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Using model theory to find ω-admissible concrete domains*

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Abstract. Concrete domains have been introduced in the area of Description Logic to enable reference to concrete objects (such as numbers) and predefined predicates on these objects (such as numerical comparisons) when defining concepts. Unfortunately, in the presence of general concept inclusions (GCIs), which are supported by all modern DL systems, adding concrete domains may easily lead to undecidability. One contribution of this paper is to strengthen the existing undecidability results further by showing that concrete domains even weaker than the ones considered in the previous proofs may cause undecidability. To regain decidability in the presence of GCIs, quite strong restrictions, in sum called ω-admissiblity, need to be imposed on the concrete domain. On the one hand, we generalize the notion of ω-admissiblity from concrete domains with only binary predicates to concrete domains with predicates of arbitrary arity. On the other hand, we relate ω-admissiblity to well-known notions from model theory. In particular, we show that finitely bounded, homogeneous structures yield ω-admissible concrete domains. This allows us to show ω-admissibility of concrete domains using existing results from model theory.

Keywords: Description logic · concrete domains · GCIs · ω-admissibility · homogeneity · finite boundedness · decidability · constraint satisfaction.

1 Introduction

Description Logics (DLs) [3,7] are a well-investigated family of logic-based knowledge representation languages, which are frequently used to formalize ontologies for application domains such as the Semantic Web [22] or biology and medicine [21]. To define the important notions of such an application domain as formal concepts, DLs state necessary and sufficient conditions for an individual to belong to a concept. These conditions can be Boolean combinations of atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). For example, the concept of a father that has only daughters can be formalized by the concept description

$$C := \neg \mathit{Female} \sqcap \exists \mathit{child}.\mathit{Human} \sqcap \forall \mathit{child}.\mathit{Female},$$

which uses the concept names *Female* and *Human* and the role name *child* as well as the concept constructors negation ($\neg$), conjunction ($\sqcap$), existential restriction ($\exists r.D$), and value restriction ($\forall r.D$). The GCIs

$$\mathit{Human} \sqsubseteq \forall \mathit{child}.\mathit{Human} \quad \text{and} \quad \exists \mathit{child}.\mathit{Human} \sqsubseteq \mathit{Human}$$

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say that humans have only human children, and they are the only ones that can have human children.

DL systems provide their users with reasoning services that allow them to derive implicit knowledge from the explicitly represented one. In our example, the above GCIs imply that elements of our concept C also belong to the concept $D := \mathit{Human} \sqcap \forall \mathit{child}.\mathit{Human}$, i.e., C is subsumed by D w.r.t. these GCIs. A specific DL is determined by which kind of concept constructors are available. A major goal of DL research was and still is to find a got compromise between expressiveness and the complexity of reasoning, i.e., to locate DLs that are expressive enough for interesting applications, but still have inference problems (like subsumption) that are decidable and preferably of a low complexity. For the DL $\mathcal{ALC}$, in which all the concept descriptions used in the above example can be expressed, the subsumption problem w.r.t. GCIs is ExpTime-complete [7].

Classical DLs like $\mathcal{ALC}$ cannot refer to concrete objects and predefined relations over these objects when defining concepts. For example, a constraint stating that parents are strictly older than their children cannot be expressed in $\mathcal{ALC}$. To overcome this deficit, a scheme for integrating certain well-behaved concrete domains, called admissible, into $\mathcal{ALC}$ was introduced in [4], and it was shown that this integration leaves the relevant inference problems (such as subsumption) decidable. Basically, admissibility requires that the set of predicates of the concrete domain is closed under negation and that the constraint satisfaction problem (CSP) for the concrete domain is decidable. However, in this setting, GCIs were not considered since they were not a standard feature of DLs then,[1] though a combination of concrete domains and GCIs would be useful in many applications. For example, using the syntax employed in [28] and also in the present paper, the above constraint regarding the age of parents and their children could be expressed by the GCI $\mathit{Human} \sqcap \exists \mathit{child}\, \mathit{age}, \mathit{age}.[>] \sqsubseteq \bot$, which says that there cannot be a human whose age is smaller than the age of one of his or her children. Here $\bot$ is the bottom concept, which is always interpreted as the empty set, *age* is a concrete feature that maps from the abstract domain populating concepts into the concrete domain of natural numbers, and $>$ is the usual greater predicate on the natural numbers.

A first indication that concrete domains might be harmful for decidability was given in [6], where it was shown that adding transitive closure of roles to $\mathcal{ALC}(\mathfrak{R})$, the extension of $\mathcal{ALC}$ by an admissible concrete domain $\mathfrak{R}$ based on real arithmetics, renders the subsumption problem undecidable. The proof of this result uses a reduction from the Post Correspondence Problem (PCP). It was shown in [26] that this proof can be adapted to the case where transitive closure of roles is replaced by GCIs, and it actually works for considerably weaker concrete domain, such as the rational numbers $\mathbb{Q}$ or the natural number $\mathbb{N}$ with a unary predicate $=_0$ for equality with zero, a binary equality predicate $=$, and a unary predicate $+_1$ for incrementation. In [7] it is shown, by a reduction from the halting

[1] Actually, they were introduced (with a different name) at about the same time as concrete domains [2,33].

problem of two-register machines, that undecidability even holds without binary equality. In the present paper, we will improve on this result by showing that, even if $=_0$ is removed as well, undecidability still holds, and that the same is true if we replace $+_1$ with $+$.

To regain decidability, one can either impose syntactic restriction on how the DL can interact with the concrete domain [17,31]. The main idea is here to disallow paths (such as *child age* in our example), which has the effect that concrete domain predicates cannot compare properties (such as the age) of different individuals. The other option is to impose stronger restrictions than admissibility on the concrete domain. After first positive results for specific concrete domains (e.g., a concrete domain over the rational numbers with order and equality [25,27]), the notion of ω-admissible concrete domains was introduced in [28], and it was shown (by designing a tableau-based decision procedure) that integrating such a concrete domain into $\mathcal{ALC}$ leaves reasoning decidable also in the presence of GCIs. In the present paper, we generalize the notion of ω-admissiblity and the decidability result from concrete domains with only binary predicates as in [28] to concrete domains with predicates of arbitrary arity. But the main contribution of this paper is to show that there is a close relationship between ω-admissiblity and well-known notions from model theory. In particular, we show that finitely bounded, homogeneous structures yield ω-admissible concrete domains. This allows us to locate new ω-admissible concrete domains using existing results from model theory.

2 Preliminaries

We write $[n]$ for the set $\{1, \dots, n\}$. Given a set A, the *diagonal relation* on A is defined as the binary relation $\Delta_A \coloneqq \{(a, a) \mid a \in A\}$. The *kernel* of a mapping $f\colon A \to B$, denoted by $\ker f$, is the equivalence relation $\{(a, a') \in A \times A \mid f(a) = f(a')\}$. For $a \in A_1 \times \cdots \times A_k$ and $i \in [k]$, we denote the ith component of the tuple a by $a[i]$. By pr_i we denote the usual *projection* function $\mathrm{pr}_i\colon A_1 \times \cdots \times A_k \to A$ with $\mathrm{pr}_i(a) = a[i]$.

From a mathematical point of view, concrete domains are relational structures. A *(relational) signature* τ is a set of *predicate symbols*, each with an associated natural number called its *arity*. A *relational τ-structure* $\mathfrak{A}$ consists of a set A (the *domain*) together with relations $R^{\mathfrak{A}} \subseteq A^k$ for each k-ary symbol $R \in \tau$. We often describe structures by listing their domain and relations, e.g., we write $\mathfrak{Q} = (\mathbb{Q}; <)$ for the relational structure whose domain is the set of rational numbers $\mathbb{Q}$, and which has the usual smaller relation $<$ on $\mathbb{Q}$ as its only relation.[2]

The *union* of two τ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the τ-structure $\mathfrak{A}_1 \cup \mathfrak{A}_2$ with domain $A_1 \cup A_2$ satisfying $R^{\mathfrak{A}_1 \cup \mathfrak{A}_2} = R^{\mathfrak{A}_1} \cup R^{\mathfrak{A}_2}$ for every $R \in \tau$. It is called *disjoint* if $A_1 \cap A_2 = \emptyset$. The definition of a union of two τ-structures has an obvious extension to arbitrary families of τ-structures. An *expansion* of the τ-structure $\mathfrak{A}$

[2] By a slight abuse of notation, we use $<$ instead of $<^{\mathfrak{Q}}$ to denote also the interpretation of the predicate symbol $<$ in $\mathfrak{Q}$.

is a σ-structure $\mathfrak{B}$ with $A = B$, $\tau \subseteq \sigma$, and $R^{\mathfrak{B}} = R^{\mathfrak{A}}$ for each relation symbol $R \in \tau$. Conversely, we call $\mathfrak{A}$ a reduct of $\mathfrak{B}$.

One possibility to obtain an expansion of a τ-structure is to use formulas of *first-order* (fo) *logic* over the signature τ to define new predicates, where a formula with k free variables defines a k-ary predicate in the obvious way. We assume that equality $=$ as well as the symbol `false` for falsity are always available when building these formulas. Thus, atomic formulas are of the form `false`, $x_i = x_j$, and $R(x_1, \ldots, x_k)$ for some k-ary $R \in \tau$ and variables $x_1, \ldots, x_k$. The *first-order theory* of a structure is the set of all first-order sentences that are true in the structure. In addition to full first-order logic, we also use standard fragments of first-order logic such as the *existential positive* ($\exists^+$), the *quantifier-free* (qf), and the *primitive positive* (pp) *fragment*. The existential positive fragment consists of formulas built using conjunction, disjunction, and existential quantification only. The quantifier-free fragment consists of Boolean combinations of atomic formulas, and the primitive positive fragment of existentially quantified conjunctions of atomic formulas. Let Σ be a set of first-order formulas and $\mathfrak{D}$ a structure. We say that a relation over D has a Σ *definition* in $\mathfrak{D}$ if it is of the form $\{t \in D^k \mid \mathfrak{D} \models \phi(t)\}$ for some $\phi \in \Sigma$. We refer to this relation by $\phi^{\mathfrak{D}}$. For example, the formula $y < x \vee x = y$ is an existential positive formula and, interpreted in the structure $\mathfrak{Q}$, it clearly defines the binary relation $\geq$ on $\mathbb{Q}$. This shows that $\geq$ is $\exists^+$ definable in $\mathfrak{Q}$. An example of a pp formula is the formula $x = x$, which defines the unary relation interpreted as the whole domain $\mathbb{Q}$.

A *homomorphism* $h\colon \mathfrak{A} \to \mathfrak{B}$ for τ-structures $\mathfrak{A}, \mathfrak{B}$ is a mapping $h\colon A \to B$ that *preserves* each relation of $\mathfrak{A}$, i.e., $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$ for some k-ary relation symbol $R \in \tau$ implies $(h(a_1), \ldots, h(a_k)) \in R^{\mathfrak{B}}$. We write $\mathfrak{A} \to \mathfrak{B}$ if $\mathfrak{A}$ homomorphically maps to $\mathfrak{B}$ and $\mathfrak{A} \not\to \mathfrak{B}$ otherwise. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *homomorphically equivalent* if $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{A}$. An *endomorphism* is a homomorphism from a structure to itself. By an *embedding* we mean an injective homomorphism that additionally satisfies the only if direction in the definition of a homomorphism, i.e., it also preserves the complements of relations. We write $\mathfrak{A} \hookrightarrow \mathfrak{B}$ if $\mathfrak{A}$ embeds into $\mathfrak{B}$. A *substructure* of $\mathfrak{A}$ is a structure $\mathfrak{B}$ over $B \subseteq A$ such that the natural inclusion map $i\colon A \to B$ is an embedding. We call $\mathfrak{A}$ an *extension* of $\mathfrak{B}$. An *isomorphism* is a surjective embedding. We say that two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* and write $\mathfrak{A} \cong \mathfrak{B}$ if there exists an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. An *automorphism* is an isomorphism from a structure into itself.

The definition of admissiblity of a concrete domain in [4] requires that the constraint satisfaction problem for this structure is decidable. Let $\mathfrak{D}$ be a structure with a finite relational signature τ. The *constraint satisfaction problem* of $\mathfrak{D}$, short CSP($\mathfrak{D}$), is the following decision problem:

INPUT: A finite τ-structure $\mathfrak{A}$.
QUESTION: Does $\mathfrak{A}$ homomorphically map to $\mathfrak{D}$?

Formally, $\mathrm{CSP}(\mathfrak{D})$ is the class of all finite τ-structures that homomorphically map to $\mathfrak{D}$. We call $\mathfrak{D}$ the *template* of $\mathrm{CSP}(\mathfrak{D})$. A *solution* for an instance $\mathfrak{A}$ of the CSP is a homomorphism $h\colon \mathfrak{A} \to \mathfrak{D}$.

It is easy to see that deciding whether a CSP instance admits a solution amounts to evaluating a pp sentences in the template and vice versa [9]. For example, verifying whether the structure $\mathfrak{A} = (\{a_1, a_2, a_3\}; <^{\mathfrak{A}})$ with $<^{\mathfrak{A}} := \{(a_1, a_2), (a_2, a_3), (a_3, a_1)\}$ homomorphically maps into $\mathfrak{Q}$ is the same as checking whether the pp sentence $\exists x_1.\exists x_2.\exists x_3.(x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1)$ is true in $\mathfrak{Q}$.

The CSP for $\mathfrak{Q}$ is in P since a structure $\mathfrak{A} = (A, <^{\mathfrak{A}})$ can homomorphically be mapped into $\mathbb{Q}$ iff it does not contain a $<$-cycle, i.e., there are is no $n \geq 1$ and elements $a_0, \ldots, a_{n-1} \in A$ such that $a_0 <^{\mathfrak{A}} \ldots <^{\mathfrak{A}} a_{n-1} <^{\mathfrak{A}} a_0$. Testing whether such a cycle exists can be done in logarithmic space since it requires solving the reachability problem in a directed graph (digraph). In the example above, we obviously have a cycle, and thus this instance of $\mathrm{CSP}(\mathfrak{Q})$ has no solution.

The definition of admissiblity in [4] actually also requires that the predicates are closed under negation and that there is a predicate for the whole domain. We have already seen that the negation $\geq$ of $<$ is $\exists^+$ definable in $\mathfrak{Q}$ and that the predicate for the whole domain is pp definable. The negation of this predicate has the pp definition $x < x$. The following lemma implies that the expansion of $\mathfrak{Q}$ by these predicates still has a decidable CSP.[3]

Lemma 1 ([9]). *Let $\mathfrak{C}, \mathfrak{D}$ be structures over the same domain with finite signatures. If the relations of $\mathfrak{C}$ have a pp ($\exists^+$) definition in $\mathfrak{D}$, then* $\mathrm{CSP}(\mathfrak{C}) \leq_{\mathrm{PT{\small IME}}} \mathrm{CSP}(\mathfrak{D})$ $\big(\mathrm{CSP}(\mathfrak{C}) \leq_{\mathrm{NPT{\small IME}}} \mathrm{CSP}(\mathfrak{D})\big)$.

3 DLs with concrete domains

As in [4] and [28], we extend the well-known DL $\mathcal{ALC}$ with concrete domains. We assume that the reader is familiar with the syntax and semantics of $\mathcal{ALC}$ (as, e.g., defined in [7]), and thus only show how both need to be extended to accommodate a concrete domain $\mathfrak{D}$. In the general definition, we allow reference to Σ definable predicates for a fragment Σ of FO rather than just the elements of τ. For technical reasons, we must, however, restrict the arities of definable predicates by a fixed upper bound d. Given a τ-structure $\mathfrak{D}$ with finite relational signature τ, a set Σ of FO formulas over the signature τ, and a bound $d \geq 1$ on the arity of the Σ-definable predicates, we obtain a DL $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$, which extends $\mathcal{ALC}$ as follows.

