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Maybe Eventually?
Towards Combining Temporal and Probabilistic Description Logics and Queries
(Extended Version)

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Abstract

We present some initial results on ontology-based query answering with description logic ontologies that may employ temporal and probabilistic operators on concepts and axioms. Specifically, we consider description logics extended with operators from linear temporal logic (LTL), as well as subjective probability operators, and an extended query language in which conjunctive queries can be combined using these operators. We first show some complexity results for the setting in which either only temporal operators or only probabilistic operators may be used, both in the ontology and in the query, and then show a 2ExpSpace lower bound for the setting in which both types of operators can be used together.

1 Introduction

Ontology-Based Query Answering (OBQA) received considerable attention in the past, as it allows to query incomplete data in the presence of an ontology providing background knowledge about the data domain. While classically, OBQA considers a setting where the data is both static and certain, there are many applications where this assumption does not hold, which lead to the development of temporal query languages for OBQA [10, 34, 11, 5], and research on OBQA for probabilistic data [22, 8, 9, 7]. Temporal OBQA has been proposed as a technique for querying historical data and to detect situations in streams of data. To describe temporal patterns in a query, *temporal queries* as in [10, 11, 5] extend conjunctive queries (CQs) with operators from linear temporal logic (LTL). Probabilistic OBQA is motivated by data sets obtained using uncertain methods such as language and image recognition, or uncertain sensor measurements. In this setting, query answers hold true with a certain probability, which may be part of the query result. As historical data can be obtained using language recognition, and situation recognition is often applied in applications that involve temporal data based on uncertain sensor measurements, there exist applications in which we want to query data that is both temporal and probabilistic in nature. Motivated by this, recently, temporal probabilistic OBQA has been investigated [24], where the temporal query language from [11] is extended with probability operators, and data are considered sequences of probabilistic ABoxes as in [22]. As an example for a probabilistic temporal query, consider a health supervision app on a

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smartphone which operates on a sequence of data obtained using motion and blood pressure sensors. The following query then detects situations in which the patient was, during the last 10 time units, with a low probability exercising, until with a high probability he had a high blood pressure, in which case the app might issue a warning:

$$q(x) \leftarrow \bigcirc^{-10}\big(P_{<0.2}\mathsf{Excercising}(x) \mathbin{\mathsf{U}} P_{>0.7}\mathsf{HighBloodPressure}(x)\big).$$

While the mentioned works allow for an extended expressivity in the query language, they only consider ontologies that are formulated using a classical (atemporal and non-probabilistic) DL. Since the role of the ontology in OBQA is to provide additional background knowledge, temporal and/or probabilistic OBQA would benefit from ontology languages that provide both temporal and probabilistic language constructs. To stay with the current example, this could for instance be used to express that if a patient starts exercising, his blood pressure is likely to remain increased until the patient takes a break:

$$\mathsf{StartsExcercising} \sqsubseteq (\mathsf{P}_{>0.7}\mathsf{IncreasedBloodPressure}) \mathbin{\mathsf{U}} \mathsf{StopsExcercising},$$

where $\mathsf{StartsExcercising}$ and $\mathsf{StopsExcercising}$ are defined in further axioms using temporal concept operators.

Temporal DLs have been well investigated in the literature, and may extend classical DLs with LTL-operators on axioms and concepts [30, 6], with MTL-operators [3, 37, 21], Halpern and Shoham's interval logic [1, 35], or temporal attributes [33]. Similarly, several probabilistic extensions to DLs have been suggested, such as the non-monotonic DL P-$\mathcal{SHIF}(\mathcal{D})$/P-$\mathcal{SHOIN}(\mathcal{D})$ [27], the DLs Prob-$\mathcal{ALC}$/Prob-$\mathcal{EL}$ for expressing subjective probabilities [20], DLs using log-linear probabilities [32] and the Bayesian DLs $\mathcal{BEL}$ and $\mathcal{BALC}$ [15, 12]. There is also research on ontology languages that combine temporal and probabilistic aspects: these consider temporal probabilistic Datalog programs [16], dynamic Bayesian DL networks [14], and temporal extensions of DL-Lite [26], but do not consider expressive query languages, or the full expressivity of temporal DLs such as LTL-$\mathcal{ALC}$ and Prob-$\mathcal{ALC}$. There is some research on answering unions of conjunctive queries in temporal DL-Lite [2], and instance retrieval in temporal extensions of $\mathcal{EL}$ [19], but not on answering temporal queries, and to the best of our knowledge, there is no research on OBQA with ontology languages that employ probabilistic concept operators.

The aim of this paper is to theoretically investigate a setting where temporal operators, as well as operators expressing subjective probability, can be used both as part of the ontology language and as part of the query language. While some complexity bounds are still open at this point, we present initial results towards understanding the complexity in such a setting. Specifically, our contributions are the following.

1. In Section 2, we combine the languages studied in [11, 30, 20] to define the syntax and semantics of *temporal probabilistic DL formulae* (TPDFs), which generalise temporal probabilistic knowledge bases and queries.
2. In Section 3, we give tight complexity bounds for TPDFs with only temporal operators.
3. In Section 4, we give upper bounds for TPDFs with only probability operators.
4. In Section 5, we show that for TPDFs that use both temporal and probability operators, satisfiability is 2ExpSpace-hard.

Details of proofs and definitions can be found in the appendix.

2 Temporal Probabilistic Description Logic Formulae

2.1 Preliminaries

We assume basic knowledge about expressive DLs. Our results concern DLs ranging from $\mathcal{ALC}$ to $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$. Details about the DLs relevant for this paper, as well as on query answering, can be found in the appendix. We assume DL concepts to be composed using the operators of the respective DL based on the pair-wise disjoint, countably infinite sets $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ of respectively *concept names*, *role names* and *individual names*. We assume DL axioms to be either *general class inclusions* (GCIs) of the form $C \sqsubseteq D$, or *assertions* of the forms $C(a)$, $r(a, b)$ with C and D being concepts in the respective DL, r, s role names, and a, b individual names. We use $C \equiv D$ as abbreviation for the two GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$. Satisfiability of sets $\mathcal{K}$ of axioms is defined in terms of *interpretations* $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a set of domain elements and $\cdot^\mathcal{I}$ is a function mapping individual names to domain elements, concepts to subsets of $\Delta^\mathcal{I}$ and roles to subsets of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. Conjunctive queries (CQs) and entailment of Boolean CQs are defined as usual (eg., see [29]): specifically, CQs can contain free variables called *answer variables*, and a Boolean CQ is a CQ without free variables. A *query answer* to a CQ ϕ in a DL KB $\mathcal{K}$ is an assignment of individual names to the free variables in ϕ such that the resulting Boolean CQ is entailed by $\mathcal{K}$.

To distinguish between different intervals relevant in this paper, we use the notation $[i, j]$ to denote closed intervals over the reals, and the notation $[\![i, j]\!]$ to denote closed intervals over the integers. A *probability measure* over a (possibly infinite) set W is a function $P : \mathcal{W} \to [0, 1]$, where $\mathcal{W} \subseteq 2^W$ is a σ-algebra (it contains W is is closed under complement and countable union), s.t. $P(\emptyset) = 0$, $P(W) = 1$, and for any countable set $\mathcal{W}' \subseteq \mathcal{W}$ of pairwise disjoint sets $W' \subseteq W$, we have $P(\bigcup_{W' \in \mathcal{W}} W') = \sum_{W' \in \mathcal{W}} P(W')$.

2.2 Syntax

We consider extensions of classical DLs which additionally allow temporal concepts of the form $\bigcirc C$ (next) and $C \mathsf{U} D$ (until), and probabilistic concepts of the form $\mathsf{P}_{\circledast \mathsf{p}} C$, where $\circledast \in \{<, =, >\}$, $p \in [0, 1]$ and C, D are concepts. These concepts may be used at any place within a concept, and we call the resulting concepts *temporal probabilistic concepts*. Here, we do not fix a particular DL as basis, but may refer to the *underlying DL* which is extended by these operators. While classically, a DL knowledge base is build using DL axioms, against which queries are evaluated, it will be convenient to study queries and DL axioms not in separation, but to allow for an integrated language in which DL axioms and CQs can be arbitrarily mixed within a formula. This further expressivity can for instance be used to specify that a certain DL axiom holds until a Boolean CQ becomes satisfied. For this reason, we collectively call DL axioms and CQs *generalised axioms*. *Temporal probabilistic DL formulae* (TPDFs) α are then built according to the following syntax rule, where X is a generalised axiom that may use temporal probabilistic concepts, $\circledast \in \{<, =, >\}$ and $p \in [0, 1]$:

$$\alpha ::= X \mid \neg\alpha \mid \alpha \wedge \alpha \mid \bigcirc\alpha \mid \alpha \mathsf{U} \alpha \mid \mathsf{P}_{\circledast \mathsf{p}}\alpha.$$

The operators $\neg$, $\wedge$, $\bigcirc$ and U are called *temporal operators*, while the operators $\mathsf{P}_{\circledast \mathsf{p}}$ are called *probability operators*. We define further operators as the usual abbreviations, that is, for TPDFs ϕ and ψ, we denote $\mathbf{true} := \psi \vee \neg\phi$ (for some ϕ), $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\Diamond\phi := \mathbf{true}\mathsf{U}\phi$ and $\Box\phi := \neg\Diamond\neg\phi$, $\phi \to \psi := \neg\phi \vee \psi$ and $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, and similar for concepts. A TPDF is called *Boolean* if every CQ in it is Boolean.

As typical for temporal reasoning with DLs, we assume a set $\mathsf{N}_{rig} \subseteq \mathsf{N_C} \cup \mathsf{N_R}$ of *rigid names*, composed of a set $\mathsf{N}_{Crig} = \mathsf{N}_{rig} \cap \mathsf{N_C}$ of *rigid concepts* and a set $\mathsf{N}_{Rrig} = \mathsf{N}_{rig} \cap \mathsf{N_R}$ of *rigid* *roles*, which denote concept and role names whose interpretation does not change over time.

2.3 Semantics

To define the semantics of TPDFs, we have to take into consideration two dimensions: the temporal dimension and the probabilistic dimension. A *temporal interpretation* is a sequence $(\mathcal{I}_i)_{i\geq 0}$ of interpretations $\mathcal{I}_i = \langle \Delta, \cdot^{\mathcal{I}_i} \rangle$ sharing the same domain $\Delta^{\mathcal{J}}$, such that for any rigid name $X \in \mathsf{N}_{rig}$ and $i, j \geq 0$, $X^{\mathcal{I}_i} = X^{\mathcal{I}_j}$. A *probabilistic temporal interpretation* is then a probability measure $\iota : \mathfrak{J} \rightarrow [0,1]$, over a set $\mathcal{J}$ of temporal interpretations $(\mathcal{I}_i)_{i\geq 0}$ sharing the same set Δ^ι of domain elements ($\mathfrak{J} \subseteq 2^{\mathcal{J}}$ is then a sigma algebra). We call $\mathcal{J}$ the *possible worlds* of ι.

To define the semantics of temporal and probabilistic operators, we define the function $\cdot^{\mathcal{I}_i,\iota}$ on concepts, where $(\mathcal{I}_j)_{j\geq 0} \in \mathcal{J}$ and $i \geq 0$. $\cdot^{\mathcal{I}_i,\iota}$ is defined as $\cdot^{\mathcal{I}_i}$ for the concept operators of the underlying DL, and for the remaining operators by

$$\begin{aligned}
(\bigcirc C)^{\mathcal{I}_i,\iota} &= C^{\mathcal{I}_{i+1},\iota} \\
(C\mathsf{U}D)^{\mathcal{I}_i,\iota} &= \{d \in \Delta^\iota \mid \exists j \geq i : d \in D^{\mathcal{I}_i,\iota}, \forall k \in [\![i, j-1]\!] : d \in C^{\mathcal{I}_k,\iota}\} \\
(\mathsf{P}_{\circledast \mathsf{p}} C)^{\mathcal{I}_i,\iota} &= \{d \in \Delta^\iota \mid \iota(\{(\mathcal{I}'_j)_{j\geq 0} \in \mathcal{J} \mid d \in C^{\mathcal{I}'_i,\iota}\}) \circledast p\}.
\end{aligned}$$

Satisfaction of Boolean TPDFs is defined by:

1. $\mathcal{I}_i, \iota \models \alpha$ iff $\mathcal{I}_i \models \alpha$, where α is a Boolean CQ or a role assertion,
2. $\mathcal{I}_i, \iota \models C \sqsubseteq D$ iff $C^{\mathcal{I}_i,\iota} \subseteq D^{\mathcal{I}_i,\iota}$,
3. $\mathcal{I}_i, \iota \models C(a)$ iff $a^{\mathcal{I}_i} \in C^{\mathcal{I}_i,\iota}$,
4. $\mathcal{I}_i, \iota \models \bigcirc\phi$ iff $\mathcal{I}_{i+1}, \iota \models \psi$,
5. $\mathcal{I}_i, \iota \models \phi\mathsf{U}\psi$ iff there exists $j \geq i$ s.t. $\mathcal{I}_j, \iota \models \psi$ and for all $k \in [\![j, i-1]\!]$, $\mathcal{I}_k, \iota \models \phi$, and
6. $\mathcal{I}_i, \iota \models \mathsf{P}_{\circledast \mathsf{p}}\phi$ iff $\iota(\{(\mathcal{I}'_j)_{j\geq 0} \in \mathcal{J} \mid \mathcal{I}'_i, \iota \models \phi\}) \circledast p$.[1]

We say that ι satisfies a Boolean TPDF ϕ, in symbols $\iota \models \phi$, if for all $(\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}$, $\mathcal{I}_0, \iota \models \phi$, in which case ι is a *model of* ϕ. A Boolean TPDF is satisfiable iff it has a model.

The paper focusses on showing complexity bounds for Boolean TPDF satisfiability. Note that other reasoning tasks that are more related to classical OBQA can be easily reduced to TPDF satisfiability. For instance, for the problem of *temporal probabilistic query answering*, we are given a Boolean TPDF ϕ, and a non-Boolean TPDF ψ that contains only CQs and no DL axioms (a temporal probabilistic query), and we want to find an assignment of individual names to the answer variables in ψ so that the resulting TPDF is logically entailed by ϕ. This problem can be reduced to deciding the unsatisfiability of Boolean TPDFs of the form $\phi \wedge \neg\psi'$, where ψ' is obtained from ψ by replacing answer variables by individual names. As from now on, we focus on Boolean TPDFs only, we will omit the "Boolean" and just call them TPDFs in the following.

