downward and upward admissible.

Theorem 4. *Subsumption $C \sqsubseteq_\mathcal{T} D$ of $\mathcal{FL}_0$ concepts C, D w.r.t. TBoxes $\mathcal{T}$ consisting of $\mathcal{FL}_0$ GCIs and global role-value maps is undecidable. This is the case even if $\mathcal{T}$ contains only GCIs and a single RVM of the form $tr \sqsubseteq rt$.*

Readers that are familiar with the undecidability proof for subsumption in $\mathcal{ALC}$ with global RVMs given in [7], which is by reduction from the tiling problem, may think that the proof of the above theorem should be an easy adaptation of the proof in [7]. A closer look at that proof reveals, however, that it makes extensive use of concept constructors not available in $\mathcal{FL}_0$ (such as negation, disjunction, and existential restrictions). In addition, it requires not only the RVM $tr \sqsubseteq rt$, but also its backward direction $rt \sqsubseteq tr$. The main new contribution of the proof sketched below is thus to show that one can obtain the undecidability results also with the seriously restricted expressive power of $\mathcal{FL}_0$.

We prove Theorem 4 by a reduction from the halting problem for deterministic Turing machines (DTMs). Without loss of generality, we consider DTMs that have a one-side infinite tape, where the left-most tape cell is marked using the special symbol \$. Whenever the machine moves to the left onto this cell, in the next step it immediately goes to the right again and leaves the symbol \$ and the state unchanged. We also assume that the machine can only go left or right (i.e., it cannot stay in place). The machine starts with an "empty" tape, i.e., a tape where the left-most cell contains \$ and all other cells contain the blank symbol B. The blank symbol and \$ cannot be written by the machine. It halts when a special halting state *halt* is reached. For all other states, there is a transition for every possible tape symbol. Clearly, the question whether such a DTM halts when started with the initial state q_0 on the empty tape is undecidable.

Let $M = (Q, \Sigma, \delta, q_0)$ be such a DTM. In order to encode the halting problem for M into a subsumption problem, we set $N_R = \{r, t\}$ (for "right" and "then") and $N_C = Q \cup \Sigma \cup \{H, N\}$ (the latter two for "halt" and "not-head"). The idea is to construct a set of GCIs that encodes the transition function δ of M such that a model should be a structure like the one shown in Figure 2, which corresponds to the unique run of the machine started with the initial state q_0 on the empty tape (where $x_{i,j}$ is the letter at position j at step i of the run, and $q_{i,j}$ is either N, if the head is not at position j at step i, or the state of the machine at step i otherwise).

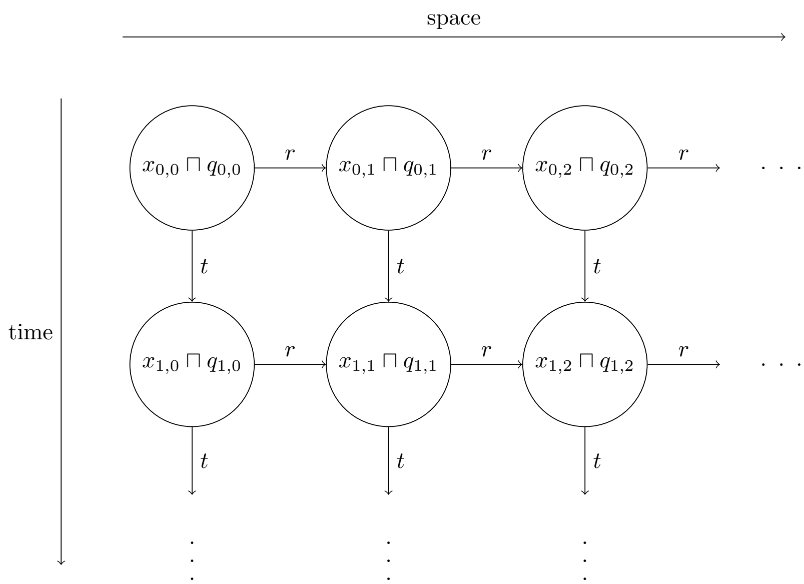

Figure 2: A model of $\mathcal{T}_M$ corresponding to the unique run of M.