From a syntactic point of view, we assume that the set of role names $\mathsf{N_R}$ contains a set of functional roles $\mathsf{N_{fR}} \subseteq \mathsf{N_R}$, and that in addition we have a set of feature names $\mathsf{N_F}$, which provide the connection between the abstract and the concrete domain. A *path* is of the form $r\,f$ or f where $r \in \mathsf{N_R}$ and $f \in \mathsf{N_F}$. In our example in the introduction, *age* is a feature name and *child age* is a path. The DL $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$

[3] The lemma actually only yields an NP decision procedure for this CSP, but it is easy to see that the above polynomial-time cycle-checking algorithm can be adapted such that it also works for the expanded structure.

extends $\mathcal{ALC}$ with two new concept constructors:

$$\exists p_1, \ldots, p_k. [\phi(x_1, \ldots, x_k)] \quad \text{and} \quad \forall p_1, \ldots, p_k. [\phi(x_1, \ldots, x_k)],$$

where $k \leq d$, $p_1, \ldots, p_k$ are paths, and $\phi(x_1, \ldots, x_k)$ is a formula in Σ with k free variables, defining a k-ary predicate on D. As usual, a TBox is defined to be a finite set of GCIs $C \sqsubseteq D$, where C, D are $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$ concept descriptions.

Regarding the semantics, we consider interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, consisting of a non-empty set $\Delta^\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$, which interprets concept names A as subsets $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and role names r as binary relations $r^\mathcal{I}$ on $\Delta^\mathcal{I}$, with the restriction that $r^\mathcal{I}$ is functional for $r \in \mathsf{N_{fR}}$, i.e., $(d, e) \in r^\mathcal{I}$ and $(d, e') \in r^\mathcal{I}$ imply $e = e'$. In addition, features $f \in \mathsf{N_F}$ are interpreted as functional binary relations $f^\mathcal{I} \subseteq \Delta^\mathcal{I} \times D$. We extend the interpretation function to paths of the form $p = r\, f$ by setting

$$p^\mathcal{I} = \{(d, d') \mid \text{there is } d'' \in \Delta^\mathcal{I} \text{ such that } (d, d'') \in r^\mathcal{I} \text{ and } (d'', d') \in f^\mathcal{I}\}.$$

For the concept constructors of $\mathcal{ALC}$, the extension of the interpretation function to complex concepts is defined in the usual way. The new concrete domain constructors are interpreted as follows:

$$\begin{aligned}
(\exists p_1, \ldots, p_k. [\phi(x_1, \ldots, x_k)])^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid {} & \text{there exist } d_1, \ldots, d_k \in D \text{ such that} \\
& (d, d_i) \in p_i^\mathcal{I} \text{ for all } i \in [k] \text{ and } \mathfrak{D} \models \phi(d_1, \ldots, d_k)\}, \\
(\forall p_1, \ldots, p_k. [\phi(x_1, \ldots, x_k)])^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid {} & \text{for all } d_1, \ldots, d_k \in D \text{ we have that} \\
& (d, d_i) \in p_i^\mathcal{I} \text{ for all } i \in [k] \text{ implies } \mathfrak{D} \models \phi(d_1, \ldots, d_k)\}.
\end{aligned}$$

As usual, an interpretation $\mathcal{I}$ is a model a TBox $\mathcal{T}$ if it satisfies all the GCIs in $\mathcal{T}$, where $\mathcal{I}$ satisfies the GCI $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds. The $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$ concept description C is satisfiable w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $C^\mathcal{I} \neq \emptyset$. Since all Boolean operators are available in $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$, the subsumption problem mentioned in the introduction and the satisfiability problem are inter-reducible in polynomial time [7].

As a convention, we write $\mathcal{ALC}(\mathfrak{D})$ instead of $\mathcal{ALC}^d_\Sigma(\mathfrak{D})$ if d is the maximal arity of the predicates in τ and Σ consists of all atomic τ-formulas not using the equality predicate.

3.1 Undecidable DLs with concrete domains

We show by a reduction from the halting problem of two-register machines that concept satisfiability in $\mathcal{ALC}(\mathfrak{D})$ is undecidable if $\mathfrak{D}$ is a structure with domain $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{N}$ whose only predicate is the binary predicate $+_1$, which is interpreted as incrementation (i.e., it consists of the tuples $(m, m+1)$ for numbers m in the respective domain).

Our proof is an an adaptation of the undecidability proof in [7] to the case where no zero test $=_0$ is available. A *two-register machine* (2RM) is a pair (Q, P) with *states* $Q = \{q_0, \ldots, q_\ell\}$ and *instructions* $I_0, \ldots, I_{\ell-1}$. By definition, q_0 is the initial state and q_ℓ the halting state. In state q_i ($i < \ell$) the instruction I_i must be applied. Instructions come in two varieties. An *incrementation instruction* is of the form $I = +(r, q)$ where $r \in \{1, 2\}$ is the register number and q is a state. This instruction increments (the content of) register r and then goes to state q. A *decrementation instruction* is of the form $I = -(r, q, q')$ where $r \in \{1, 2\}$ and q, q' are states. This instruction decrements register r and goes to state q if the content of register r is not zero; otherwise, it leaves register r as it is and goes to state q'. It is well-known that the problem of deciding whether a given 2RM halts on input $(0, 0)$ is undecidable [30].

Proposition 1. *If $\mathfrak{D}$ is $(\mathbb{Q}, +_1)$, $(\mathbb{Z}, +_1)$, or $(\mathbb{N}, +_1)$, then concept satisfiability in $\mathcal{ALC}(\mathfrak{D})$ w.r.t. TBoxes is undecidable.*

Proof. Let (Q, P) be an arbitrary 2RM. We define a concept C and a TBox $\mathcal{T}$ in such a way that every model of $\mathcal{T}$ in which C is non-empty represents the computation of (Q, P) on the input $(0, 0)$. For every state q_i we introduce a concept name Q_i. We also introduce two concept names Z_1, Z_2 to indicate a positive zero test for the first and second register, respectively. In addition, we introduce a functional role $g \in \mathsf{N_{fR}}$ representing the transitions between configurations of the 2RM. For $p \in \{1, 2\}$, we have features $r_p \in \mathsf{N_F}$ representing the content of register p. However, since our concrete domain does not have the predicate $=_0$, we cannot enforce that, in our representation of the initial configuration, r_1 and r_2 have value zero. What we can ensure, though, is that their value is the same number, which we can store in a concrete feature $z \in \mathsf{N_F}$. The idea is now that register p of the machine actually contain the value of r_p offset with the value of z. We also need auxiliary concrete features $s_1, s_2, s \in \mathsf{N_F}$, which respectively refer to the successor values of r_1, r_2, z. They are needed to express equality using $+_1$.

The following GCI ensures that the elements of C represent the initial configuration together with appropriate values for the auxiliary features:

$$C \sqsubseteq Q_0 \sqcap \exists r_1, s_1. [+_1] \sqcap \exists r_2, s_2. [+_1] \sqcap \exists z, s_1. [+_1] \sqcap \exists z, s_2. [+_1] \sqcap \exists z, s. [+_1].$$

Next, the GCI $\top \sqsubseteq \exists gz, s. [+_1] \sqcap \exists gz, gs. [+_1]$ guarantees that the value z of an individual carries over to its g-successor. We denote the second value in $\{1, 2\}$ beside p by $\widehat{p}$, i.e., $\widehat{p} = 3 - p$. To enforce that the incrementation instructions are executed correctly, for every instruction $I_i = +(p, q_j)$, we include in $\mathcal{T}$ the GCI

$$Q_i \sqsubseteq \exists r_p, gr_p. [+_1] \sqcap \exists gr_{\widehat{p}}, s_{\widehat{p}}. [+_1] \sqcap \exists s_p, gs_p. [+_1] \sqcap \exists r_{\widehat{p}}, gs_{\widehat{p}}. [+_1] \sqcap \exists g. Q_j$$

The GCIs $Z_p \sqsubseteq \exists z, s_p. [+_1]$, $\exists z, s_p. [+_1] \sqsubseteq Z_p$ ensure that Z_p represents a positive zero test for register p. Note that, for individuals for which values for s, z, s_p, r_p are defined, the negation of Z_p is semantically equivalent to a negative zero test for register p. To enforce that decrementation is executed correctly, for every instruction $I_i = -(p, q_j, q_k)$, we include in $\mathcal{T}$ the GCIs

$$Q_i \sqcap Z_p \sqsubseteq \exists gr_p, s_p. [+_1] \sqcap \exists gr_{\widehat{p}}, s_{\widehat{p}}. [+_1] \sqcap \exists r_p, gs_p. [+_1] \sqcap \exists r_{\widehat{p}}, gs_{\widehat{p}}. [+_1] \sqcap \exists g. Q_k,$$

$$Q_i \sqcap \neg Z_p \sqsubseteq \exists gr_p, r_p. [+_1] \sqcap \exists gr_{\widehat{p}}, s_{\widehat{p}}. [+_1] \sqcap \exists gs_p, s_p. [+_1] \sqcap \exists r_{\widehat{p}}, gs_{\widehat{p}}. [+_1] \sqcap \exists g. Q_j.$$

Finally, we include the GCI $Q_\ell \sqsubseteq \bot$, which states that the halting state is never reached. It is now easy to see that the computation of (Q, P) on $(0, 0)$ does not reach the halting state iff C is satisfiable w.r.t. $\mathcal{T}$. □

It turns out that undecidability also holds if we use the ternary predicate $+$ rather than the binary predicate $+_1$. Intuitively, with $+$ we can easily test for 0 since m is 0 iff $m + m = m$. Instead of incrementation by 1, we can then use addition of a fixed non-zero number.

Proposition 2. *If $\mathfrak{D}$ is $(\mathbb{Q}, +)$, $(\mathbb{Z}, +)$, or $(\mathbb{N}, +)$, then concept satisfiability in $\mathcal{ALC}(\mathfrak{D})$ w.r.t. TBoxes is undecidable.*

Proof. Similarly as in the proof of Proposition 1, we reduce the halting problem of two-register machines to concept satisfiability in in $\mathcal{ALC}(\mathfrak{D})$. This time we closely follow the proof of Theorem 5.24 in [7]. For this reason, we only provide the GCIs that encode the run of an arbitrary 2RM on the input $(0, 0)$, the rest is obvious. As before, $g \in \mathsf{N_{fR}}$ represents the transition function, and $r_1, r_2 \in \mathsf{N_F}$ represent the contents of the two registers initialized with the value 0. Additionally, $z \in \mathsf{N_F}$ is an auxiliary feature that assumes the value 0, and $u \in \mathsf{N_F}$ is an auxiliary feature that assumes the value of some non-zero number. The initial configuration is represented by the following GCI which, in particular, prevents u from assuming the value 0:

$$C \sqsubseteq Q_0 \sqcap \exists z, z, z. [+] \sqcap \exists r_1, r_1, r_1. [+] \sqcap \exists r_2, r_2, r_2. [+] \sqcap \neg(\exists u, u, u. [+])$$

The GCI

$$\top \sqsubseteq \exists z, z, z. [+] \sqcap \exists u, z, gu. [+]$$

ensures that z has the value 0 everywhere, and it simultaneously transfers the value of u to g-successors. Consequently, u has a fixed non-zero value on the g-paths starting with our initial element of C.

The incrementation instruction $I_i = +(p, q_j)$ is represented by the GCI

$$Q_i \sqsubseteq \exists g. Q_j \sqcap \exists r_p, u, gr_p. [+] \sqcap \exists r_{\widehat{p}}, z, gr_{\widehat{p}}. [+],$$

and the decrementation instruction $I_i = -(p, q_j, q_k)$ is represented by the GCIs

$$\begin{aligned} Q_i \sqcap \exists r_p, z, z. [+] &\sqsubseteq \exists g. Q_k \sqcap \exists r_p, z, gr_p. [+] \sqcap \exists r_{\widehat{p}}, z, gr_{\widehat{p}}. [+] \\ Q_i \sqcap \neg(\exists r_p, z, z. [+]) &\sqsubseteq \exists g. Q_j \sqcap \exists gr_p, u, r_p. [+] \sqcap \exists r_{\widehat{p}}, z, gr_{\widehat{p}}. [+] \end{aligned}$$

The non-termination is, again, represented by the GCI $Q_\ell \sqsubseteq \bot$. □

3.2 ω-admissible concrete domains

To regain decidability in the presence of GCIs and concrete domains, the notion of ω-admissible concrete domains was introduced in [28]. We generalize this notion and the decidability result from concrete domains with only binary predicates as in [28] to concrete domains with predicates of arbitrary arity.

We say that a countable structure $\mathfrak{D}$ has *homomorphism compactness* if, for every countable structure $\mathfrak{B}$, it holds that $\mathfrak{B} \to \mathfrak{D}$ iff $\mathfrak{A} \to \mathfrak{D}$ for every finite structure $\mathfrak{A}$ with $\mathfrak{A} \hookrightarrow \mathfrak{B}$. A relational τ-structure $\mathfrak{D}$ satisfies

(JE) if, for some $d \geq 2$, $\bigcup \left\{ R^{\mathfrak{D}} \;\middle|\; R \in \tau,\ R^{\mathfrak{D}} \subseteq D^k \right\} = D^k$ if $k \leq d$ and $\emptyset$ else;
(PD) if $R^{\mathfrak{D}} \cap R'^{\mathfrak{D}} = \emptyset$ for all pairwise distinct $R, R' \in \tau$;
(JD) if $\bigcup\{R^{\mathfrak{D}} \;\big|\; R \in \tau,\ R^{\mathfrak{D}} \subseteq \triangle_D\} = \triangle_D$.

Here JE stands for "jointly exhaustive," PD for "pairwise disjoint," and JD for "jointly diagonal." Note that JD was not considered in [28]. We include it here since it makes the comparison with known notion from model theory easier. In addition, all the ω-admissible concrete domains considered in [28] satisfy JD.

A relational τ-structure $\mathfrak{D}$ is a *patchwork* if it is JDJEPD, and for all finite JEPD τ-structures $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ with $e_1 \colon \mathfrak{A} \hookrightarrow \mathfrak{B}_1$, $e_2 \colon \mathfrak{A} \hookrightarrow \mathfrak{B}_2$, $\mathfrak{B}_1 \to \mathfrak{D}$ and $\mathfrak{B}_2 \to \mathfrak{D}$, there exist $f_1 \colon \mathfrak{B}_1 \to \mathfrak{D}$ and $f_2 \colon \mathfrak{B}_2 \to \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$.

Definition 1. *The relational structure* $\mathfrak{D}$ *is* ω-admissible *if* $\mathrm{CSP}(\mathfrak{D})$ *is decidable,* $\mathfrak{D}$ *has homomorphism compactness, and* $\mathfrak{D}$ *is a patchwork.*

The idea is now that one can use disjunctions of atomic formulas of the same arity within concrete domain restrictions. We refer to the set of all FO τ-formulas of the form $R_1(x_1, \dots, x_k) \vee \cdots \vee R_m(x_1, \dots, x_k)$ for $R_1, \dots, R_m$ k-ary predicates in τ by $\vee^+$. The following theorem is proved in the appendix by extending the tableau-based decision procedure given in [28] to our more general definition of ω-admissibility. Note that the proof of correctness of this procedure does not depend on JD.