Remark. There is a subtle difference between our semantics and that of Prob-$\mathcal{ALC}$/Prob-$\mathcal{EL}$ as introduced in [20], in that we do not require the set of possible worlds to be countable. We believe that, especially if we add a temporal dimension, considering only countable sets of possible worlds is too restrictive. For instance, if we allow a domain element to arbitrarily

[1] Note that we implicitly require that $\mathfrak{J}$ contains all subsets of $2^{\mathcal{J}}$ relevant to these definitions.

switch betweeing satisfying a concept A and not satisfying it there are uncountably many possible sequences for this, each corresponding to a real number in between 0 and 1. There is no real reason why some of these sequences should be excluded. As we show in the appendix, there are TPDFs even without temporal operators that are only satisfiable in interpretations with an uncountable set of possible worlds, which means that our results do not directly transfer to the setting considered in [20].

3 Only Temporal Operators

We first consider the purely temporal case of TPDFs without probability operators. This problem has so far only been studied for *temporal queries* and *temporal DLs*, but not for the combination of both. Our first result concerns TPDFs without temporal concepts, that is, temporal operators can be used on CQs and on axioms, but not within concepts. Here, complexity upper bounds directly follow from the complexity of temporal query entailment with classical ontologies, as studied in [11, 5, 4] . Let ϕ be a TPDF of which we want to determine satisfiability. We define a set $\mathcal{T}_\phi$ of classical DL axioms that contains the following axiom for every GCI $C \sqsubseteq D$ occurring in ϕ:

$$A_{\neg(C\sqsubseteq D)} \equiv C \sqcap \neg D.$$

Instances of $A_{\neg(C\sqsubseteq D)}$ witness the non-entailment of $C \sqsubseteq D$, so that we can use the CQ $\exists x.A_{\neg(C\sqsubseteq D)}(x)$ to express that the GCI does not hold. We then define a TPDF ϕ' that is obtained from ϕ by replacing every GCI $C \sqsubseteq D$ with $\neg\exists x.A_{\neg(C\sqsubseteq D)}(x)$. ϕ' contains no GCIs, and we have $\Box \bigwedge_{\alpha\in\mathcal{T}_\phi} \alpha \models \neg\phi'$ iff ϕ is unsatisfiable. We thus get the following theorem directly from results on satisfiability of temporal CQs in [11, 5, 4].

Theorem 1. *Satisfiability of TPDFs without probability operators and temporal concepts, and with underlying DL $\mathcal{L}$, is*

- PSPACE-*complete for $\mathcal{L} = \mathcal{EL}$ and $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$,*
- EXPTIME-*complete for $\mathcal{L} \in \{\mathcal{ALC}, \mathcal{ALCQ}\}$ and $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$,*
- NEXPTIME-*complete for $\mathcal{L} \in \{\mathcal{EL}, \mathcal{ALC}, \mathcal{ALCQ}\}$ and $\mathsf{N}_{Rrig} = \emptyset$,*
- NEXPTIME-*complete $\mathcal{L} = \mathcal{EL}$ and $\mathsf{N}_{Rrig} \neq \emptyset$,*
- 2EXPTIME-*complete for $\mathcal{L} \in \{\mathcal{ALCI}, \mathcal{ALCIQ}, \mathcal{ALCOQ}, \mathcal{ALCOI}\}$, and*
- *decidable for $\mathcal{ALCOIQ}$.*

If we also allow for temporal concept operators, we have to do a bit more. We first note that with rigid roles, using temporal operators on the level of concepts leads to undecidability already if the underlying DL is $\mathcal{EL}$ [30]. We thus only have to consider the case where $\mathsf{N}_{Rrig} = \emptyset$. To show upper bounds for this case, we extend the method from [39] for temporal DLs based on quasimodels to also incorporate CQs. Namely, we abstract temporal interpretations using sequences of *quasistates*, which each contain a set of CQs and GCIs that hold or do not hold at the corresponding time point, together with a set of *concept types*, which represent the current states of domain elements.

Given a TPDF ϕ, let $\mathsf{con}(\phi)$ denote the set of (sub-)concepts occurring in ϕ, $\mathsf{form}(\phi)$ denote the set of sub-formulae of ϕ, and $\mathsf{ind}(\phi)$ denote the set of individual names occurring in ϕ. Furthermore, define $\mathsf{t_c}(\phi) = \{C, \neg C \mid C \in \mathsf{con}(\phi)\} \cup \{\{a\} \mid a \in \mathsf{ind}(\phi)\}$ and $\mathsf{t_f}(\phi) = \{\psi, \neg\psi \mid \psi \in \mathsf{form}(\phi)\}$. A *concept type* is then a subset $t \subseteq \mathsf{t_c}(\phi)$ s.t.

C1 for every $\neg C \in \mathsf{t_c}(\phi)$, $\neg C \in t$ iff $C \notin t$, and

C2 for every $C \sqcap D \in \mathsf{t_c}(\phi)$, $C \sqcap D \in t$ iff $C, D \in t$.

If a concept type t contains a concept of the form $\{a\}$, we call t a *nominal type*. A *formula type* is a subset $t \subseteq \mathsf{t_f}(\phi)$ s.t.

F1 for every $\neg\psi \in \mathsf{t_c}(\phi)$, $\neg\psi \in t$ iff $\psi \notin t$, and

F2 for every $\psi_1 \wedge \psi_2 \in \mathsf{t_c}(\phi)$, $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$.

A *quasistate* is a set S of formula and concept types s.t. S contains exactly one formula type t_S.

If the formula type only contains GCIs and their negation, there are easy syntactic conditions for when a quasistate can correspond to an element of a temporal interpretation. This becomes however more difficult when it can also contain CQs, which is why we instead formulate a semantic admissibility condition for quasistates. We first introduce the notion of a conceptual abstraction. Since quasistates will also be used in Section 4, we define them here more general for quasistates that may also contain probability operators. Given a concept or TPDF X, its *conceptual abstraction* X^{ca} is obtained by replacing every outermost concept C of the forms $\bigcirc D$, $D_1 \mathsf{U} D_2$, and $\mathsf{P}_{\circledast \mathsf{p}} D$ by the fresh concept name A_C, and every outermost TPDF ψ of the forms $\bigcirc\psi_1$, $\psi_1 \mathsf{U} \psi_2$ and $\mathsf{P}_{\circledast \mathsf{p}}\psi_1$ by $A_\psi(a)$, where A_ψ is fresh. A quasistate S is then *admissible* iff there exists a (classical) interpretation $\mathcal{I}$ s.t.

S1 for every TPDF $\alpha \in \mathsf{form}(\phi)$, $\mathcal{I} \models \alpha^{\mathsf{ca}}$ iff $\alpha \in t_S$, and

S2 for every concept type $t \subseteq \mathsf{t_c}(\phi)$, $\bigcap_{C \in t}(C^{\mathsf{ca}})^{\mathcal{I}} \neq \emptyset$ iff $t \in S$.

While a quasistate can contain up to exponentially many concept types, we show in the appendix that for $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$, it can still be decided in 2ExpTime wrt. to the input formula whether a given quasistate is admissible, while this can be done in ExpTime for $\mathcal{ALCQ}$.

It remains to represent the temporal dimension, which we do in terms of *runs* and *temporal quasimodels*.

A *concept/formula run* is a sequence $\sigma : \mathbb{N} \to \mathsf{t_c}(\phi)/\mathsf{t_f}(\phi)$ of concept/formula types s.t. for all $i \geq 0$,

R1 for every $\bigcirc\alpha \in \mathsf{t_c}(\phi)/\mathsf{t_f}(\phi)$, $\bigcirc\alpha \in \sigma(i)$ iff $\alpha \in \sigma(i+1)$,

R2 for every $\alpha \mathsf{U} \beta \in \mathsf{t_c}(\phi)/\mathsf{t_f}(\phi)$, $\alpha \mathsf{U} \beta \in \sigma(i)$ iff there exists $j \geq i$ s.t. $\beta \in \sigma(i)$ and for all $k \in [\![i, j-1]\!]$, $\alpha \in \sigma(i)$,

R3 for every $j \in \mathbb{N}$, $\sigma(i) \cap \mathsf{N}_{Crig} = \sigma(j) \cap \mathsf{N}_{Crig}$, and

R4 for every $j \in \mathbb{N}$ and $a \in \mathsf{N_I}$, $\{a\} \in \sigma(i)$ iff $\{a\} \in \sigma(j)$.

A *temporal quasimodel for* ϕ is a tuple $\langle Q, \mathfrak{R} \rangle$, where Q is a sequence mapping each natural number to an admissible quasistate $Q(i)$, and $\mathfrak{R}$ is a set of runs s.t.

Q1 $\phi \in t_{Q(0)}$,

Q2 for each $i \geq 0$ and $t \in Q(i)$, there exists a run $\sigma \in \mathfrak{R}$ s.t. $\sigma(i) = t$, and

Q3 for each run $\sigma \in \mathfrak{R}$, and $i \geq 0$, $\sigma(i) \in Q(i)$.

Temporal quasimodels witness the satisfiability of TPDFs without probability operators. Furthermore, we can use a regularity argument as in [36] to limit the shape of these quasimodels. This is summarized in the following lemma.

Lemma 1. *If the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$, then ϕ is satisfiable iff there exists a quasimodel $\langle Q, \mathfrak{R}\rangle$ for ϕ where Q is of the form*

$$Q(0)\dots Q(n)\big(Q(n+1)\dots Q(n+m)\big)^{\omega},$$

with n and m double exponentially bounded in the size of ϕ.

The proof of the lemma makes use of the fact that, in a classical DL interpretation, if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$, we can arbitrarily extend the set of domain elements that belong to a given concept type without affecting entailment of CQs or the extension of other types. This is not so easily possible for DLs that support both inverse roles and counting quantifiers, which is why we do not have results for $\mathcal{ALCIQ}$. Using lower bounds for CQ entailment in $\mathcal{ALCI}$ [28] and $\mathcal{ALCO}$ [31], and for TPDFs with temporal operators only on concepts and GCIs [39], we obtain the following completeness results.

Theorem 2. *Satisfiability of TPDFs without probability operators is undecidable if $\mathsf{N}_{Rrig} \neq \emptyset$. Otherwise, it is* 2ExpTime*-complete if the underlying DL is $\mathcal{ALCO}$, $\mathcal{ALCI}$ or $\mathcal{ALCOI}$, and* ExpSpace*-complete if the underlying DL is $\mathcal{ALC}$ or $\mathcal{ALCQ}$.*

4 Only Probability Operators

We next consider the purely probabilistic case, that is, we allow probability operators on the level of concepts, axioms and queries, but no temporal operators. While [20] consider extensions of $\mathcal{ALC}$ and $\mathcal{EL}$ with probability operators on concepts and assertions, they do not consider these operators on GCIs. We extend this setting by allowing probability operators also on GCIs, and additionally allowing CQs.

Our method for deciding entailment of those TPDFs is again based on quasistates and types, over which we this time define probability measures.

A *probabilistic quasistate* is a probability measure $PS : 2^{\mathfrak{S}} \to [0,1]$ over a set $\mathfrak{S}$ of quasistates. It is *admissible* iff for every quasistate $S \in \mathfrak{S}$:

PS1 S is admissible, and

PS2 for every $\mathsf{P}_{\circledast \mathsf{p}}\psi \in \mathsf{t}_{\mathsf{f}}(\phi)$, $\mathsf{P}_{\circledast}\psi \in t_S$ iff $PS(\{S \in \mathfrak{S} \mid \psi \in t_S\}) \circledast p$.

While every quasistate contains a set of concept types, we might need a more fine-grained probability measure for each concept type to verify the probabilistic concepts in them. For this, we define *probabilistic concept types*. A probabilistic concept type $pt : 2^T \to [0,1]$ is a probability measure over a set T of concept types s.t

PT for every $\mathsf{P}_{\circledast \mathsf{p}}C \in \mathsf{t}_{\mathsf{c}}(\phi)$ and $t \in T$, $\mathsf{P}_{\circledast \mathsf{p}}C \in t$ iff $pt(\{t \in T \mid C \in t\}) \circledast p$.

It is *compatible to a probabilistic quasistate* $PS : 2^{\mathfrak{S}} \to [0,1]$ iff there exists a probability measure $P_{PS,pt} : 2^{W_{PS,pt}} \to [0,1]$ over some set $W_{PS,pt} \subseteq \mathfrak{S} \times T$ s.t.

PC1 $\langle S, t\rangle \in W_{PS,pt}$ implies $t \in S$,

PC2 for every $S \in \mathfrak{S}$, $P_{PS,pt}(\{\langle S', t\rangle \in W_{PS,pt} \mid S' = S\}) = PS(\{S\})$, and

PC3 for every $t \in T$, $P_{PS,pt}(\{\langle S, t'\rangle \in W_{PS,pt} \mid t' = t\}) = pt(\{t\})$.

We call $P_{PS,pt}$ a *joined probability measure for PS and pt*. A probabilistic quasimodel for ϕ is now a tuple $\langle PS, \mathfrak{PT}\rangle$ of a probabilistic quasistate $PS : 2^{\mathfrak{S}} \to [0, 1]$ and a set $\mathfrak{PT}$ of probabilistic concept types s.t.

PQ1 for every $S \in \mathfrak{S}$, $\phi \in t_S$,

PQ2 every probabilistic concept type $pt \in \mathfrak{PT}$ is compatible to PS, and

PQ3 for every quasistate $S \in \mathfrak{S}$ and concept type $t \in S$, there exists a joined probability measure for PS and some $pt \in \mathfrak{PT}$ s.t. $\langle S, t\rangle \in W_{PS,pt}$.

Note that in Condition **PQ3**, we only require $\langle S, t\rangle \in W_{PS,pt}$, but not $P_{PS,pt}(\{\langle S, t\rangle\}) > 0$. This is still sufficient to ensure that the type t can be instantiated in every possible world corresponding to S, and in fact necessary to ensure completeness, because we allow for uncountable sets of possible worlds in our semantics.

Lemma 2. *A TPDF ϕ without temporal operators, with underlying DL $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$, is satisfiable iff there exists a probabilistic quasimodel $\langle PT, \mathfrak{PT}\rangle$ for ϕ.*

Probabilistic quasimodels are similar to temporal quasimodels, where instead of sequences, we use probability measures. For some DLs, this difference in structure can be exploited to gain better complexity bounds. While there can be in general double exponentially many quasistates and probabilistic concept types, if the underlying DL is $\mathcal{ALCQ}$, only exponentially many of each are needed. In contrast to the temporal quasimodels in Section 3, which indeed may always require a double exponential number of quasistates, probabilistic quasimodels benefit from a lack of *order*: this allows us to merge quasistates that agree on their formula type and nominal types, which is the reason why we can bound the size of probabilistic quasimodels for $\mathcal{ALCQ}$.

