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Projection in a Description Logic of Context with Actions (Extended Version)

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Contents

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Abstract

Projection is the problem of checking whether the execution of a given sequence of actions will achieve its goal starting from some initial state. In this paper, we study a setting where we combine a two-dimensional Description Logic of context (ConDL) with an action formalism. We choose a well-studied ConDL where both: the possible states of a dynamical system itself (object level) and also different context-dependent views on this system state (context level) are organised in relational structures and can be described using usual DL constructs. To represent how such a system and its views evolve we introduce a suitable action formalism. It allows to describe change on both levels. Furthermore, the observable changes on the object level due to an action execution can also be context-dependent. We show that the formalism is well-behaved in the sense that projection has the same complexity as standard reasoning tasks in case $\mathcal{ALCO}$ is the underlying DL.

1 Introduction

The *role-based* paradigm of modelling languages has been introduced for the design of adaptive and context-sensitive software systems. The concept of roles has been used at different levels of abstraction, for example in data models [5], in a formal high-level modelling language [9] for dynamical systems, and as an extension of more low-level object-oriented programming languages [10]. Unlike in a classical object-oriented setting, where an object has a fixed number of methods attached to it, in a role-based setting an object adapts its behaviour dynamically according to the roles it can play in different contexts. For example, in the conference management system used for this workshop the concept of roles is quite prominent. In the context of this workshop a researcher might play the role of an author whereas in context of some other conference also the role of a program committee member can be played by the same researcher. Both the view on submissions and the abilities to change something are context-dependent and can change over time in this scenario.

How to deal with explicit context extensions of modelling languages efficiently is a well-studied research topic in different areas (e.g. [8, 10, 6]). In [7] Böhme and Lippmann studied a family of *contextualized Description Logics* (ConDLs). For this family a reasoning tool has been implemented and it has been used for translating and checking consistency of models of a role-based modelling language for software systems [9, 6].

However, ConDLs are only suitable for expressing *static* context-dependent knowledge. In this paper, we focus on an extension with *dynamic aspects* and introduce a *ConDL-based action formalism* for reasoning about change in context models. To talk about particular states we consider the ConDL $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ from [7]. It is a two-sorted logic with a *meta level* signature for describing contexts and an *object level* signature for the object domain. $\mathcal{ALCO}$ is the outer *meta level* logic and is used to describe sets of contexts and relations among them. Each context element of the meta level domain corresponds to a relational structure of the inner object level, which is represented using $\mathcal{ALCO}$ as well. Both levels are connected with a modality that allows to access the object level from the meta level. In the example of the conference management system one could think of a model, where we talk about researcher accounts with their properties (for example, being PC member or author) and relations (like conflict of interest with someone else) on the meta level, and where each account corresponds to an individual view on concrete submissions and reviews on the object level. The action formalism we introduce allows to describe changes on both levels. For example, the meta level can change if someone becomes a PC member or declares conflict of interest with someone. An object level action could be if a particular review is entered for a submission. The observable changes of this action from the perspective of a particular account depend on its meta level properties. As a reasoning task we consider the *projection problem*. Projection is the problem of checking whether the execution of a given sequence of actions will achieve its goal starting from some initial state. In our example, a typical projection query could ask whether, after a subreviewer has been assigned by a PC member to some submission sub and after a review has been entered by some other PC member for this submission sub, the subreviewer is able to see this review or not. To solve projection we reduce it polynomially to consistency in the underlying logic $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ by applying techniques that have been used before for reasoning in DL-based action formalisms [4].

The remainder of this paper is structured as follows. In the next section we recall the definitions of $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$. Section 3 introduces our action formalism and defines projection. In Section 4 we present our reduction method for deciding projection and we finish with a conclusion in Section 5.

2 The Description Logic of Context

For representing context-dependent knowledge we choose $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$, a simple member of the family of ConDLs studied in [7]. To keep this part as simple as possible we focus only on the standard DL $\mathcal{ALCO}$ on both levels. Before defining the two dimensional DL $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ we first briefly recall the basic definitions of standard $\mathcal{ALCO}$. For a thorough introduction to DLs we refer to [1, 2].

Definition 1 (Syntax and semantics of $\mathcal{ALCO}$)**.** *Let* $\mathsf{N} = (\mathsf{N_C}, \mathsf{N_R}, \mathsf{N_I})$ *be a* signature *of disjoint sets of* concept names*,* role names *and* individual names*, respectively. Let* $A \in \mathsf{N_C}$*,* $r \in \mathsf{N_R}$ *and* $a \in \mathsf{N_I}$*. An* $\mathcal{ALCO}$-concept C *is built according to the following syntax rule*

$$C ::= \top \mid A \mid \{a\} \mid C \sqcap C \mid \neg C \mid \exists r.C.$$

Let C *and* D *be* $\mathcal{ALCO}$*-concept. A* general concept inclusion *(GCI) is of the form* $C \sqsubseteq D$*. An* $\mathcal{ALCO}$-KB φ *is a Boolean combination of GCIs.*

The semantics *is defined in terms of an* interpretation $\mathcal{I} = (\Delta_\mathcal{I}, \cdot^\mathcal{I})$ *over* N*, where* $\Delta_\mathcal{I}$ *is the non-empty domain of* $\mathcal{I}$ *and* $\cdot^\mathcal{I}$ *is a mapping that maps each* $A \in \mathsf{N_C}$ *to a set* $A^\mathcal{I} \subseteq \Delta_\mathcal{I}$*, each* $r \in \mathsf{N_R}$ *to a relation* $r^\mathcal{I} \subseteq \Delta_\mathcal{I} \times \Delta_\mathcal{I}$ *and each* $a \in \mathsf{N_I}$ *to an element* $a^\mathcal{I} \in \Delta_\mathcal{I}$*. We make the unique name assumption, that is, all individual names refer to different domain elements. Furthermore, the mapping* $\cdot^\mathcal{I}$ *is extended to complex concepts* C, D *as follows:* $\top^\mathcal{I} := \Delta_\mathcal{I}$*,* $(\{a\})^\mathcal{I} := \{a^\mathcal{I}\}$*,* $(C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}$*,* $(\neg C)^\mathcal{I} := \Delta_\mathcal{I} \setminus C^\mathcal{I}$*,* $(\exists r.C)^\mathcal{I} := \{d \in \Delta_\mathcal{I} \mid \text{ there is } e \in \Delta_\mathcal{I} \text{ with } (d,e) \in r^\mathcal{I} \text{ and } e \in C^\mathcal{I}\}$*. The interpretation* $\mathcal{I}$ *is a model of a GCI* $C \sqsubseteq D$ *iff* $C^\mathcal{I} \subseteq D^\mathcal{I}$*. The definition of a model of a KB* φ *as a Boolean combination of GCIs is defined as usual.*

Assume in an example domain about conference management we have a concept name *Subs* (set of submissions), a role name *has-review* and an individual $\mathsf{sub_1}$. We can describe the set of submissions without a review as the $\mathcal{ALCO}$-concept: $\mathit{Subs} \sqcap \neg(\exists \mathit{has\text{-}review}.\top)$ and a GCI like $\{\mathsf{sub_1}\} \sqsubseteq \mathit{Subs} \sqcap \neg(\exists \mathit{has\text{-}review}.\top)$ expresses that the individual $\mathsf{sub_1}$ is an instance of this concept. In the extended logic $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ we are going to define next, one can add an additional level on top which in our example domain could be the level where we talk about PC members, authors their potential conflicts and their different views on the level of submissions and reviews.

The logic is *two-sorted* with a *meta level signature* $\mathsf{M} = (\mathsf{M_C}, \mathsf{M_R}, \mathsf{M_I})$ and an *object level signature* $\mathsf{O} = (\mathsf{O_C}, \mathsf{O_R}, \mathsf{O_I})$. We call $\mathsf{M_C}$, $\mathsf{M_R}$ and $\mathsf{M_I}$ the set of *meta concept names*, *role names*, and *individual names* respectively. Analogously, $\mathsf{O_C}$, $\mathsf{O_R}$, $\mathsf{O_I}$ is called the set of *object concept names*, *role names*, and *individual names* respectively. All sets are assumed to be pairwise disjoint.

We assume the standard definition of the syntax of $\mathcal{ALCO}$-*concepts*, *general concept inclusions (GCIs)* and KBs (see [1, 2] for details) and the corresponding semantics in terms of interpretations over some signature (either M or O in our case).

Definition 2 (Syntax)**.** *Let φ be an $\mathcal{ALCO}$-KB over the object level signature O and $A \in \mathsf{M_C}$, $r \in \mathsf{M_R}$ and $a \in \mathsf{M_I}$ meta level names. An $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$-*meta level concept description *C over M and O (*m-concept *for short) is built according to the following syntax rule*

$$C ::= A \mid \{a\} \mid [\![\varphi]\!] \mid C \sqcap C \mid \neg C \mid \exists r.C.$$

Further constructors are defined as abbreviations: $\top := \neg(A \sqcap \neg A)$ and $\bot := (A \sqcap \neg A)$ (for some $A \in \mathsf{M_C}$), $C \sqcup D := \neg(C \sqcap \neg D)$ and $\forall r.C := \neg\exists r.\neg C$.

*Let C and D be m-concepts. An $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$-*Boolean meta level knowledge base *ψ over M and O (m-*KB *for short) is built according to the following syntax rule*

$$\psi ::= C \sqsubseteq D \mid \psi \wedge \psi \mid \neg\psi.$$

Notation for concept assertions *and* role assertions *is used as abbreviations: $(a : C) := \{a\} \sqsubseteq C$ and $((a, b) : r) := \{a\} \sqsubseteq \exists r.\{b\}$. Further Boolean connectives like $\vee$ and $\rightarrow$ are defined as usual.*

The semantics of $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ is defined in terms of *nested interpretations*. The structure consists of a single meta level interpretation over M where each domain element is associated with an object level interpretation over O.

Definition 3 (Nested Interpretation)**.** *A* nested interpretation *$\mathfrak{I}$ (over M and O) is a tuple of the form $\mathfrak{I} := (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$, where*

- $(\mathbb{C}, \cdot^{\mathfrak{I}})$ *is an M-interpretation, and*
- $\mathcal{I}_c := (\Delta, \cdot^{\mathcal{I}_c})$ *is an O-interpretation for each $c \in \mathbb{C}$.*

Definition 4 (Semantics)**.** *Let $\mathfrak{I} = (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c\in\mathbb{C}})$ be a nested interpretation. The* extension of the mapping *$\cdot^{\mathfrak{I}}$* to complex m-concepts *is defined by induction on the structure of m-concepts C and D as follows:*

$$\begin{aligned}
(\{a\})^{\mathfrak{I}} &:= \{a^{\mathfrak{I}}\};\\
([\![\varphi]\!])^{\mathfrak{I}} &:= \{c \in \mathbb{C} \mid \mathcal{I}_c \models \varphi\};\\
(C \sqcap D)^{\mathfrak{I}} &:= C^{\mathfrak{I}} \cap D^{\mathfrak{I}};\\
(\neg C)^{\mathfrak{I}} &:= \mathbb{C} \setminus C^{\mathfrak{I}}\\
(\exists r.C)^{\mathfrak{I}} &:= \{c \in \mathbb{C} \mid \text{ there exists } c' \in \mathbb{C} \text{ such that } (c, c') \in r^{\mathfrak{I}} \text{ and } c' \in C^{\mathfrak{I}}\},
\end{aligned}$$

where $a \in \mathsf{M_I}$, $r \in \mathsf{M_R}$ and φ is an $\mathcal{ALCO}$-KB over O.