Theorem 1. *Let* $\mathfrak{D}$ *be an* ω*-admissible* τ*-structure with at most* d*-ary relations for some* $d \geq 2$*. Then concept satisfiability in* $\mathcal{ALC}^d_{\vee^+}(\mathfrak{D})$ *w.r.t. TBoxes is decidable.*

The main motivation for the definition of ω-admissible concrete domains in [28] was that they can capture qualitative calculi of time and space. In particular, it was shown in [28] that Allen's interval logic [1] as well as the region connection calculus RCC8 [32] can be represented as ω-admissible concrete domains. To the best of our knowledge, no other ω-admissible concrete domains have been exhibited in the literature since then.

4 A model-theoretic approach towards ω-admissibility

In this section, we introduce several model-theoretic properties of relational structures and show their connection to ω-admissibility. This allows us to formulate sufficient conditions for ω-admissibility using well-know notions from model theory, and thus to use existing model-theoretic results to find new ω-admissible concrete domains.

ω-categoricity. We start with introducing ω-categoricity since it gives us homomorphism compactness "for free." A structure is *ω-categorical* if its first-order theory has exactly one countable model up to isomorphism. For example, it is well-known that $\mathfrak{Q}$ is, up to isomorphism, the only countable dense linear order without lower or upper bound. This result, which clearly implies that $\mathfrak{Q}$ is ω-categorical, is due to Cantor.

For every structure $\mathfrak{A}$, the set of all its automorphisms forms a permutation group, which we denote by $\mathrm{Aut}(\mathfrak{A})$ (see Theorem 1.2.1 in [20]). Every relation with an first-order definition in $\mathfrak{A}$ is easily seen to be preserved by $\mathrm{Aut}(\mathfrak{A})$. For ω-categorical structures, the other direction holds as well.

Theorem 2 (Engeler, Ryll-Nardzewski and Svenonius [20]). *For a countably infinite structure $\mathfrak{A}$ with a countable signature, the following are equivalent:*

1. *$\mathfrak{A}$ is ω-categorical.*
2. *For every $k \geq 1$, only finitely many k-ary relations are first-order definable in $\mathfrak{A}$.*
3. *Every relation over A preserved by $\mathrm{Aut}(\mathfrak{A})$ is first-order definable in $\mathfrak{A}$.*

The following corollary to this theorem establishes the first link between model theory and ω-admissibility.

Corollary 1 (Lemma 3.1.5 in [9]). *Every ω-categorical structure has homomorphism compactness.*

In order to obtain JDJEPD, we replace the original relations of a given ω-categorical τ-structure $\mathfrak{A}$ with appropriate first-order definable ones, using the results of Theorem 2. The *orbit* of a tuple $(a_1, \dots, a_k) \in A^k$ under the natural action of $\mathrm{Aut}(\mathfrak{A})$ on A^k is the set $\Big\{\Big(g(a_1), \dots, g(a_n)\Big) \mid g \in \mathrm{Aut}(\mathfrak{A})\Big\}$. By Theorem 2, the set of all at most k-ary relations definable in $\mathfrak{A}$ is finite for every $k \in \mathbb{N}$. Since every such set is closed under intersections, it contains finitely many minimal non-empty relations. Since every relation over A that is preserved by all automorphisms of $\mathfrak{A}$ is first-order definable in $\mathfrak{A}$, these minimal elements are precisely the orbits of tuples over A under the natural action of $\mathrm{Aut}(\mathfrak{A})$.

Definition 2. *For a given arity bound $d \geq 2$, the d-*reduct *of the ω-categorical τ-structure $\mathfrak{A}$, denoted by $\mathfrak{A}^{\leqslant d}$, is the relational structure over A whose relations are all orbits of at most d-ary tuples over A under* $\mathrm{Aut}(\mathfrak{A})$*. We denote the signature of $\mathfrak{A}^{\leqslant d}$ by $\tau^{\leqslant d}$.*

It is easy to see that $\mathfrak{A}^{\leqslant d}$ is JDJEPD, and that every at most d-ary relation over A first-order definable in $\mathfrak{A}$ can be obtained as a disjunction of atomic formulas built using the symbols in $\tau^{\leqslant d}$. As an example, consider the ω-categorical structure $\mathfrak{Q}$. The orbits of k-tuples of elements of $\mathbb{Q}$ can be defined by quantifier-free formulas that are conjunctions of atoms of the form $x_i = x_j$ or $x_i < x_j$. For example, the orbit of the tuple $(2, 3, 2, 5)$ consists of all tuples $(q_1, q_2, q_3, q_4) \in \mathbb{Q}^4$ that satisfy the formula $x_1 < x_2 \wedge x_1 = x_3 \wedge x_2 < x_4$ if x_i is replaced by q_i for $i = 1, \dots, 4$. The first-order definable k-ary relations in $\mathfrak{Q}$ are obtained as unions of these orbits, where the defining formula is then the disjunction of the formulas defining the respective orbits. Since these formulas are quantifier-free, this also shows that $\mathfrak{Q}$ admits quantifier elimination. Recall that a τ-structure admits *quantifier elimination* if for every first-order τ-formula there exists a quantifier-free (qf) τ-formula that defines the same relation over this structure.

Homogeneity. To obtain the patchwork property, we restrict the attention to homogeneous structures. A structure $\mathfrak{A}$ is *homogeneous* if every isomorphism between finite substructures of $\mathfrak{A}$ extends to an automorphism of $\mathfrak{A}$.

Theorem 3 ([20]). *A countable relational structure with a finite signature is homogeneous iff it is ω-categorical and admits quantifier elimination.*

Since $\mathfrak{Q}$ is ω-categorical and admits quantifier elimination, it is thus homogeneous. This can, however, also easily be shown directly without using the theorem. In fact, given finite substructures $\mathfrak{B}$ and $\mathfrak{C}$ of $\mathfrak{Q}$ and an isomorphism between them, we know that B consists of finitely many elements $p_1, \ldots, p_n$ and C of the same number of elements $q_1, \ldots, q_n$ such that $p_1 < \ldots < p_n$, $q_1 < \ldots < q_n$, and the isomorphism maps p_i to q_i (for $i = 1, \ldots, n$). It is now easy to see that $<$ is also a dense linear order without lower or upper bound on the sets $\{p \mid p < p_1\}$ and $\{q \mid q < q_1\}$, and thus there is an order isomorphism between these sets. The same is true for the pairs of sets $\{p \mid p_i < p < p_{i+1}\}$ and $\{q \mid q_i < q < q_{i+1}\}$, and for the pair $\{p \mid p_n < p\}$ and $\{q \mid q < q_n\}$. Using the isomorphisms between these pairs, we can clearly put together an isomorphisms from $\mathfrak{Q}$ to $\mathfrak{Q}$ that that extends the original isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$.

Countable homogeneous structures can be obtained as Fraïssé limits of amalgamation classes. A class $\mathcal{K}$ of relational τ-structures has the *amalgamation property* (AP) if, for every $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ with $e_1 \colon \mathfrak{A} \hookrightarrow \mathfrak{B}_1$ and $e_2 \colon \mathfrak{A} \hookrightarrow \mathfrak{B}_2$ there exists $\mathfrak{C} \in \mathcal{K}$ with $f_1 \colon \mathfrak{B}_1 \hookrightarrow \mathfrak{C}$ and $f_2 \colon \mathfrak{B}_2 \hookrightarrow \mathfrak{C}$ such that $f_1 \circ e_1 = f_2 \circ e_2$. A class $\mathcal{K}$ of finite relational structures with a countable signature τ is called an *amalgamation class* if it has AP, is closed under taking isomorphisms and substructures, and contains only countably many structures up to isomorphism. We denote by $\mathrm{Age}(\mathfrak{A})$ the class of all finite structures that embed into the structure $\mathfrak{A}$.

Theorem 4 (Fraïssé [20]). *Let $\mathcal{K}$ be an amalgamation class of τ-structures. Then there exists a homogeneous countable τ-structure $\mathfrak{A}$ with $\mathrm{Age}(\mathfrak{A}) = \mathcal{K}$. The structure $\mathfrak{A}$ is unique up to isomorphism and referred to as the* Fraïssé limit *of $\mathcal{K}$. Conversely, $\mathrm{Age}(\mathfrak{A})$ for a countable homogeneous structure $\mathfrak{A}$ with a countable signature is an amalgamation class.*

For our running example $\mathfrak{Q} = (\mathbb{Q}, <)$, we have that $\mathrm{Age}(\mathfrak{Q})$ consists of all finite linear orders, and thus by Fraïssé's theorem this class of structures is an amalgamation class. In addition, $\mathfrak{Q}$ is the Fraïssé limit of this class. Proposition 3 below shows that there is a close connection between AP and the patchwork property. Its proof uses the following lemma.

Lemma 2. *Let $\mathfrak{A}, \mathfrak{B}$ be two JEPD τ-structures, such that $\mathfrak{B}$ is JD, and $f: \mathfrak{A} \to \mathfrak{B}$ a homomorphism. Then f preserves the complements of all relations of $\mathfrak{A}$ and* $\ker f = \bigcup\{R^{\mathfrak{A}} \mid R \in \tau$ *and* $R^{\mathfrak{B}} \subseteq \Delta_B\}$.

Proof. Since $\mathfrak{A}$ is JEPD, for every $R \in \tau$ and every $(x_1, \ldots, x_k) \notin R^{\mathfrak{B}}$, there exists exactly one $R' \in \tau \backslash \{R\}$ with $(x_1, \ldots, x_k) \in R'^{\mathfrak{A}}$. As f is a homomorphism, we have $(f(x_1), \ldots, f(x_k)) \in R'^{\mathfrak{B}}$. But then $(f(x_1), \ldots, f(x_k)) \notin R^{\mathfrak{B}}$ because $\mathfrak{B}$ is JEPD as well. For the second part, we clearly have $\bigcup\{R^{\mathfrak{A}} \mid R \in \tau \text{ and } R^{\mathfrak{B}} \subseteq \Delta_B\} \subseteq \ker f$. On the other hand, if $f(x) = f(y)$ for some $x, y \in A$, then $(f(x), f(y)) \in R^{\mathfrak{B}}$ for some binary $R \in \tau$ with $R^{\mathfrak{B}} \subseteq \Delta_B$. But then assuming that $(x, y) \notin R^{\mathfrak{A}}$ would yield a contradiction due to the first part of the lemma. Thus $\ker f \subseteq \bigcup\{R^{\mathfrak{A}} \mid R \in \tau \text{ and } R^{\mathfrak{B}} \subseteq \Delta_B\}$. ☐

Proposition 3. *Let $\mathfrak{D}$ be a JDJEPD τ-structure. Then $\mathfrak{D}$ is a patchwork iff* $\mathrm{Age}(\mathfrak{D})$ *has AP.*

Proof. For simplification purposes, every statement indexed by i is suppose to hold for both $i \in \{1, 2\}$. First, suppose that $\mathrm{Age}(\mathfrak{D})$ has AP. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be finite JEPD τ-structures with $e_i: \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ and $h_i: \mathfrak{B}_i \to \mathfrak{D}$. We must show that there exist $f_i: \mathfrak{B}_i \to \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$. Let $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ be the substructures of $\mathfrak{D}$ on $(h_1 \circ e_1)(A)$ and $(h_2 \circ e_2)(A)$, respectively. Clearly both $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ are JDJEPD, because they are substructures of $\mathfrak{D}$. Due to Lemma 2, we have $\tilde{\mathfrak{A}}_1 \cong \tilde{\mathfrak{A}}_2$, because both $h_1 \circ e_1$ and $h_2 \circ e_2$ preserve the complements of all relations of $\mathfrak{A}$ and $\ker h_1 \circ e_1 = \bigcup\{R^{\mathfrak{A}} \mid R \in \tau \text{ and } R^{\mathfrak{D}} \subseteq \Delta_D\} = \ker h_2 \circ e_2$.

However, what we want is an isomorphism that commutes with $h_1 \circ e_1$ and $h_2 \circ e_2$. Consider the map $g: \tilde{A}_1 \to \tilde{A}_2$ given by $g\big((h_1 \circ e_1)(a)\big) := (h_2 \circ e_2)(a)$. It is well defined, because $\ker h_1 \circ e_1 = \ker h_2 \circ e_2$. Now, for every $R \in \tau$ and $((h_1 \circ e_1)(a_1), \ldots, (h_1 \circ e_1)(a_k)) \in R^{\tilde{\mathfrak{A}}_1}$, we have $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$, because $h_1 \circ e_1$ preserves the complements of all relations of $\mathfrak{A}$ due to Lemma 2. But this implies $((h_2 \circ e_2)(a_1), \ldots, (h_2 \circ e_2)(a_k)) \in R^{\tilde{\mathfrak{A}}_2}$, because $h_2 \circ e_2$ is a homomorphism. By Lemma 2, g preserves the complements of all relations of $\tilde{\mathfrak{A}}_1$ and

$$\ker g = \bigcup \left\{ R^{\tilde{\mathfrak{A}}_1} \;\middle|\; R \in \tau \text{ and } R^{\tilde{\mathfrak{A}}_2} \subseteq \Delta_{\tilde{A}_2} \right\} = \Delta_{\tilde{A}_1}.$$

Hence g is an isomorphism that additionally satisfies $g \circ h_1 \circ e_1 = h_2 \circ e_2$ by its definition. Let $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ be the substructures of $\mathfrak{D}$ on $h_1(B_1)$ and $h_2(B_2)$, respectively. Now consider the inclusions $\tilde{e}_i: \tilde{\mathfrak{A}}_i \hookrightarrow \tilde{\mathfrak{B}}_i$. Since $\mathrm{Age}(\mathfrak{D})$ has AP, there exists $\mathfrak{C} \in \mathrm{Age}(\mathfrak{D})$ together with $\tilde{f}_i: \tilde{\mathfrak{B}}_i \hookrightarrow \mathfrak{C}$ and $e: \mathfrak{C} \hookrightarrow \mathfrak{D}$ such that $\tilde{f}_1 \circ \tilde{e}_1 = \tilde{f}_2 \circ \tilde{e}_2 \circ g$. We define the homomorphisms $f_i: \mathfrak{B}_i \to \mathfrak{D}$ by $f_i := e \circ \tilde{f}_i \circ h_i$. Then, for every $a \in A$, we have

$$\begin{aligned} f_1 \circ e_1(a) &= e \circ \tilde{f}_1 \circ h_1 \circ e_1(a) = e \circ \tilde{f}_1 \circ \tilde{e}_1 \circ h_1 \circ e_1(a) \\ &= e \circ \tilde{f}_2 \circ \tilde{e}_2 \circ g \circ h_1 \circ e_1(a) = e \circ \tilde{f}_2 \circ \tilde{e}_2 \circ h_2 \circ e_2(a) \\ &= e \circ \tilde{f}_2 \circ h_2 \circ e_2(a) = f_2 \circ e_2(a). \end{aligned}$$

Note that, as inclusions, the mappings $\tilde{e}_i$ are the identity on the elements for which they are defined. The above identities show that $\mathfrak{D}$ is a patchwork.

For the other direction, suppose that $\mathfrak{D}$ is a patchwork. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be finite τ-structures with $e_i \colon \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ and $h_i \colon \mathfrak{B}_i \hookrightarrow \mathfrak{D}$. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are isomorphic to substructures of $\mathfrak{D}$, they are clearly JEPD. Thus, as $\mathfrak{D}$ is a patchwork, there exist homomorphisms $f_i \colon \mathfrak{B}_i \to \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$. By Lemma 2, the f_i preserve the complements of all relations of $\mathfrak{B}_i$, and

$$\ker f_i = \bigcup \left\{ R^{\mathfrak{B}_i} \;\middle|\; R \in \tau \text{ and } R^{\mathfrak{D}} \subseteq \Delta_D \right\} = \ker h_i = \Delta_{B_i}.$$

This means that f_i are embeddings. We obtain AP for Age($\mathfrak{D}$) by choosing $\mathfrak{C}$ to be the substructure of $\mathfrak{D}$ on $f_2(B_1) \cup f_1(B_2)$. □

Recall that, to obtain JDJEPD, we actually need to take the d-reduct of a given ω-categorical structure, rather than the structure itself. Fortunately, homogeneity transfers from $\mathfrak{D}$ to $\mathfrak{D}^{\leqslant d}$.