Our decision procedure for TPDF satisfiability consists of guessing and verifying a probabilistic quasimodel of the appropriate size. Here, we make use of a result from [17], which is also used in [20] to provide the complexity bounds of Prob-$\mathcal{ALC}$, to limit the required precision used in the probability measures.

Theorem 3. *Satisfiability of TPDFs without temporal operators is in* NExpTime *if the underlying DL is $\mathcal{ALCQ}$, and in* N2ExpTime *if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$.*

Since satisfiability of Prob-$\mathcal{ALC}$ is still in ExpTime [20], the only known complexity lower bounds stem from the complexity of Boolean query entailment. We leave it as future work to investigate whether our complexity bounds can be optimised.

5 Temporal and Probability Operators

If we allow both temporal and probability operators, satisfiability of TPDFs becomes 2ExpSpace-hard, even if we disallow rigid names. We show this by a reduction of the double-exponential corridor tiling problem. This problem is formalised as follows. We are given a set T of tiles containing an *initial tile type* $t_0 \in T$ and a *final tile type* $t_f \in T$, two sets $H \subseteq T \times T$ and $V \subseteq T \times T$ of respectively *horizontal* and *vertical tiling conditions*, and a natural number n. The problem is then to decide whether there exists a natural number m and a *tiling* $t : [\![1, 2^{2^n}]\!] \times [\![1, m]\!] \to T$ s.t. $t(1, 1) = t_0$, $t(1, m) = t_f$, for every $i \in [\![1, 2^{2^n} - 1]\!]$

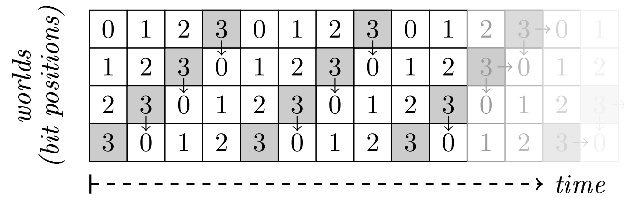

Figure 1: Illustration of how counters are used to identify neighbouring possible worlds.

and $j \in [\![1, m]\!]$, we have $\langle t(i, j), t(i + 1, j)\rangle \in H$, and for every $i \in [\![1, 2^{2^n}]\!]$ and $j \in [\![1, m - 1]\!]$, we have $\langle t(i, j), t(i, j + 1)\rangle \in V$. It follows from the relationship between corridor-tilings and space-bounded Turing machines shown in [38] that the double-exponential tiling problem is 2ExpSpace-complete.

While the full reduction is shown in the appendix, we sketch the main ideas here. We use 2^{2^n} domain elements to represent the vertical dimension of the tiling, and the time line to represent the horizontal dimension. The probabilistic dimension is used to implement a double exponential counter on each domain element, which is used to identify which row of the tiling it represents. Here, we use temporal and probabilistic concepts to force the existence of exponentially many possible worlds per domain element, which at each time point store the different bit values of the double exponential counter using a concept Bit. Specifically, the individual satisfies Bit in the ith possible world iff the ith bit of the double exponential counter has the value 1.

A main challenge in the construction is the lack of order in temporal probabilistic models. The set of possible worlds in an interpretation is unordered, which means we cannot directly refer to the "ith" or "next" possible world. This is however neccessary to implement a double exponential counter, since we have to transfer information about carrier bits from one possible world to another. Furthermore, since we do not allow rigid roles, we cannot keep the relationship between the different domain elements stable throughout the time line. As a result, we cannot directly refer to the domain element that refers to the next row in order to test the vertical tiling conditions. For both challenges, we use a similar trick.

For the double-exponential counter, we need to be able to identify possible worlds for the respective domain element that correspond to neighbouring bit positions. To do this, we implement a single-exponential counter in each possible world, which is incremented along the time line, so that in each world, the counter has a different value. This is visualised in Figure 1. To implement these counters, we use concept names $A_1, \ldots, A_n$ representing the bit value at the positions 1 to n of this counter. At each time point, two neighbouring possible worlds can be identified easily: the one with a counter value of $2^n - 1$ satisfies $\bigsqcap_{i \in [\![1,n]\!]} A_i$, and unless it corresponds to the last bit position, the next bit position corresponds to the world with a counter value of 0, which satisfies $\bigsqcap_{i \in [\![1,n]\!]} \neg A_i$. Using this mechanism, we can for instance transfer the information on whether the current bit has to be flipped using the following GCIs:

$$\bigsqcap_{i \in [\![1,n]\!]} A_i \sqcap \mathsf{Flip} \sqcap \mathsf{Bit} \sqsubseteq \mathsf{P}_{=1}((\bigsqcap_{i \in [\![1,n]\!]} \neg A_i) \to \mathsf{Flip})$$

$$(\bigsqcap_{i \in [\![1,n]\!]} A_i) \sqcap (\neg\mathsf{Flip} \sqcup \neg\mathsf{Bit}) \sqsubseteq \mathsf{P}_{=1}((\bigsqcap_{i \in [\![1,n]\!]} \neg A_i) \to (\mathsf{FirstBit} \sqcup \neg\mathsf{Flip})).$$

Using further axioms, this allows us to implement a double exponential counter on each domain element, which is incremented every 2^n time points.

The same technique is used on a different level to identify which domain elements correspond to neighbouring rows in the ceiling. We make sure that eventually, we have at each time point a different double exponential counter value represented by some domain element. At each time point, we can then identify two neighbouring domain elements easily: the one with a counter value of 0 satisfies $\mathsf{P}_{=1}\neg\mathsf{Bit}$, and the one with a counter value of $2^{2^n} - 1$ satisfies $\mathsf{P}_{=1}\mathsf{Bit}$. We can thus test the vertical tiling conditions with the following axiom:

$$\Box \bigwedge_{t \in T} \Big(\neg (t \sqcap \mathsf{P}_{=1}\mathsf{Bit} \sqsubseteq \bot) \rightarrow \bigvee_{\langle t, t' \rangle \in V} (\mathsf{P}_{=1}\neg\mathsf{Bit} \sqsubseteq t') \Big).$$

The reduction allows us to establish the following theorem.

Theorem 4. *Satisfiability of TPDFs is* 2ExpSpace*-hard. This already holds if*

- *no CQs are used,*
- *the underlying DL is* $\mathcal{ALC}$*,*
- *probabilistic operators are only used on the level of concepts,*
- $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$*, and*
- *on the axiom level, we only use Boolean connectives and the operator* $\Box$*, which does not occur under a negation operator.*

6 Conclusion

In the context of description logics, temporal and probabilistic extensions have mostly been investigated in isolation, and similarly, such extensions on DL languages and query languages have not been investigated in combination. In this paper, we presented several results towards filling these gaps. First, we showed tight complexity bounds for a setting where temporal operators are used on the level of axioms and queries, as well as on queries, axioms and concepts in combination, showing that the overall complexity does not increase by such a combination for any DL between $\mathcal{ALC}$ and $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$. Second, we considered the setting where probability operators may be used on the level of concepts, axioms and queries, obtaining an NExpTime upper bound if the underlying DL is $\mathcal{ALCQ}$, and an N2ExpTime upper bound if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$. Finally, we showed that the combination of both temporal and probabilistic operators on concepts and axioms results in 2ExpSpace-hardness. We believe that it might be possible to obtain matching upper bounds by a combination of the structures we used in this paper to show our upper bound.

While temporal ABoxes can be easily encoded into TPDFs, our results do not generalise the settings with probabilistic ABoxes studied in [22], or in the work in [24] on temporal probabilistic query answering, since these works assume the probability measure on the possible worlds to be fixed, which is not the case with our semantics. We believe however that extending to such settings does not have an impact on the complexity, as our languages are all already ExpTime-hard. Another possible direction is investigating special operators that are both temporal and probabilistic in nature, such as the probabilistic diamond-operator introduced in [25, 26].

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A Temporal Probabilistic Description Logics

A.1 Preliminaries

We recall the DLs studied in the paper, as well as conjunctive query answering, in detail.

Description Logics. Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be pair-wise countably infinite sets of respectively *concept names*, *role names* and *individual names*. A *role* is an expression of the forms r, r^-, where $r \in \mathsf{N_R}$. *Concepts* are of the following forms, where $A \in \mathsf{N_C}$, R is a role, C, D are concepts, $n \in \mathbb{N}$ and $a \in \mathsf{N_I}$:

$$A \mid C \sqcap D \mid \exists R.C \mid \forall R.C \mid {\geq} nR.C \mid \{a\}.$$

A *knowledge base* (KB) is a set of *DL axioms*, which are either *GCIs* of the form $C \sqsubseteq D$, where C, D are concepts, or *assertions* of the forms $A(a)$ and $r(a,b)$, $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$.

Different DLs are differentiated based on the operators allowed: $\mathcal{EL}$ only supports concepts of the form A, $C \sqcap D$ and $\exists R.C$ and axioms of the form $C \sqsubseteq D$, and no roles of the form r^-. $\mathcal{ALC}$ extends $\mathcal{EL}$ with concepts of the form $\neg C$. More expressive DLs are denoted by attaching a letter to the DL, where we use $\mathcal{I}$ for support of roles r^-, $\mathcal{O}$ for concepts of the form $\{a\}$, $\mathcal{Q}$ for concepts of the form ${\geq}nR.C$. For example, $\mathcal{ALCI}$ extends $\mathcal{ALC}$ with inverse roles, whereas $\mathcal{ALCOQ}$ extends $\mathcal{ALC}$ with concepts of the form $\{a\}$ and ${\geq}nR.C$.

The semantics of KBs is defined in terms of *interpretations* $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a set of *domain elements* and $\cdot^\mathcal{I}$ maps each concept name $A \in \mathsf{N_C}$ to a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, each role name $r \in \mathsf{N_R}$ to a relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, each individual name $a \in \mathsf{N_I}$ to a domain element $a^\mathcal{I} \in \Delta^\mathcal{I}$, and each role r^- to $(r^-)^\mathcal{I} = (r^\mathcal{I})^-$. It is extended to concepts as follows.

$$(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}, \qquad (\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}, \qquad \{a\}^\mathcal{I} = \{a^\mathcal{I}\},$$
$$(\exists R.C)^\mathcal{I} = \{d \in \mathcal{I} \mid \exists e \in \Delta^\mathcal{I} : \langle d, e \rangle \in R^\mathcal{I}, e \in C^\mathcal{I}\},$$
$$(\geq nR.C)^\mathcal{I} = \{d \in \mathcal{I} \mid \#\{e \in \Delta^\mathcal{I} \mid \langle d, e \rangle \in R^\mathcal{I}, e \in C^\mathcal{I}\} \geq n\}$$

We say that an interpretation $\mathcal{I}$ *satisfies* an axiom/assertion α, in symbols $\mathcal{I} \models \alpha$, if $\alpha = C \sqsubseteq D$ and $C^\mathcal{I} \subseteq D^\mathcal{I}$; $\alpha = A(a)$ and $a^\mathcal{I} \in A^\mathcal{I}$; and $\alpha = r(a,b)$ and $\langle a^\mathcal{I}, b^\mathcal{I} \rangle \in r^\mathcal{I}$. $\mathcal{I}$ is a *model* of a KB iff it satisfies all axioms in it. Finally, a KB $\mathcal{K}$ *entails* an DL axiom α iff α is satisfied by every model of $\mathcal{K}$.

Conjunctive Queries. A *conjunctive query* (CQ) takes the form $q = \exists \vec{y}.\phi(\vec{x}, \vec{y})$, where $\vec{x}$, $\vec{y}$ are vectors of variables and $\phi(\vec{x}, \vec{y})$ is a conjunction over atoms of the forms $A(t_1)$ and $r(t_1, t_2)$, where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and t_1 and t_2 are *terms* taken from $\mathsf{N_I}$, $\vec{x}$ and $\vec{y}$. $\vec{x}$ are the *answer variables* of q. Given an interpretation $\mathcal{I}$ and a CQ q with answer variables $x_1, \ldots, x_n$, the vector $a_1 \ldots a_n \subseteq {\mathsf{N_I}}^n$ is an *answer of* q *in* $\mathcal{I}$ if there exists a mapping $\pi : \mathsf{term}(q) \to \Delta^\mathcal{I}$ s.t. $\pi(x_i) = a_i$ for $i \in [\![1, n]\!]$, $\pi(b) = b^\mathcal{I}$ for $b \in \mathsf{N_I}$, $\pi(t) \in A^\mathcal{I}$ for every $A(t)$ in q, and $\langle \pi(t_1), \pi(t_2) \rangle \in r^\mathcal{I}$ for every $r(t_1, t_2)$ in q. A vector $a_1 \ldots a_n$ is a *certain answer* of q in a KB $\mathcal{K}$ if it is an answer in every model of $\mathcal{K}$. If a query does not contain any answer variables, it is a *Boolean CQ*, and we say it is *entailed* by a KB $\mathcal{K}$ (interpretation $\mathcal{I}$) if it has the empty vector as answer.

A.2 Semantics

We prove the claim made in the remark after the definition of the semantics of TPDFs.

Theorem 5. *There is a TPDF of the form $C \sqsubseteq D$ without temporal operators which is satisfiable only in interpretations that have an uncountable set of possible worlds and an uncountable domain.*

Proof. The TPDF in question is

$$\phi = \top \sqsubseteq \mathsf{P}_{=0}A \sqcap \exists r.A.$$

We first show that it is satisfiable with an interpretation that has an uncountable domain and an uncountable set of possible worlds. We then show that it cannot be satisfied by an interpretation in which both are countable.

We first define the model $\iota : \mathfrak{J} \to [0,1]$ of ϕ, where $\mathfrak{J} \subseteq 2^{\mathcal{J}}$. The domain Δ is defined as $\Delta = [0,1]$, that is, we have a domain element for every real number between 0 and 1. Note that this set is uncountable. Next, we define the possible worlds in $\mathcal{J}$. Since ϕ contains no temporal operators, it is sufficient to focus on the first interpretation $\mathcal{I}_0$ in each sequence $(\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}$. We use one such interpretation $\mathcal{I}_q$ for each $q \in [0,1]$, on which the interpretation function $\cdot^{\mathcal{I}_q}$ is then defined by:

$$\begin{aligned} A^{\mathcal{I}_q} &= \{q\} \\ r^{\mathcal{I}_q} &= \{\langle q', q\rangle \mid q' \in [0,1]\}. \end{aligned}$$

Clearly, every interpretation satisfies $\top \sqsubseteq \exists r.A$.