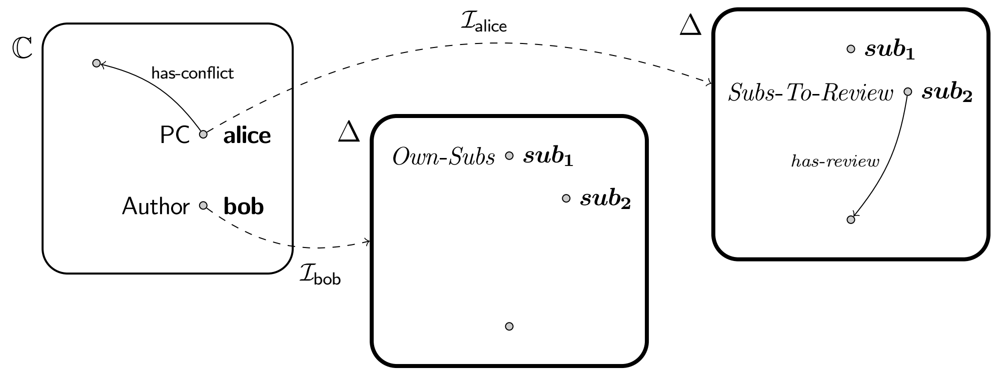

Figure 1: A model of axioms 1-5

Let ψ be an m-KB. Satisfaction of ψ in $\mathfrak{I}$, *written as* $\mathfrak{I} \models \psi$ *(*$\mathfrak{I}$ is a model of ψ*), is defined by induction on the structure of* ψ *as follows:*

$$\begin{array}{ll} \mathfrak{I} \models C \sqsubseteq D & \textit{iff } C^{\mathfrak{I}} \subseteq D^{\mathfrak{I}}; \\ \mathfrak{I} \models \psi_1 \wedge \psi_2 & \textit{iff } \mathfrak{I} \models \psi_1 \textit{ and } \mathfrak{I} \models \psi_2; \\ \mathfrak{I} \models \neg\psi_1 & \textit{iff } \mathfrak{I} \not\models \psi_1. \end{array}$$

Example 5. *We describe some aspects of a conference management domain. On the meta level we talk about accounts that can be PC members of a conference (meta level concept name* PC*) with possibly conflict of interest (meta level role* has-conflict*) to authors (concept name* Author*). Each account has a particular view on the object level where we have a domain of submissions and reviews. The object level concept names Subs-To-Review and Own-Subs describe the assigned submissions for reviewing and their own written submissions, respectively. The object level role has-review relates submissions to their reviews. We describe an initial situation using the meta level individual names* bob's-account *and* alice's-account *and the object level name* sub_1 *denoting a concrete submission. Intuitively, in this model the meta level concept*

$$[\![\mathsf{sub}_1 : \mathit{Own\text{-}Subs}]\!]$$

describes the set of accounts (meta level domain elements) in which sub_1 *is an instance of Own-Subs. Therefore, it represents the set of author accounts of* sub_1*. The following meta level axioms represent some initial knowledge:*

$$\mathsf{alice\text{'}s\text{-}account} : (\forall \mathsf{has\text{-}conflict}.\neg[\![\mathsf{sub}_1 : \mathit{Own\text{-}Subs}]\!]) \tag{1}$$

$$\mathsf{bob\text{'}s\text{-}account} : [\![\mathsf{sub}_1 : \mathit{Own\text{-}Subs}]\!] \tag{2}$$

$$\top \sqsubseteq [\![\mathsf{sub}_1 : \forall \mathit{has\text{-}review}.\bot]\!] \tag{3}$$

$$\mathsf{Author} \equiv \neg[\![\mathit{Own\text{-}Subs} \sqsubseteq \bot]\!] \tag{4}$$

$$\neg[\![\mathit{Subs\text{-}To\text{-}Review} \sqsubseteq \bot]\!] \sqsubseteq \mathsf{PC} \tag{5}$$

Alice has no conflict of interest with an author of the submission sub_1 (1)*. Bob is an author of* sub_1 (2)*, which has not received any reviews yet* (3)*. Author accounts are defined as those accounts with own submissions* (4)*. Only PC members are allowed to review* (5)*.*

We show a model (nested interpretation) of axioms (1) - (5) *in Figure 1. It shows the meta level with PC members and authors on the left labelled with* $\mathbb{C}$*. For the sake of conciseness, we use* alice *and* bob *to denote* alice's-account *and* bob's-account*. These are the two named accounts in our domain. In this particular model* alice *is a* PC *member with a conflict of interest to some unnamed account. In the middle and on the right of the figure the particular views of* bob *and* alice *on submissions and reviews are shown.*

3 Representing Context-dependent Change

We define separate *action descriptions* for the object level and the meta level. Action descriptions are complex expressions with constructs for describing conditional and simultaneous execution. Semantically, actions update interpretations by changing the membership of named individuals to concept names or of of pairs of named individuals to role names.

Definition 6. *Let ψ be an m-KB and $A \in \mathsf{M_C}$, $r \in \mathsf{M_R}$ and $a, b \in \mathsf{M_I}$ meta level names. An* M-action description α *(*M-action *for short) is built according to the following syntax rule:*

$$\alpha := \langle A \oplus a\rangle \mid \langle A \ominus a\rangle \mid \langle r \oplus (a,b)\rangle \mid \langle r \ominus (a,b)\rangle \mid (\psi \triangleright \alpha) \mid (\alpha \parallel \alpha).$$

Let C be an m-concept and $B \in \mathsf{O_C}$, $s \in \mathsf{O_R}$ and $o, o' \in \mathsf{O_I}$ object level names. An O-action description β *(*O-action *for short) is built according to the following syntax rule:*

$$\beta := \langle B \oplus o\rangle \mid \langle B \ominus o\rangle \mid \langle s \oplus (o,o')\rangle \mid \langle s \ominus (o,o')\rangle \mid (C \triangleright \beta) \mid (\beta \parallel \beta).$$

We write just action *if we do not distinguish between* M*-actions and* O*-actions. Actions of the form $\langle A \oplus a\rangle$, $\langle r \oplus (a,b)\rangle$, $\langle B \oplus o\rangle$ or $\langle s \oplus (o,o')\rangle$ are called* atomic effects*.*

An atomic effect like $\langle A \oplus a\rangle$ over some signature N change an N-interpretation $\mathcal{I}$ by adding $a^{\mathcal{I}}$ to $A^{\mathcal{I}}$ and $\langle r \ominus (a,b)\rangle$ deletes $(a^{\mathcal{I}}, b^{\mathcal{I}})$ from $r^{\mathcal{I}}$. M-actions have m-KBs as conditions and O-actions have m-concepts as conditions. A conditional M-action $(\psi \triangleright \alpha_1)$ takes effect in the meta-level interpretation only if ψ is satisfied and a conditional O-action of the form $C \triangleright \beta_1$ means that an O-interpretation $\mathcal{I}_c$ in a nested interpretation $\mathfrak{I}$ is only updated with β_1 if c belongs to C in $\mathfrak{I}$. The construct $(\alpha_1 \circ \alpha_2)$ means that α_1 and α_2 are executed simultaneously.

For the first step of the definition of the execution semantics we define how a set of atomic M-effects (or atomic O-effects) updates a non-nested M-interpretation (or O-interpretation).

Definition 7 (Update)**.** *Let $\mathsf{N} \in \{\mathsf{M}, \mathsf{O}\}$ denote either the meta-level or object-level signature, and let $\mathcal{I} := (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an* N*-interpretation and $\mathfrak{E}$ a set of atomic* N*-effects. The* update of $\mathcal{I}$ with $\mathfrak{E}$ *is an interpretation denoted by $\mathcal{I}^{\mathfrak{E}}$ and is defined for all $A \in \mathsf{N_C}$, all $r \in \mathsf{N_R}$ and all $a \in \mathsf{N_I}$ as follows*

$$\begin{aligned}
\Delta_{\mathcal{I}^{\mathfrak{E}}} &:= \Delta_{\mathcal{I}};\\
A^{\mathcal{I}^{\mathfrak{E}}} &:= \big(A^{\mathcal{I}} \setminus \{a^{\mathcal{I}} \mid \langle A \ominus a\rangle \in \mathfrak{E}\}\big) \cup \{b^{\mathcal{I}} \mid \langle A \oplus b\rangle \in \mathfrak{E}\}\\
r^{\mathcal{I}^{\mathfrak{E}}} &:= \big(r^{\mathcal{I}} \setminus \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid \langle r \ominus (a,b)\rangle \in \mathfrak{E}\}\big) \cup \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid \langle r \oplus (a,b)\rangle \in \mathfrak{E}\}\\
a^{\mathcal{I}^{\mathfrak{E}}} &:= a^{\mathcal{I}}.
\end{aligned}$$

Next, we define the atomic effects of the execution of a complex action in a (nested) interpretation.

Definition 8 (Effects)**.** *Let $\mathfrak{I} = (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c \in \mathbb{C}})$ be a nested interpretation and α an* M*-action. The* set of atomic M-effects for $\mathfrak{I}$ and α*, denoted by $\mathcal{E}(\alpha, \mathfrak{I})$, is defined by induction on the structure of α as follows*

$$\begin{aligned}
&\mathcal{E}(\langle A \oplus a\rangle, \mathfrak{I}) := \{\langle A \oplus a\rangle\} \textit{ and } \mathcal{E}(\langle r \oplus (a,b)\rangle, \mathfrak{I}) := \{\langle r \oplus (a,b)\rangle\}\\
&\mathcal{E}(\psi \triangleright \alpha_1, \mathfrak{I}) \;:= \begin{cases} \mathcal{E}(\alpha_1, \mathfrak{I}) & \textit{if } \mathfrak{I} \models \psi,\\ \emptyset & \textit{otherwise;} \end{cases}\\
&\mathcal{E}(\alpha_1 \parallel \alpha_2, \mathfrak{I}) := \mathcal{E}(\alpha_1, \mathfrak{I}) \cup \mathcal{E}(\alpha_2, \mathfrak{I}).
\end{aligned}$$

Let β be an O*-action and* $c \in \mathbb{C}$*. The* set of sets of atomic O-effects for $\mathfrak{I}$, c and β, *denoted by* $\mathcal{E}(\beta, c, \mathfrak{I})$*, is defined by induction on the structure of* β *as follows*

$$\mathcal{E}(\langle B \oplus o\rangle, c, \mathfrak{I}) := \{\langle B \oplus o\rangle\} \textit{ and } \mathcal{E}(\langle s \oplus (o, o')\rangle, c, \mathfrak{I}) := \{\langle s \oplus (o, o')\rangle\}$$

$$\mathcal{E}(C \triangleright \beta_1, c, \mathfrak{I}) \ := \begin{cases} \mathcal{E}(\beta_1, c, \mathfrak{I}) & \textit{if } c \in C^{\mathfrak{I}}, \\ \emptyset & \textit{otherwise}; \end{cases}$$

$$\mathcal{E}(\beta_1 \parallel \beta_2, c, \mathfrak{I}) \ := \mathcal{E}(\beta_1, c, \mathfrak{I}) \cup \mathcal{E}(\beta_2, c, \mathfrak{I}).$$

An M-action only updates the outer meta level interpretation of a nested interpretation and an O-action leaves the meta level interpretation unchanged and updates all object level interpretation simultaneously.