Lemma 3. *Let $\mathfrak{D}$ be a countable homogeneous structure with a finite relational signature τ. Then $\mathfrak{D}^{\leqslant d}$ is homogeneous for every d that exceeds or is equal to the maximal arity of a symbol from τ.*

Proof. By Theorem 3, $\mathfrak{D}$ has quantifier elimination. Note that the relations of $\mathfrak{D}^{\leqslant d}$ and $\mathfrak{D}$ are first-order interdefinable, which implies $\mathrm{Aut}(\mathfrak{D}^{\leqslant d}) = \mathrm{Aut}(\mathfrak{D})$ by Theorem 2. This shows in particular that $\mathfrak{D}^{\leqslant d}$ is ω-categorical. Every first-order $\tau^{\leqslant d}$-formula ϕ defines a relation in $\mathfrak{D}^{\leqslant d}$ that has a first-order definition ϕ' in $\mathfrak{D}$. We can assume that ϕ' is quantifier-free due to Theorem 3. We replace every atomic formula $\psi(x_1, \ldots, x_k)$ in ϕ' by $\bigvee_{i=1}^{n} R_i(x_1, \ldots, x_k)$ with $R_1, \ldots, R_n \in \tau^{\leqslant d}$, where $R_1^{\mathfrak{D}^{\leqslant d}} \cup \cdots \cup R_n^{\mathfrak{D}^{\leqslant d}}$ is the unique decomposition of $\psi^{\mathfrak{D}}$ into orbits of k-tuples over D under $\mathrm{Aut}(\mathfrak{D})$. The resulting formula is a quantifier-free first-order definition of $\phi^{\mathfrak{D}^{\leqslant d}}$ in $\mathfrak{D}^{\leqslant d}$. Thus $\mathfrak{D}^{\leqslant d}$ has quantifier elimination as well, which means that it is homogeneous due to Theorem 3. □

Finite boundedness. The only property of ω-admissible structures we have not yet considered in this section is the decidability of the CSP. One possibility to achieve this is to consider finitely bounded structures. For a class $\mathcal{N}$ of τ-structures (called *bounds*), we denote by $\mathrm{Forb}_e(\mathcal{N})$ the class of all finite τ structures not embedding any member of $\mathcal{N}$. Following [14], we say that a structure $\mathfrak{A}$ is *finitely bounded* if its signature is finite and $\mathrm{Age}(\mathfrak{A}) = \mathrm{Forb}_e(\mathcal{N})$ for a finite set of bounds $\mathcal{N}$.

Note that the structure $\mathfrak{A}$ is finitely bounded iff there exists a universal first-order sentence $\Phi(\mathfrak{A})$ s.t. $\mathfrak{B} \in \mathrm{Age}(\mathfrak{A})$ iff $\mathfrak{B} \models \Phi(\mathfrak{A})$. In fact, for every bound $\mathfrak{C} \in \mathcal{N}$ of $\mathfrak{A}$ with domain $\{c_1, c_2, \ldots\}$ we can write down a quantifier-free formula $\phi_{\mathfrak{C}}$ with free variables $x_{c_1}, x_{c_2}, \ldots$ that describes $\mathfrak{C}$ up to isomorphism. Then we set $\Phi(\mathfrak{A}) \coloneqq \bigwedge_{\mathfrak{C} \in \mathcal{N}} \forall x_{c_1}, x_{c_2} \cdots . \neg \phi_{\mathfrak{C}}(x_{c_1}, x_{c_2} \ldots)$. Conversely, assume that we have a sentence $\Phi(\mathfrak{A})$ that defines $\mathrm{Age}(\mathfrak{A})$. If we define the set $\mathcal{N}$ to consist (up to

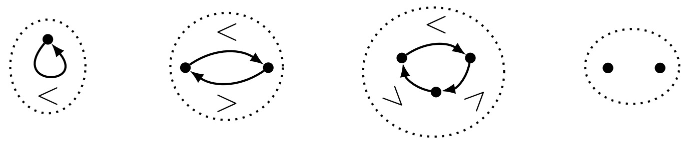

Fig. 1. A set of four forbidden substructures for $\mathfrak{Q} = (\mathbb{Q}; <)$.

isomorphism) of all τ-structures of size at most equal to the number of variables of $\Phi(\mathfrak{A})$ that do not satisfy $\Phi(\mathfrak{A})$, then $\mathcal{N}$ is a set of bounds for $\mathfrak{A}$.

The structure $\mathfrak{Q}$ is finitely bounded. To show this, we can use the set $\mathcal{N}$ consisting of the four structures depicted in Fig. 1: the self loop, the 2-cycle, the 3-cycle, and two isolated vertices. We must show that $\mathrm{Age}(\mathfrak{Q}) = \mathrm{Forb}_e(\mathcal{N})$. Clearly, none of the structures in $\mathcal{N}$ embeds into a linear order, which shows $\mathrm{Age}(\mathfrak{Q}) \subseteq \mathrm{Forb}_e(\mathcal{N})$. Conversely, assume that $\mathfrak{A}$ is an element of $\mathrm{Forb}_e(\mathcal{N})$. We must show that $<^{\mathfrak{A}}$ is a linear order. Since $\mathcal{N}$ contains the self loop, we have $(a, a) \notin <^{\mathfrak{A}}$ for all $a \in A$, which shows that $<^{\mathfrak{A}}$ is irreflexive. For distinct elements $a, b \in A$, we must have $a <^{\mathfrak{A}} b$ or $b <^{\mathfrak{A}} a$ since otherwise the structure consisting of two isolated vertices could be embedded into $\mathfrak{A}$. This shows that any two distinct elements are comparable w.r.t. $<^{\mathfrak{A}}$. To show that $<^{\mathfrak{A}}$ is transitive, assume that $a <^{\mathfrak{A}} b$ and $b <^{\mathfrak{A}} c$ holds. Since the 2-cycle does not embed into $\mathfrak{A}$, a and c must be distinct, and are thus comparable. We cannot have $c <^{\mathfrak{A}} a$ since then we could embed the 3-cycle into $\mathfrak{A}$. Consequently, we must have $a <^{\mathfrak{A}} c$, which proves transitivity. This show that $\mathfrak{A}$ is a linear order. As formula $\Phi(\mathfrak{Q})$ we can take the conjunction of the usual axioms defining linear orders.

Finitely bounded structures are interesting since their CSP and their first-order theory are decidable. The first result can, e.g., be found in [13] (Theorem 4) and the second result is stated in [23,24].

Proposition 4. *Let $\mathfrak{D}$ be a finitely bounded homogeneous structure with $|D| > 1$. Then* CSP($\mathfrak{D}$) *is in* NP *and the first-order theory of $\mathfrak{D}$ is* PSPACE*-complete.*

Proof. We only prove the second part of the statement as a detailed proof of it does not appear in the literature. Let τ be the signature of $\mathfrak{D}$. We first show PSPACE-hardness by a reduction from the *quantified Boolean formula* (QBF) *satisfiability* problem. Let ϕ be a QBF of the form $Q_1 X_1. \cdots Q_k X_k. \psi(X_1, \ldots, X_k)$ such that ψ is quantifier-free. We introduce a fresh auxiliary first-order variable h. Then, for every propositional variable X_i, we introduce a fresh first-order variable x_i and replace every occurrence of the variable X_i in ψ with the literal $(x_i = h)$ and denote the resulting τ-formula by ψ'. Now we set ϕ' to be the first-order sentence $\exists h. Q_1 x_1. \cdots Q_k x_k. \psi'(h, x_1, \ldots, x_k)$. Since $\mathfrak{D}$ contains at least two elements, it is easy to see that the following holds: ϕ is satisfiable as a QBF iff $\mathfrak{D} \models \phi'$.

Next we describe a PSPACE algorithm that decides the first-order theory of $\mathfrak{D}$. It is based on the algorithm from the proof of Proposition 3.5 in [24], for which an exponential time complexity is shown in [24]. Note that, since D is possibly infinite, we cannot simply substitute all elements from D, one after the other, for a particular quantified variable. First recall that, since $\mathfrak{D}$ is finitely bounded, there exists a universal first-order sentence $\Phi(\mathfrak{D})$ that defines $\mathrm{Age}(\mathfrak{D})$, i.e., a finite

τ-structure can be embedded into $\mathfrak{D}$ iff it satisfies $\Phi(\mathfrak{D})$. Since the structure $\mathfrak{D}$ is fixed, this sentence is also fixed, which means that it has constant size.

Now, let $b_1, b_2, \dots$ be a countably infinite sequence of pairwise distinct symbols. For an first-order τ-formula ϕ with free variables $x_1, \dots, x_n$, let $[\phi]_{\mathfrak{D}}$ denote the set of all τ-structures $\mathfrak{B}$ with domain $\{b_1, \dots, b_n\}$ for which there exists an embedding $h \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \phi(h(b_1), \dots, h(b_n))$. Every such embedding $h \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$ represents an injective[4] substitution of elements from D for the variables $x_1, \dots, x_n$. We claim that $[\phi]_{\mathfrak{D}}$ does not depend on the choice of h. To see this, consider two embeddings $h_1, h_2 \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \phi(h_1(b_1), \dots, h_1(b_n))$. For each $i \in [2]$, let $\mathfrak{B}_i$ be the substructure of $\mathfrak{D}$ on the image of $\{b_1, \dots, b_n\}$ under h_i. Consider the map $\widetilde{f} \colon B_1 \to B_2$ that sends, for every $j \in [n]$, $h_1(b_j)$ to $h_2(b_j)$. Using the definition of an embedding, it is easy to show that $\widetilde{f}$ is an isomorphism from $\mathfrak{B}_1$ to $\mathfrak{B}_2$. By homogeneity of $\mathfrak{D}$, there exists an automorphism f of $\mathfrak{D}$ that extends $\widetilde{f}$. Since ϕ is an first-order formula, $\phi^{\mathfrak{D}}$ is preserved by f, which shows that $\mathfrak{D} \models \phi(h_2(b_1), \dots, h_2(b_n))$ holds as well.

We show by *induction* on the structure of an first-order τ-formula ϕ with free variables $x_1, \dots, x_n$ that, given a τ-structure $\mathfrak{B}$ with domain $\{b_1, \dots, b_n\}$, it can be decided in PSpace in the size of ϕ whether $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$. This proves the PSpace upper bound claimed in the proposition because, if ϕ has no free variables, then testing whether the empty structure is contained in $[\phi]_{\mathfrak{D}}$ is equivalent to answering $\mathfrak{D} \models \phi$.

In the *base case*, we consider an atomic formula $\phi(x_1, \dots, x_n)$, that is, ϕ is either a positive R-literal for some $R \in \tau$, or the equality between two variables. Suppose that $\mathfrak{B}$ is a τ-structure with domain $\{b_1, \dots, b_n\}$. If $\mathfrak{B} \models \neg\phi(b_1, \dots, b_n)$, then clearly $\mathfrak{B} \notin [\phi(x_1, \dots, x_n)]_{\mathfrak{D}}$ because embeddings preserve complements of relations. If $\mathfrak{B} \models \phi(b_1, \dots, b_n)$, then $\mathfrak{D} \models \phi(h(b_1), \dots, h(b_n))$ holds for every embedding $h \colon \mathfrak{B} \to \mathfrak{D}$. Consequently, testing whether $\mathfrak{B} \in [\phi(x_1, \dots, x_n)]_{\mathfrak{D}}$ boils down to testing whether $\mathfrak{B} \hookrightarrow \mathfrak{D}$, which is the case iff $\mathfrak{B} \models \Phi(\mathfrak{D})$. This can be done in PSpace in the size of ϕ because it is well-known that first-order model checking with a fixed first-order sentence can be done in polynomial time in the size of the input structure.

For the *induction step*, we can restrict the attention to formulas ϕ of the form $\psi_1 \vee \psi_2$, $\neg\psi$ and $\exists x.\, \psi$. Suppose that ϕ is of the form $\psi_1 \vee \psi_2$ such that the induction hypothesis applies to both ψ_1 and ψ_2. For each $i \in [2]$, let $\mathfrak{B}_i$ be the substructure of $\mathfrak{B}$ on those b_js that correspond to the free variables of ψ_i. We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ iff $\mathfrak{B} \models \Phi(\mathfrak{D})$ and $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ for $i = 1$ or $i = 2$. The forward direction is trivial. Now suppose that $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ for $i = 1$ or $i = 2$ and $\mathfrak{B} \models \Phi(\mathfrak{D})$. Then we have an embedding $h_i \colon \mathfrak{B}_i \hookrightarrow \mathfrak{D}$ witnessing $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$, and we also have an embedding $h \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$. But then $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ is also witnessed by $h|_{B_i}$ because $[\psi_i]_{\mathfrak{D}}$ does not depend on the choice of the embedding. This shows that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ is witnessed by h. Testing whether $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ can be done in PSpace in the size

[4] In our proof we will ensure that injective substitutions are sufficient, by appropriately identifying variables.

of ψ_i by the induction hypothesis, and we have already seen in the base case that testing whether $\mathfrak{B} \models \Phi(\mathfrak{D})$ can be done in polynomial time in the size of ϕ.

Suppose that ϕ is of the form $\neg\psi$ such that the induction hypothesis applies to ψ. We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ iff $\mathfrak{B} \notin [\psi]_{\mathfrak{D}}$. Suppose that there exists $h \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \neg\psi(h(b_1), \dots, h(b_n))$. Then there cannot be an embedding $h' \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \psi(h'(b_1), \dots, h'(b_n))$ because containment in $[\phi]_{\mathfrak{D}}$ does not depend on the choice of the embedding. The backward direction is analogous. By the induction hypothesis, testing whether $\mathfrak{B} \in [\psi]_{\mathfrak{D}}$ can be done in PSpace in the size of ψ and thus also in the size of ϕ.

Now suppose that ϕ is of the form $\phi(x_1, \dots, x_n) = \exists x_{n+1}. \psi(x_1, \dots, x_{n+1})$ such that the induction hypothesis applies to ψ. We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ iff one of the following is true

1. there exists an extension $\mathfrak{B}'$ of $\mathfrak{B}$ by b_{n+1} such that $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$,
2. there exists $i \in [n]$ such that $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$ holds for the formula ψ_i obtained from ψ by replacing each occurrence of the variable x_{n+1} in ψ by x_i.

First, suppose that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ is witnessed by some embedding $h \colon \mathfrak{B} \hookrightarrow \mathfrak{D}$. Then there exists $d \in D$ such that $\mathfrak{D} \models \psi(h(b_1), \dots, h(b_n), d)$. If d is distinct from $h(b_1), \dots, h(b_n)$, then we are in the case (1) and consider the extension h' of h that maps b_{n+1} to d. We define $\mathfrak{B}'$ as the τ-structure with the domain $\{b_1, \dots, b_{n+1}\}$ such that, for every k-ary symbol $R \in \tau$, we have $t \in R^{\mathfrak{B}'}$ iff $h'(t) \in R^{\mathfrak{D}}$. Clearly h' is an embedding that witnesses $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$. Otherwise we have $d = h(b_i)$ for some $i \in [n]$. We consider the formula ψ_i from (2). Then h is an embedding that witnesses $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$. Since the backward direction is obvious, it remains to show that the tests required by (1) and (2) can be performed in PSpace.