ι is now defined based on the Lebesgue measure on $[0,1]$, and intuitively corresponds to a uniform probability distribution over $[0,1]$. Specifically, $\mathfrak{J} \subseteq 2^{\mathcal{J}}$ is the smallest set containing the set $\{\mathcal{I}_q \mid q \in [i,j]\}$ for every interval $[i,j] \subseteq [0,1]$, and that is closed under complement and countable union. ι is defined such that it satisfies

$$\iota(\{\mathcal{I}_q \mid q \in [i,j]\}) = j - i,$$

and is extended to $\mathfrak{J}$ so that is satisfies the properties of a probability measure. It is not hard to see that for every domain element $d \in \Delta$,

$$\iota(\{\mathcal{I}_q \in \mathcal{J} \mid d \in A^{\mathcal{I}_q}\}) = 0,$$

and therefore $d \in (\mathsf{P}_{=0}A)^{\mathcal{I}_q,\iota}$ for every $\mathcal{I}_q \in \mathcal{J}$. It follows that ι is a model of ϕ.

It remains to show that ϕ is not satisfiable by interpretations $\iota : \mathfrak{J} \to [0,1]$ in which $\mathfrak{J}$ or the common domain Δ are countable. Let $\iota : \mathfrak{J} \to [0,1]$ be a probabilistic temporal model of ϕ, where $\mathfrak{J} \subseteq 2^{\mathcal{J}}$. In every interpretation $(\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}$, there must exist some domain element $d \in \Delta$ s.t. $d \in A^{\mathcal{I}_0}$. Since each $d \in \Delta$ satisfies $\mathsf{P}_{=0}(A)$, we have $\iota(\{(\mathcal{I}_i)_{i\geq 0}\}) = 0$ for every interpretation $(\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}$. By the definition of probability measure spaces, we have for every countable set $\mathfrak{J}' \subseteq \mathcal{J}$ of pairwise disjoint subsets of $\mathcal{J}$ that

$$\iota\Big(\bigcup_{\mathcal{J}' \in \mathfrak{J}'} \mathcal{J}'\Big) = \sum_{\mathcal{J}' \in \mathfrak{J}'} \iota(\mathcal{J}').$$

Set

$$\mathfrak{J}' = \{\{(\mathcal{I}_i)_{i\geq 0}\} \mid (\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}\}.$$

If $\mathfrak{J}$ is countable, then so is $\mathfrak{J}'$, and we obtain $\iota(\mathcal{J}) = 0$, which contradicts $\iota(\mathcal{J}) = 1$. As a consequence, ι cannot be a probability measure if $\mathcal{J}$ is countable.

We obtain that ϕ is only satisfiable in models with an uncountable set of possible worlds. This further implies that ϕ is only satisfiable in models with an uncountable domain: otherwise, since the number of concept and role names occurring in ϕ is finite, we could always find a model that also has a countable set of possible worlds. $\square$

B Only Temporal Operators

Theorem 1. *Satisfiability of TPDFs without probability operators and temporal concepts, and with underlying DL $\mathcal{L}$, is*

- PSpace-*complete for $\mathcal{L} = \mathcal{EL}$ and $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$,*
- ExpTime-*complete for $\mathcal{L} \in \{\mathcal{ALC}, \mathcal{ALCQ}\}$ and $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$,*
- NExpTime-*complete for $\mathcal{L} \in \{\mathcal{EL}, \mathcal{ALC}, \mathcal{ALCQ}\}$ and $\mathsf{N}_{Rrig} = \emptyset$,*
- NExpTime-*complete $\mathcal{L} = \mathcal{EL}$ and $\mathsf{N}_{Rrig} \neq \emptyset$,*
- 2ExpTime-*complete for $\mathcal{L} \in \{\mathcal{ALCI}, \mathcal{ALCIQ}, \mathcal{ALCOQ}, \mathcal{ALCOI}\}$, and*
- *decidable for $\mathcal{ALCOIQ}$.*

Proof. Let ϕ be a TPDF of which we want to determine satisfiability. We define a set $\mathcal{T}_\phi$ of classical DL axioms that contains the following axiom for every GCI $C \sqsubseteq D$ occurring in ϕ:

$$A_{\neg(C \sqsubseteq D)} \equiv C \sqcap \neg D.$$

Instances of $A_{\neg(C \sqsubseteq D)}$ witness the non-entailment of $C \sqsubseteq D$, so that we can use the CQ $\exists x.A_{\neg(C \sqsubseteq D)}(x)$ to express that the GCI does not hold. We then define a TPDF ϕ' that is obtained from ϕ by replacing every GCI $C \sqsubseteq D$ with $\neg\exists x.A_{\neg(C \sqsubseteq D)}(x)$. ϕ' contains no GCIs, and we have $\Box \bigwedge_{\alpha \in \mathcal{T}_\phi} \alpha \models \neg\phi'$ iff ϕ is unsatisfiable. The theorem thus directly follows from results on satisfiability of temporal CQs in [11, 5, 4]. □

Before we prove Lemma 1, we show that a TPDF ϕ without probability operators is satisfiable iff there exists a quasimodel verifying it. We prove both directions of this statement in separate lemmas. In the following, we assume the underlying DL is included in $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$.

Lemma 3. *If a TPDF ϕ without probability operators is satisfiable, then there exists a quasimodel $\langle Q, \mathfrak{R} \rangle$ for it.*

Proof. Assume that ϕ is satisfiable. There then exists a model model $\iota : \mathfrak{J} \to [0, 1]$ of ϕ, where $\mathfrak{J} \subseteq 2^{\mathcal{J}}$. Since ϕ does not contain any probability operators, we can assume wlog. that $\mathcal{J}$ contains exactly one sequence $(\mathcal{I}_i)_{i>0}$ of classical interpretations. We construct a quasimodel $\langle Q, \mathfrak{R} \rangle$ based on this sequence. For a classical interpretation $\mathcal{I}_i$ and a domain element d, we denote by $\mathsf{type}(\mathcal{I}_i, d) = \{C \in \mathsf{t_c}(\phi) \mid d \in C^{\mathcal{I}_i, \iota}\}$ the *concept type of d in $\mathcal{I}_i$*, and by $\mathsf{type}(\mathcal{I}_i) = \{\psi \in \mathsf{t_f}(\phi) \mid \mathcal{I}_i, \iota \models \psi\}$ the *formula type of $\mathcal{I}_i$*.

We define the sequence Q of quasistates by setting for all $i \in \mathbb{N}$:

$$Q(i) = \{\mathsf{type}(\mathcal{I}_i)\} \cup \{\mathsf{type}(\mathcal{I}_i, d) \mid d \in \Delta\}.$$

It follows directly from construction that every $Q(i)$ is admissible, and that $\phi \in t_{Q(0)}$ (the witnessing interpretation $\mathcal{I}'$ is obtained from $\mathcal{I}_i$ by setting $A_C^{\mathcal{I}'} = C^{\mathcal{I}_i, \iota}$, where C is a temporal concept and A_C is the conceptual abstraction of C).

We associate to every $d \in \Delta$ a run σ_d defined by $\sigma_d(i) = \{\mathsf{type}(\mathcal{I}_i, d)\}$. By checking the Conditions **R1**–**R4** and comparing to the definition of the semantics of the temporal operators, one sees that $\sigma_d(i)$ is indeed a run for every $d \in \Delta$. The set $\mathfrak{R}$ of runs is then defined as $\mathfrak{R} = \{\sigma_d \mid d \in \Delta\}$.

It follows directly from the construction that $\langle Q, \mathfrak{R} \rangle$ satisfies Conditions **Q1** – **Q3**, and is thus a quasimodel. □

Before we show how to construct an interpretation based on a quasimodel, we need a small auxiliary lemma concerning the DLs $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$, which will also be used in the proofs for Section 4.

For a concept type $t \subseteq \mathsf{t_c}(\phi)$, a probabilistic temporal interpretation $\iota : 2^{\mathcal{J}} \to [0,1]$ with domain Δ, a possible world $(\mathcal{I}_i)_{i\geq 0} \in \mathcal{J}$ and $i \geq 0$, define

$$t^{\mathcal{I}_i,\iota} = \bigcap_{C \in t} C^{\mathcal{I}_i,\iota}$$

as the *extension of* t in $\mathcal{I}_i, \iota$.

Furthermore, we define the extension of t in a classical DL interpretation $\mathcal{I}$ as

$$t^{\mathcal{I}} = \bigcap_{C \in t} (C^{\mathsf{ca}})^{\mathcal{I}}.$$

Lemma 4. *Let t be a concept type with $t \cap \mathsf{N_I} = \emptyset$, and $\mathcal{I}$ an interpretation with $t^{\mathcal{I}} \neq \emptyset$. Then, there exists an interpretation $\mathcal{I}'$ s.t. $t^{\mathcal{I}'} = t^{\mathcal{I}} \cup \{d\}$ for some fresh domain element d, for every type $t_2 \neq t$, $t_2^{\mathcal{I}'} = t_2^{\mathcal{I}}$, and for every CQ q, $\mathcal{I} \models q$ iff $\mathcal{I}' \models q$.*

Proof. For $\mathcal{ALCOI}$, we can duplicate some domain element $d \in t^{\mathcal{I}}$ to a new fresh element d_2 that satisfies the same concept names and has the same role successors and role predecessors. For $\mathcal{ALCOQ}$, we duplicate $d \in t^{\mathcal{I}}$ to a new domain element d_2 that satisfies the same concept names and has the same role successors. It is then not hard to see that the resulting interpretation satisfies the same CQs as the initial one, that $d_2 \in t^{\mathcal{I}}$, and that the extensions of the other types remain unchanged. □

Lemma 5. *If there exists a quasimodel $\langle Q, \mathfrak{R} \rangle$ for ϕ, then ϕ is satisfiable.*

Proof. Given a quasimodel $\langle Q, \mathfrak{R} \rangle$ that verifies a temporal formula ϕ, we construct a temporal interpretation $(\mathcal{I}_i)_{i>0}$ such that for the probabilistic temporal interpretation $\iota : \{\emptyset, \{(\mathcal{I}_i)_{i\geq 0}\}\} \to [0,1]$ with $\iota(\{(\mathcal{I}_i)_i\}) = 1$, we have $\mathcal{I}_0, \iota \models \phi$.

Since every quasistate $Q(i)$ is admissible, there is a sequence $\mathcal{I}_1, \mathcal{I}_2 \ldots$ of interpretations that witness the admissibility of the quasistates $Q(1), Q(2), \ldots$. Since $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$ have the finite model property, we can assume wlog. that every such interpretation has finitely many elements.

Note that for every named individual $a \in \mathsf{N_I}$ and $i > 0$, $\mathcal{I}_i$ contains exactly one domain element $d = a^{\mathcal{I}}$, which as a consequence implies that each quasistate $Q(i)$ contains at most one type t s.t. $\{a\} \in t$. Furthermore, by Condition **R4**, if there exists some type $t \in Q(i)$ s.t. $\{a\} \in t$, then there is such a type in every $Q(j)$, $j > 0$.

We extend these interpretation $\mathcal{I}_i$ such that the resulting sequence is a model of ϕ. We first define the domain Δ. For every individual name a occuring in ϕ, Δ contains a domain element d_a. Let n be the maximal number of domain elements that occur in the extension $t^{\mathcal{I}_i}$ of ant type t in any interpretation $\mathcal{I}_i$ in our sequence. For every run $\sigma \in \mathfrak{R}$ that is neither a formula run nor contains any nominal runs, Δ contains n domain elements $d_{\sigma,i}$, where $i \in [\![1, n]\!]$. This concludes the definition of Δ.

For each interpretation $\mathcal{I}_i$, by successive application of Lemma 4 and renaming of domain elements, we can transform $\mathcal{I}_i$ into an interpretation such that for every domain element $d_{\sigma,j}$, $j \in [\![1, n]\!]$, $d_{\sigma,j} \in \sigma(i)^{\mathcal{I}'_i}$, and that furthermore, for the each run σ s.t. $\{a\} \in \sigma(i)$, $a^{\mathcal{I}'} = d_a$. The final temporal interpretation is then the sequence $\mathcal{J} = \langle \mathcal{I}'_i \rangle_{i>0}$ of all so obtained interpretations. It can now be shown by structual induction on the concept operators, by comparing Conditions **R1**–**R4** with the semantics of temporal operators, that for every domain element $d \in \Delta$, every interpretation $\mathcal{I}'_i$, every temporal concept C, $d \in A_C^{\mathcal{I}_i}$ iff $d \in C^{\mathcal{I}_i,\iota}$, and for every concept, $d \in (C^{\mathsf{ca}})^{\mathcal{I}_i}$ iff $d \in C^{\mathcal{I}_i,\iota}$. Similarly, we can show by induction over the structure of ϕ and using $\phi \in t_{Q(0)}$, that $\iota \models \phi$. Hence, ϕ is satisfiable. □

Lemma 1. *If the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$, then ϕ is satisfiable iff there exists a quasimodel $\langle Q, \mathfrak{R}\rangle$ for ϕ where Q is of the form*

$$Q(0)\dots Q(n)\big(Q(n+1)\dots Q(n+m)\big)^{\omega},$$

with n and m double exponentially bounded in the size of ϕ.

Proof. Since the other direction directly follows from Lemma 5, we only need to prove that, if a TPDF ϕ without probability operators is satisfiable, then there exists a quasimodel as required by the lemma. Assume therefore that ϕ is satisfiable. By Lemma 3, there then exists a quasimodel $\langle Q_0, \mathfrak{R}_0\rangle$ for ϕ, which we step-wise transform into the required form.

Central for our construction is the following claim.

Claim 1. Let $\langle Q, \mathfrak{R}\rangle$ be a quasimodel for ϕ and $i, j \geq 0$ be such that $Q(i) = Q(j)$. Then, there exists a quasimodel $\langle Q', \mathfrak{R}'\rangle$ for ϕ in which Q' is of the following form:

$$Q(0), \dots Q(i), Q(j+1), \dots$$

Proof of claim. By Condition **Q3**, for every $t \in Q(i)$, there exists a run $\sigma \in \mathfrak{R}$ with $\sigma(i) = t$, and a run $\sigma' \in \mathfrak{R}$ with $\sigma'(j) = t$. We can thus define $\mathfrak{R}'$ by setting:

$$\mathfrak{R}' = \{\sigma(0), \dots, \sigma(i), \sigma'(j+1), \dots \mid \sigma, \sigma' \in \mathfrak{R}, \sigma(i) = \sigma(j)\}.$$

every sequence in $\mathfrak{R}'$ satisfies Conditions **R1**–**R4**, and $\langle Q', \mathfrak{R}'\rangle$ satisfies Conditions **Q1**–**Q3**. This finishes the proof of the claim.