Definition 9 (Nested Update)**.** *Let* $\mathfrak{I} := (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{I}_c\}_{c \in \mathbb{C}})$ *be a nested interpretation,* α *an* M*-action and* β *an* O*-action. The* update of $\mathfrak{I}$ with α *is the nested interpretation*

$$\mathfrak{I}^{\alpha} := (\mathbb{C}, \cdot^{\mathfrak{I}^{\alpha}}, \Delta, \{\mathcal{I}_c\}_{c \in \mathbb{C}}),$$

where $(\mathbb{C}, \cdot^{\mathfrak{I}^{\alpha}})$ *is the update of* $(\mathbb{C}, \cdot^{\mathfrak{I}})$ *with* $\mathcal{E}(\alpha, \mathfrak{I})$ *and all other components are unchanged. The* update of $\mathfrak{I}$ with β *is the nested interpretation*

$$\mathfrak{I}^{\beta} := (\mathbb{C}, \cdot^{\mathfrak{I}}, \Delta, \{\mathcal{J}_c\}_{c \in \mathbb{C}}),$$

where for each $c \in \mathbb{C}$ *the* O*-interpretation* $\mathcal{J}_c := (\Delta, \cdot^{\mathcal{J}_c})$ *is the update of* $\mathcal{I}_c$ *with* $\mathcal{E}(\beta, c, \mathfrak{I})$*.*

Let σ *be a sequence of* M*-actions and* O*-actions the update* $\mathfrak{I}^{\sigma}$ *is defined in the obvious way by induction on* σ*.*

Note that it is possible to write an action that adds and deletes an object (or a pair of objects) to and from a name simultaneously. The semantics of updates gives precedence to add effects but we want to exclude those descriptions. In the following we assume that for any M-action α, any O-action β and any nested interpretation $\mathfrak{I}$ and meta level domain element c the sets $\mathcal{E}(\alpha, \mathfrak{I})$ and $\mathcal{E}(\beta, c, \mathfrak{I})$ are non-contradictory.

We are interested to check whether a certain consequence formulated as an m-KB holds after executing a sequence of actions given an incomplete representation of the initial state in terms of an m-KB.

Definition 10 (Projection Problem)**.** *Let* ψ, ψ' *be m-KBs and* σ *a sequence of actions. We say that* ψ' *is a consequence of executing* σ *in* ψ *iff for all models* $\mathfrak{I}$ *of* ψ*, we have that* $\mathfrak{I}^{\sigma} \models \psi'$*. The projection problem is then to decide whether* ψ' *is a consequence of executing* σ *in* ψ*.*

We continue our example about the conference management system.

Example 11 (Example 5 continued)**.** *An* M*-action for adding Alice as a PC Member of DL is given by*

$$\mathsf{add\text{-}pc} := \langle \mathsf{DL\text{-}PC} \oplus \mathsf{alice's\text{-}account}\rangle.$$

Alice gets assigned the submission sub *for reviewing under the condition that she has no conflict of interest with an author of this submission. It is defined as an* O*-action:*

$$\begin{aligned} \mathsf{add\text{-}sub} := \ & (\{\mathsf{alice's\text{-}account}\} \sqcap \neg\exists \mathsf{has\text{-}conflict}.[\![\mathsf{sub} : \mathit{Own\text{-}Subs}]\!]) \triangleright \\ & \langle \mathit{Subs\text{-}To\text{-}Review} \oplus \mathsf{sub}\rangle \end{aligned}$$

Note that only the account of Alice is affected. The action only updates the object level interpretation associated with $\mathsf{alice's\text{-}account}$ *by adding* sub_1 *to the review set (Subs-To-Review).*

Let rev *be the name for the review Alice has written for* sub_1*. We define an* O*-action that enters this review to the system and removes* sub_1 *from the review list of Alice simultaneously.*

$$\begin{aligned}\texttt{finish} &:= \texttt{enter} \parallel \texttt{remove};\\ \texttt{enter} &:= (\mathsf{PC} \sqcap \neg\exists \mathsf{has\text{-}conflict}.[\![\mathsf{sub}_1 : \mathit{Own\text{-}Subs}]\!]) \rhd \langle \mathit{has\text{-}review} \oplus (\mathsf{sub}_1, \mathsf{rev})\rangle;\\ \texttt{remove} &:= \{\mathsf{alice\text{'}s\text{-}account}\} \rhd \langle \mathit{Subs\text{-}To\text{-}Review} \ominus s\rangle.\end{aligned}$$

The review rev *is only visible for PC members with no conflict of interest with someone that is an author of* sub_1*.*

Assume initially we have axioms (1)-(5) *from Example 5. After performing the sequence* $\texttt{add-pc}; \texttt{add-sub}$ *it holds that*

$$\mathsf{alice\text{'}s\text{-}account} : [\![\mathsf{sub}_1 : \mathit{Subs\text{-}To\text{-}Review}]\!]$$

is true and the constraint

$$\neg[\![\mathit{Subs\text{-}To\text{-}Review} \sqsubseteq \bot]\!] \sqsubseteq \mathsf{PC}$$

is preserved. Furthermore, after $\texttt{add-pc}; \texttt{add-sub}; \texttt{finish}$ *we have that*

$$\mathsf{alice\text{'}s\text{-}account} : [\![(\mathsf{sub}_1, \mathsf{rev}) : \mathit{has\text{-}review}]\!]$$

is true.

4 Deciding the Projection Problem in $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$

The approach of solving the projection problem in a DL-based action formalism by reducing it to a standard consistency problem in the underlying DL has been applied already in several settings (e.g. [4, 3, 11]). The overall idea we use here is similar to previous techniques extended to nested structures in our case.

As a first step we introduce a normal form of action descriptions by conjoining conditions and pushing them inside. We say that an N-action μ with $\mathsf{N} \in \{\mathsf{M}, \mathsf{O}\}$ is in *normal form* if it is of the form

$$(\psi_1 \rhd e_1) \parallel ... \parallel (\psi_n \rhd e_n),$$

where each e_i, for any i, $1 \leq i \leq n$ is an atomic N-effect and ψ_i is either an m-KB (in case of an M-action) or an m-concept (in case of an O-action). We normalize an arbitrary N-action by applying exhaustively the following rules:

$$\psi_1 \rhd (\psi_2 \rhd \mu) \rightsquigarrow (\psi_1 \star \psi_2) \rhd \mu \qquad \psi \rhd (\mu_1 \parallel \mu_2) \rightsquigarrow (\psi \rhd \mu_1) \parallel (\psi \rhd \alpha_2)\,,$$

where $\star \in \{\wedge, \sqcap\}$ stands for $\wedge$ in case of an M-action and for $\sqcap$ in case of an O-action. W.l.o.g., we assume from now on that any action is in the normal form. For convenience, we denote a normal form of an N-action μ as a set of atomic N-effects with a single condition attached: $\mu = \{(\psi_1 \rhd e_1), ..., (\psi_n \rhd e_n)\}$.

Let an m-KB ψ (initial state), ψ' (goal state) and a sequence of actions $\sigma = \mu_1, ..., \mu_n$ in normal form be the input of the projection problem. Our goal is to construct a *reduction m-KB* that is consistent iff ψ' is a consequence of executing σ in ψ.

We say that concepts, roles, and individuals are relevant if they occur in the input of the projection problem. For the reduction we use fresh concept names and role names of the corresponding sort. For each execution step $0 \leq i \leq n$, we introduce fresh time-stamped copies $A^{(i)}$ of all relevant concept names, $r^{(i)}$ of all relevant role names, and fresh time-stamped concept names $T_C^{(i)}$ for every relevant complex subconcept C. $A^{(0)}$ refers to the initial content of A and the further copies $A^{(j)}$, $1 \leq j \geq n$ refer only to the set of named individual names of the corresponding sort that are instance of A after the jth execution step. This holds for both concept and role names. The copies of the form $T_C^{(i)}$ represent the content (both named and unnamed) of the complex concept C after the ith execution step. The distinction between named and unnamed is made because actions only affect named individuals.

Furthermore, for the set of all named individuals of sort object in the input (denoted by $\mathrm{Obj}_{\mathsf{O}}$) and for the set of all named meta level individuals in the input ($\mathrm{Obj}_{\mathsf{M}}$) two fresh concept names N_{O} and N_{M}, respectively, are introduced.

The meaning of the new names is now axiomatized using meta level axioms as follows. For N_{O} and N_{M} we have

$$\psi_{\mathrm{obj}} = (N_{\mathsf{M}} \equiv \bigsqcup_{c \in \mathrm{Obj}_{\mathsf{M}}} \{c\}) \wedge (\top \sqsubseteq [\![N_{\mathsf{O}} \equiv \bigsqcup_{a \in \mathrm{Obj}_{\mathsf{O}}} \{a\}]\!]).$$

We use $\tau(C, i)$ to denote the concept definition we introduce to define the names of the form $T_C^{(i)}$. It is defined by induction on the structure of C as follows:

$$\begin{aligned}
\top \sqsubseteq [\![T_A^{(i)} &\equiv (N_{\mathsf{O}} \sqcap A^{(i)}) \sqcup (\neg N_{\mathsf{O}} \sqcap A^{(0)})]\!];\\
\top \sqsubseteq [\![T_{\{a\}}^{(i)} &\equiv \{a\}]\!];\\
\top \sqsubseteq [\![T_{\neg C}^{(i)} &\equiv \neg T_C^{(i)}]\!];\\
\top \sqsubseteq [\![T_{C_1 \sqcap C_2}^{(i)} &\equiv T_{C_1}^{(i)} \sqcap T_{C_2}^{(i)}]\!];\\
\top \sqsubseteq [\![T_{\exists r.C}^{(i)} &\equiv (N_{\mathsf{O}} \sqcap ((\exists r^{(0)}.(\neg N_{\mathsf{O}} \sqcap T_C^{(i)})) \sqcup (\exists r^{(i)}.(N_{\mathsf{O}} \sqcap T_C^{(i)})))) \sqcup\\
&\quad (\neg N_{\mathsf{O}} \sqcap \exists r^{(0)}.T_C^{(i)})]\!].
\end{aligned}$$

Given an O-GCI $\gamma = C \sqsubseteq D$ and a timestamp $0 \leq i \leq n$, we define the timestamped copy $\gamma^{(i)} := T_C^{(i)} \sqsubseteq T_D^{(i)}$, and $\varphi^{(i)}$ as the result of replacing every O-GCI γ in φ by $\gamma^{(i)}$.

Similarly, we define $\psi_{\mathrm{def}_{\mathsf{M}}}^{(i)}$ as a conjunction of m-concept definitions for every relevant m-concept G, depends on the form of G.

$$\begin{aligned}
T_E^{(i)} &\equiv (N_{\mathsf{M}} \sqcap E^{(i)}) \sqcup (\neg N_{\mathsf{M}} \sqcap E^{(0)});\\
T_{\{c\}}^{(i)} &\equiv \{c\};\\
T_{\neg G}^{(i)} &\equiv \neg T_G^{(i)};\\
T_{G_1 \sqcap G_2}^{(i)} &\equiv T_{G_1}^{(i)} \sqcap T_{G_2}^{(i)};\\
T_{\exists s.G}^{(i)} &\equiv (N_{\mathsf{M}} \sqcap ((\exists s^{(0)}.(\neg N_{\mathsf{M}} \sqcap T_G^{(i)})) \sqcup (\exists s^{(i)}.(N_{\mathsf{M}} \sqcap T_G^{(i)})))) \sqcup\\
&\quad (\neg N_{\mathsf{M}} \sqcap \exists s^{(0)}.T_G^{(i)});\\
T_{[\![\varphi]\!]}^{(i)} &\equiv [\![\varphi^{(i)}]\!].
\end{aligned}$$

Analogously, $\zeta^{(i)} = T_G^{(i)} \sqsubseteq T_H^{(i)}$ for an m-GCI $\zeta = G \sqsubseteq H$ and a timestamp i, $0 \leq i \leq n$. Furthermore, given an m-KB ψ and an i, $0 \leq i \leq n$, we denote by $\psi^{(i)}$ the result of replacing every m-GCI ζ in ψ by $\zeta^{(i)}$.

We simply put timestamp zero for the initial knowledge base ψ, i.e., we include $\psi^{(0)}$ as a conjunct of the reduction m-KB.