More formally, the TBox $\mathcal{T}_M$ consists of the global RVM $tr \sqsubseteq rt$ together with the following GCIs, whose rôle will be explained later:

(1) $\forall t.H \sqsubseteq H$
(2) $\forall r.H \sqsubseteq H$
(3) $B \sqsubseteq \forall r.(B \sqcap N)$
(4) $N \sqcap a \sqsubseteq \forall t.a$ for all $a \in \Sigma$
(5) $N \sqcap \forall rr.N \sqsubseteq \forall tr.N$
(6) $\$ \sqcap \forall r.N \sqsubseteq \forall t.N$
(7) $\forall r.(q \sqcap a) \sqsubseteq \forall t.(N \sqcap \forall r.(b \sqcap \forall r.q'))$
if $\delta(q, a) = (q', b, \rightarrow)$
(8) $\forall r.(q \sqcap a) \sqsubseteq \forall t.(q' \sqcap \forall r.(b \sqcap \forall r.N))$
if $\delta(q, a) = (q', b, \leftarrow)$
(9) $(q \sqcap a) \sqsubseteq H$ if $\delta(q, a) = \mathit{halt}$
(10) $(q \sqcap \$) \sqsubseteq \forall t.(\$ \sqcap \forall r.q)$

Intuitively, these GCIs have the following meaning:

- The first two GCIs propagate the information that the machine has reached the halting state backwards through time and space.
- The third GCI reflects the fact that, if there is a B on a cell, then the machine never went in a position further than the one of this cell (since the

machine never writes B). Thus, the letter of any cell to the right of a B should be B too, and the head of the machine cannot be there.

- The fourth GCI reflects the fact that, if the head is not on a cell at step n, then the letter on this cell should be the same at step $n + 1$.
- The fifth GCI reflects the fact that the head of the tape can only move one cell at a time, so if the head is not directly to the left or to the right of a cell at step n, it cannot be on this cell at step $n + 1$.
- The sixth GCI reflects the same kind of idea: if the head is not directly to the right of the leftmost cell at step n, then it cannot be on this cell at step $n + 1$.
- The seventh and eighth GCIs describe the behavior of the machine when it makes a transition (where the head is and is not, what letter changes, etc.).
- The ninth GCI recognizes the fact that the machine halts. This information is then propagate backwards by the first two GCIs.
- The tenth GCI describes the fact that, when the head is on the \$ symbol, the machine has to go right and stay in the same state.

The following lemma shows the correctness of the reduction, and thus yields the undecidability result stated in Theorem 4.

Lemma 2. *The DTM M halts when started with the initial state q_0 on the empty tape iff*

$$\$ \sqcap \forall r.(B \sqcap q_0) \sqsubseteq_{\mathcal{T}_M} H.$$

Proof. "$\Leftarrow$" If M does not halt then one can use its run to create a model of $\mathcal{T}_M$ that looks like the structure depicted in Figure 2, and where the element in the upper left corner belongs to the left-hand side $\$ \sqcap \forall r.(B \sqcap q_0)$ of the subsumption statement. Formally, it is defined as follows:

- $\Delta^{\mathcal{I}} = \mathbb{N}^2$,
- $r^{\mathcal{I}} = \{\big((i, j), (i + 1, j)\big) \mid (i, j) \in \mathbb{N}^2\}$,
- $t^{\mathcal{I}} = \{\big((i, j), (i, j + 1)\big) \mid (i, j) \in \mathbb{N}^2\}$,
- $H^{\mathcal{I}} = \emptyset$,
- $(i, j) \in N^{\mathcal{I}}$ iff the head is not on position i at step j,
- $(i, j) \in q^{\mathcal{I}}$ iff the head is on position i and the state is q at step j,
- $(i, j) \in a^{\mathcal{I}}$ iff the letter on position i at step j is a.