There can be at most double exponentially many different quasistates, since each quasistate is a set of types, and there are only exponentially many different types. It follows that for some $i \geq 0$, for every $j > i$ there are infinitely many $k > j$ s.t. $Q_0(i) = Q_0(k)$. Based on this, we can find two values $0 \leq n \leq m$ s.t. $Q_0(n) = Q_0(m)$ and for every run $\sigma \in \mathfrak{R}_0$,

1. $\sigma(n) = \sigma(m)$, and
2. for every run $\alpha \mathsf{U} \beta \in \sigma(n)$, $\beta \in \sigma(k)$ for some $k \in [\![n, m]\!]$.

Such indices n and m can be found as follows: starting from some quasistate $Q_0(n)$ that occurs infinitely often, initialise $m = n$; then iterate over all runs $\sigma \in \mathfrak{R}$ and check whether Condition 2 is satisfied for σ and the current value m, and otherwise replace m by the next value $m' > m$ s.t. $Q_0(m') = Q_0(m)$. Note that once Condition 2 is satisfied for some run σ and $j > i$, it is satisfied for all runs σ' with $\sigma(k) = \sigma'(k)$ for all $k \in [i, j]$, and for all larger indices $m' > m$, so that this procedure finally gives us the required indices.

To keep track of the satisfaction of the until formulae between n and m, we mark the runs with fresh concept, obtaining the set $\mathfrak{R}_1$ of new runs. For every run σ and $\alpha \mathsf{U} \beta \in \sigma(n)$, there is a smallest number $k \in [\![n, m]\!]$ with $\beta \in \sigma(k)$. If σ is a concept run in $\mathfrak{R}_0$, we add to $\mathfrak{R}_1$ the run σ' obtained from σ by adding the fresh concept name $B_{C\mathsf{U}D}$ to every element $\sigma(l)$ where $l \leq k$, and $\neg B_{C\mathsf{U}D}$ to every element $\sigma(l)$, where $l > k$. We do the same for every formula run σ and $\psi_1 \mathsf{U} \psi_2 \in \sigma(n)$, where we use the formula $B_{\psi_1\mathsf{U}\psi_2}(a)$ instead. The sequence Q_1 of quasistates is obtained from Q_0 by extending the types accordingly, so that Condition **Q3** is still satisfied. Note that in the resulting quasimodel, for every type $t \in Q_1(m)$ contains the fresh concept names only in negated form, and $Q_1(n)$ contains the type obtained from t by adding for every $\alpha \mathsf{U} \beta \in t$ the corresponding marker.

We now apply Claim 1 to "merge" any repeated quasistates between $Q_1(0)$ and $Q_1(n)$, and between $Q_1(n)$ and $Q_1(m)$, resulting in a new quasistate $\langle Q_2, \mathfrak{R}_2 \rangle$ and two new indices n' and m' s.t. $Q_2(n') = Q_1(n)$ and $Q_2(m') = Q_1(m)$. Note that our fresh concept names make sure that each merging step preserves Conditions 1 (modulo the fresh symbols) and 2, so that $\langle Q_2, \mathfrak{R}_2 \rangle$ satisfies these conditions as well, albeit for the indices n' and m'. Furthermore, since Q_2 still contains at most double exponentially many quasistates, and no quasistate occurs twice between $Q(0)$ and $Q(n)$, or twice between $Q(n)$ and $Q(m)$, n' and m' are double exponentially bounded. To obtain a quasimodel for our input formula ϕ, we remove all occurences of the fresh concept names again and result in a quasimodel $\langle Q_3, \mathfrak{R}_3 \rangle$ which satisfies Conditions 1 and 2 for the indices n' and m'. We can now use a similar argument as we used to prove Claim 1 to show that there exists a quasimodel $\langle Q, \mathfrak{R} \rangle$ in which Q is of the form

$$Q_3(0), \ldots, Q_3(n')\big(Q_3(n'+1), \ldots, Q_3(m')\big)^{\omega},$$

which is now of the form as required. □

Next, we establish the complexity of deciding admissibility of quasistates.

Lemma 6. *If the underlying DL is $\mathcal{ALCOI}$ or $\mathcal{ALCOQ}$, whether a given quasistate S is admissible can be decided in time double-exponential in the size of the input formula ϕ. If the underlying DL is $\mathcal{ALCQ}$, it can be decided in time single-exponential in the size of the input formula ϕ.*

Proof. Note hat S contains at most exponentially many elements, as there can be at most exponentially many types. We reduce our problem to a query entailment problem of the form $\mathcal{K} \not\models Q$, where $\mathcal{K}$ is a set of DL axioms and Q is a disjunction of CQs. $\mathcal{K}$ is the union of two KBs, $\mathcal{K}_1$ and $\mathcal{K}_2$. $\mathcal{K}_1$ is defined based on the formula type t_S, and contains the following axioms:

1. for every DL axiom $\alpha \in t_S$, it contains α^{ca},
2. for every $\exists \vec{x}.q(\vec{x}) \in t_S$, the set of assertions $q(\vec{a})$ obtained from $q(\vec{x})$ by replacing each variable by a fresh individual name, and
3. for every $\neg(C \sqsubseteq D) \in t_S$, it contains the assertion $(C^{\mathsf{ca}} \sqcap \neg D^{\mathsf{ca}})(a)$, where a is fresh.

In models of $\mathcal{K}_1$, the conceptual abstractions of every GCI, negated GCI and CQ in t_T are satisfied. $\mathcal{K}_1$ contains polynomially many elements, since the formula type t_S contains one element per generalised axiom occurring in ϕ.

$\mathcal{K}_2$ is defined based on the concept types in S, and contains

1. for every concept type $t \in S$ that is not a nominal type, the assertion $(\bigsqcap_{C \in t} C^{\mathsf{ca}})(a)$, where a is a fresh individual name,
2. for every nominal type $t \in S$, where $\{a\} \in t$, the assertion $(\bigsqcap_{C \in t \setminus \{a\}} C^{\mathsf{ca}})(a)$, and
3. for every concept type $t \subseteq \mathsf{t_c}(\phi)$ s.t. $t \notin S$ and t is not a nominal type, the GCI $\bigsqcap_{C \in t} C^{\mathsf{ca}} \sqsubseteq \bot$.

In models $\mathcal{I}$ of $\mathcal{K}_2$, there is an a domain element for every type $t \in S$ that satisfies the conceptual abstractions of the concepts in t. Furthermore, for every concept type t not in S, there is no domain element in $\mathcal{I}$ that satisfies all of the conceptual abstractions of the concepts in t. $\mathcal{K}_2$ contains exponentially many elements, since there are exponentially many different concept types.

$\mathcal{K}$ is the union $\mathcal{K}_1 \cup \mathcal{K}_2$ of both KBs, and thus also of exponential size. The UCQ Q is a disjunction over all CQs $q \in \mathsf{form}(\phi)$ for which $\neg q \in t_S$, and all assertions $\alpha \in \mathsf{form}(\phi)$ for which $\neg\alpha \in t_S$. If $\mathcal{K} \not\models Q$, then there is a model $\mathcal{I}$ of $\mathcal{K}$ that does not satisfy any of the disjuncts in Q. From the above observations, it follows that this model witnesses the admissibility of S. It us further not hard to see that any interpretation that witnesses the admissibility of S can be transformed into a model $\mathcal{I}$ of $\mathcal{K}$ s.t. $\mathcal{I} \not\models Q$. As a result, we have that S is admissible iff $\mathcal{K} \not\models Q$.

While $\mathcal{K}$ is exponential in the size ϕ, Q is polynomial in the size of ϕ. As shown in [23, Lemma 16,17] based on results in [18, 13] (and, separately, within proofs for results in [4]), query entailment from $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$-KBs can be decided in time double exponential in the size of the query and single exponential in the size of the KB. We obtain that, if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$, the required entailment test can be performed in time double exponential in the size of ϕ.

For $\mathcal{ALCQ}$, we need to decide this entailment in single exponential time, for which we use the technique for query entailment from $\mathcal{SHQ}$-KBs presented in [29]. The author shows that, in order to decide entailment of a UCQ Q, one can construct, based on Q, a series $\mathcal{K}'_1, \ldots, \mathcal{K}'_n$ of exponentially many, polynomially sized $\mathcal{ALCQ}^\sqcap$ KBs (so called spoilers), so that $\mathcal{K} \not\models Q$ iff for some $i \in [\![1, n]\!]$, $\mathcal{K} \cup \mathcal{K}'_i$ is satisfiable [29, Lemma 3]. $\mathcal{ALCQ}^\sqcap$ extends $\mathcal{ALCQ}$ by conjunctions over role names, and satisfiability of $\mathcal{ALCQ}^\sqcap$ KBs can be decided in time exponential wrt. of the size of the KB. However, in our case, $\mathcal{K}$ is already of exponential size, so that this result alone would only give us a double exponential upper bound. We therefore have show that for each $i \in [\![1, n]\!]$, the satisfiability of $\mathcal{K} \cup \mathcal{K}'_i = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}'_i$ can still be decided in time exponential in the size of ϕ.

We show how to decide satisfiability of each KBs $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}'_i$ in time exponential in the size of ϕ, using classical type elimination. In type elimination, the aim is to compute a set of concept types that corresponds to a model of the KB, where each concept type is represented by some domain element. Specifically, we apply this technique for the KB $\mathcal{K}_1 \cup \mathcal{K}_2$, and additionally ensure that only the types encoded in $\mathcal{K}'_i$ are represented. Because there are only exponentially many types to consider, this allows us to establish the required bound.

We describe this in detail. We consider the set $T_0 \subseteq 2^{\mathsf{t_c}(\mathcal{K}_1 \cup \mathcal{K}'_i)}$ of concept types for $\mathcal{K}_1 \cup \mathcal{K}'_i$. Since $\mathcal{K}_1$ and $\mathcal{K}'_i$ are both polynomial in ϕ, there are at most exponentially many such types. Note furthermore that $\mathsf{t_c}(\phi) \subseteq \mathsf{t_c}(\mathcal{K}_1 \cup \mathcal{K}'_i)$, and that $\mathcal{K}_2$ simply states which types from $2^{\mathsf{t_c}(\phi)}$ need to occur in a model, which are exactly those in our quasistate S. We thus remove from T_0 all types t for which there is no type $t' \in S$ s.t. $t' \subseteq t$. Next, we have to choose one type in $t_a \in T_0$ for each individual name a occurring in $\mathcal{K}_1 \cup \mathcal{K}'_i$, since each concept type represents a distinct domain element. As there are exponentially many choices for this, this accounts to iterating the type elimination algorithm at most exponentially many times, each time with a different choice, until it is successful.

The type elimination algorithm now proceeds by eliminating concept types that cannot be represented by any domain element in models of the KB, which may depend on role restrictions in that type and other concept types in the current set that can be picked to satisfy these role restrictions. In the following, denote by t_r the set of all subsets of role names that occur in ϕ. A *successor-mapping* for a concept type t wrt. to a set T_i of types is a mapping $m : t_r \times T_i \to \mathbb{N}$. We call m valid if

1. for every $\geq i(r_1 \sqcap \ldots \sqcap r_m).C \in t$, we have $\sum_{\langle \mathcal{R},t\rangle \in M} m(\mathcal{R},t) \geq i$, where $M = \{\langle \mathcal{R},t\rangle \mid r_1,\ldots,r_m \in \mathcal{R}, C \in t\}$,
2. for every $\neg{\geq} i(r_1 \sqcap \ldots \sqcap r_m).C \in t$, we have $\sum_{\langle \mathcal{R},t\rangle \in M} m(\mathcal{R},t) < i$, where $M = \{\langle \mathcal{R},t\rangle \mid r_1,\ldots,r_m \in \mathcal{R}, C \in t\}$,
3. for every $\exists(r_1 \sqcap \ldots \sqcap r_m).C \in t$, we have $\sum_{\langle \mathcal{R},t\rangle \in M} m(\mathcal{R},t) \geq 1$, where $M = \{\langle \mathcal{R},t\rangle \mid r_1,\ldots,r_m \in \mathcal{R}, C \in t\}$,
4. for every $\neg\exists(r_1 \sqcap \ldots \sqcap r_m).C \in t$, we have $\sum_{\langle \mathcal{R},t\rangle \in M} m(\mathcal{R},t) = 0$, where $M = \{\langle \mathcal{R},t\rangle \mid r_1,\ldots,r_m \in \mathcal{R}, C \in t\}$.

In order to decide whether a concept type t has a valid successor-mapping wrt. to a set T_i of types, we can see these conditions as a set of k linear inequations, with the values of the function m variables, so that we obtain at most one inequation per concept in t. By Carathéodory's theorem, if such an inequation system has a positive solution, then it has a solution in which at most $k+1$ variables have a value different from 0, which is polynomial in the size of t. We can thus decide whether there exists a valid successor-mapping in exponential time by iterating over all subsets $M' \subseteq t_r \times T_i$ of size $k+1$, and then checking in polynomial time whether assigning a positive number to each element in M' gives us a solution to the inequation system.

The type elimination now proceeds by successively eliminating from the current set T_i of concept types each concept type $t \in T_i$ that does have a valid successor mapping in the current set of types. Since in each step, exponentially many types have to be checked, each check can be performed in exponential time, and we can eliminate at most exponentially many types, this process takes at most exponential time.

We then check whether the resulting set T_* of types satisfies the following conditions:

- for every individual name a occurring in $\mathcal{K}_1 \cup \mathcal{K}_i'$, T_* contains a type t with $\{a\} \in s$, and
- for every type $t \in S$, T_* contains a type t' with $t \subseteq t'$.