Then, we encode the effect of each action μ_i of σ. We define an m-KB $\psi^{(i)}_{\text{act}}$ that encode M-effects of action μ_i. We distinguish two cases, whether the action is an M-action or an O-action. First, we consider the case of μ_i is an M-action α_i. Intuitively, we make sure if the condition is satisfied, then corresponding unconditional effects are applied to the next step.

$$\begin{aligned}
\psi^{(i)}_{\text{act}_\mathsf{M}} := & \bigwedge_{\psi \triangleright \langle E \ominus c\rangle \in \alpha_i} (\psi^{(i-1)} \rightarrow (c : \neg E^{(i)})) \wedge \\
& \bigwedge_{\psi \triangleright \langle E \oplus c\rangle \in \alpha_i} (\psi^{(i-1)} \rightarrow (c : E^{(i)})) \wedge \\
& \bigwedge_{\psi \triangleright \langle s \ominus (c,d)\rangle \in \alpha_i} (\psi^{(i-1)} \rightarrow ((c,d) : \neg s^{(i)})) \wedge \\
& \bigwedge_{\psi \triangleright \langle s \oplus (c,d)\rangle \in \alpha_i} (\psi^{(i-1)} \rightarrow ((c,d) : s^{(i)}))
\end{aligned}$$

We encode the O-effects similarly, with taking the context into account. Instead of having an m-KB, we have a timestamped m-concept as the condition. The O-effects are propagated using referring meta concept for those contexts.

$$\begin{aligned}
\psi^{(i)}_{\text{act}_\mathsf{O}} := & \bigwedge_{G \triangleright \langle A \ominus a\rangle \in \beta_i} (T^{(i-1)}_G \sqsubseteq [\![(a : \neg A^{(i)})]\!]) \wedge \\
& \bigwedge_{G \triangleright \langle A \oplus a\rangle \in \beta_i} (T^{(i-1)}_G \sqsubseteq [\![(a : A^{(i)})]\!]) \wedge \\
& \bigwedge_{G \triangleright \langle r \ominus (a,b)\rangle \in \beta_i} (T^{(i-1)}_G \sqsubseteq [\![((a,b) : \neg r^{(i)})]\!]) \wedge \\
& \bigwedge_{G \triangleright \langle r \oplus (a,b)\rangle \in \beta_i} (T^{(i-1)}_G \sqsubseteq [\![((a,b) : r^{(i)})]\!])
\end{aligned}$$

In case of the other type of action happens at timestamp i, the corresponding $\psi^{(i)}_{\text{act}}$ is simply $\top$. For example, $\psi^{(i)}_{\text{act}_\mathsf{M}} = \top$ if μ_i is an O-action.

Then, we make sure a change only happens if there is an effect that enforces it. The $\psi^{(i)}_{\text{min}_\mathsf{M}}$ ensures a minimization of changes to the names individuals on the meta level. For every i, $1 \leq i \leq n$, we define $\psi^{(i)}_{\text{min}_\mathsf{M}}$:

$$\begin{aligned}
\psi^{(i)}_{\text{min}_\mathsf{M}} := & \bigwedge_{c \in \mathrm{Obj}_\mathsf{M}, E \in \mathsf{M}_\mathsf{C}} (((c : E^{(i)}) \wedge \bigwedge_{\psi \triangleright \langle E \ominus c\rangle \in \alpha_i} \neg\psi^{(i-1)}) \rightarrow (c : E^{(i)})) \wedge \\
& \bigwedge_{c \in \mathrm{Obj}_\mathsf{M}, E \in \mathsf{M}_\mathsf{C}} (((c : \neg E^{(i)}) \wedge \bigwedge_{\psi \triangleright \langle E \oplus c\rangle \in \alpha_i} \neg\psi^{(i-1)}) \rightarrow (c : \neg E^{(i)})) \wedge \\
& \bigwedge_{c,d \in \mathrm{Obj}_\mathsf{M}, s \in \mathsf{M}_\mathsf{R}} (((c,d) : s^{(i)}) \wedge \bigwedge_{\psi \triangleright \langle s \ominus (c,d)\rangle \in \alpha_i} \neg\psi^{(i-1)}) \rightarrow (c,d) : s^{(i)})) \wedge \\
& \bigwedge_{c,d \in \mathrm{Obj}_\mathsf{M}, s \in \mathsf{M}_\mathsf{R}} (((c,d) : \neg s^{(i)}) \wedge \bigwedge_{\psi \triangleright \langle s \oplus (c,d)\rangle \in \alpha_i} \neg\psi^{(i-1)}) \rightarrow (c,d) : \neg s^{(i)}).
\end{aligned}$$

Similarly, we ensure a minimization of changes on the object level. For every i, $1 \leq i \leq n$

$$\begin{aligned}
\psi^{(i)}_{\mathrm{min}_{\mathsf{O}}} := & \bigwedge_{a \in \mathrm{Obj}_{\mathsf{O}}, A \in \mathsf{O}_{\mathsf{C}}} ((\llbracket (a : A^{(i)}) \rrbracket \sqcap \bigsqcap_{G \rhd \langle A \ominus a \rangle \in \beta_i} \neg T^{(i-1)}_G) \sqsubseteq \llbracket (a : A^{(i)}) \rrbracket) \wedge \\
& \bigwedge_{a \in \mathrm{Obj}_{\mathsf{O}}, A \in \mathsf{O}_{\mathsf{C}}} ((\llbracket (a : \neg A^{(i)}) \rrbracket \sqcap \bigsqcap_{G \rhd \langle A \oplus a \rangle \in \beta_i} \neg T^{(i-1)}_G) \sqsubseteq \llbracket (a : \neg A^{(i)}) \rrbracket) \wedge \\
& \bigwedge_{a, b \in \mathrm{Obj}_{\mathsf{O}}, r \in \mathsf{O}_{\mathsf{R}}} ((\llbracket ((a, b) : r^{(i)}) \rrbracket \sqcap \bigsqcap_{G \rhd \langle r \ominus (a,b) \rangle \in \beta_i} \neg T^{(i-1)}_G) \sqsubseteq \llbracket ((a, b) : r^{(i)}) \rrbracket) \wedge \\
& \bigwedge_{a, b \in \mathrm{Obj}_{\mathsf{O}}, r \in \mathsf{O}_{\mathsf{R}}} ((\llbracket ((a, b) : \neg r^{(i)}) \rrbracket \sqcap \bigsqcap_{G \rhd \langle r \oplus (a,b) \rangle \in \beta_i} \neg T^{(i-1)}_G) \sqsubseteq \llbracket ((a, b) : \neg r^{(i)}) \rrbracket).
\end{aligned}$$

In case of the other type of action happens at timestamp i, the corresponding $\psi^{(i)}_{\mathrm{act}}$ is simply $\top$. For example, $\psi^{(i)}_{\mathrm{act}_{\mathsf{M}}} = \top$ if μ_i is an O-action.

Finally, we define the complete reduction:

$$\begin{aligned}
\psi_{\mathrm{red}} := \quad & \psi_{\mathrm{init}} \wedge \psi_{\mathrm{obj}} \wedge \bigwedge_{0 \leq i \leq n} \psi^{(i)}_{\mathrm{def}_{\mathsf{O}}} \wedge \bigwedge_{0 \leq i \leq n} \psi^{(i)}_{\mathrm{def}_{\mathsf{M}}} \wedge \\
& \bigwedge_{1 \leq i \leq n} \psi^{(i)}_{\mathrm{act}_{\mathsf{O}}} \wedge \bigwedge_{1 \leq i \leq n} \psi^{(i)}_{\mathrm{act}_{\mathsf{M}}} \wedge \bigwedge_{1 \leq i \leq n} \psi^{(i)}_{\mathrm{min}_{\mathsf{O}}} \wedge \bigwedge_{1 \leq i \leq n} \psi^{(i)}_{\mathrm{min}_{\mathsf{M}}}
\end{aligned}$$

Lemma 12. *Let ψ be an m-KB, $\sigma = \mu_1, ..., \mu_n$ be a sequence of actions, and ψ_{red} be defined as above. The following properties hold:*

1. *For every sequence of nested-interpretations $\mathfrak{I}_0, ..., \mathfrak{I}_n$ such that $\mathfrak{I}_0 \models \psi$ and $\mathfrak{I}_i = \mathfrak{I}^{\alpha_i}_{i-1}$ for each i, $1 \leq i \leq n$ there exists an interpretation $\mathfrak{L} \models \psi_{\mathrm{red}}$ such that:*
   (a) *for every i, $0 \leq i \leq n$, and every relevant concept G, we have $G^{\mathfrak{I}_i} = (T^{(i)}_G)^{\mathfrak{L}}$;*
   (b) *for every i, $0 \leq i \leq n$, and every relevant GCI β, we have $\mathfrak{I}_i \models \beta$ iff $\mathfrak{L} \models \beta^{(i)}$; and*
   (c) *for every i, $0 \leq i \leq n$, and every relevant m-KB ψ, we have $\mathfrak{I}_i \models \psi$ iff $\mathfrak{L} \models \psi^{(i)}$.*
2. *For every nested-interpretation $\mathfrak{L} \models \psi_{\mathrm{red}}$, there exists a sequence of nested-interpretations $\mathfrak{I}_0, ..., \mathfrak{I}_n$ such that $\mathfrak{I}_0 \models \psi$ and $\mathfrak{I}_i = \mathfrak{I}^{\alpha_i}_{i-1}$ for every i, $1 \leq i \leq n$ such that:*
   (a) *for every i, $0 \leq i \leq n$, and every relevant concept G, we have $G^{\mathfrak{I}_i} = (T^{(i)}_G)^{\mathfrak{L}}$;*
   (b) *for every i, $0 \leq i \leq n$, and every relevant GCI β, we have $\mathfrak{I}_i \models \beta$ iff $\mathfrak{L} \models \beta^{(i)}$; and*
   (c) *for every i, $0 \leq i \leq n$, and every relevant m-KB ψ, we have $\mathfrak{I}_i \models \psi$ iff $\mathfrak{L} \models \psi^{(i)}$.*

Proof. It is easy to see that ψ_{red} is of size polynomial in the size of σ, $\mathcal{R}_{\mathsf{M}}$ and $\mathcal{R}_{\mathsf{O}}$. We first prove Property (1). Let $\mathfrak{I}_0 = (\mathbb{C}^{\mathfrak{I}_0}, \cdot^{\mathfrak{I}_0}, \Delta^{\mathfrak{I}_0}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{I}_0}_{c \in \mathbb{C}}), ..., \mathfrak{I}_n = (\mathbb{C}^{\mathfrak{I}_n}, \cdot^{\mathfrak{I}_n}, \Delta^{\mathfrak{I}_n}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{I}_n}_{c \in \mathbb{C}})$ such that $\mathfrak{I}_0 \models \psi$ and $\mathfrak{I}_i = \mathfrak{I}^{\alpha_i}_{i-1}$ for each i, $1 \leq i \leq n$. We define the interpretation $\mathfrak{L} = (\mathbb{C}^{\mathfrak{L}}, \cdot^{\mathfrak{L}}, \Delta^{\mathfrak{L}}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{L}}_{c \in \mathbb{C}})$ as follows:

- $\mathbb{C}^{\mathfrak{L}} := \mathbb{C}^{\mathfrak{I}_0}$;
- $c^{\mathfrak{L}} := c^{\mathfrak{I}_0}$ for every $c \in \mathsf{M}_{\mathsf{I}}$;
- $N^{\mathfrak{L}}_{\mathsf{M}} := \{c^{\mathfrak{L}} \mid c \in \mathrm{Obj}_{\mathsf{M}}\}$
- $(E^{(i)})^{\mathfrak{L}} := E^{\mathfrak{I}_i}$ for every $G \in \mathcal{R}_{\mathsf{M}} \cap \mathsf{M}_{\mathsf{C}}$ and every i, $0 \leq i \leq n$;
- $(s^{(i)})^{\mathfrak{L}} := s^{\mathfrak{I}_i}$ for every $s \in \mathcal{R}_{\mathsf{M}} \cap \mathsf{M}_{\mathsf{R}}$ and every i, $0 \leq i \leq n$;

- $(T_G^{(i)})^{\mathfrak{L}} := G^{\mathfrak{I}_i}$ for every m-concept $G \in \mathcal{R}_{\mathsf{M}}$ and every i, $0 \leq i \leq n$.
- for each $c \in \mathbb{C}$:
  - $\Delta^{\mathcal{I}_c^{\mathfrak{L}}} := \Delta^{\mathcal{I}_c^{\mathfrak{I}_0}}$;
  - $a^{\mathcal{I}_c^{\mathfrak{L}}} := a^{\mathcal{I}_c^{\mathfrak{I}_0}}$ for every $a \in \mathsf{O_I}$;
  - $N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} := \{a^{\mathcal{I}_c^{\mathfrak{I}_i}} \mid a \in \mathrm{Obj}_{\mathsf{O}}\}$;
  - $(A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} := A^{\mathcal{I}_c^{\mathfrak{I}_i}}$ for every $A \in \mathcal{R}_{\mathsf{O}} \cap \mathsf{O_C}$ and every i, $0 \leq i \leq n$;
  - $(r^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} := r^{\mathcal{I}_c^{\mathfrak{I}_i}}$ for every $r \in \mathcal{R}_{\mathsf{O}} \cap \mathsf{O_R}$ and every i, $0 \leq i \leq n$;
  - $(T_C^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} := C^{\mathcal{I}_c^{\mathfrak{I}_i}}$ for every O-concept $C \in \mathcal{R}_{\mathsf{O}}$ and every i, $0 \leq i \leq n$.