Since the machine does not halt, one can interpret H as the empty set without violating GCI (9). Since all elements have an r-successor and a t-successor, this also ensures that GCIs (1) and (2) are verified. All the others GCIs are also verified because they only encode the properties of a run of a Turing machine, which are obviously satisfied here. Thus, we have a counterexample to the subsumption, since (0,0) does not satisfy it in this interpretation.

"$\Rightarrow$" Proving the converse direction is a bit more tricky. Basically, we show that a counterexample $(x, \mathcal{I})$ to the subsumption contains the structure induced by the run of M and depicted in Figure 2 as a kind of substructure. To be more precise, we can show by induction on n that, for all i, if $(x, y) \in (t^n r^i)^\mathcal{I}$, then

- $y \in a^\mathcal{I}$ where a is the letter on the i-th cell at the n-th step of the run,
- $y \in N^\mathcal{I}$ if the head of the machine is not on the i-th cell at the n-th step, and
- $y \in q^\mathcal{I}$ if the head is on the i-th cell and the state of the machine is q at the n-th step.

Note that the implications in the other direction need not hold, i.e., it can well be that y also belongs to other states q' or letters a' in $\mathcal{I}$, and that y may also belong to N if the head is actually there.

The base case ($n = 0$) can be easily deduced from GCI (3) and the hypothesis that x satisfies the left-hand side of the subsumption $\$ \sqcap \forall r.(B \sqcap q_0)$.

For the induction step, let $n \in \mathbb{N}$ and $i \in \mathbb{N}$.

The three bullet points below show that, for all $z \in (t^{n+1} r^i)^\mathcal{I}(x)$, we have $z \in a^\mathcal{I}$ where a is the letter on the i-th cell at the n-th step. This is achieved by a case distinction regarding the position of the head.

- *Head is on position $i \geq 1$ at step n.* Let a the letter on cell i at step n, q the state of the machine at step n, and b the letter on cell i at step $n + 1$. By the induction hypothesis, we have $y \in (q \sqcap a)^\mathcal{I}$ for all $y \in (t^n r^i)^\mathcal{I}(x)$. Moreover, the transition taken at step n is of the form $\delta(q, a) = (q', b, d)$, for some state q' and direction d. Hence, by GCI (7) or GCI (8), we know that $z \in b^\mathcal{I}$ holds for all $z \in (t^n r^{i-1} tr)^\mathcal{I}(x)$ (and thus for all $z \in (t^{n+1} r^i)^\mathcal{I}(x)$ since $t \circ r \sqsubseteq r \circ t$ holds in $\mathcal{I}$).
- *Head is on position $i = 0$ at step n.* By the induction hypothesis, if q is the state of the machine at step n, then $y \in (q \sqcap \$)^\mathcal{I}$ for all $y \in (t^n)^\mathcal{I}(x)$. By GCI (10), we then also have $z \in \$^\mathcal{I}$ for all $z \in (t^{n+1})^\mathcal{I}(x)$.
- *Head is not on position i at step n.* Let a be the letter on cell i at step n. Then, by the induction hypothesis, we have $y \in (N \sqcap a)^\mathcal{I}$ for all $y \in$

$(t^n r^i)^{\mathcal{I}}(x)$. Since the head is not on cell i at step n, we know that the letter on cell i at step $n+1$ is still a. By GCI (4), we have $z \in a^{\mathcal{I}}$ that for all $z \in (t^n r^i t)^{\mathcal{I}}(x) \supseteq (t^{n+1} r^i)^{\mathcal{I}}(x)$.

The following three bullet points show that, for all $z \in (t^{n+1} r^i)^{\mathcal{I}}(x)$, we have $z \in N^{\mathcal{I}}$ if the head of the machine is not on the i-th cell at the n-th step; and $z \in q^{\mathcal{I}}$ if the head is on the i-th cell and the state of the machine is q at the n-th step.