In case (1), we generate all extensions $\mathfrak{B}'$ of $\mathfrak{B}$ by b_{n+1} and test, using the induction hypothesis, whether $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$ for some such extension. This can clearly be done in PSpace because τ is fixed and finite, and for each extension $\mathfrak{B}'$ we can test $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$ within PSpace due to the induction hypothesis. In case (2) we guess any such $i \in [n]$ and test, using the induction hypothesis, whether $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$. This completes the proof. ☐

The following proposition implies that Proposition 4 applies not only to a given finitely bounded homogeneous structure $\mathfrak{D}$, but also to its d-reduct $\mathfrak{D}^{\leqslant d}$.

Proposition 5. *Let $\mathfrak{A}$ be a finitely bounded homogeneous structure and $\mathfrak{B}$ a structure with the same domain and finitely many relations that are first-order definable in $\mathfrak{A}$. Then $\mathfrak{B}$ is a reduct of a finitely bounded homogeneous structure.*

Proof. Let $\tilde{\mathfrak{A}}$ be the expansion of $\mathfrak{A}$ by the relations of $\mathfrak{B}$, where we assume that the signatures of $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint. By Theorem 3, each of the new relations has a quantifier-free definition in $\mathfrak{A}$. Consequently, we can extend the sentence $\Phi(\mathfrak{A})$ with universal sentences defining the relations of $\mathfrak{B}$, which yields a universal formula that shows finite boundedness of $\tilde{\mathfrak{A}}$. The structure $\tilde{\mathfrak{A}}$ is homogeneous since an isomorphism between two finite substructures of $\tilde{\mathfrak{A}}$ induces an isomorphism between their reducts to the signature of $\mathfrak{A}$, which extends to an automorphism of $\mathfrak{A}$ by homogeneity of $\mathfrak{A}$. This is also an automorphism of $\tilde{\mathfrak{A}}$ since automorphisms preserve first-order definable relations. Now we are done as $\mathfrak{B}$ is a reduct of $\tilde{\mathfrak{A}}$. $\square$

We are now ready to formulate our first sufficient condition for ω-admissibility.

Theorem 5. *Let $\mathfrak{D}$ be a finitely bounded homogeneous relational structure with at most d-ary relations for some $d \geq 2$. Then $\mathfrak{D}^{\leqslant d}$ is ω-admissible.*

Proof. It follows directly from the definition of d-reducts that $\mathfrak{D}^{\leqslant d}$ is JDJEPD. By Lemma 3, $\mathfrak{D}^{\leqslant d}$ is homogeneous. By Theorem 3, $\mathfrak{D}^{\leqslant d}$ is ω-categorical. Thus $\mathfrak{D}$ has homomorphism compactness by Corollary 1. By Theorem 4, $\mathrm{Age}(\mathfrak{D}^{\leqslant d})$ has AP. Thus $\mathfrak{D}^{\leqslant d}$ is a patchwork by Proposition 3. By Proposition 5, Lemma 1, and Proposition 4, $\mathrm{CSP}(\mathfrak{D}^{\leqslant d})$ is in NP. Hence $\mathfrak{D}^{\leqslant d}$ is ω-admissible. $\square$

This theorem, together with Theorem 1, immediately yields decidability for $\mathcal{ALC}^d_{\vee+}(\mathfrak{D}^{\leqslant d})$. The following corollary shows that we can even allow for arbitrary first-order definable relations with arity bounded by d in the concrete domain. The idea for proving this result is to reduce concept satisfiability in $\mathcal{ALC}^d_{\mathrm{fo}}(\mathfrak{D})$ to concept satisfiability in $\mathcal{ALC}^d_{\vee+}(\mathfrak{D}^{\leqslant d})$. We know that every at most d-ary relation over D first-order definable in $\mathfrak{D}$ can be obtained as a disjunction of atomic formulas built using the signature of $\mathfrak{D}^{\leqslant d}$. What still needs to be shown is that, given a first-order formula in the signature of $\mathfrak{D}$ with at most d free variables, this disjunction can effectively be computed.

Corollary 2. *Let $\mathfrak{D}$ be a finitely bounded homogeneous relational structure with at most d-ary relations for some $d \geq 2$. Then concept satisfiability in $\mathcal{ALC}^d_{\mathrm{fo}}(\mathfrak{D})$ w.r.t. TBoxes is decidable.*

Proof. Let τ be the signature of $\mathfrak{D}$. We claim that satisfiability of $\mathcal{ALC}^d_{\mathrm{fo}}(\mathfrak{D})$ concepts w.r.t. TBoxes can be reduced to satisfiability of $\mathcal{ALC}^d_{\vee+}(\mathfrak{D}^{\leqslant d})$ concepts w.r.t. TBoxes. For this purpose, we need to replace first-order τ-formulas ϕ in concrete domain constructors $\forall p_1, \dots, p_k. [\phi]$ or $\exists p_1, \dots, p_k. [\phi]$ with disjunctions ψ of atomic formulas in the signature $\tau^{\leqslant d}$ of $\mathfrak{D}^{\leqslant d}$. By Theorem 3 together with Theorem 2, the (finitely many) relations in $\tau^{\leqslant d}$ have quantifier-free definitions in $\mathfrak{D}$. Since d and $\mathfrak{D}$ are fixed, we can make a list consisting of the quantifier-free definitions for each of them in constant time. Given an first-order τ-formulas ϕ with k free variables, let $\psi_1, \dots, \psi_m$ be the quantifier-free definitions in $\mathfrak{D}$ for all the k-ary relations of $\tau^{\leqslant d}$ that we have listed before. We test, for every $i \in [m]$, whether $\mathfrak{D} \models \exists y_1, \dots, y_k. \phi(y_1, \dots, y_k) \wedge \psi_i(y_1, \dots, y_k)$, which is possible in PSPACE by Proposition 4. By selecting those $\psi_{i_1}, \dots, \psi_{i_s}$ that tested positively, we know for all $d_1, \dots, d_k \in D$ that $\mathfrak{D} \models \phi(d_1, \dots, d_k)$ iff $\mathfrak{D} \models \bigvee_{r=1}^{s} \psi_{i_r}(d_1, \dots, d_k)$. Now we replace each $\psi_{i_r}(y_1, \dots, y_k)$ with $R(y_1, \dots, y_k)$, where R is the unique k-ary relation symbol from $\tau^{\leqslant d}$ for which we have $\mathfrak{D} \models \psi_{i_r}(d_1, \dots, d_k)$ iff $\mathfrak{D}^{\leqslant d} \models R(d_1, \dots, d_k)$. This yields the desired formula ψ that replaces ϕ. Now the claim follows from Theorem 5 and Theorem 1. $\square$

Cores. Finally, we consider the situation where we have a homogeneous relational structure $\mathfrak{D}$ with finitely many at most d-ary relations that is not finitely bounded, but which we know (by some other means) to have a decidable CSP. In this situation, we can show decidability for $\mathcal{ALC}^d_{\exists+}(\mathfrak{D})$ under one additional assumption. A structure $\mathfrak{D}$ is called a *core* if every endomorphism of $\mathfrak{D}$ is an self-embedding of $\mathfrak{D}$. It was shown in [8] that, if $\mathfrak{D}$ is a homogeneous core, then the orbits of tuples over D under $\mathrm{Aut}(\mathfrak{D})$ are pp definable in $\mathfrak{D}$. As an easy consequence of this result, we obtain our second sufficient condition for ω-admissibility.

Theorem 6. *Let $\mathfrak{D}$ be a homogeneous core with finitely many at most d-ary relations for some $d \geq 2$ and decidable CSP. Then $\mathfrak{C}^{\leqslant d}$ is ω-admissible.*

Proof. It follows directly from the definition of d-reducts that $\mathfrak{D}^{\leqslant d}$ is JDJEPD. By Lemma 3, $\mathfrak{D}^{\leqslant d}$ is homogeneous. By Theorem 3, $\mathfrak{D}^{\leqslant d}$ is ω-categorical. Thus $\mathfrak{D}$ has homomorphism compactness by Corollary 1. By Theorem 4, $\mathrm{Age}(\mathfrak{D}^{\leqslant d})$ has AP. Thus $\mathfrak{D}^{\leqslant d}$ is a patchwork by Proposition 3. By the results of [8], $\mathfrak{D}$ is model-complete and orbits of tuples over D under $\mathrm{Aut}(\mathfrak{D})$ are pp definable in $\mathfrak{D}$. Then Lemma 1 implies $\mathrm{CSP}(\mathfrak{D}^{\leqslant d}) \leq_{\text{PTIME}} \mathrm{CSP}(\mathfrak{D})$. Hence $\mathfrak{D}^{\leqslant d}$ is ω-admissible. □

Let $\mathfrak{D}$ be a structure as in the above theorem. By showing that concept satisfiability in $\mathcal{ALC}^d_{\exists+}(\mathfrak{D})$ can be reduced to concept satisfiability in $\mathcal{ALC}^d_{\vee+}(\mathfrak{D}^{\leqslant d})$, we obtain the following decidability result.

Corollary 3. *Let $\mathfrak{D}$ be a homogeneous core with finitely many at most d-ary relations for some $d \geq 2$ and a decidable CSP. Then concept satisfiability in $\mathcal{ALC}^d_{\exists+}(\mathfrak{D})$ w.r.t. TBoxes is decidable.*

Proof. To show the corollary, it is thus sufficient to prove that concept satisfiability w.r.t. TBoxes is decidable in $\mathcal{ALC}^d_{\exists+}(\mathfrak{D})$. We achieve this by reducing this problem to satisfiability of $\mathcal{ALC}^d_{\vee+}(\mathfrak{D}^{\leqslant d})$ concepts w.r.t. TBoxes. As in the proof of Corollary 2, we do this by showing how existential positive formulas ϕ occurring in concrete domain constructors can be replaced by disjunctions ψ of atomic formulas in the signature of $\mathfrak{D}^{\leqslant d}$.

By the results of [8], the relations of $\mathfrak{D}^{\leqslant d}$ have pp definitions in $\mathfrak{D}$. Since d and $\mathfrak{D}$ are fixed, we can make a list consisting of the pp definitions for each of them in constant time. Given an existential positive τ-formula ϕ with k free variables, let $\psi_1, \dots, \psi_m$ be the pp definitions in $\mathfrak{D}$ for all the k-ary relations of $\mathfrak{D}^{\leqslant d}$ that we have listed before. Since $\mathrm{CSP}(\mathfrak{D})$ is decidable, we can decide for $i \in [n]$ whether $\mathfrak{D} \models \exists y_1, \dots, y_k.\, \psi_i \wedge \phi$. In fact, deciding whether an existential positive sentence is true in $\mathfrak{D}$ only differs from solving $\mathrm{CSP}(\mathfrak{D})$ in a non-deterministic step that deals with disjunction. By selecting those $\psi_{i_1}, \dots, \psi_{i_s}$ that tested positively, we know that $\mathfrak{D} \models \phi(x_1, \dots, x_k)$ iff $\mathfrak{D} \models \bigvee_{r=1}^{s} \psi_{i_r}(x_1, \dots, x_k)$ holds for all $x_1, \dots, x_k \in D$. Now we replace each $\psi_{i_r}(y_1, \dots, y_k)$ with $R(y_1, \dots, y_k)$, where R is the unique k-ary relation symbol from the signature of $\mathfrak{D}^{\leqslant d}$ that satisfies $\mathfrak{D} \models \psi_{i_r}(x_1, \dots, x_k)$ iff $\mathfrak{D}^{\leqslant d} \models R(x_1, \dots, x_k)$. This yields the desired formula ψ, which completes the reduction. Now the claim follows from Theorem 6 and Theorem 1. □

5 Application and discussion

In this section, we discuss how our results can be used to obtain specific ω-admissible concrete domains. But let us first start with a caveat.

5.1 Finiteness of signature matters.

In Corollary 2 and Corollary 3, the signature of the structure $\mathfrak{D}$ is required to be finite. This restriction is needed to obtain decidability. For instance, the expansion of the structure $(\mathbb{Z}; +_1)$ from Section 3.1 by all relations $+_k = \{(m, n) \in \mathbb{Z}^2 \mid m + k = n\}$ for $k \in \mathbb{Z}$ is homogeneous, and satisfiability of finite conjunctions of constraints is decidable in this structure. However, we have seen in Proposition 1 that reasoning with $(\mathbb{Z}; +_1)$ as a concrete domain w.r.t. TBoxes is undecidable.

5.2 (Un)decidability of the conditions.

If one intends to use Theorem 5 to obtain an ω-admissible concrete domain, one could start with selecting a finite set $\mathcal{N}$ of bound, i.e., forbidden τ-substructures, for a finite signature τ. The question is then whether $\mathcal{N}$ really induces a finitely bounded structure, i.e., whether there is a τ-structure $\mathfrak{D}$ such that $\mathrm{Age}(\mathfrak{D}) = \mathrm{Forb}_e(\mathcal{N})$. The bad news is that this question is in general undecidable. In fact, it is shown in [16] that the *joint embedding property (JEP)* is undecidable for classes of structures that are definable by finitely many bounds. In addition, it is known that a class of structures definable by finitely many bounds has JEP iff this class is the age of some structure [10].

However, if one restricts the attention to signatures containing only binary relations, then it is decidable whether a class of the form $\mathrm{Forb}_e(\mathcal{N})$ has AP [12]. If this is the case, then the Fraïssé limit $\mathfrak{D}$ of $\mathrm{Forb}_e(\mathcal{N})$ is a finitely bounded homogeneous structure satisfying $\mathrm{Age}(\mathfrak{D}) = \mathrm{Forb}_e(\mathcal{N})$ by Theorem 4.

5.3 Scope of the homogeneity condition

The decidability results given in Corollaries 2 and 3 presuppose that the given concrete domain $\mathfrak{D}$ is a homogeneous structure. One might ask whether this restriction precludes some ω-admissible concrete domains to be covered by these corollaries. We can show that every countable ω-admissible structure is equivalent to an up to isomorphism unique homogeneous structure in the sense that one can be embedded into each other (see Proposition 6), but in general it is not the case that both structures satisfy the same first-order sentences (see Example 2). We leave it as an open question whether such structures can lead to distinct extensions of $\mathcal{ALC}$ by concrete domains.