Property (a) follows from the definition of $\mathfrak{L}$, and consequently Property (b) and (c). It remains for us to show that $\mathfrak{L} \models \psi_{\text{red}}$.

It is easy to see that $\mathfrak{L} \models \psi_{\text{obj}}$ follows directly from the definition of both $N_{\mathsf{M}}^{\mathfrak{L}}$ and $(N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}})$ (of each $c \in \mathbb{C}$). Furthermore, $\mathfrak{L} \models \psi_{\text{init}}$ follows immediately from the Property (c) due to $\psi_{\text{init}} = \psi^{(0)}$ and $\mathfrak{I}_0 \models \psi$.

First, we show a claim that unnamed individual membership of concepts and roles can be found in the initial state:

Claim 13. *For every $c \in \mathbb{C}$, and every i, $0 \leq i \leq n$:*

1. *for every $A \in \mathsf{O_C}$, we have $A^{\mathcal{I}_c^{\mathfrak{I}_0}} \setminus \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} = A^{\mathcal{I}_c^{\mathfrak{I}_i}} \setminus \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}$;*
2. *for every $r \in \mathsf{O_R}$, we have $r^{\mathcal{I}_c^{\mathfrak{I}_0}} \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \times \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}) = r^{\mathcal{I}_c^{\mathfrak{I}_i}} \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \times \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}})$.*

First, we show point 1 by induction on i. Take arbitrary $c \in \mathbb{C}$ and $A \in \mathsf{O_C}$. For the base case $i = 0$, the claim is trivially satisfied. Assume that the claim holds for i, i.e., $(A^{\mathcal{I}_c^{\mathfrak{I}_0}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}) = (A^{\mathcal{I}_c^{\mathfrak{I}_i}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}})$. Then, we show $(A^{\mathcal{I}_c^{\mathfrak{I}_i}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}) = (A^{\mathcal{I}_c^{\mathfrak{I}_{i+1}}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}})$. Since $\mathfrak{I}_{i+1} = \mathfrak{I}_i^{\alpha_{i+1}}$, we have that $A^{\mathcal{I}_c^{\mathfrak{I}_{i+1}}} \setminus \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} = (A^{\mathcal{I}_c^{\mathfrak{I}_i}} \setminus \{a^{\mathcal{I}} \mid \langle A \ominus a \rangle \in \mathcal{E}\} \cup \{b^{\mathcal{I}} \mid \langle A \oplus b \rangle \in \mathcal{E}\}) \setminus \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}$ where $\mathcal{E} = \{e \mid C \triangleright e \in \beta, c \in C^{\mathfrak{I}}\}$. We know that $\{a^{\mathcal{I}} \mid \langle A \ominus a \rangle \in \mathcal{E}\} \subseteq \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}$ and $\{b^{\mathcal{I}} \mid \langle A \oplus b \rangle \in \mathcal{E}\} \subseteq \mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}$ by the definition of $\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}$. Thus, we have $(A^{\mathcal{I}_c^{\mathfrak{I}_{i+1}}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}}) = (A^{\mathcal{I}_c^{\mathfrak{I}_i}}) \setminus (\mathsf{N}_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}})$. Property 2 of the claim for O-role can be shown analogously.

We have a similar claim for the meta level:

Claim 14. *For every $c \in \mathbb{C}$, and every i, $0 \leq i \leq n$:*

1. *for every $E \in \mathsf{M_C}$, we have $E^{\mathfrak{I}_0} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} = E^{\mathfrak{I}_i} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$;*
2. *for every $S \in \mathsf{M_R}$, we have $s^{\mathfrak{I}_0} \setminus (\mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} \times \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}) = s^{\mathfrak{I}_i} \setminus (\mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} \times \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}})$.*

We show point 1 by induction on i. Take any $E \in \mathsf{M_C}$. For the base case $i = 0$, the claim is trivially satisfied. Assume that the claim holds for i, i.e. $E^{\mathfrak{I}_0} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} = E^{\mathfrak{I}_i} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$. Then, we show $E^{\mathfrak{I}_i} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} = E^{\mathfrak{I}_{i+1}} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$. Since $\mathfrak{I}_{i+1} = \mathfrak{I}_i^{\alpha}$, we have that $E^{\mathfrak{I}_{i+1}} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} = (E^{\mathfrak{I}_i} \setminus \{c^{\mathcal{I}} \mid \langle E \ominus c \rangle \in \mathcal{E}\} \cup \{d^{\mathcal{I}} \mid \langle E \oplus d \rangle \in \mathcal{E}\}) \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$ where $\mathcal{E} = \{e \mid (\varphi \triangleright e) \in \mathcal{E}, \mathcal{I} \models \varphi\}$. We know that $\{c^{\mathcal{I}} \mid \langle E \ominus c \rangle \in \mathcal{E}\} \subseteq \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$ and $\{d^{\mathcal{I}} \mid \langle E \oplus d \rangle \in \mathcal{E}\}) \subseteq \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$. Thus, we have $E^{\mathfrak{I}_{i+1}} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}} = E^{\mathfrak{I}_i} \setminus \mathsf{N}_{\mathsf{M}}^{\mathfrak{L}}$. Again, Property 2 of the claim can be shown analogously.

Now, we show that $\mathfrak{L} \models \psi_{\text{def}_{\mathsf{O}}}$, for every concept $C \in \mathcal{R}_{\mathsf{O}}$ and every i, $0 \leq i \leq n$. We prove by case distinction of the form of C:

- $\top \sqsubseteq [\![T^{(i)}_A \equiv (N_O \sqcap A^{(i)}) \sqcup (\neg N_O \sqcap A^{(0)})]\!]$. Then, it is enough to show that for any context $c \in \mathbb{C}$, $(T^{(i)}_A)^{\mathcal{I}^{\mathfrak{L}}_c} = ((N_O \sqcap A^{(i)}) \sqcup (\neg N_O \sqcap A^{(0)}))^{\mathcal{I}^{\mathfrak{L}}_c}$ holds. We have:

$$\begin{aligned}
(T^{(i)}_A)^{\mathcal{I}^{\mathfrak{L}}_c} &= A^{\mathcal{I}^{\mathfrak{I}_i}_c} && \text{by definition}\\
&= (N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap A^{\mathcal{I}^{\mathfrak{I}_i}_c}) \cup (\neg N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap A^{\mathcal{I}^{\mathfrak{I}_i}_c}) && \text{by the semantics}\\
&= (N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap A^{\mathcal{I}^{\mathfrak{I}_i}_c}) \cup (\neg N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap A^{\mathcal{I}^{\mathfrak{I}_0}_c}) && \text{by Claim 13}\\
&= (N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap (A^{(i)})^{\mathcal{I}^{\mathfrak{L}}_c}) \cup (\neg N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap (A^{(0)})^{\mathcal{I}^{\mathfrak{L}}_c}) && \text{by definition}\\
&= (N_{\mathsf{O}} \sqcap (A^{(i)})) \sqcup (\neg N_{\mathsf{O}} \sqcap (A^{(0)}))^{\mathcal{I}^{\mathfrak{L}}_c} && \text{by the semantics}
\end{aligned}$$

- $\top \sqsubseteq [\![T^{(i)}_{\{a\}} \equiv \{a\}]\!]$. Then, it is enough to show that for any context $c \in \mathbb{C}$ and any $a \in \mathrm{Obj}_{\mathsf{O}}$, we have that $(T^{(i)}_{\{a\}})^{\mathcal{I}^{\mathfrak{L}}_c} = \{a\}^{\mathcal{I}^{\mathfrak{L}}_c}$ holds. By definition, in any context $c \in \mathbb{C}$, we have $(T^{(i)}_{\{a\}})^{\mathcal{I}^{\mathfrak{L}}_c} = \{a\}^{\mathcal{I}^{\mathfrak{I}_i}_c} = \{a^{\mathcal{I}^{\mathfrak{I}_i}_c}\} = \{a^{\mathcal{I}^{\mathfrak{L}}_c}\} = \{a\}^{\mathcal{I}^{\mathfrak{L}}_c}$.

- $\top \sqsubseteq [\![T^{(i)}_{\neg C} \equiv \neg T^{(i)}_C]\!]$. Then, it is enough to show that for any context $c \in \mathbb{C}$, we have that $(T^{(i)}_{\neg C})^{\mathcal{I}^{\mathfrak{L}}_c} = (\neg T^{(i)}_C)^{\mathcal{I}^{\mathfrak{L}}_c}$ holds. By definition, in any context $c \in \mathbb{C}$, we have that $(T^{(i)}_{\neg C})^{\mathcal{I}^{\mathfrak{L}}_c} = (\neg C)^{\mathcal{I}^{\mathfrak{I}_i}_c} = \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus C^{\mathcal{I}^{\mathfrak{I}_i}_c} = \Delta^{\mathcal{I}^{\mathfrak{L}}_c} \setminus (T^{(i)}_C)^{\mathcal{I}^{\mathfrak{L}}_c} = (\neg T^{(i)}_C)^{\mathcal{I}^{\mathfrak{L}}_c}$.