It is standard to show that, if the procedure is successful, we can build a model of $\mathcal{K} \cup \mathcal{K}_i'$ based on the types in T_*. On the other hand, the procedure is successful if there exists a model of $\mathcal{K} \cup \mathcal{K}'$, since we can build the set T_* based on the types that occur in this model. We obtain that satisfiability of $\mathcal{K} \cup \mathcal{K}_i'$ can be decided in exponential time, which means that we can decide $\mathcal{K} \models Q$ in exponential time, which means that, if the underlying DL is $\mathcal{ALCQ}$, the admissibility of a quasistate can be decided in time single-exponential in ϕ. □

Theorem 2. *Satisfiability of TPDFs without probability operators is undecidable if* $\mathsf{N}_{Rrig} \neq \emptyset$*. Otherwise, it is* 2ExpTime*-complete if the underlying DL is* $\mathcal{ALCO}$*,* $\mathcal{ALCI}$ *or* $\mathcal{ALCOI}$*, and* ExpSpace*-complete if the underlying DL is* $\mathcal{ALC}$ *or* $\mathcal{ALCQ}$*.*

Proof. By Lemma 1, we can reduce the satisfiability of a TPDF without probability operators to the existence of a regular quasimodel. We describe a non-deterministic procedure that guesses and verifies such a structure, and then argue that it is in the targeted complexity classes. In the procedure, we first guess the numbers n and m form Lemma 1, which are double exponentially bounded by ϕ and can thus be stored in binary using only exponentially many bits. We then guess the quasistates $Q(0), \ldots, Q(n), Q(n+1), \ldots, Q(n+m)$ one after the other, keeping only the current quasistate as well as $Q(n+1)$ in memory. For each quasistate, by Lemma 6 we can verify its admissibility in ExpTime ($\mathcal{ALCQ}$) respectively 2ExpTime (for $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$). In addition, we keep a set of "unresolved" concepts and formulae of the form $\alpha \mathsf{U} \beta$ for each run until they have been verified. Clearly, apart from the admissibility test, all of these operations only take exponential space. Finally, we verify that $Q(n+m) = Q(n+1)$, and that every expression of the form $\alpha \mathsf{U} \beta$ in $Q(n+1)$ has been satisfied before $Q(n+m)$. This procedure runs in ExpSpace if the underlying DL is $\mathcal{ALCQ}$, and in $\text{ExpSpace}^{2\text{ExpTime}} = 2\text{ExpTime}$ if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$. □

C Only Probability Operators

We prove both directions of Lemma 2 in separate lemmas.

Lemma 7. *If a TPDF ϕ without temporal operators is satisfiable, then there exists a probabilistic quasimodel $\langle PS, \mathfrak{PT} \rangle$ for ϕ.*

Proof. Let $\iota : \mathfrak{J} \to [0,1]$, where $\mathfrak{J} \subseteq 2^{\mathcal{J}}$, be a probabilistic model of ϕ with domain Δ. Since ϕ contains no temporal operators, only the first interpretation $\mathcal{I}_0$ in each sequence $(\mathcal{I}_i)_{i \geq 0} \in \mathcal{J}$ is relevant, which is why, in the following, we leave out the subscripts of the interpretations and treat $\mathcal{J}$ as a set of classical interpretations (each time referring to the first interpretation of the sequence).

Based on ι, we build a full quasistate. First, we associate to every interpretation $\mathcal{I} \in \mathcal{J}$ an admissible quasistate $S_{\mathcal{I}}$ by setting:

1. $t_{S_{\mathcal{I}}} = \{\psi \in \mathsf{t_f}(\phi) \mid \mathcal{I}, \iota \models \psi\}$, and
2. $S_{\mathcal{I}} = \{t_{S_{\mathcal{I}}}\} \cup \{t \subseteq \mathsf{t_c}(\phi) \mid t^{\mathcal{I},\iota} \neq \emptyset\}$.

It follows from the definition of admissible quasistates that $T_{\mathcal{I}}$ is admissible: we just have to replace every probabilistic concept C in every $\mathcal{I}$ by its conceptual abstraction C^{ca} to obtain a witnessing interpretation. Note that several interpretations may have the same admissible quasistate. We set $\mathfrak{S} = \{S_{\mathcal{I}} \mid \mathcal{I} \in \mathcal{J}\}$, and define $PS : 2^{\mathfrak{S}} \to [0,1]$ by setting, for every $S \in \mathfrak{S}$,

$$PS(\{S\}) = P(\{\mathcal{I} \in \mathcal{J} \mid S = S_{\mathcal{I}}\}).$$

PS is a probabilistic quasistate. Furthermore, we have $\phi \in t_T$ for every $S \in \mathfrak{S}$, since by assumption, $\iota \models \phi$, and therefore, for every $\mathcal{I} \in \mathcal{J}$, $\mathcal{I}, \iota \models \phi$. Consequently, $\mathfrak{S}$ satisfies Condition **PQ1**.

We continue by constructing the probabilistic concept types. For every domain element $d \in \Delta$ and interpretation $\mathcal{I} \in \mathcal{J}$, we define the concept type $\mathsf{type}(d, \mathcal{I}) = \{C \in \mathsf{t_c}(\phi) \mid d \in C^{\mathcal{I},\iota}\}$. To every domain element $d \in \Delta$, we assign the probabilistic concept type $pt_d : 2^{T_d} \to [0,1]$, where

$$T_d = \{\mathsf{type}(d, \mathcal{I}) \mid \mathcal{I} \in \mathcal{J}\}$$

and for every $t \in T_d$,

$$pt_d(\{t\}) = \iota(\{\mathcal{I} \in \mathcal{J} \mid t = t_{d,\mathcal{I}}\}).$$

Our set of probabilistic concept types is then $\mathfrak{PT} = \{pt_d \mid d \in \Delta\}$. We show that $\langle PS, \mathfrak{PT} \rangle$ is a probabilistic quasimodel. We have already shown that Condition **PQ1** holds. It remains to show that the remaining two conditions of probabilistic quasimodels are also satisfied.

We first show that Condition **PQ2** is satisfied, that is, every $pt \in \mathfrak{PT}$ is compatible to PS. Let $pt : 2^T \to [0,1] \in \mathfrak{PT}$ be some probabilistic type in $\mathfrak{PT}$. Note that there is some domain element $d \in \Delta$ s.t. $pt = pt_d : 2^{T_d} \to [0,1]$. We define the domain of the joined probability measure $P_{PS,pt} : 2^{W_{PS,pt}} \to [0,1]$ for PS and pt by setting $W_{PS,pt} = \{\langle S_{\mathcal{I}}, t_{d,\mathcal{I}} \rangle \mid \mathcal{I} \in \mathcal{J}\}$. By construction, $W_{PS,pt} \subseteq \mathfrak{S} \times T_d$. Furthermore, $\langle S_{\mathcal{I}}, t_{d,\mathcal{I}} \rangle \in W_{PS,pt}$ implies $t_{d,\mathcal{I}} \in S_{\mathcal{I}}$, since by construction, $S_{\mathcal{I}}$ contains $\mathsf{type}(e, \mathcal{I})$ for every $e \in \Delta$. Therefore, $W_{PS,pt}$ satisfies Condition **PC1**.

The measure $P_{PS,pt}$ is now defined for every $\langle S,t\rangle \in W_{P,pt}$ by

$$P_{PS,pt}(\{\langle S,t\rangle\}) = \iota(\{\mathcal{I} \in \mathcal{J} \mid S_{\mathcal{I}} = S, t_{d,\mathcal{I}} = t\}).$$

To see that $P_{PS,pt}$ is indeed a probability measure, note that for every interpretation $\mathcal{I} \in \mathcal{J}$, there exists some tuple $\langle S,t\rangle \in W_{PS,pt}$ s.t. $S_{\mathcal{I}} = T$ and $t_{\mathcal{I}} = t$ (by construction), and that for any two distinct tuples $\langle S_1,t_1\rangle$, $\langle S_2,t_2\rangle \in W_{P,pt}$, the corresponding sets of interpretations are disjoint. (Otherwise, there would be some interpretation $\mathcal{I} \in \mathcal{J}$ s.t. both $S_1 = S_{\mathcal{I}} = S_2$ and $t_1 = t_{d,\mathcal{I}} = t_2$, which means the tuples would not be distinct.) Regarding Condition **PC2**, we have for every $S \in \mathfrak{S}$

$$\begin{aligned}
&P_{PS,pt}(\{\langle S,t\rangle \mid (S,t) \in W_{PS,pt}\})\\
=&P_{PS,pt}(\{\langle S_{\mathcal{I}},t_{d,\mathcal{I}}\rangle \mid \mathcal{I} \in \mathcal{J}, S = S_{\mathcal{I}}\})\\
=&\iota(\{\mathcal{I} \in \mathcal{J} \mid S = S_{\mathcal{I}}\})\\
=&PS(\{S\}),
\end{aligned}$$

and regarding Condition **PC3**, we have for every $t \in T$

$$\begin{aligned}
&P_{PS,pt}(\{\langle S,t\rangle \mid (S,t) \in W_{PS,pt}\})\\
=&P_{PS,pt}(\{\langle S_{\mathcal{I}},t_{d,\mathcal{I}}\rangle \mid \mathcal{I} \in \mathcal{J}, t_{d,\mathcal{I}} = t\})\\
=&\iota(\{\mathcal{I} \in \mathcal{J} \mid t = t_{d,\mathcal{I}}\})\\
=&pt(\{t\}).
\end{aligned}$$

Consequently, $P_{pt,PS}$ satisfies Conditions **PC1**–**PC3**, and thus witnesses the compatibility of pt with PS, which means that Condition **PQ2** is satisfied for $\langle PS,\mathfrak{PT}\rangle$.

Regarding Condition **PQ3**, we have to show that for every quasistate $S \in \mathfrak{S}$ and every concept type $t \in S$, there exists a probabilistic concept type $pt : 2^T \to [0,1] \in \mathfrak{PT}$ s.t. $\langle S,t\rangle \in W_{PS,pt}$. For every such S and t, there exists an interpretation $\mathcal{I} \in \mathcal{J}$ and a domain element $d \in \Delta$ s.t. $S = S_{\mathcal{I}}$ and $t_{d,\mathcal{I}}$, so that this condition directly follows. We obtain that $\langle PS,\mathfrak{PT}\rangle$ satisfies all Conditions **PQ1**–**PQ3**, and thus that it is a probabilistic quasimodel for ϕ. □

Lemma 8. *Given a probabilistic formula ϕ, if there exists a probabilistic quasimodel $\langle PS,\mathfrak{PT}\rangle$ for ϕ, then ϕ is satisfiable.*

Proof. Let $\langle PS,\mathfrak{PT}\rangle$ be a probabilistic quasimodel, where $PS : \mathfrak{S} \to [0,1]$. Note that $\langle PS,\mathfrak{PT}\rangle$ remains a quasimodel if we remove every quasistate $S \in \mathfrak{S}$ for which $PS(S) = 0$, so that we can assume wlog. that S does not contain such quasistates.

We show how to construct, based on the probabilistic quasimodel, a probabilistic temporal interpretation that is a model of ϕ. Note for every quasistate $S \in \mathfrak{S}$, there exists an interpretation $\mathcal{I}_S$ witnessing its admissibility. Based on these interpretations, we build a probabilistic model of ϕ. Recall also that by the example given at the beginning of the appendix, the model of ϕ might have an uncountable set of possible worlds. To keep the following simple, our construction does not treat this as a special case, but always yields such a model. Specifically, it will contain a possible world $\mathcal{I}_q$ for every real number $q \in [0,1]$:

$$\mathcal{J} = \{\mathcal{I}_q \mid q \in [0,1]\},$$

on which we define a probability measure based on the Lebesgue measure over $[0,1]$. Specifically, ι is generated based on all intervals $[l,u] \subseteq [0,1]$, for which it satisfies:

$$\iota(\{\mathcal{I}_q \in \mathcal{J} \mid q \in [l,u]\}) = u - l.$$

We use the real numbers q to assign possible worlds to quasistates, and later to also assign types to domain elements. Assume the sets of possible worlds to be enumerated: $\mathfrak{S} = \{S_1,\ldots,S_n\}$, and assign to each quasistate S_i, $i \in [\![1,n]\!]$, an interval $I(S_i) = [l,u)$ for $i < n$ and $[l,u]$ for $i = n$, where

- $l = \sum_{i \in [\![1,i]\!]} PS(\{S_i\})$, and
- $u = l + PS(\{S_n\})$.

For each $q \in [0,1]$, $\mathcal{I}_q$ will be a model corresponding to the quasistate S_i for which $q \in I(S_i)$. Note that this ensures that for each $S_i \in \mathfrak{S}$, the probability $PS(\{S_i\})$ will be respected by ι.

Additionally, we have to make sure that for every $S \in \mathfrak{S}$, $q \in I(S_i)$ and $t \in S_i$, $\mathcal{I}_q$ has at least one domain element corresponding to t, and which satisfies all probabilistic concepts in t. We do so based on the joined probability measures $P_{S,t} : 2^{W_{S,t}} \to [0,1]$, which exist due to Condition **PQ3** for every $S \in \mathfrak{S}$ and $t \in S$. As for the probabilistic quasistate, we may assume wlog. that for every such measure, the only tuple $\langle S', t'\rangle \in W_{S,t}$ for which we can have $P_{S,t}(\langle S', t'\rangle) = 0$ is $\langle S, t\rangle$.

For every pair $\langle S, t\rangle$ with $S \in \mathfrak{S}$ and $t \in S$, the domain Δ contains domain elements which again correspond to an interval over the real numbers:

$$\begin{aligned} \Delta = \{d_{S,t,q} \mid S \in \mathfrak{S}, t \in S, q \in [0,u), u = P_{S,t}(\{\langle S,t\rangle\})\} \\ \cup \{d_{S,t,0} \mid S \in \mathfrak{S}, t \in S, P_{S,t} = 0\} \end{aligned}$$

For each joined probability measure $P_{S,t} : W_{S,t} \to [0,1]$, we define a set of intervals that reflect the probabilities in $P_{S,t}$. For this, we assume a total order $<_t$ on the elements in $W_{S,t}$, which satisfies $\langle S_i, t'\rangle <_t \langle S_j, t''\rangle$ if $i < j$. Based on this order, we assign to each tuple $\langle S', t'\rangle \in W_{S,t}$ an interval $I_{S,t}(S', t')$ based on the bounds

- $l = \sum_{\langle S'', t''\rangle \in W_{S,t}, \langle S'', t''\rangle <_t \langle S', t'\rangle} P_{S,t}(\{\langle S'', t''\rangle\})$, and
- $u = l + P_{S,t}(\{\langle S', t'\rangle\})$.

We assign to $I_{S,t} = [l, l]$ if $P_{S,t}(\{\langle S', t'\rangle\}) = 0$. Otherwise, we assign one of the intervals $[l, u]$, $[l, u)$, $(l, u]$ or (l, u), where

- the interval is open on the left if $P_{S,t}(\{\langle S'', t''\rangle\}) = 0$ for the largest tuple $\langle S'', t''\rangle$ lower than $\langle S', t'\rangle$, according to $<_t$, and otherwise, it is closed on the left, and
- the interval is open on the right unless $u = 1$, in which case it is closed on the right.