Proposition 6. *Let $\mathfrak{A}$ be a countable ω-admissible structure. There exists an up to isomorphism unique countable homogeneous core $\mathfrak{B}$ with $\mathfrak{A} \hookrightarrow \mathfrak{B}$ and $\mathfrak{B} \hookrightarrow \mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a countable ω-admissible structure. Then, by Proposition 3, Age($\mathfrak{A}$) has AP. Since the signature of $\mathfrak{A}$ is finite, Age($\mathfrak{A}$) is an amalgamation class. By Theorem 4, there exists a countable homogeneous structure $\mathfrak{B}$ with Age($\mathfrak{B}$) = Age($\mathfrak{A}$). By Theorem 3, $\mathfrak{B}$ is ω-categorical. Then, by Corollary 1, $\mathfrak{B}$ has homomorphism compactness. Since both $\mathfrak{A}$ and $\mathfrak{B}$ are countable, have homomorphism compactness, and we have the same age, we conclude that $\mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{B} \to \mathfrak{A}$. It is known that ω-categorical structures satisfy an even stronger property than homomorphism compactness, which we refer to as *embedding compactness* (Lemma 3.1.5 in [9]). This property guarantees an embedding from a given countable structure if there exists an embedding from all of its finite substructures. This means that $\mathfrak{A} \hookrightarrow \mathfrak{B}$ because Age($\mathfrak{A}$) $\subseteq$ Age($\mathfrak{B}$). Since $\mathfrak{B}$ has quantifier elimination and inherits JEPD from $\mathfrak{A}$, it is an ω-categorical *model-complete* core by Theorem 3.6.11 in [9]. Now if we had two such ω-categorical model-complete cores $\mathfrak{B}$ and $\mathfrak{B}'$, then they would be homomorphically equivalent by homomorphism compactness, which means that the have to be isomorphic by a result in [8]. Let $e\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ and $f\colon \mathfrak{B} \to \mathfrak{A}$ be the embedding and the homomorphism from above, respectively. We claim that f is an embedding as well. Since $\mathfrak{B}$ is a core and $e \circ f$ is an endomorphism of $\mathfrak{B}$, it is a self-embedding of $\mathfrak{B}$. Suppose that f is not an embedding. Then there exists a tuple $\boldsymbol{t} \notin R^{\mathfrak{B}}$ such that $f(\boldsymbol{t}) \in R^{\mathfrak{A}}$. Since e is a homomorphism, we have $(e \circ f)(\boldsymbol{t}) \in R^{\mathfrak{B}}$. But then $e \circ f$ is not an embedding, a contradiction. Thus f is an embedding. ◻

5.4 Reproducing known results.

The examples for ω-admissible concrete domains given in [28] were RCC8 and Allen's interval algebra, for which the patchwork property is proved "by hand" in [28]. Given our Theorem 5, we obtain these results as a consequence of known results from model theory. It was shown in [15] that RCC8 has a representation by a homogeneous structure $\mathfrak{R}$ with a finite relational signature (see Theorem 2 in [15]). Since Age($\mathfrak{R}$) has a finite universal axiomatization (see Definition 3 in [15]), $\mathfrak{R}$ is finitely bounded. For Allen's interval algebra, it was shown in [19] that it has a representation by a homogeneous structure $\mathfrak{A}$ with a finite relational signature (see the second example on page 270 in [19]). Since Age($\mathfrak{A}$) has a finite universal axiomatization (see the composition table from Figure 4 in [1]), $\mathfrak{A}$ is finitely bounded. The structure $\mathfrak{Q} = (\mathbb{Q}, <)$ we used as our running example also satisfies the preconditions of Theorem 5, and thus Corollary 2 yields decidability of $\mathcal{ALC}^{d}_{\mathrm{fo}}(\mathfrak{Q})$ with TBoxes. For $\mathfrak{Q}$ extended just with $>, \leq, \geq, =, \neq$, decidability was proved in [25], using an automata-based procedure. Our results show that there is also a tableau-based decision procedure for this logic.

5.5 Expansions, disjoint unions, and products.

When modelling concepts in a DL with concrete domain $\mathfrak{D}$, it is often useful to be able to refer to specific elements c of the domain, i.e., to have unary predicate symbols $=_c$ that are interpreted as $\{c\}$. Adding finitely many such predicates is harmless since we can show that the class of reducts of finitely bounded homogeneous structures is closed under expansion by finitely many predicates of the form $=_c$. This follows from the following result, which was stated (without proof) in [13].

Proposition 7. *Let $\mathfrak{A}$ be a finitely bounded homogeneous structure. Any expansion of $\mathfrak{A}$ by a relation of the form $\{c\}$ for $c \in A$ is a reduct of a finitely bounded homogeneous structure.*

Proof. Let τ be the signature of $\mathfrak{A}$ and $\tau' = \tau \cup \{=_c\}$. Consider the τ'-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ where $=_c$ is interpreted as $\{c\}$. By Theorem 2 and Theorem 3, there exists a unary qf τ-formula ϕ_c that defines the orbit of c in $\mathfrak{A}$. Let $\Phi(\mathfrak{A})$ be a universal sentence that defines $\mathrm{Age}(\mathfrak{A})$. We set

$$\Phi(\mathfrak{A}') := \Phi(\mathfrak{A}) \wedge \Big(\forall x. =_c(x) \Rightarrow \phi_c(x)\Big) \wedge \Big(\forall x, y. =_c(x) \wedge =_c(y) \Rightarrow x = y\Big).$$

It is easy to see that $\Phi(\mathfrak{A}')$ is a universal sentence that defines $\mathrm{Age}(\mathfrak{A}')$. This shows that $\mathfrak{A}'$ is finitely bounded. Unfortunately, $\mathfrak{A}'$ need not be homogeneous. To obtain homogeneity, we need to add further relations. For each k-ary symbol $R \in \tau$ with $k \geq 1$ and index set $\emptyset \neq X \subseteq [k]$, we introduce a $(k - |X|)$-ary symbol $R_{X \to c}$. Then we consider the expansion $\mathfrak{A}''$ of $\mathfrak{A}'$ to the signature τ'' that extends τ' by these new symbols where, for each new symbol $R_{X \to c}$, its interpretation in $\mathfrak{A}''$ is defined as follows. If $[k] \setminus X = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$, then

$$(a_1, \dots, a_s) \in R^{\mathfrak{A}''}_{X \to c} \quad \text{iff} \quad (b_1, \dots, b_k) \in R^{\mathfrak{A}},$$

where $b_i = c$ if $i \in X$ and $b_{i_j} = a_j$ for $j \in [s]$. Since these new relations can be defined in $\mathfrak{A}'$ by universal sentences, $\mathfrak{A}''$ is still finitely bounded. We claim that it is also homogeneous. Let $f \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be a τ''-isomorphism between two finite substructures of $\mathfrak{A}''$. If $c \in B_1$, then $c \in B_2$ and f maps c to c. We know that f extends to a τ-automorphism of $\mathfrak{A}$ since $\mathfrak{A}$ is homogeneous. Since $f(c) = c$, this automorphism also respects $=_c$ and the new relation symbols $R_{X \to c}$, i.e., it is also a τ''-automorphism of $\mathfrak{A}''$.

Now, suppose that $c \notin B_1$. Then we also have $c \notin B_2$ since isomorphisms preserve also the complements of relations. For $i \in [2]$, let $\mathfrak{B}'_i$ is the substructure of $\mathfrak{A}''$ on $B'_i := B_i \cup \{c\}$. We extend the isomorphism f to a function $f' : B'_1 \to B'_2$ by mapping c to c, and claim that this is again a τ''-isomorphism. Clearly, it preserves $=_c$.

Now, let $R \in \tau$ be a relation symbol and consider a k-tuple of elements of B'_1. If none of the components of the tuple is c, then we have $(a_1, \dots, a_k) \in R^{\mathfrak{B}'_1}$ iff $(a_1, \dots, a_k) \in R^{\mathfrak{B}_1}$ iff $(f(a_1), \dots, f(a_k)) \in R^{\mathfrak{B}_2}$ iff $(f'(a_1), \dots, f'(a_k)) \in R^{\mathfrak{B}'_2}$.

Otherwise, let $X := \{i \mid a_i = c\}$ and assume that $[k] \setminus X = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$. Using the definition of the interpretation of $R_{X \to c}$ in $\mathfrak{A}''$ and the fact that f is a τ''-isomorphism, we then have $(a_1, \dots, a_k) \in R^{\mathfrak{B}'_1}$ iff $(a_{i_1}, \dots, a_{i_s}) \in R^{\mathfrak{B}_1}_{X \to c}$ iff $(f(a_{i_1}), \dots, f(a_{i_s})) \in R^{\mathfrak{B}_2}_{X \to c}$ iff $(f'(a_1), \dots, f'(a_k)) \in R^{\mathfrak{B}'_2}$.

Summing up, we have shown that f' is a τ''-isomorphism between $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$. Clearly, f' is also a τ-isomorphism between these two structures that extends f and satisfies $f'(c) = c$. By homogeneity of $\mathfrak{A}$, f' can be extended to a τ-automorphism of $\mathfrak{A}$. Since $f(c) = c$, this automorphism also respects $=_c$ and the new relation symbols $R_{X\to c}$, i.e., it is also a τ''-automorphism of $\mathfrak{A}''$. Thus, we have shown that $\mathfrak{A}''$ is homogeneous.

Now the statement of the proposition follows since the expansion $\mathfrak{A}'$ of $\mathfrak{A}$ by $\{c\}$ is a reduct of $\mathfrak{A}''$. □

It would also be useful to be able to refer to predicates of different concrete domains (say RCC8 *and* Allen) when defining concepts. In [5], it was shown that admissible concrete domains are closed under disjoint union. We can show the corresponding result for finitely bounded homogeneous structures. In the proposition formulated below, we assume that the component structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ have the same signature, but disjoint domains. In [5], the signatures of the structures are assumed to be disjoint as well (see the example of combining RCC8 and Allen). The case of disjoint signatures can, however, be reduced to the case of a common signature: we simply expand the structures to the union of their signatures by interpreting relation symbols not belonging to their signature as the empty set. Since empty relations can be defined by first-order formulas, such an expansion by empty relations leaves homogeneity and finite boundedness intact (see Proposition 5). The disjoint union of two structures was formally defined in Section 2 for the case of two structures. The extension of this definition to $n > 2$ structures is done in the obvious way.

Proposition 8. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be finitely bounded homogeneous structures over a common signature τ, but with disjoint domains. Then their disjoint union $\bigcup_{i=1}^{k} \mathfrak{A}_i$ is a reduct of a finitely bounded homogeneous structure.*

Proof. For brevity we write $\mathfrak{A}$ for the disjoint union $\bigcup_{i=1}^{k} \mathfrak{A}_i$. Let σ be the signature τ extended by a unary symbol D_i for each $i \in [k]$. Consider the σ-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ where $D_i^{\mathfrak{A}'} = A_i$ for each $i \in [k]$.

To show that $\mathfrak{A}'$ is homogeneous, we first observe the following. If, for each $i \in [k]$, f_i is an automorphism of $\mathfrak{A}_i$, then the map $f\colon A \to A$ satisfying $f|_{A_i} \coloneqq f_i$ is an automorphism of $\mathfrak{A}'$ since it additionally preserves $D_i^{\mathfrak{A}'}$ for each $i \in [k]$. Conversely, if f is an automorphism of $\mathfrak{A}'$, then $f|_{A_i}$ is an automorphism of $\mathfrak{A}_i$ for each $i \in [k]$. Now, let $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be an isomorphism between two finite substructures of $\mathfrak{A}'$. Since f preserves $D_i^{\mathfrak{B}_1} = B_1 \cap A_i$ for each $i \in [k]$, the restrictions $f|_{B_1 \cap A_i}$ are isomorphisms, and thus extend to automorphism of $\mathfrak{A}_i$ for each $i \in [k]$ by homogeneity of the structures $\mathfrak{A}_i$. By the observation about automorphisms above, this implies that f itself extends to an automorphism of $\mathfrak{A}'$.

Next we show that $\mathfrak{A}'$ is finitely bounded. For each $i \in [k]$, let

$$\Phi(\mathfrak{A}_i) = \forall x_{i_1}, \ldots, x_{i_{n_i}}.\, \phi_i(x_{i_1}, \ldots, x_{i_{n_i}}) \quad \text{with } \phi_i \text{ quantifier-free}$$

be a universal sentence that defines $\mathrm{Age}(\mathfrak{A}_i)$. Now consider the universal sentence

$$\begin{aligned}\Phi(\mathfrak{A}') :=& \Big(\forall x. \bigwedge_{i \neq j} \neg\big(D_i(x) \wedge D_j(x)\big)\Big) \wedge \Big(\forall x. \bigvee_{i=1}^{k} D_i(x)\Big) \\ &\wedge \bigwedge_{i=1} \forall x_{i_1}, \dots, x_{i_{n_i}}. \Big(\bigwedge_{j=1}^{n_i} D_i(x_{i_j}) \implies \phi_i(x_{i_1}, \dots, x_{i_{n_i}})\Big).\end{aligned}$$

Let $\mathfrak{B}$ be a finite σ-structure that satisfies $\Phi(\mathfrak{A}')$. By the first line in $\Phi(\mathfrak{A}')$, the unary relations $D_i^{\mathfrak{B}}$ are pairwise disjoint and exhaustive. By the second line in $\Phi(\mathfrak{A}')$, the τ-reduct of the substructure of $\mathfrak{B}$ on $D_i^{\mathfrak{B}}$ is contained in $\mathrm{Age}(\mathfrak{A}_i)$ for each $i \in [k]$. Hence $\mathfrak{B}$ is a substructure of $\mathfrak{A}'$. Conversely, every finite substructure of $\mathfrak{A}'$ must satisfy $\Phi(\mathfrak{A}')$. This completes the proof as $\mathfrak{A}$ is the τ-reduct of $\mathfrak{A}'$. □

Using disjoint union to refer to several concrete domain works well if the paths employed in concrete domain constructors contain only functional roles, which is the case considered in [5], but it is not appropriate if we allow for non-functional roles in paths.

Example 1. For example, if we want to refer to time and space of an event, we can use the disjoint union of RCC8 and Allen, employing two feature names *time* and *space*. If *succ* is a functional role, then the concept description

$$\mathit{Event} \sqcap \exists \mathit{succ}.\mathit{Event} \sqcap \exists \mathit{time}, \mathit{succ}\ \mathit{time}.[\mathsf{Before}] \sqcap \exists \mathit{space}, \mathit{succ}\ \mathit{space}.[\mathsf{EC}]$$

describes an event e that has a unique successor event e' that takes place after e and in a region that is externally connected to e. However, if *succ* is not functional, then the above concept description does not express that e has a successor event e' being temporally after and spatially externally connected. Instead, there could be two successor events, one satisfying the temporal constraint and the other the spatial one.

To overcome this problem, we propose to use the so-called full product [9].[5] Let $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ be finitely many structures with disjoint relational signatures $\tau_1, \dots, \tau_k$. Furthermore, let $=_1, \dots, =_d$ be fresh binary symbols such that, for every $i \in [k]$, $=_i$ interprets as $\triangle_{A_i}$ over $\mathfrak{A}_i$. We assume in the following that the relation $=_i$ is part of the signature of $\mathfrak{A}_i$. This assumption is without loss of generality since the equality predicate is pp definable, and thus extending a homogeneous structure with an explicit relation symbol for it leaves the structure homogeneous (see the proof of Proposition 5).

The *full product* of $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, denoted by $\mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$, has as its domain the Cartesian product $A := A_1 \times \dots \times A_k$ and as its signature the union of the signatures τ_i. Recall that, for $a \in A_1 \times \dots \times A_k$ and $i \in [k]$, we denote the ith

[5] We will show below that employing the usual direct product of the structures does not work since it does not preserve homogeneity.

component of the tuple a by $a[i]$. The relations of $\mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$ are defined as follows:

$$R^{\mathfrak{A}} := \{(a_1, \ldots, a_n) \in A^n \mid (a_1[i], \ldots, a_n[i]) \in R^{\mathfrak{A}_i}\}$$

for every $i \in [k]$ and every n-ary relation $R \in \tau_i$.

The *product action* of $\mathrm{Aut}(\mathfrak{A}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{A}_k)$ on $A_1 \times \cdots \times A_k$ is given by $g(a) := (g[1](a[1]), \ldots, g[k](a[k]))$. Beside giving us access to all of the original relations through projections, full products have the following important property.