- $\top \sqsubseteq [\![T^{(i)}_{C_1 \sqcap C_2} \equiv T^{(i)}_{C_1} \sqcap T^{(i)}_{C_2}]\!]$. Then, it is enough to show that for any context $c \in \mathbb{C}$, we have that $(T^{(i)}_{C_1 \sqcap C_2})^{\mathcal{I}^{\mathfrak{L}}_c} = (T^{(i)}_{C_1} \sqcap T^{(i)}_{C_2})^{\mathcal{I}^{\mathfrak{L}}_c}$. By definition, in any context $c \in \mathbb{C}$, we have that $(T^{(i)}_{C_1 \sqcap C_2})^{\mathcal{I}^{\mathfrak{L}}_c} = (C_1 \sqcap C_2)^{\mathcal{I}^{\mathfrak{I}_i}_c} = C_1^{\mathcal{I}^{\mathfrak{I}_i}_c} \cap C_2^{\mathcal{I}^{\mathfrak{I}_i}_c} = (T^{(i)}_{C_1})^{\mathcal{I}^{\mathfrak{L}}_c} \cap (T^{(i)}_{C_1})^{\mathcal{I}^{\mathfrak{L}}_c} = (T^{(i)}_{C_1} \sqcap T^{(i)}_{C_2})^{\mathcal{I}^{\mathfrak{L}}_c}$

- $\top \sqsubseteq [\![T^{(i)}_{\exists r.C} \equiv (N_{\mathsf{O}} \sqcap (\exists r^{(0)}.(\neg N_{\mathsf{O}} \sqcap T^{(i)}_C) \sqcup \exists r^{(i)}.(N_{\mathsf{O}} \sqcap T^{(i)}_C))) \sqcup (\neg N_{\mathsf{O}} \sqcap \exists r^{(0)}.T^{(i)}_C)]\!]$. Then, it is enough to show that for any context $c \in \mathbb{C}$, we have that $(T^{(i)}_{\exists r.C})^{\mathcal{I}^{\mathfrak{L}}_c} = ((N_{\mathsf{O}} \sqcap (\exists r^{(0)}.(\neg N_{\mathsf{O}} \sqcap T^{(i)}_C) \sqcup \exists r^{(i)}.(N_{\mathsf{O}} \sqcap T^{(i)}_C))) \sqcup (\neg N_{\mathsf{O}} \sqcap \exists r^{(0)}.T^{(i)}_C))^{\mathcal{I}^{\mathfrak{L}}_c}$. By definition, in any context $c \in \mathbb{C}$, we have that

$$\begin{aligned}
&(T^{(i)}_{\exists r.C})^{\mathcal{I}^{\mathfrak{L}}_c}\\
&\quad = (\exists r.C)^{\mathcal{I}^{\mathfrak{I}_i}_c}\\
&\quad = \{a \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\}\\
&\quad = \{a \in N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\} \cup\\
&\qquad \{a \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \mid \exists b. b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\\
&\quad = \{a \in N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus N^{\mathcal{I}^{\mathfrak{L}}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\} \cup\\
&\qquad \{a \in N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in N^{\mathcal{I}^{\mathfrak{L}}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\} \cup\\
&\qquad \{a \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \mid \exists b. b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\}\\
&\quad = \{a \in N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus N^{\mathcal{I}^{\mathfrak{L}}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_0}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\} \cup\\
&\qquad \{a \in N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{I}_i}_c} \mid \exists b, b \in N^{\mathcal{I}^{\mathfrak{L}}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\} \cup\\
&\qquad \{a \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \setminus N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \mid \exists b. b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in r^{\mathcal{I}^{\mathfrak{I}_0}_c} \text{and } b \in C^{\mathcal{I}^{\mathfrak{I}_i}_c}\}\\
&\quad = (N_{\mathsf{O}}^{\mathcal{I}^{\mathfrak{L}}_c} \cap\\
&\qquad (\{a \in \Delta^{\mathcal{I}^{\mathfrak{L}}_c} \mid \exists b, b \in \Delta^{\mathcal{I}^{\mathfrak{I}_i}_c} \text{with } (a,b) \in (r^{(0)})^{\mathcal{I}^{\mathfrak{L}}_c} \text{and } b \in (\neg N_{\mathsf{O}} \sqcap T^{(i)}_C)^{\mathcal{I}^{\mathfrak{L}}_c}\} \cup
\end{aligned}$$

$$
\begin{aligned}
&\{a \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \mid \exists b, b \in N^{\mathcal{I}_c^{\mathfrak{L}}} \text{with } (a,b) \in (r^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{and } b \in (N_{\mathsf{O}} \sqcap T_C^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}\})) \cup \\
&((\neg N_{\mathsf{O}})^{\mathcal{I}_c^{\mathfrak{L}}} \cap \\
&\{a \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \mid \exists b, b \in \Delta^{\mathcal{I}_c^{\mathfrak{I}_i}} \text{with } (a,b) \in (r^{(0)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{and } b \in (T_C^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}\}) \\
= &((N_{\mathsf{O}} \sqcap (\exists r^{(0)}.(\neg N_{\mathsf{O}} \sqcap T_C^{(i)}) \sqcup \exists r^{(i)}.(N_{\mathsf{O}} \sqcap T_C^{(i)}))) \sqcup (\neg N_{\mathsf{O}} \sqcap \exists r^{(0)}.T_C^{(i)}))^{\mathcal{I}_c^{\mathfrak{L}}}.
\end{aligned}
$$

This ends the proof that $\mathfrak{L} \models \psi^{(i)}_{\mathrm{def}_{\mathsf{O}}}$, for every concept $C \in \mathcal{R}_{\mathsf{O}}$ and for any i, $0 \le i \le n$.

Analogously, we can use a similar proof for the meta level, i.e., to show that $\mathfrak{L} \models \psi_{\mathrm{def}_{\mathsf{M}}}$ for every $G \in \mathcal{R}_{\mathsf{M}}$ and every i, $0 \le i \le n$. However, we show the proof of extension for referring meta concept. First we show a claim, for any $c \in \mathbb{C}$ we have $\mathcal{I}_c^{\mathfrak{I}_i} \models (C \sqsubseteq D)$ iff $\mathcal{I}_c^{\mathfrak{L}} \models (T_C^{(i)} \sqsubseteq T_D^{(i)})$.

We have that

$$
\begin{aligned}
\mathcal{I}_c^{\mathfrak{I}_i} \models (C \sqsubseteq D) &\text{ iff } C^{\mathcal{I}_c^{\mathfrak{I}}} \subseteq D^{\mathcal{I}_c^{\mathfrak{I}}} \\
&\text{ iff } (T_C^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (T_D^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \\
&\text{ iff } \mathcal{I}_c^{\mathfrak{L}} \models (T_C^{(i)} \sqsubseteq T_D^{(i)}).
\end{aligned}
$$

We have to show that for any referring meta concept $[\![\alpha]\!] \in \mathcal{R}_{\mathsf{M}}$, $\mathfrak{L} \models T^{(i)}_{[\![\alpha]\!]} \equiv [\![\alpha^{(i)}]\!]$. By definition, we have that $(T^{(i)}_{[\![\alpha]\!]})^{\mathfrak{L}} = [\![\alpha]\!]^{\mathfrak{I}_i} \doteq [\![\alpha^{(i)}]\!]^{\mathfrak{L}}$, where $\doteq$ follows from the previous claim. This finishes the proof that $\mathfrak{L} \models \bigwedge_{1 \le i \le n} \psi^{(i)}_{\mathrm{def}_{\mathsf{M}}}$.

Now we show that $\mathfrak{L} \models \psi^{(i)}_{\mathrm{act}_{\mathsf{O}}}$ for any i, $1 \le i \le n$. We prove by a case distinction of each conjunction. Assume it is of the form $T_G^{(i-1)} \sqsubseteq [\![(a : \neg A^{(i)})]\!]$. It is enough to show that if $c \in (T_G^{(i-1)})^{\mathfrak{L}}$ then $c \in [\![(a : A^{(i)})]\!]^{\mathfrak{L}}$. Assume that $c \in (T_G^{(i-1)})^{\mathfrak{L}}$, and consequently $c \in G^{\mathfrak{I}_{i-1}}$. Due to the construction definition, there exists $G \triangleright \langle A \ominus a \rangle \in \beta_i$. Then, we have $\langle A \ominus a \rangle \in \mathcal{E}(\mathfrak{I}_{i-1})$ due to the semantics. Since $\mathcal{I}_c^{\mathfrak{I}_i} = (\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\alpha_i}$, we have $a \notin A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Then, we have $a \in \Delta \setminus A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Then we have $\{a^{\mathcal{I}_c^{\mathfrak{I}_i}}\} \subseteq \Delta \setminus A^{\mathcal{I}_c^i}$. Then, we have $\{a^{\mathcal{I}_c^{\mathfrak{I}_i}}\} \subseteq \Delta \setminus A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Then, we have $\{a\}^{\mathcal{I}_c^{\mathfrak{I}_i}} \subseteq \Delta \setminus A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Then, we have $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq \Delta \setminus (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Finally, we have $(\{a\} \sqsubseteq A^{(i)})^{\mathfrak{L}}$. Thus, we show that $c \in [\![\{a\} \sqsubseteq A^{(i)}]\!]^{\mathfrak{L}}$. All other three cases can be proven analogously. Furthermore, in case of α_i is an M-action, we have a trivial case where $\psi^{(i)}_{\mathrm{act}_{\mathsf{O}}} = \top$.

Now we show that $\mathfrak{L} \models \psi^{(i)}_{\mathrm{act}_{\mathsf{M}}}$ for any i, $1 \le i \le n$. We prove by a case distinction of each conjunction. Assume it is of the form $\varphi^{(i-1)} \to \{c\} \sqsubseteq \neg E^{(i)}$. Assume that $\mathfrak{L} \models \varphi^{(i-1)}$. Due to the construction definition, there exists $\varphi \triangleright \langle E \ominus c \rangle \in \mathsf{Eff}_{\mathsf{M}}(\alpha_i)$. Then, we have $\langle E \ominus c \rangle \in \mathcal{E}(\mathfrak{I}_{i-1})$ due to the semantics. Since $\mathfrak{I}_i = (\mathfrak{I}_{i-1})^{\alpha_i}$, we have $c \notin E^{\mathfrak{I}_i}$. Then, we have $e \in \mathbb{C} \setminus E^{\mathfrak{I}_i}$. Then, we have $\{c^{\mathfrak{I}_i}\} \subseteq \mathbb{C} \setminus E^{\mathfrak{I}_i}$. Then, we have $\{c\}^{\mathfrak{I}_i} \subseteq \mathbb{C} \setminus E^{\mathfrak{I}_i}$. Then, we have $\{c\}^{\mathfrak{L}} \subseteq \mathbb{C} \setminus E^{\mathfrak{L}}$. Thus, we show that $\mathfrak{L} \models \{c\} \sqsubseteq \neg E$. All other three cases can be proven analogously. Furthermore, in case of α_i is an O-action, we have a trivial case where $\psi^{(i)}_{\mathrm{act}_{\mathsf{M}}} = \top$.

We now prove that $\mathfrak{L} \models \psi^{(i)}_{\mathrm{min}_{\mathsf{O}}}$ for any i, $1 \le i \le n$. We prove by a case distinction of each conjunction. Assume it is of the form $(([\![(a : A^{(i-1)})]\!] \sqcap \bigsqcap_{G \triangleright \langle A \ominus a \rangle \in \beta_i} \neg T_G^{(i)}) \sqsubseteq [\![(a : A^{(i)})]\!]$ for some $a \in \mathsf{Ind}_{\mathsf{O}}$ and $A \in \mathsf{O}_{\mathsf{C}}$. Then, we have to show for any $c \in \mathbb{C}$ such that $c \in [\![(a : A^{(i-1)})]\!]^{\mathfrak{L}}$ and $c \in (\neg T_G^{(i)})^{\mathfrak{L}}$ for any $G \triangleright \langle A \ominus a \rangle \in \beta_i$, we have that $c \in [\![(a : A^{(i)})]\!]^{\mathfrak{L}}$. We use a proof by contradiction. For arbitrary $c \in \mathbb{C}$, assume that:

1. $c \in [\![a : A^{(i-1)}]\!]^{\mathfrak{L}}$;
2. $c \in (\bigsqcap_{G \triangleright \langle A \ominus a \rangle \in \mathrm{Eff}_{\mathsf{O}}(\alpha_i)} \neg T_G^{(i-1)})^{\mathfrak{L}}$; and

3. $c \notin ([\![(a : A^{(i)})]\!]^{\mathfrak{L}}$.

It is easy to see that $a \in (A^{(i-1)})^{\mathcal{I}_c^{\mathfrak{L}}}$ follows from (1). Then, $a \in A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$. Since $\mathcal{I}_c^{\mathfrak{I}_i} = (\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\alpha_i}$, we have that $A^{\mathcal{I}_c^{\mathfrak{I}_i}} := A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \setminus \{a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \mid \langle A \ominus a\rangle \in \mathcal{E}\} \cup \{b^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \mid \langle A \oplus b\rangle \in \mathcal{E}\}$. Since we have (3), then $a \notin (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ and consequently, $a \notin A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Then, there exists some $\langle A \ominus a\rangle \in \mathcal{E}$. By definition, there exists some $G \triangleright \langle A \ominus a\rangle \in \beta_i$ such that $c \in G^{\mathfrak{I}_{i-1}}$. By definition of $\mathfrak{L}$, we have that $c \in (T_G^{(i-1)})^{\mathfrak{L}}$ for some $G \triangleright \langle A \ominus a\rangle \in \beta_i$. This contradicts (2).