- *Head is not on position $i-1$ or $i+1$ at step n for $i \geq 1$.* Then the head cannot be on position i at step $n+1$. Moreover, by the induction hypothesis, we have that $y \in (N \sqcap \forall rr.N)^{\mathcal{I}}$ for all $y \in (t^n r^{i-1})^{\mathcal{I}}(x)$. Then, by GCI (5), we have $z \in N^{\mathcal{I}}$ for all $z \in (t^{n+1} r^i)^{\mathcal{I}}(x)$.
- *Head is not on position $i+1 = 1$ at step n.* Then, it cannot be on position $i = 0$ at step $n+1$. Moreover, by the induction hypothesis, we have $y \in (\$ \sqcap \forall r.N)^{\mathcal{I}}$ for all $y \in (t^n)^{\mathcal{I}}(x)$. Then, by GCI (6), we obtain $z \in N^{\mathcal{I}}$ for all $z \in (t^{n+1})^{\mathcal{I}}(x)$.
- *Head is on position $i-1$ or $i+1$ at step n.* There are a number of subcases to be considered here, which we will not do in detail since they are rather tedious and can be treated similarly to the previous cases. The main idea is that GCIs (7), (8) and (10) will ensure that all $z \in (t^{n+1} r^i)^{\mathcal{I}}(x)$ will be in N or in some $q \in Q$ when necessary.

Let us now suppose that the machine halts at some point. Then, there is a step n_0, a position i_0, a state q, and a letter a such that at step n_0 the head is at position i_0, the state of the machine is q, the letter on position i_0 is a, and $\delta(q, a) = \mathit{halt}$. By what we have just shown, we know that $y \in (q \sqcap a)^{\mathcal{I}}$ holds for all $y \in (t^{n_0} r^{i_0})^{\mathcal{I}}(x)$. Thus, GCI (9) yields $y \in H^{\mathcal{I}}$ for all $y \in (t^{n_0} r^{i_0})^{\mathcal{I}}(x)$. Using the GCIs (1) and (2) it is easy to show that this implies $x \in H^{\mathcal{I}}$, which is a contradiction to our assumption that x is a counterexample to the subsumption. Thus, we have shown that the machine does not halt. □

This completes the proof of Theorem 4. Using the fact that GCIs and local RVMs can express global RVMs, or Lemma 1, we can transfer the undecidability result stated in Theorem 4 also to local RVMs and TBoxes of a restricted form.

Corollary 3. *Subsumption $C \sqsubseteq_{\mathcal{T}} D$ in $\mathcal{FL}_0$ extended with local role-value maps is undecidable even if*

1. *C, D are $\mathcal{FL}_0$ concepts and $\mathcal{T}$ contains GCIs between $\mathcal{FL}_0$ concepts and a single GCI of the form $\top \sqsubseteq (tr \sqsubseteq rt)$ involving a local RVM, or*
2. *D is an $\mathcal{FL}_0$ concept, $\mathcal{T}$ contains only GCIs between $\mathcal{FL}_0$ concepts, and $C = C' \sqcap E$ for an $\mathcal{FL}_0$ concept C' and a fixed concept E of $\mathcal{FL}_0$ extended with local role-value maps.*

6 Conclusion

In this paper we have, on the one hand, given a more direct proof of the known fact that subsumption in $\mathcal{FL}_0$ w.r.t. GCIs is ExpTime-hard. We believe that the ideas underlying the reduction employed in this proof may turn out to be helpful for showing ExpTime-hardness for other inexpressive DLs. On the other hand, we have determined decidable and undecidable cases for $\mathcal{FL}_0$ extended with role-value maps. For the case without a TBox, we have shown that admissible global RVMs leave the subsumption problem decidable. What remains open is the question whether the same is true for admissible local RVMs. For the decidable cases, it would also be interesting to investigate the complexity of the subsumption problem, depending on the form of the available RVMs.

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