These conditions make sure that the intervals fill the complete range of real numbers in $[0, 1]$ without gaps, while taking special care of intervals of size 0.

Based on these intervals, we assign a type to each domain element in each interpretation. Sepcifically, the function $\mathsf{type}(\mathcal{I}_q, d_{S,t,q'})$ assigns to each interpretation $\mathcal{I}_q$ and domain element $d_{S,t,q'} \in \Delta$ a type $t' \in S$ as follows.

- If $q \notin I_{S,t}(\langle S, t\rangle)$, then $\mathsf{type}(\mathcal{I}_q, d_{S,t,q'}) = t'$, where t' is the unique type for which $q \in I_{S,t}(\langle S', t'\rangle)$. Note that in this case, $q \in I(S')$, which means that $\mathcal{I}_q$ corresponds to the quasistate S'.
- If $q \in I_{S,t}(\langle S, t\rangle)$ and $q + q' \in I(S)$, then $\mathsf{type}(\mathcal{I}_q, d_{S,t,q'}) = t'$, where t' is the unique type for which $(q + q') \in I_{S,t} \in I_{S,t}(S, t')$. Again we have $q \in I(S)$, so that $\mathcal{I}_q$ corresponds to the quasistate S.
- If $q \in I_{S,t}(\langle S, t\rangle)$ and $q + q' \notin I(S)$, then $q + q' - P_{S,t}(\{\langle S, t\rangle\}) \in I(S)$, and we set $\mathsf{type}(\mathcal{I}_q, d_{S,t,q'}) = t'$, where t' is the unique type for which

$$(q + q' - P_{S,t}(\{\langle S, t'\rangle\})) \in I_{S,t}(\langle S, t'\rangle).$$

It is again not hard to see that the function $\mathsf{type}()$ reflects the joined probability measures $P_{S,t}$, in the sense that for each $d_{S,t,q} \in \Delta$ and $\langle S', t' \rangle \in W_{S,t}$, we have

$$\iota(\{\mathcal{I}_q \mid \mathsf{type}(\mathcal{I}_q, d_{S,t,q}) = t'\}) = P_{S,t}(\{\langle S', t' \rangle\}).$$

This further ensures that the probabilities in the probabilistic concept type $pt : 2^T \to [0,1]$ corresponding to $P_{S,t}$ are taken into account, so hat for every $t' \in T$ and $d_{S,t,q} \in \Delta$,

$$\iota(\{\mathcal{I}_q \mid \mathsf{type}(\mathcal{I}_q, d_{S,t,q}) = t'\}) = pt(\{t'\}),$$

which in turn ensures that all probabilistic concepts in t' are taken into consideration.

We now specify how the interpretations $\mathcal{I}_q$ are constructed. For each $\mathcal{I}_q$, there exists a quasistate S s.t. $q \in I(S)$. Since S is admissible, there exists an interpretation $\mathcal{I}_S$ which witnesses the admissibility of S. Furthermore, the above construction ensures that for every concept type $t \in S$, there exists at least one domain element $d_{S,t,q'} \in \Delta$ s.t. $\mathsf{type}(\mathcal{I}_q, d_{S,t,q'}) = t$. We collect those domain elements in the set $d(\mathcal{I}_q, t)$:

$$d(\mathcal{I}_q, t) = \{d_{S,t,q'} \in \Delta \mid \mathsf{type}(\mathcal{I}_q, d_{S,t,q'}) = t\}.$$

We now extend $\mathcal{I}_S$ to the interpretation $\mathcal{I}_q$ by replacing, for each type $t \in S$, the domain elements in $t^{\mathcal{I}_S}$ by $d(\mathcal{I}_q, t)$, which we can do in a way that preserves the set of entailed CQs and GCIs according to Lemma 4. Based on the previous observations, it is now standard to show by structural induction on the concepts that $t^{\mathcal{I}_q} = t^{\mathcal{I}_q,\iota}$ for all $q \in [0,1]$ and $t \subseteq \mathsf{t_c}(\phi)$, and that $\mathcal{I}_q \models \psi$ for every $\psi \in S$, where S is such that $q \in I_S$. We obtain that $\iota \models \phi$, and therefore that ϕ is satisfiable. □

Lemma 2 is now a direct consequence of Lemma 8 and 7

We next show that, if the underlying DL is $\mathcal{ALCQ}$, then it suffices to consider quasimodels that only have an exponential number of quasistates and probabilistic concept types. We first show the following auxiliary lemma.

Lemma 9. *Assume the underlying DL is $\mathcal{ALCQ}$, and let S_1 and S_2 be two admissible quasistates s.t every formula type and every nominal type in $S_1 \cup S_2$ is also contained in $S_1 \cap S_2$. Then, their union $S_1 \cup S_2$ is also an admissible quasistate.*

Proof. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the interpretations witnessing the admissibility of S_1 and S_2. Without loss of generality, we may assume that $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \emptyset$. We define the interpretation $\mathcal{I}$ witnessing $S_1 \cup S_2$ as follows:

- $\Delta^{\mathcal{I}_1} \cup \Delta^{\mathcal{I}_2} \setminus \{a^{\mathcal{I}_2} \mid a \text{ occurs in } \phi\}$,
- for every $a \in \mathsf{N_I}$ occurring in ϕ, $a^{\mathcal{I}} = a^{\mathcal{I}_1}$,
- for every $A \in \mathsf{N_C}$: $A^{\mathcal{I}} = (A^{\mathcal{I}_1} \cup a^{\mathcal{I}_2}) \cap \Delta^{\mathcal{I}}$, and
- for every $r \in \mathsf{N_R}$; $r^{\mathcal{I}} = (r^{\mathcal{I}_1} \cup r^{\mathcal{I}_2}) \cap (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})$.

Every CQ entailed by $\mathcal{I}_1$ is still entailed by $\mathcal{I}$, and every GCI not entailed by $\mathcal{I}_1$ is still not entailed by $\mathcal{I}$. Furthermore, note that no domain elements from $\Delta^{\mathcal{I}_1}$ are connected to any domain elements from $\Delta^{\mathcal{I}_2}$. It follows that every CQ not entailed by $\mathcal{I}_1$ nor $\mathcal{I}_2$ is still not entailed by $\mathcal{I}$, and that every GCI entailed by $\mathcal{I}_1$ and by $\mathcal{I}_2$ is also entailed by $\mathcal{I}$. From here it follows that $\mathcal{I}$ witnesses the admissibility of $S_1 \cup S_2$. □

Lemma 10. *Assume the underlying DL is $\mathcal{ALCQ}$. Then, if there exists a probabilistic quasimodel for ϕ, then there exists a probabilistic quasimodel $\langle PS, \mathfrak{PT}\rangle$ with $PS : 2^{\mathfrak{S}} \to [0,1]$ in which $\mathfrak{S}$ contains at most exponentially many elements.*

Proof. We show that every probabilistic quasimodel can be transformed into one such that for every formula type $t \subseteq \mathsf{t}_{\mathsf{f}}(\phi)$ and set $T \subseteq 2^{\mathsf{t}_{\mathsf{c}}(\phi)}$ of nominal types, $\mathfrak{S}$ contains at most one quasistate S s.t. $T \subseteq S$ and $t \in T$. As there are at most exponentially many possible types, ϕ contains at most polynomially many individual names and every admissible quasistate contains at most one nominal type per individual name in ϕ, this implies the lemma.

Let $\langle PS_1, \mathfrak{PT}_1\rangle$ with $PS_1 : 2^{\mathfrak{S}_1} \to [0,1]$ be a probabilistic quasimodel s.t. for some formula type $t \subseteq \mathsf{t}_{\mathsf{f}}(\phi)$, there exists two distinct quasistates $S_1, S_2 \in \mathfrak{S}_1$ with $t_{S_1} = t_{S_2} = t$, and every nominal type in S_1 is also contained in S_2 and vice versa. We construct a new probabilistic quasimodel $\langle PS, \mathfrak{PT}\rangle$ in which S_1 and S_2 are replaced by their union. By doing so exhaustively, we obtain a probabilistic quasimodel as in the lemma.

By Lemma 9, $S_1 \cup S_2$ is also an admissible quasistate. We can therefore define $PS : \mathfrak{S} \to [0,1]$ by setting $\mathfrak{S} = (\mathfrak{S}_1 \setminus \{S_1, S_2\}) \cup \{S_1 \cup S_2\}$. Note that it is possible that $(S_1 \cup S_2) \in \mathfrak{S}_1$, in which case $\mathfrak{S}$ has two quasistates less than $\mathfrak{S}_1$, while otherwise, it has only one quasistate less. We define PS by setting

- for all $S \in \mathfrak{S}$ with $S \neq S_1 \cup S_2$: $PS(S) = PS_1(\{S\})$,
- if $S_1 \cup S_2 \notin \mathfrak{S}_1$: $PS(\{S_1 \cup S_2\}) = PS_1(\{S_1\}) + PS_1(\{S_2\})$, and
- if $S_1 \cup S_2 \in \mathfrak{S}_1$: $PS(\{S_1 \cup S_2\}) = PS_1(\{S_1\}) + PS_1(\{S_2\}) + PS_1(\{S_1 \cup S_2\})$.

PS is an admissible probabilistic quasistate, and we set $\mathfrak{PT} = \mathfrak{PT}_1$ to obtain the set of probabilistic concept types. To show that $\langle PS, \mathfrak{PT}\rangle$ is a probabilistic quasimodel, it remains to show that every probabilistic concept type $pt : 2^T \to [0,1] \in \mathfrak{PT}$ is compatible to PS. We do so by defining a the joined probability measure $P_{PS,pt} : W_{PS,pt} \to [0,1]$ of PS and pt based on the joined measure $P_{PS_1,pt} : W_{PT_1,pt} \to [0,1]$ of PS_1 and pt. $W_{PS,pt}$ is obtained from $W_{PS,pt}$ by replacing every $\langle S_1, t\rangle$ and $\langle S_2, t\rangle$ by $\langle S_1 \cup S_2, t\rangle$. Clearly, $W_{PS,pt}$ satisfies Condition **PC1**. $P_{PS,pt}$ is defined by setting for all $t \in T$:

- for all $S' \in \mathfrak{S}$ s.t. $S' \neq S_1 \cup S_2$:
$$P_{PS,pt}(\{\langle S, t\rangle\}) = P_{PS_2,pt}(\{\langle S, t\rangle\}),$$
- if $S_1 \cup S_2 \notin \mathfrak{S}_1$:
$$P_{PS,pt}(\{\langle S_1 \cup S_2, t\rangle\}) = P_{PS_1,pt}(\{\langle S_1, t\rangle\}) + P_{PS_1,pt}(\{\langle S_2, t\rangle\}),$$
- if $S_1 \cup S_2 \in \mathfrak{S}_1$:
$$\begin{aligned} P_{PS,pt}(\{\langle S_1 \cup S_2, t\rangle\}) &= P_{PS_1,pt}(\{\langle S_1, t\rangle\}) + P_{PS_1,pt}(\{\langle S_2, t\rangle\}) \\ &\quad + P_{PS_1,pt}(\{\langle S_1 \cup S_2, t\rangle\}). \end{aligned}$$

$P_{PS,pt}$ satisfies both Condition **PC2** and Condition **PC3**, which means that pt is compatible to PS. It follows that every $pt \in \mathfrak{PT}$ is compatible to PS, and that $\langle PS, \mathfrak{PT}\rangle$ satisfies Condition **PQ2**. It is not hard to see that it also satisfies Conditions **PQ1** and **PQ3**, and therefore is a quasistate for ϕ. □

Lemma 11. *If the underlying DL is $\mathcal{ALCQ}$, then ϕ is satisfiable iff it has a probabilistic quasimodel $\langle PS, \mathfrak{PT} \rangle$ with $PS : \mathfrak{S} \to [0, 1]$, where both $\mathfrak{S}$ and $\mathfrak{PT}$ contain at most exponentially many elements.*

Proof. Assume the underlying DL is $\mathcal{ALCQ}$. By Lemma 10, if ϕ is satisfiable, then it is satisfiable in a probabilistic quasimodel $\langle PS, \mathfrak{PT} \rangle$ with $PS : \mathfrak{S} \to [0, 1]$ where $\mathfrak{S}$ contains at most exponentially many elements. For $\mathfrak{PT}$, we only require one probabilistic concept type per pair S, t, where $S \in \mathfrak{S}$ and $t \in S$ (Condition **PQ3**). Since $\mathfrak{S}$ and each $S \in \mathfrak{S}$ contain at most exponentially many elements, we obtain that we can find a $\mathfrak{PT}$ that contains at most exponentially many elements. □

We have now all that is needed to prove our complexity upper bounds.

Theorem 3. *Satisfiability of TPDFs without temporal operators is in* NExpTime *if the underlying DL is $\mathcal{ALCQ}$, and in* N2ExpTime *if the underlying DL is $\mathcal{ALCOQ}$ or $\mathcal{ALCOI}$.*

Proof. By Lemma 2, it suffices to show that we can decide the existence of a probabilistic quasimodel for ϕ in N2ExpTime, respectively in NExpTime if the underlying DL is $\mathcal{ALCQ}$. The first step is to guess and verify the domains of all probability measures involved.

- We guess the set $\mathfrak{S}$ of quasistates to be used in the probability measure $PS : 2^{\mathfrak{S}} \to [0, 1]$. There are at most double exponentially many possible, and by Lemma 6, admissibility of each of them can be decided in 2ExpTime. In case of $\mathcal{ALCQ}$, by Lemma 11 we only need to guess exponentially many, the admissibility of each of which can be guessed in ExpTime.
- For each pair $\langle S, t \rangle$ of a quasisate $S \in \mathfrak{S}$ and a type $t \in S$, we guess a set T of types with $t \in T$, to be used in the probability measure $pt : T \to [0, 1]$, and the set $W_{PS,pt} \subseteq \mathfrak{S} \times T$ satisfying $\langle S, t \rangle \in W_{PS,pt}$, to be used in the joined probability measure $P_{PS,pt} : 2^{W_{PS,pt}} \to [0, 1]$.

To determine the probability measures, we construct a set of inequations.

We first point out some criteria of probabilistic quasimodels $\langle PS, \mathfrak{PT} \rangle$, where $PS : \mathfrak{S} \to [0, 1]$.