All other three cases can be proven analogously. This ends the proof of $\mathfrak{L} \models \bigwedge_{1\leq i\leq n} \psi^{(i)}_{\text{min}_\mathsf{O}}$.

We now prove that $\mathfrak{L} \models \psi^{(i)}_{\text{min}_\mathsf{M}}$ for any i, $1 \leq i \leq n$. We prove again by a case distinction of each conjunction. Assume it is of the form $((c : E^{(i-1)}) \wedge \bigwedge_{\varphi\triangleright\langle E\ominus c\rangle\in\alpha_i} \neg\varphi^{(i)}) \rightarrow (\{c\} \sqsubseteq E^{(i)})$ for some $c \in \mathsf{Ind}_\mathsf{M}$ and $E \in \mathsf{M_C}$. Then, we show that if $\mathfrak{L} \models (\{c\} \sqsubseteq E^{(i-1)})$ and $\mathfrak{L} \models \bigwedge_{\varphi\triangleright\langle E\ominus c\rangle\in\alpha_i} \neg\varphi^{(i)}$ then $(c : E^{(i)})$. We use a proof by contradiction. Assume that:

1. $\mathfrak{L} \models \{c\} \sqsubseteq E^{(i-1)}$;
2. $\mathfrak{L} \models \bigwedge_{\varphi\triangleright\langle E\ominus c\rangle\in\text{Eff}_\mathsf{M}(\alpha_i)} \neg\varphi^{(i)}$; and
3. $\mathfrak{L} \models \{c\} \sqsubseteq E^{(i)}$.

It is easy to see that $c^{\mathfrak{L}} \in (E^{(i-1)})^{\mathfrak{L}}$ follows from (1). Then, $c \in (E^{\mathfrak{I}_{i-1}})$. Since $\mathfrak{I}^i = (\mathfrak{I}^{(i-1)})^{\alpha_i}$, we have that $E^{\mathfrak{I}_i} := E^{\mathfrak{I}_{i-1}} \setminus \{c^{\mathfrak{I}_{i-1}} \mid \langle E \ominus e\rangle \in \mathcal{E}\} \cup \{d^{\mathfrak{I}_{i-1}} \mid \langle E \oplus c\rangle \in \mathcal{E}\}$. Since we have (3), then $c \notin (E^{(i)})^{\mathfrak{L}}$ and consequently, $c \notin E^{\mathfrak{I}_i}$. Then, there exists some $\langle E \ominus c\rangle \in \mathcal{E}$. By definition, there exists some $\varphi \triangleright \langle E \ominus c\rangle$ such that $\mathfrak{I}_{i-1} \models \varphi$. By definition of $\mathfrak{L}$, we have that $\mathfrak{L} \models \varphi^{(i)}$ for some $\varphi \triangleright \langle E \ominus c\rangle \in \text{Eff}_\mathsf{M}(\alpha_i)$. This contradicts (2).

All other three cases can be proven analogously. This ends the proof of $\mathfrak{L} \models \bigwedge_{1\leq i\leq n} \psi^{(i)}_{\text{min}_\mathsf{O}}$.

This finishes the proof that $\mathfrak{L} \models \psi_{\text{red}}$. Thus, we have shown Property (1) of Lemma 12.

Now we show Property (2) of Lemma 12. Let $\mathfrak{L} = (\mathbb{C}^{\mathfrak{L}}, \cdot^{\mathfrak{L}}, \Delta^{\mathfrak{L}}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{L}}_{c\in\mathbb{C}})$ be an interpretation such that $\mathfrak{L} \models \psi_{\text{red}}$. We define the interpretations $\mathfrak{I}_0 = (\mathbb{C}^{\mathfrak{I}_0}, \cdot^{\mathfrak{I}_0}, \Delta^{\mathfrak{I}_0}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{I}_0}_{c\in\mathbb{C}}), ..., \mathfrak{I}_n = (\mathbb{C}^{\mathfrak{I}_n}, \cdot^{\mathfrak{I}_n}, \Delta^{\mathfrak{I}_n}, \{\cdot^{\mathcal{I}_c}\}^{\mathfrak{I}_n}_{c\in\mathbb{C}})$ as follows:

- $\mathbb{C}^{\mathfrak{I}_i} := \mathbb{C}^{\mathfrak{L}}$ for every i, $0 \leq i \leq n$;
- $c^{\mathfrak{I}_i} := c^{\mathfrak{L}}$ for every $c \in \mathsf{M_I}$ and every i, $0 \leq i \leq n$;
- $E^{\mathfrak{I}_i} := (T_E^{(i)})^{\mathfrak{L}}$ for every $E \in \mathcal{R}_\mathsf{M} \cap \mathsf{M_C}$ and every i, $0 \leq i \leq n$; and
- $s^{\mathfrak{I}_i} := ((s^{(i)})^{\mathfrak{L}} \cap (N_\mathsf{M}^{\mathfrak{L}} \times N_\mathsf{M}^{\mathfrak{L}})) \cup ((s^{(0)})^{\mathfrak{L}} \cap ((\mathbb{C}^{\mathfrak{L}} \times (\neg N_\mathsf{M})^{\mathfrak{L}}) \cup ((\neg N_\mathsf{M})^{\mathfrak{L}} \times \mathbb{C}^{\mathfrak{L}})))$ for every $s \in \mathcal{R}_\mathsf{M} \cap \mathsf{M_R}$ and every i, $0 \leq i \leq n$.
- for each $c \in \mathbb{C}^{\mathfrak{I}_i}$ for every i, $0 \leq i \leq n$:
  - $\Delta^{\mathcal{I}_c^{\mathfrak{I}_i}} := \Delta^{\mathcal{I}_c^{\mathfrak{L}}}$;
  - $a^{\mathcal{I}_c^{\mathfrak{I}_i}} := a^{\mathcal{I}_c^{\mathfrak{L}}}$ for every $a \in \mathsf{O_I}$ and every i, $0 \leq i \leq n$;
  - $A^{\mathcal{I}_c^{\mathfrak{I}_i}} := (T_A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ for every $A \in \mathcal{R}_\mathsf{O} \cap \mathsf{O_C}$ and every i, $0 \leq i \leq n$; and
  - $r^{\mathcal{I}_c^{\mathfrak{I}_i}} := ((r^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \cap (N_\mathsf{O}^{\mathcal{I}_c^{\mathfrak{L}}} \times N_\mathsf{O}^{\mathcal{I}_c^{\mathfrak{L}}})) \cup ((r^{(0)})^{\mathcal{I}_c^{\mathfrak{L}}} \cap ((\Delta^{\mathcal{I}_c^{\mathfrak{L}}} \times (\neg N_\mathsf{O})^{\mathcal{I}_c^{\mathfrak{L}}}) \cup ((\neg N_\mathsf{O})^{\mathcal{I}_c^{\mathfrak{L}}} \times \Delta^{\mathcal{I}_c^{\mathfrak{L}}})))$ for every $r \in \mathcal{R}_\mathsf{O} \cap \mathsf{O_R}$ and every i, $0 \leq i \leq n$.

Now, we show Property (a). Take any i, $0 \leq i \leq n$. However, we prove a similar claim for the object level first.

Claim 15. *For every $c \in \mathbb{C}$ and every i, $0 \leq i \leq n$, and every relevant O-concept C, we have $C^{\mathcal{I}_c^{\mathfrak{J}_i}} = (T_C^{(i)})^{\mathfrak{L}}$, and consequently $\mathcal{I}_c^{\mathfrak{L}} \models T_C^{(i)} \sqsubseteq T_D^{(i)}$ iff $\mathcal{I}_c^{\mathfrak{J}_i} \models C \sqsubseteq D$*

Take any $c \in \mathbb{C}$, and any i, $0 \leq i \leq n$. We prove it by the induction on the structure of C.

- $C = A \in \mathsf{O_C}$.
  Follows from the definition.
- $C = \{a\}$, where $a \in \mathrm{Obj}_{\mathsf{O}}$.
  Then $\{a\}^{\mathcal{I}_c^{\mathfrak{J}_i}} = \{a^{\mathcal{I}_c^{\mathfrak{J}_i}}\} = \{a^{\mathcal{I}_c^{\mathfrak{L}}}\} = \{a\}^{\mathcal{I}_c^{\mathfrak{L}}} = (T_{\{a\}}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$.
- $C = \neg C_1$
  Then $(\neg C_1)^{\mathcal{I}_c^{\mathfrak{J}_i}} = \Delta^{\mathcal{I}_c^{\mathfrak{J}_i}} \setminus C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} = \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \setminus (T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} = (\neg T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} = (T_{\neg C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$.
- $C = C_1 \sqcap C_2$
  Then $(C_1 \sqcap C_2)^{\mathcal{I}_c^{\mathfrak{J}_i}} = C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} \cap C_2^{\mathcal{I}_c^{\mathfrak{J}_i}} = (T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \cap (T_{C_2}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} = (T_{C_1}^{(i)} \sqcap T_{C_2}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} = (T_{C_1 \sqcap C_2}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$.
- $C = \exists r.C_1$

$$
\begin{aligned}
& a \in (\exists r.C_1)^{\mathcal{I}_c^{\mathfrak{J}_i}} \\
\text{iff } & a \in \Delta^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } b \in C^{\mathcal{I}_c^{\mathfrak{J}_i}} \\
\text{iff } & a \in N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } b \in C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ OR} \\
& a \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \setminus N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } e \in C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} \\
\text{iff } & a \in N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \setminus N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } b \in C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ OR} \\
& a \in N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } b \in C_1^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ OR} \\
& a \in \Delta^{\mathcal{I}_c^{\mathfrak{J}}} \setminus N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in r^{\mathcal{I}_c^{\mathfrak{J}_i}} \text{ and } b \in C_1^{\mathcal{I}_c^{\mathfrak{J}_I}} \\
\text{iff } & a \in N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in (r^{(0)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and } b \in (T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{ OR} \\
& a \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \setminus N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in (r^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and } b \in (T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{ OR} \\
& a \in \Delta^{\mathcal{I}_c^{\mathfrak{J}}} \setminus N_{\mathsf{O}}^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and there exists } b \in \Delta^{\mathcal{I}_c^{\mathfrak{L}}} \text{ such that } (a,b) \in (r^{(0)})^{\mathcal{I}_c^{\mathfrak{L}}} \text{ and } b \in (T_{C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \\
\text{iff } & a \in (N_{\mathsf{O}} \sqcap ((\exists r^{(0)}.(\neg N_{\mathsf{O}} \sqcap T_C^{(i)})) \sqcup (\exists r^{(i)}.(N_{\mathsf{O}} \sqcap T_C^{(i)})))) \sqcup (\neg N_{\mathsf{O}} \sqcap \exists r^{(0)}.T_C^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \\
\text{iff } & a \in (T_{\exists r.C_1}^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \quad \text{(due to } \mathfrak{L} \models \psi_{\mathrm{def}_{\mathsf{O}}}^{(i)} \text{ for case concept } \exists r.C_1).
\end{aligned}
$$

This ends the proof of the claim.

Analogously, we can use a similar proof for the meta level to finally show Property (a) holds. However, we show the extension for the referring meta concept. We show that $[\![\psi]\!]^{\mathfrak{J}_i} = (T_{[\![\psi]\!]}^{(i)})^{\mathfrak{L}}$. We have $[\![\psi]\!]^{\mathfrak{J}_i} = [\![\psi^{(i)}]\!]^{\mathfrak{L}} \doteq (T_{[\![\psi]\!]}^{(i)})^{\mathfrak{L}}$, where $\doteq$ holds due to $\mathfrak{L} \models T_{[\![\psi]\!]}^{(i)} \equiv [\![\psi^{(i)}]\!]$. Thus, we have shown Property (a) holds. Obviously, Property (b) is an easy consequence of Property (a) and Property (c) follows from Property (b) inductively.