Proposition 9 ([9], Prop. 3.3.13). *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be structures with disjoint relational signatures $\tau_1, \ldots, \tau_k$ such that, for $i \in [k]$, τ_i contains the symbol $=_i$, which is defined in $\mathfrak{A}_i$ as Δ_{A_i}. Then $\mathrm{Aut}(\mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k)$ is equal to $\mathrm{Aut}(\mathfrak{A}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{A}_k)$ in its product action on $A_1 \times \cdots \times A_k$.*

We can show that the full product preserves homogeneity and finite boundedness. First, we we consider homogeneity.

Proposition 10. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be homogeneous structures with disjoint relational signatures $\tau_1, \ldots, \tau_k$ such that, for $i \in [k]$, τ_i contains the symbol $=_i$, which is defined in $\mathfrak{A}_i$ as Δ_{A_i} . Then $\mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$ is a homogeneous structure.*

Proof. For brevity we set $\mathfrak{A} := \mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$ and denote the signature of $\mathfrak{A}$ by τ. We must show that $\mathfrak{A}$ is homogeneous. Let $f \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be an isomorphism between finite substructures of $\mathfrak{A}$. For every $i \in [k]$, we define $\mathfrak{B}_{1,i}$ and $\mathfrak{B}_{2,i}$ as the substructure of $\mathfrak{A}_i$ on $\mathrm{pr}_i(B_1)$ and $\mathrm{pr}_i(B_2)$, respectively. For every $i \in [k]$ and $R \in \tau_i \cup \{=_i\}$, the relation $R^{\mathfrak{B}_1}$ is preserved by f. Consider the map $f_i \colon \mathfrak{B}_{1,i} \to \mathfrak{B}_{2,i}$ given by $f_i(b[i]) := f(b)[i]$. It is well defined, since for any $b, b' \in B_1$ with $b[i] = b'[i]$, we have $(b, b') \in =_i^{\mathfrak{B}_1}$, which implies $f(b)[i] = f(b')[i]$, because $=_i$ is preserved by f. Since f is an isomorphism, the previous argument can also be read backwards, which implies that f_i is injective. It follows directly from the definition of f_i that it is surjective, because f is surjective. Finally, f_i is an isomorphism since, for every $R \in \tau_i \cup \{=_i\}$, we have

$$\begin{aligned}
(a_1[i], \ldots, a_k[i]) \in R^{\mathfrak{B}_{1,i}} \text{ iff } & (a_1[i], \ldots, a_k[i]) \in R^{\mathfrak{A}_i} \cap \mathrm{pr}_i(B_1)^k \\
\text{iff } & (a_1, \ldots, a_k) \in R^{\mathfrak{A}} \cap B_1^k \\
\text{iff } & (a_1, \ldots, a_k) \in R^{\mathfrak{B}_1} \\
\text{iff } & (f(a_1), \ldots, f(a_k)) \in R^{\mathfrak{B}_2} \\
\text{iff } & (f(a_1), \ldots, f(a_k)) \in R^{\mathfrak{A}} \cap B_2^k \\
\text{iff } & (f(a_1)[i], \ldots, f(a_k)[i]) \in R^{\mathfrak{A}_i} \cap \mathrm{pr}_i(B_2)^k \\
\text{iff } & (f(a_1)[i], \ldots, f(a_k)[i]) \in R^{\mathfrak{B}_{2,i}} \\
\text{iff } & (f_i(a_1[i]), \ldots, f_i(a_k[i])) \in R^{\mathfrak{B}_{2,i}}.
\end{aligned}$$

Each f_i extends to an automorphism f'_i of $\mathfrak{A}_i$, because $\mathfrak{A}_i$ is homogeneous. By Proposition 9, the product action of $(f'_1, \ldots, f'_k)$ on $A_1 \times \cdots \times A_k$ specifies an automorphism f' of $\mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$. It follows from the definition of f_i that f' extends f. $\square$

Finite boundedness is also preserved under building the full product, and thus the prerequisites for Theorem 5 and Corollary 2 are preserved under building the full product.

Proposition 11. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be finitely bounded structures with disjoint relational signatures $\tau_1, \ldots, \tau_k$ such that, for $i \in [k]$, τ_i contains the symbol $=_i$, which is defined in $\mathfrak{A}_i$ as Δ_{A_i} . Then $\mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$ is finitely bounded.*

Proof. For each $i \in [k]$, let $\Phi(\mathfrak{A}_i)$ be the universal sentence that defines $\mathrm{Age}(\mathfrak{A}_i)$. Let $\Phi'(\mathfrak{A}_i)$ be the sentence obtained from $\Phi(\mathfrak{A}_i)$ by replacing each occurrence of a literal of the form $(x = y)$ in $\Phi'(\mathfrak{A}_i)$ by the literal $(x =_i y)$. Furthermore, for each symbol $R \in \tau_i$ of arity n other than $=_i$, let ψ_R be the sentence

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n. \Big(\bigwedge_{j=1}^{n} x_j =_i y_j \Big) \implies \Big(R(x_1, \ldots, x_n) \iff R(y_1, \ldots, y_n) \Big).$$

Now consider the τ-sentence

$$\begin{aligned} \Phi(\mathfrak{A}) \coloneqq \bigwedge_{i=1}^{k} \Big(& \forall x, y, z. \, (x =_i x) \wedge \Big(x =_i y \iff y =_i x \Big) \\ & \wedge \Big(x =_i y \wedge y =_i z \implies x =_i z \Big) \Big) \\ & \wedge \Big(\forall x, y. \, x = y \iff \bigwedge_{i=1}^{k} x =_i y \Big) \wedge \bigwedge_{i=1}^{k} \Phi'(\mathfrak{A}_i) \wedge \bigwedge_{R \in \tau \setminus \{=_1, \ldots, =_k\}} \psi_R. \end{aligned}$$

We claim that $\Phi(\mathfrak{A})$ defines $\mathrm{Age}(\mathfrak{A})$.

For the forward direction, let $\mathfrak{B}$ be a finite substructure of $\mathfrak{A} \coloneqq \mathfrak{A}_1 \boxtimes \cdots \boxtimes \mathfrak{A}_k$. By the definition of $\boxtimes$, the relation $=_i^{\mathfrak{A}}$ is an equivalence relation for each $i \in [k]$, because $=_i^{\mathfrak{A}_i}$ is an equivalence relation. Since $\mathfrak{B}$ is a substructure of $\mathfrak{A}$, $=_i^{\mathfrak{B}}$ is an equivalence relation for each $i \in [k]$ as well. Thus $\mathfrak{B}$ satisfies the first line in $\Phi(\mathfrak{A})$. For all $x, y \in B$ we have $x = y$ iff $x =_i^{\mathfrak{B}} y$ for each $i \in [k]$, because $=_i^{\mathfrak{B}}$ stands for the equality in the i-th coordinate. Thus $\mathfrak{B}$ satisfies the first clause on the second line in $\Phi(\mathfrak{A})$. Let $\mathfrak{B}_i$ be the substructure of $\mathfrak{A}_i$ on $\mathrm{pr}_i(B)$. As a substructure of $\mathfrak{A}_i$, $\mathfrak{B}_i$ satisfies $\Phi(\mathfrak{A}_i)$ because $\Phi(\mathfrak{A}_i)$ defines $\mathrm{Age}(\mathfrak{A}_i)$. But then $\mathfrak{B}_i$ must also satisfy $\Phi'(\mathfrak{A}_i)$ because $=_i$ interprets as the binary equality predicate in $\mathfrak{B}_i$. We claim that $\mathfrak{B}$ satisfies $\Phi'(\mathfrak{A}_i)$ for each $i \in [k]$. Let $b_1, \ldots, b_m \in B$ be any tuples to be substituted for the universally quantified variables $x_1, \ldots, x_m$ of $\Phi'(\mathfrak{A}_i)$. Let $\psi'(x_1, \ldots, x_m)$ be a formula in DNF equivalent to the quantifier-free part of $\Phi'(\mathfrak{A}_i)$. Let $\psi^\star$ be a disjunct in ψ' such that $\mathfrak{B}_i \models \psi^\star(b_1[i], \ldots, b_m[i])$. Recall that $\Phi'(\mathfrak{A}_i)$ contains no $=$-literals. Also recall that, for every n-ary symbol $R \in \tau_i$, we have $(b_{i_1}, \ldots, b_{i_n}) \in R^{\mathfrak{B}}$ iff $(b_{i_1}[i], \ldots, b_{i_n}[i]) \in R^{\mathfrak{B}_i}$ by the definition of $\boxtimes$. This means that, if $\psi^\star$ contains a positive literal of the form $R(x_{i_1}, \ldots, x_{i_n})$ for some n-ary symbol $R \in \tau_i$, then we have $\mathfrak{B}_i \models R(b_{i_1}[i], \ldots, b_{i_n}[i])$ iff $\mathfrak{B} \models R(b_{i_1}, \ldots, b_{i_n})$. Likewise we have $\mathfrak{B}_i \models \neg R(b_{i_1}[i], \ldots, b_{i_n}[i])$ iff $\mathfrak{B} \models \neg R(b_{i_1}, \ldots, b_{i_n})$. Since $\mathfrak{B} \models \psi^\star(b_1, \ldots, b_m)$ and $b_1, \ldots, b_n$ were chosen arbitrarily, we conclude that $\mathfrak{B} \models \Phi'(\mathfrak{A}_i)$.

It follows directly from the argumentation above and the fact that $=_i$ interprets as the binary equality predicate in $\mathfrak{A}_i$ that $\mathfrak{B} \models \psi_R$ for each $R \in \tau \setminus \{=_1, \ldots, =_k\}$. Hence $\mathfrak{B} \models \Phi(\mathfrak{A})$.

For the backward direction, let $\mathfrak{B}$ be a finite τ-structure that satisfies $\Phi(\mathfrak{A})$. Then $=_i^{\mathfrak{B}}$ is an equivalence relation for each $i \in [k]$. For each $i \in [k]$, consider the following τ_i-structure $\mathfrak{B}_i$. The domain of $\mathfrak{B}_i$ consists of the equivalence classes w.r.t. $=_i^{\mathfrak{B}}$. Moreover, for each n-ary symbol $R \in \tau_i$, we have $(X_1, \ldots, X_n) \in R^{\mathfrak{B}_i}$ iff $(b_1, \ldots, b_n) \in R^{\mathfrak{B}}$ for some representatives $b_i \in X_i$. The relations of $\mathfrak{B}_i$ are well-defined because $\mathfrak{B} \models \psi_R$ for each $R \in \tau \setminus \{=_1, \ldots, =_k\}$. We claim that $\mathfrak{B}_i \models \Phi'(\mathfrak{A}_i)$ for each $i \in [k]$. Recall that $\Phi'(\mathfrak{A}_i)$ contains no $=$-literals. Let $X_1, \ldots, X_m$ be any equivalence classes of elements from B w.r.t. $=_i^{\mathfrak{B}}$ to be substituted for the universally quantified variables $x_1, \ldots, x_m$ of $\Phi'(\mathfrak{A}_i)$, and $b_1, \ldots, b_m$ any representatives of these equivalence classes, respectively. Let $\psi'(x_1, \ldots, x_m)$ be a formula in DNF equivalent to the quantifier-free part of $\Phi'(\mathfrak{A}_i)$. Since $\mathfrak{B} \models \Phi'(\mathfrak{A}_i)$, we have that $\mathfrak{B} \models \psi'(b_1, \ldots, b_m)$. Let $\psi^\star$ be a disjunct in ψ' such that $\mathfrak{B} \models \psi^\star(b_1, \ldots, b_m)$.

If $\psi^\star$ contains a positive literal of the form $(x_{i_1} =_i x_{i_2})$, then we have $\mathfrak{B} \models (b_{i_1} =_i b_{i_2})$. This means that b_{i_1} and b_{i_2} are contained in the same equivalence class w.r.t. $=_i^{\mathfrak{B}}$, that is, $X_{i_1} = X_{i_2}$. We conclude that $\mathfrak{B}_i \models (X_{i_1} =_i X_{i_2})$ because the symbol $=_i$ interprets in $\mathfrak{B}_i$ as the binary equality predicate. If $\psi^\star$ contains a negative literal of the form $\neg(x_{i_1} =_i x_{i_2})$, then we have $\mathfrak{B} \models \neg(b_{i_1} =_i b_{i_2})$ which means that b_{i_1} and b_{i_2} are contained in distinct equivalence classes. Then clearly $\mathfrak{B}_i \models \neg(X_{i_1} =_i X_{i_2})$.

If $\psi^\star$ contains a positive literal of the form $R(x_{i_1}, \ldots, x_{i_n})$ for some n-ary symbol $R \in \tau_i \setminus \{=_i\}$, then we have $\mathfrak{B} \models R(b_{i_1}, \ldots, b_{i_n})$. It follows directly from the definition of $\mathfrak{B}_i$ that $\mathfrak{B}_i \models R(X_{i_1}, \ldots, X_{i_n})$. If $\psi^\star$ contains a negative literal of the form $\neg R(x_{i_1}, \ldots, x_{i_n})$ for some n-ary symbol $R \in \tau_i$, then we have $\mathfrak{B} \models \neg R(b_{i_1}, \ldots, b_{i_n})$. Suppose that $(X_{i_1}, \ldots, X_{i_n}) \in R^{\mathfrak{B}_i}$. Then $(b'_{i_1}, \ldots, b'_{i_n}) \in R^{\mathfrak{B}}$ for some representatives b'_{i_ℓ} of X_{i_ℓ}. But then $(b_{i_1}, \ldots, b_{i_n}) \in R^{\mathfrak{B}}$ because $\mathfrak{B} \models \psi_R$, a contradiction. Thus $\mathfrak{B}_i \models \neg R(X_{i_1}, \ldots, X_{i_n})$.

Since $\mathfrak{B}_i \models \psi'(X_1, \ldots, X_m)$ and $X_1, \ldots, X_m$ were chosen arbitrarily, we conclude that $\mathfrak{B}_i \models \Phi'(\mathfrak{A}_i)$. Since the symbol $=_i$ interprets in $\mathfrak{B}_i$ as the binary equality predicate, we have that $\mathfrak{B}_i \models \Phi(\mathfrak{A}_i)$. Thus $\mathfrak{B}_i \in \mathrm{Age}(\mathfrak{A}_i)$ for each $i \in [k]$. For each $i \in [k]$, let e_i be an embedding from $\mathfrak{B}_i$ into $\mathfrak{A}_i$. For each $b \in B$ and each $i \in [k]$, we denote by $[b]_{=_i^{\mathfrak{B}}}$ the equivalence class of $b \in B$ w.r.t. $=_i^{\mathfrak{B}}$. Now let $e \colon B \to A_1 \times \cdots \times A_k$ be defined by $e(b) \coloneqq \big(e_1([b]_{=_1^{\mathfrak{B}}}), \ldots, e_k([b]_{=_k^{\mathfrak{B}}})\big)$. This function is well-defined because we map from elements to their equivalence classes and not the other way around. By the first clause on the second line in $\Phi(\mathfrak{A})$, for all $x, y \in B$, we have $x = y$ iff $x =_i^{\mathfrak{B}} y$ for each $i \in [k]$. This means that e is injective. For every $i \in [k]$ and every n-ary symbol $R \in \tau_i$, we have

$$\begin{aligned}(b_1, \ldots, b_n) \in R^{\mathfrak{B}} & \text{ iff } ([b_1]_{=_i^{\mathfrak{B}}}, \ldots, [b_n]_{=_i^{\mathfrak{B}}}) \in R^{\mathfrak{B}_i} \\ & \text{ iff } \Big(e_i([b_1]_{=_i^{\mathfrak{B}}}), \ldots, e_i([b_n]_{=_i^{\mathfrak{B}}})\Big) \in R^{\mathfrak{A}_i} \\ & \text{ iff } \Big(e(b_1)[i], \ldots, e(b_n)[i]\Big) \in R^{\mathfrak{A}_i} \\ & \text{ iff } \Big(e(b_1), \ldots, e(b_n)\Big) \in R^{\mathfrak{A}}.\end{aligned}$$

Hence e is an embedding from $\mathfrak{B}$ into $\mathfrak{A}$. This completes the proof. □

Coming back to Example 1, we can use a feature *time&space* that maps into the full product of Allen and RCC8 to describe an event e that has at least one successor event that that takes place after e and in a region that is externally connected to e as follows:

$$\mathit{Event} \sqcap \exists \mathit{succ}.\mathit{Event} \sqcap \exists \mathit{time\&space}, \mathit{succ}\ \mathit{time\&space}.[\mathsf{Before}(x,y) \wedge \mathsf{EC}(x,y)].$$

The following example shows, on the one hand, that the direct product of two homogeneous structures need not be homogeneous. On the other hand, it also provides us with an example of an ω-admissible structure that is not homogeneous.