- For every formula of the form $\mathsf{P}_{\circledast \mathsf{p}}\psi \in \mathsf{t_f}(\phi)$, we either have $\mathsf{P}_{\circledast \mathsf{p}}\psi \in t_S$ for all $S \in \mathfrak{S}$, or $\mathsf{P}_{\circledast \mathsf{p}}\psi \notin t_S$ for all $S \in \mathfrak{S}$. We can thus collect the set of all common formulae of this form into a set PF.
- The same holds for every probabilistic type $pt : T \to [0, 1]$, and concepts of the form $\mathsf{P}_{\circledast \mathsf{p}}C \in \mathsf{t_c}(\phi)$, so that we can collect the set of all common probabilistic concepts in pt into the set $PC(pt)$.

The Condition **PS2** can thus be captured by the set of inequations that, for every $\mathsf{P}_{\circledast \mathsf{p}}\psi \in PF$, contains

$$\sum_{S \in \mathfrak{S}, \psi \in t_S} PS(\{S\}) \circledast p. \tag{1}$$

This gives us a polynomial number of inequations of double exponential length each (since there are at most double exponentially many quasistates in $\mathfrak{S}$). If the underlying DL is $\mathcal{ALCQ}$, each inequation is of single exponential length.

Similarly, the condition of all $pt : T \to [0,1] \in \mathfrak{PT}$ being a probabilistic concept type, can be captured by the following inequation for every $\mathsf{P}_{\circledast\mathsf{p}}C \in PC(pt)$:

$$\sum_{t \in T, C \in t} pt(\{t\}) \circledast p, \tag{2}$$

which gives us, per probabilistic concept type, a polynomial number of inequations of exponential length each, and, as there are at most double exponentially many probabilistic concept types, a double exponential number of such inequations (single exponential in the case of $\mathcal{ALCQ}$).

Finally, the compatibility conditions **PC2** and **PC3** for each probabilistic concept type $pt : T \to [0,1] \in \mathfrak{PT}$ correspond to the following inequations:

1. for every $S \in \mathfrak{S}$: $\left(\sum_{\langle S,t\rangle \in W_{PS,pt}} P_{PS,pt}(\{t\})\right) - PS(\{S\}) = 0$, and
2. for every $t \in T$: $\left(\sum_{\langle S,t\rangle \in W_{PS,pt}} P_{PS,pt}(\{t\})\right) - pt(\{t\}) = 0$.

Finally, we need inequations stating that the probabilities in each probability measure add up to 1, which is established by the following inequations:

1. $\sum_{S \in \mathfrak{S}} PS(\{S\}) = 1$,
2. for every $pt : T \to [0,1]$: $\sum_{t \in T} pt(\{t\}) = 1$, and
3. for every $pt : T \to [0,1]$: $\sum_{\langle S,t\rangle \in W_{PS,pt}} P_{PS,pt}(\{\langle S,t\rangle\}) = 1$.

We end up with a set of double exponentially many inequations with up to double exponentially many elements each. In the case of $\mathcal{ALCQ}$, these are only exponentially many inequations with up to exponentially many elements each.

We now make use of a result from [17], which was also used to prove upper bounds for Prob-$\mathcal{ALC}$ in [20]. This result states that, if we are given a system $\mathcal{E}$ of r linear inequations, each having integer coefficients that can be represented using ℓ bits, then $\mathcal{E}$ has a non-negative solution iff it has a solution in which each member can be represented using at most $O(r\ell \cdot r \log(r))$ bits. Our system of inequations can be transformed into one with integer coefficients by multiplying each coefficient by a common divisor, so that we obtain such a system $\mathcal{E}$ in which r and ℓ are double exponentially bounded, or exponentially bounded in the case of $\mathcal{ALCQ}$. Consequently, we can guess a solution in N2ExpTime, and in NExpTime in the case of $\mathcal{ALCQ}$. We obtain that the complete procedure can be implemented by a non-deterministic Turing machine that runs in double exponential time for $\mathcal{ALCOQ}$ and $\mathcal{ALCOI}$, while it runs in exponential time for $\mathcal{ALCQ}$, and that satisfiability of TPDFs without temporal operators is in N2ExpTime, respectively NExpTime if the underlying DL is $\mathcal{ALCQ}$. □

D Temporal and Probabilistic Operators

Theorem 4. *Satisfiability of TPDFs is 2ExpSpace-hard. This already holds if*

- *no CQs are used,*
- *the underlying DL is $\mathcal{ALC}$,*
- *probabilistic operators are only used on the level of concepts,*

- $\mathsf{N}_{Crig} = \mathsf{N}_{Rrig} = \emptyset$, *and*
- *on the axiom level, we only use Boolean connectives and the operator* $\Box$*, which does not occur under a negation operator.*

Proof. We detail out the construction sketched in the main text.

Step 1: Create ordered bit positions. We define a concept Init that will later initialise a double-exponential counter on a domain element. This counter is implemented using 2^n "time lines" in different possible worlds, each of which will store a bit value of this counter. As a first step, we make sure that these time lines are enforced for instances of this concept, and that they are "ordered" in a way that allows us to refer to the "next" possible world. This ordering is crucial for implementing the double exponential counter.

The different time lines are identified using a single-exponential counter represented using concept names A_i, $i \in [\![1, n]\!]$, which we in the following call the *A-counter.* The following axiom ensures that there exists a time line with an A-counter value of 0, which corresponds to the first bit position:

$$\Box\Big(\mathsf{Init} \sqsubseteq \mathsf{P}_{>0}\big(\bigsqcap_{i\in[\![1,n]\!]} \neg A_i\big)\Big)$$
$$\Box\Big(\mathsf{Init} \sqsubseteq \mathsf{P}_{=1}\big(\bigsqcap_{i\in[\![1,n]\!]} \neg A_i \to \mathsf{FirstBit}\big)\Big)$$
$$\Box\Big(\mathsf{FirstBit} \sqsubseteq \bigcirc\mathsf{FirstBit}\Big).$$

The following axioms for every $i \in [\![1, n]\!]$ ensure that the A-counter is incremented at each following time point.

$$\Box\Big(\bigsqcap_{j\in[\![1,i-1]\!]} A_j \equiv A_i \leftrightarrow \bigcirc\neg A_i\Big).$$

The axiom states that we flip the ith bit if all lower bit values are 1, and otherwise we do not flip it. We follow here the common convention that an empty conjunction corresponds to $\top$.

For each time line with a counter value of 0, we force the existence of another time line with a counter value of $2^n - 1$:

$$\Box\Big(\bigsqcap_{i\in[\![1,n]\!]} \neg A_i \sqsubseteq \mathsf{P}_{>0} \bigsqcap_{i\in[\![1,n]\!]} A_i\Big).$$

Since the counters are incremented, this ensures that eventually, at each time point, every possibe A-counter value is present in some possible world, so that we have indeed 2^n possible "time lines" for the domain element. At each time point, there are now two neighbouring time lines that are easy to identify: in the one where the A-counter value is 0, the concept $\bigsqcap_{i\in[\![1,n]\!]} \neg A_i$ is satisfied, and in the one where the A-counter value is $2^n - 1$, the concept $\bigsqcap_{i\in[\![1,n]\!] A_i}$ is satisfied.

Step 2: Implement double exponential counter. We implement a double-exponential counter, which we call B-counter, that starts counting on individuals satisfying Init.

We use a concept name Bit to represent bit values within the possible worlds. Initially, the counter stores the value 0:

$$\Box\Big(\mathsf{Init} \sqsubseteq \mathsf{P}_{=1}\neg\mathsf{Bit}\Big).$$

We have to flip a bit at position i exactly if bit with a lower position has a value of 1. We use a concept Flip to represent whether this is the case. We keep the current Flip and Bit value until the A-counter reaches $2^n - 1$, and then flip the bit value if required.

$$\Box\Big(\bigsqcup_{i\in[\![1,n]\!]} \neg A_i \sqsubseteq \mathsf{Bit} \leftrightarrow \bigcirc\mathsf{Bit}\Box\Big)$$
$$\Box\Big(\bigsqcup_{i\in[\![1,n]\!]} \neg A_i \sqsubseteq \mathsf{Flip} \leftrightarrow \bigcirc\mathsf{Flip}\Big)$$
$$\Box\Big(\bigsqcap_{i\in[\![1,n]\!]} A_i \sqcap \mathsf{Flip} \sqsubseteq \mathsf{Bit} \leftrightarrow \bigcirc\neg\mathsf{Bit}\Big)$$
$$\Box\Big(\bigsqcap_{i\in[\![1,n]\!]} A_i \sqcap \neg\mathsf{Flip} \sqsubseteq \mathsf{Bit} \leftrightarrow \bigcirc\mathsf{Bit}\Big)$$

It remains to implement the behaviour of the concept Flip. The first bit always flips:

$$\Box\Big(\mathsf{FirstBit} \sqsubseteq \mathsf{Flip}\Big).$$

If the bit at position i has to be flipped, and its value is 1, then the bit at position $i+1$ also has to be flipped. In order to identify the time line that corresponds to the next bit position, we wait until the current A-counter reaches $2^n - 1$, so that we can identify the next world in which the A-counter now has a value of 0. This behaviour is captured by the following axioms:

$$\Box\Big(\bigsqcap_{i\in[\![1,n]\!]} A_i \sqcap \mathsf{Flip} \sqcap \mathsf{Bit} \sqsubseteq \mathsf{P}_{=1}((\bigsqcap_{i\in[\![1,n]\!]} \neg A_i) \to \mathsf{Flip})\Big)$$
$$\Box\Big((\bigsqcap_{i\in[\![1,n]\!]} A_i) \sqcap (\neg\mathsf{Flip} \sqcup \neg\mathsf{Bit}) \sqsubseteq \mathsf{P}_{=1}((\bigsqcap_{i\in[\![1,n]\!]} \neg A_i) \to (\mathsf{FirstBit} \sqcup \neg\mathsf{Flip}))\Big).$$

The disjunction with $\mathsf{FirstBit}$ in the last axiom ensures we do not conflict with the first bit, which always has to be flipped.

This completes the specification of the double exponential counter. Now every individual satisfying Init will initialise its B-counter value with 0, which is increased every 2^n time points, and goes down back to 0 after it reached its maximal value of $2^{2^n} - 1$.

Step 3: Enforce Tiling Conditions. The double exponential corridor tiling is now implemented using 2^{2^n} individuals, each carrying a double exponential B-counter, which represent the rows of the tiling, while the columns are represented along the time line. Note that, since we do not have rigid roles, it is not possible to keep the connection between these individuals stable using roles. Instead, we use GCIs to transfer tile information from one individual to the next. For this, we use a similar trick as for the double exponential counter. We enforce the existence of 2^{2^n} different individuals which, at each time point, carry a different B-counter value. At each time point, we can identify two individuals easily: the individual with counter value 0 satisfies $\mathsf{P}_{=1}\neg\mathsf{Bit}$, and the individual whose counter value is $2^{2^n} - 1$ satisfies $\mathsf{P}_{=1}\mathsf{Bit}$. We keep the current tile type on an individual until its counter value reaches $2^{2^n} - 1$, and then enforce the tiling conditions using this fact.

First, we make sure that at each time point, each domain element represents exacly one tiling type $t \in T$, represented using the concept name t:

$$\Box\Big(\top \sqsubseteq \bigsqcup_{t\in T} t)$$
$$\Box \bigwedge_{t,t'\in T, ti\neq j} (t \sqcap t' \sqsubseteq \bot).$$

Next, we initialise the first row, which represents the inital tile type t_0, and is marked with the special concept $\mathsf{FirstRow}$:

$$\neg(\mathsf{Init} \sqsubseteq \bot)$$
$$\mathsf{Init} \sqsubseteq \mathsf{FirstRow} \sqcap t_0.$$

Note that these axioms are not under a $\Box$-operator, and thus only have to be satisfied at the first time point of the interpretation.

We use role-successors to step-wise enforce the existence of the remaining rows:

$$\Box(\mathsf{P}_{=1}\mathsf{Bit} \sqsubseteq \exists r.\mathsf{P}_{=1}\mathsf{Init}).$$

We use a special concept $\mathsf{Success}$ to mark the success of the tiling, which is step-wise transferred along all other rows.

$$\Box(\mathsf{FirstRow} \sqsubseteq t_f \leftrightarrow \mathsf{Success})$$
$$\Box(\mathsf{Success} \sqsubseteq \bigcirc\mathsf{Success})$$
$$\Box\Big(\neg(\mathsf{Success} \sqcap \mathsf{P}_{=1}\mathsf{Bit} \sqsubseteq \bot) \leftrightarrow (\mathsf{P}_{=1}\neg\mathsf{Bit} \sqsubseteq \bigcirc\mathsf{Success})\Big)$$

The negated GCI in the last formulae expresses the existence of some individual satisfying $\mathsf{Success} \sqcap \mathsf{P}_{=1}\mathsf{Bit}$. Note that these axioms also ensure that individuals can *only* satisfy $\mathsf{Success}$ *after* a successful tiling has been completed, since otherwise, the concept would be back-propagated to the first row, which then has to satisfy t_f.

Each individual keeps its tile type until its B-counter reaches $2^{2^n} - 1$, and then enforces the horizontal and the vertical tiling conditions. For the horizontal condition, we just compare with the next time point. In each case, tiling conditions have only to be checked if we have not already compeleted the tiling.

$$\Box \bigwedge_{t\in T} (t \sqcap \mathsf{P}_{<1}\mathsf{Bit} \sqsubseteq \bigcirc t)$$
$$\Box \bigwedge_{t\in T} \bigvee_{\langle t,t'\rangle\in H} \Big(t \sqcap \mathsf{P}_{=1}\mathsf{Bit} \sqsubseteq \mathsf{Success} \sqcup \bigcirc t'\Big)$$

For the vertical condition, we identify the next row via its B-counter value.

$$\Box \bigwedge_{t\in T} \Big(\neg(t \sqcap \mathsf{P}_{=1}\mathsf{Bit} \sqsubseteq \mathsf{Success}) \to \bigvee_{\langle t,t'\rangle\in V} (\mathsf{P}_{=1}\neg\mathsf{Bit} \sqsubseteq t')\Big)$$

Again, the negated GCI is used to express the existence of some individual that satisfying $t \sqcap \mathsf{P}_{=1}\mathsf{Bit}$, that does not also satisfy $\mathsf{Success}$.

To complete the construction, we use the following axiom to express that existence of a successful tiling:

$$\mathsf{Init} \sqsubseteq \Diamond\mathsf{Success}.$$

The final TPDF ϕ is now a conjunction of all TPDFs. It is standard to verify that ϕ is of the required form, and is satisfiable iff the tiling problem has a solution. We obtain that satisfiability of TPDFs is 2ExpSpace-hard. $\Box$