Now we show that $\mathfrak{J}_0 \models \psi$, and for every i, $0 \leq i \leq n$ we have that $\mathfrak{J}_i = \mathfrak{J}_{i-1}^{\alpha_i}$. It is easy to see that $\mathfrak{J}_0 \models \psi$ due to $\psi_{\mathrm{init}} = \psi^{(0)}$ and $\mathfrak{L} \models \psi_{\mathrm{init}}$. Next, we show that $\mathfrak{J}_i = \mathfrak{J}_{i-1}^{\alpha_i}$, i.e., the conditions in Definition 8 are satisfied. First, we show the second condition: for any $c \in \mathbb{C}$, $\mathcal{I}_c^{\mathfrak{J}_i} = (\mathcal{I}_c^{\mathfrak{J}_{i-1}})^{\alpha_i}$. By definition, we have $(\mathcal{I}_c^{\mathfrak{J}_{i-1}})^{\alpha_i} = (\mathcal{I}_c^{\mathfrak{J}_{i-1}})^{\mathcal{E}}$, where $\mathcal{E} = \{e \mid C \triangleright e \in \beta_i, c \in C_{i-1}^{\mathfrak{J}}\}$. Now, for every i, $1 \leq i \leq n$ we have

- $\Delta^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} = \Delta^{\mathcal{I}_c^{\mathfrak{L}}} = \Delta^{\mathcal{I}_c^{\mathfrak{I}_i}}$;
- Let $A \in \mathcal{R}_{\mathsf{O}} \cap \mathsf{O}_{\mathsf{C}}$, we show that $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \in A^{\mathcal{I}_c^{\mathfrak{I}_i}}$ iff $a^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}} \in A^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}}$.
  $\Leftarrow$ Assume $a^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}} \in A^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}}$. We consider two cases:
  - $\langle A \oplus a\rangle \in \mathcal{E}$. Then, there exists $G \triangleright \langle A \oplus a\rangle \in \beta_i$ such that $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. By definition, we have $\mathfrak{L} \models T_G^{(i-1)} \sqsubseteq [\![(a : A^{(i)})]\!]$ and $c \in (T_G^{(i-1)})^{\mathfrak{L}}$. Then, $c \in [\![(a : A^{(i)})]\!]^{\mathfrak{L}}$. Then, $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \subseteq (A^{\mathcal{I}_c^{\mathfrak{I}_i}})$. Finally, $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \in A^{\mathcal{I}_c^{\mathfrak{I}_i}}$.
  - $\langle A \oplus a\rangle \notin \mathcal{E}$. Then, $a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \in A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ and there does not exist $G \triangleright \langle A \ominus a\rangle \in \beta_i$ such that $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. Notice that

  $$\mathfrak{L} \models ([\![(a : A^{(i)})]\!] \sqcap \bigsqcap_{G\triangleright\langle A\ominus a\rangle\in\beta_i} \neg T_G^{(i-1)}) \sqsubseteq [\![(a : A^{i})]\!] \tag{6}$$

  We have $a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \in A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ iff $\{a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}\} \subseteq A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ iff $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ iff $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ iff $c \in [\![(a : A^{(i)})]\!]^{\mathfrak{L}}$ (2). Assume $c \notin \bigsqcap_{G\triangleright\langle A\ominus a\rangle\in\beta_i} \neg T_G^{(i-1)}$. Then, there exists some $G \triangleright \langle A \ominus a\rangle \in \beta_i$ such that $c^{\mathfrak{L}} \notin (T_G^{(i-1)})^{\mathfrak{L}}$ iff $c^{\mathfrak{L}} \in (T_G^{(i-1)})^{\mathfrak{L}}$ iff $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$, which contradicts $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. Thus, we have $c^{\mathfrak{L}} \in (\bigsqcap_{G\triangleright\langle A\ominus a\rangle\in\beta_i} \neg T_G^{(i-1)})^{\mathfrak{L}}$ (3). It is easy to see from (1), (2), and (3), we have $c^{\mathfrak{L}} \in ([\![(a : A^{(i)})]\!]^{\mathfrak{L}}$. Then, $(\{a\} \sqsubseteq A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{I}_i}}\} \subseteq A^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Finally, $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \in A^{\mathcal{I}_c^{\mathfrak{I}_i}}$.

  $\Rightarrow$ Proof by contraposition. Assume $a^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}} \notin A^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}}$. We show that $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \notin (A^{\mathcal{I}_c^{\mathfrak{I}_i}})$. We consider two cases:
  - $\langle A \ominus a\rangle \in \mathcal{E}$. Then, there exists $G \triangleright \langle A \ominus a\rangle \in \beta_i$ such that $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. By definition, we have $\mathfrak{L} \models T_G^{(i-1)} \sqsubseteq [\![(a : \neg A^{(i)})]\!]$ and $c \in (T_G^{(i-1)})^{\mathfrak{L}}$. Then, $c \in [\![(a : \neg A^{(i)})]\!]^{\mathfrak{L}}$. Then, $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \sqsubseteq (\neg A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \sqsubseteq (\neg A^{\mathcal{I}_c^{\mathfrak{I}_i}})$. Finally, $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \notin A^{\mathcal{I}_c^{\mathfrak{I}_i}}$.
  - $\langle A \ominus a\rangle \notin \mathcal{E}$. Then, $a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \notin A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ and there does not exist $G \triangleright \langle A \oplus a\rangle \in \beta_i$ such that $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. Notice that

  $$\mathfrak{L} \models ([\![(a : \neg A^{(i)})]\!] \sqcap \bigsqcap_{G\triangleright\langle A\oplus a\rangle\in\mathrm{Eff}_{\mathsf{O}}(\alpha_i)} \neg T_G^{(i-1)}) \sqsubseteq [\![(a : \neg A^{(i)})]\!] \tag{7}$$

  We have $a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \notin A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ iff $\{a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}\} \subseteq A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}} \setminus \Delta^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ iff $\{a^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}\} \subseteq \neg A^{\mathcal{I}_c^{\mathfrak{I}_{i-1}}}$ iff $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \subseteq (\neg A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ iff $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (\neg A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$ iff $c \in [\![\{a\} \sqsubseteq \neg A^{(i-1)}]\!]^{\mathfrak{L}}$ (2). Assume $c \notin \bigsqcap_{G\triangleright\langle A\oplus a\rangle\in\mathrm{Eff}_{\mathsf{O}}(\alpha_i)} \neg T_G^{(i-1)}$. Then, there exists some $G \triangleright \langle A \oplus a\rangle \in \beta_i$ such that $c^{\mathfrak{L}} \notin (T_G^{(i-1)})^{\mathfrak{L}}$ iff $c^{\mathfrak{L}} \in (T_G^{(i-1)})^{\mathfrak{L}}$ iff $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$, which contradicts $c^{\mathfrak{I}_{i-1}} \in G^{\mathfrak{I}_{i-1}}$. Thus, we have $c^{\mathfrak{L}} \in (\bigsqcap_{G\triangleright\langle A\oplus a\rangle\in\mathrm{Eff}_{\mathsf{O}}(\alpha_i)} \neg T_G^{(i-1)})^{\mathfrak{L}}$ (3). It is easy to see from (1), (2), and (3), we have $c^{\mathfrak{L}} \in ([\![(a : \neg A^{(i)})]\!])^{\mathfrak{L}}$. Then, $(\{a\} \sqsubseteq \neg A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a\}^{\mathcal{I}_c^{\mathfrak{L}}} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \setminus \Delta^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{L}}}\} \subseteq (A^{(i)})^{\mathcal{I}_c^{\mathfrak{L}}} \setminus \Delta^{\mathcal{I}_c^{\mathfrak{L}}}$. Then, $\{a^{\mathcal{I}_c^{\mathfrak{I}_i}}\} \subseteq A^{\mathcal{I}_c^{\mathfrak{I}_i}} \setminus \Delta^{\mathcal{I}_c^{\mathfrak{I}_i}}$. Finally, $a^{\mathcal{I}_c^{\mathfrak{I}_i}} \notin A^{\mathcal{I}_c^{\mathfrak{I}_i}}$.
- Let $r \in \mathcal{R}_{\mathsf{O}} \cap \mathsf{O}_{\mathsf{R}}$, we can show that $r^{\mathcal{I}_c^{\mathfrak{I}_i}} \in r^{\mathcal{I}_c^{\mathfrak{I}_i}}$ iff $r^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}} \in r^{(\mathcal{I}_c^{\mathfrak{I}_{i-1}})^{\mathcal{E}}}$ with a similar proof as for concepts.

This ends the proof of Lemma 12. $\square$

Lemma 12 shows the correspondence between the models of encoding and the original CKB. Now, we encode the consequence to be checked with a similar translation. Given a consequence ψ' and the length of σ, denoted by n, we encode them in $\psi'^{(n)}$ by replacing any concept C in ψ' with a timestamped counterpart $T_C^{(n)}$. Then, we can solve the projection problem using consistency problem in $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$.

Lemma 16. *Let ψ, ψ' be CKBs, and σ be a sequence of actions. ψ' is a consequence of executing σ to ψ iff $\psi_{\text{red}} \wedge \neg(\psi'^{(n)})$ is inconsistent.*

Proof. $\Rightarrow$ We use a proof by contradiction. Assume that $\psi_{\text{red}} \wedge \neg(\psi'^{(n)})$ is consistent. Then, there is a model $\mathfrak{L}$ such that $\mathfrak{L} \models \psi_{\text{red}}$ and $\mathfrak{L} \models \neg(\psi'^{(n)})$. However, since all $\mathfrak{I}_n \models \psi'$ then we have $\mathfrak{L} \models \psi'^{(n)}$ from Lemma 12.2, hence a contradiction.

$\Leftarrow$ We use a proof by contradiction. Assume there exists an interpretation $\mathfrak{I}_0 \models \psi$ such that $\mathfrak{I}_0^\sigma = \mathfrak{I}_n \not\models \psi'$, and consequently $\mathfrak{I}_n \models \neg\psi'$. By Lemma 12.1, we have that $\mathfrak{L} \models \psi_{\text{red}}$ and $\mathfrak{L} \models \psi'^{(n)}$, hence a contradiction. $\square$

This gives us a complexity result for the projection problem.

Theorem 17. *The projection problem in $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ is* ExpTime*-complete.*

Proof. We have shown that ψ_{red} is polynomial in the size of the input, and obviously $\neg(\psi'^{(n)})$ as well. Since the consistency problem in $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ is ExpTime-complete, we can use a following procedure: build $\psi_{\text{red}} \wedge \neg(\psi'^{(n)})$ as defined, and check using an $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ consistency checker that runs in exponential time. Hence, we get an ExpTime procedure.

For the hardness, we can reduce the inconsistency problem in $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ to the projection problem. It is easy to see that an m-KB ψ is inconsistent iff $\{a\} \sqsubseteq \bot$ is a consequence of executing $\langle\rangle$ in ψ. $\square$

5 Conclusion

We have introduced an action formalism for reasoning about context and object level change in the ConDL $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$. The formalism is well-behaved in sense that the projection problem has the same complexity as standard reasoning in $\mathcal{ALCO}$.

From a practical point of view choosing $\mathcal{ALCO}[\![\mathcal{ALCO}]\!]$ has the advantage that an efficient reasoning tool for checking consistency already exists [6]. The reasoner even supports the more expressive combination $\mathcal{SHOIQ}[\![\mathcal{SHOIQ}]\!]$.

For future work we plan to investigate whether our action formalism offers sufficient expressiveness for capturing also the dynamic features of the role-based modelling language in [9].

Furthermore, we would like to study the complexity of reasoning in several extensions of the action formalism. This for example includes operators for non-determinism in the action dimension and temporal specifications for possibly infinite action sequences.

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