Example 2. The *random graph* is the unique countably infinite simple graph $\mathfrak{G} = (G; E^{\mathfrak{G}})$ such that $\mathrm{Age}(\mathfrak{G})$ consists of all finite simple graphs [20]. It is well-known that $\mathfrak{G}$ is homogeneous and finitely bounded. It also has the so-called *extension property*: if X and Y are disjoint finite subsets of G, then there exists a vertex $v \in G \setminus (X \cup Y)$ that has an edge in $\mathfrak{G}$ to each vertex from X and to none from Y. Consider the direct product $\mathfrak{H}$ of $\mathfrak{G}$ with itself. It is easy to see that $\mathrm{Age}(\mathfrak{G}) = \mathrm{Age}(\mathfrak{H})$. This means that $\mathrm{Age}(\mathfrak{H})$ has AP. Also, $\mathfrak{H}$ is ω-categorical because it has a *two-dimensional first-order interpretation*[6] in $\mathfrak{G}$ and first-order interpretations preserve ω-categoricity by Theorem 6.3.5 in [20]. However, $\mathfrak{H}$ does not have the extension property. To see this, let a, b, c be any distinct vertices in G, $X \coloneqq \{(a,b), (b,c)\}$, and $Y \coloneqq \{(a,c)\}$. Suppose that there exists $(u,v) \in H$ that has an edge in $\mathfrak{H}$ to each vertex from X and to none from Y. By the definition of $\mathfrak{H}$, there is an edge in $\mathfrak{G}$ from u to a and from v to c. But then, by the definition of $\mathfrak{H}$ as the direct product of $\mathfrak{G}$ with itself, there is an edge from (u,v) to (a,c), a contradiction to our previous assumption. This means that $\mathfrak{G}$ and $\mathfrak{H}$ are not isomorphic, otherwise any isomorphism would yield a witness for the extension property for $\mathfrak{H}$. We conclude that $\mathfrak{H}$ is not homogeneous since homogeneous structures are uniquely determined up to isomorphism by their age due to Theorem 4. Thus, we have shown that homogeneous structures are not closed under building direct products.

Now consider the $\{R_1, R_2, R_3, E\}$-expansion $\mathfrak{A}$ of $\mathfrak{G}$ where

- R_1 interprets as the full unary relation,

[6] See [20] for all necessary information about first-order interpretations.

– R_2 interprets as the the diagonal relation $\triangle_G$, and
– R_3 interprets as the complement relation $G^2 \setminus (\triangle_G \cup E^{\mathfrak{G}})$.

Likewise we construct the $\{R_1, R_2, R_3, E\}$-expansion $\mathfrak{B}$ from $\mathfrak{H}$. Let $\mathfrak{C}$ be a substructure of $\mathfrak{A}$ and $\mathfrak{C}'$ its $\{E\}$-reduct. Since $\mathrm{Age}(\mathfrak{G}) = \mathrm{Age}(\mathfrak{H})$, there exists an isomorphism f from $\mathfrak{C}'$ to some substructure $\mathfrak{D}'$ of $\mathfrak{H}$. Let $\mathfrak{D}$ be the substructure of $\mathfrak{B}$ on the domain D' of $\mathfrak{D}'$. We claim that f is also an isomorphism from $\mathfrak{C}$ to $\mathfrak{D}$. We have $x \in R_1^{\mathfrak{C}}$ iff $f(x) \in R_1^{\mathfrak{D}}$ because $R_1^{\mathfrak{C}} = C$, $R_1^{\mathfrak{D}} = D$, and f is bijective. For the same reason we have $(x, y) \in R_2^{\mathfrak{C}} = \triangle_C$ iff $(f(x), f(y)) \in R_2^{\mathfrak{D}} = \triangle_D$. Moreover, we have

$$\begin{aligned}(x, y) \in R_3^{\mathfrak{C}} &\text{ iff } (x, y) \notin (E^{\mathfrak{C}'} \cup \triangle_C) \\ &\text{ iff } (f(x), f(y)) \notin (E^{\mathfrak{D}'} \cup \triangle_D) \\ &\text{ iff } (f(x), f(y)) \in R_3^{\mathfrak{D}}.\end{aligned}$$

We conclude that $\mathrm{Age}(\mathfrak{A}) \subseteq \mathrm{Age}(\mathfrak{B})$. Using an analogous argument, we can show $\mathrm{Age}(\mathfrak{A}) \supseteq \mathrm{Age}(\mathfrak{B})$, and thus $\mathrm{Age}(\mathfrak{A}) = \mathrm{Age}(\mathfrak{B})$. Since every homomorphism from a finite structure has a finite range, $\mathrm{Age}(\mathfrak{A}) = \mathrm{Age}(\mathfrak{B})$ implies $\mathrm{CSP}(\mathfrak{A}) = \mathrm{CSP}(\mathfrak{B})$. Since the relations $R_1^{\mathfrak{A}}$, $R_2^{\mathfrak{A}}$ and $R_3^{\mathfrak{A}}$ are first-order definable in $\mathfrak{G}$, we have that:

– $\mathfrak{A}$ is homogeneous since $\mathfrak{G}$ is homogeneous and first-order definable relations are preserved by automorphisms, thus $\mathrm{Age}(\mathfrak{A})$ has AP by Theorem 4,
– $\mathfrak{A}$ is finitely bounded by Proposition 5, and thus $\mathrm{CSP}(\mathfrak{A})$ is in NP by Proposition 4.

By definition, $\mathfrak{B}$ is JDJEPD. Since $\mathrm{Age}(\mathfrak{B})$ has AP, we have that $\mathfrak{B}$ is a patchwork by Proposition 3. By Corollary 1, $\mathfrak{B}$ has homomorphism compactness since it is ω-categorical. This is the case since expansions by first-order definable relations do not change the automorphism group. We conclude that $\mathfrak{B}$ is a structure that is ω-admissible but not homogeneous.

5.6 Henson digraphs.

A directed graph is a *tournament* if every two distinct vertices in it are connected by exactly one directed edge. In [18], Henson proved that there are uncountably many homogeneous directed graphs by showing that, for any set $\mathcal{N}$ of finite tournaments (plus the loop and the 2-cycle) such that no member of $\mathcal{N}$ is embeddable into any other member of $\mathcal{N}$, $\mathrm{Forb}_e(\mathcal{N})$ is an amalgamation class whose Fraïssé limit is a homogeneous directed graph. Furthermore, the Fraïssé limits for two distinct sets of such tournaments are distinct as well. In the literature, such directed graphs are often called *Henson digraphs* [29]. If $\mathfrak{G}$ is a Henson digraph, then $\mathrm{Age}(\mathfrak{G}) = \mathrm{CSP}(\mathfrak{G})$.[7] Clearly, only countably many Henson digraphs can have a decidable CSP. Beside the finitely bounded ones (see Proposition 4), there is an interesting

[7] One direction is obvious, the other holds because homomorphisms between directed graphs cannot contract any edges.

example constructed using the infinite set of non-isomorphic tournaments from Henson's original proof of uncountability. Consider the tournaments $\mathfrak{T}_1, \mathfrak{T}_2, \dots$ with domains $[2], [3], \dots$ such that the edge relation of $\mathfrak{T}_n$ consists of the edges (i, j) for every $j = i + 1$ with $0 \leq i \leq n$, $(0, n + 1)$, and (j, i) for every $j > i + 1$ with $(i, j) \neq (0, n + 1)$. It was shown in [11] that the CSP of the Henson digraph corresponding to $\mathcal{N} \coloneqq \{\mathfrak{T}_1, \mathfrak{T}_2, \dots\}$ is coNP-complete. This digraph is homogeneous and ω-categorical by Theorem 4, and its CSP is decidable. Thus, it satisfies the requirements of Corollary 3. However, it is clearly not finitely bounded. This example demonstrates that Corollary 3 indeed covers a larger class of structures than Corollary 2.

6 Conclusion

We have shown that ω-admissibility, which was introduced in the DL community to obtain decidable extensions of DLs by concrete domains, is closely related to well-known notions from model theory. This has allowed us to find sufficient conditions for ω-admissibility of a concrete domain, and thus conditions under which reasoning in DLs with concrete domains is decidable also in the presence of TBoxes. Given the fact that a large number of homogeneous structures are known from the literature [29] and that homogeneous and finitely bounded structures play an important rôle in the CSP community, we believe that these condition will turn out to be very useful for locating new ω-admissible concrete domains.

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A A proof of Theorem 1

Theorem 1. *Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations for some $d \geq 2$. Then concept satisfiability in $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ w.r.t. TBoxes is decidable.*

Proof. Using Lemma 4, we can reduce satisfiability of $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ concepts w.r.t. general TBoxes in PTIME to satisfiability of $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ concepts in NNF w.r.t. TBoxes in NNF. The rules (1), (2) and (3) are standard, and the rules (4) and (5) can be performed in constant time as there are only finitely many at most d-ary formulas in $\vee^+$. Then the claim follows from Lemma 5, Lemma 6 and Lemma 7. $\square$

A.1 A negation normal form

Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. Recall the definition of $\vee^+$. For every $k \leq d$ and k-ary formula ϕ from $\vee^+$, we denote by $\phi^\neg$ the unique k-ary formula from $\vee^+$ that defines $D^k \setminus \phi^{\mathfrak{D}}$ in $\mathfrak{D}$. Note that the binary equality predicate $=$ is also a valid $\vee^+$ formula, because $\mathfrak{D}$ is JD.

An $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ concept C_0 is in *negation normal form* (NNF) if negation occurs only in front of concept names. A *path p of length m* is a tuple $(\ell_1, \ldots, \ell_m)$ where $\ell_i \in \mathsf{N_R}$ for each $i < m$ and $\ell_m \in \mathsf{N_F}$. For a path $p = (\ell_1, \ldots, \ell_m)$, we set $\mathrm{def}(p) \coloneqq \exists \ell_1. \cdots \exists \ell_{m-1}. \exists \ell_m, \ell_m. [=]$ and $\mathrm{ud}(p) \coloneqq \forall \ell_1. \cdots \forall \ell_{m-1}. \forall \ell_m, \ell_m. [=^\neg]$. An $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ TBox $\mathcal{T}$ is in NNF if it is a singleton of the form $\{\top \sqsubseteq C_{\mathcal{T}}\}$ such that $C_{\mathcal{T}}$ is in NNF. We use $\mathrm{Sub}(C_0)$ to denote the set of subconcepts of a concept C_0 and, for a TBox $\mathcal{T}$ in NNF, we set $\mathrm{Sub}(\mathcal{T}) \coloneqq \mathrm{Sub}(C_{\mathcal{T}})$. We additionally define $\mathrm{Sub}(C_0, \mathcal{T}) \coloneqq \mathrm{Sub}(C_0) \cup \mathrm{Sub}(\mathcal{T})$. The following lemma is a straightforward consequence of the definitions.

Lemma 4. *Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. For every $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ TBox $\mathcal{T}$ there exists an equivalent one in NNF obtainable by an exhaustive application of the following rules.*

1. $\{C_1 \sqsubseteq D_1, \ldots, C_k \sqsubseteq D_k\} \rightsquigarrow \{\top \sqsubseteq (\neg C_1 \sqcup D_1) \sqcap \cdots \sqcap (\neg C_k \sqcup D_k)\}$.
2. $\neg\neg C \rightsquigarrow C$, $\neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D$, $\neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D$.
3. $\neg \exists r. C \rightsquigarrow \forall r. \neg C$, $\neg \forall r. C \rightsquigarrow \exists r. \neg C$.
4. $\neg(\forall p_1, \ldots, p_k. [\phi]) \rightsquigarrow \exists p_1, \ldots, p_k. [\phi^\neg]$ *if* $\phi^{\mathfrak{D}} \neq D^k$ *and* $\bot$ *else.*
5. $\neg(\exists p_1, \ldots, p_k. [\phi]) \rightsquigarrow \forall p_1, \ldots, p_k. [\phi^\neg]$ *if* $\phi^{\mathfrak{D}} \neq D^k$ *and* $\bigsqcup_{i=1}^k \mathrm{ud}(p_i)$ *else.*

A.2 Notation

Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. Fix an $\mathcal{ALC}^d_{\vee\!+}(\mathfrak{D})$ concept C_0 and a TBox $\mathcal{T}$.

Definition 3 (Completion system). *Let O_{a} and O_{c} be disjoint countably infinite sets of* abstract nodes *and* concrete nodes. *A* completion tree *for $(C_0, \mathcal{T})$ is a pair $(\mathfrak{T}, \mathcal{L})$ where $\mathfrak{T} = (T; E)$ is a finite directed tree $(T; E)$ with $T \subseteq O_{\mathsf{a}} \cup O_{\mathsf{c}}$, $E \subseteq O_{\mathsf{a}} \times (O_{\mathsf{a}} \cup O_{\mathsf{c}})$, and $\mathcal{L} \colon (T \cap O_{\mathsf{a}}) \cup E \to \mathrm{Pow}(\mathrm{Sub}(C_0, \mathcal{T})) \cup \mathsf{N_R} \cup \mathsf{N_F}$ is a mapping such that*

$$\mathcal{L}(T \cap O_{\mathsf{a}}) \subseteq \mathrm{Pow}(\mathrm{Sub}(C_0, \mathcal{T})), \ \mathcal{L}(E \cap (O_{\mathsf{a}} \times O_{\mathsf{a}})) \subseteq \mathsf{N_R}, \ \text{and } \mathcal{L}(E \cap (O_{\mathsf{a}} \times O_{\mathsf{c}})) \subseteq \mathsf{N_F}.$$

We say that $w \in T$ is an ℓ-successor *of $v \in T$ if $\mathcal{L}(v,w) = \ell$. This notion extends to paths in an obvious way. A* completion system *for $(C_0, \mathcal{T})$ is a triple $(\mathfrak{T}, \mathfrak{N}, \mathcal{L})$, where $(\mathfrak{T}, \mathcal{L})$ is a completion tree for $(C_0, \mathcal{T})$ and $\mathfrak{N}$ is a τ-structure with domain $N = T \cap O_{\mathsf{c}}$. We call $\mathfrak{N}$ the* constraint system *of $(\mathfrak{T}, \mathfrak{N}, \mathcal{L})$. We say that $\mathcal{S}$ contains a* clash *if there is $a \in T \cap O_{\mathsf{a}}$ and $A \in \mathsf{N_C}$ such that $\{A, \neg A\} \subseteq \mathcal{L}(a)$, or if $\mathfrak{N} \not\to \mathfrak{D}$. We call $\mathcal{S}$* complete *if no tableau rule from Algorithm 2 is applicable. A* completion *of $\mathfrak{N}$ is any JEPD τ-structure $\mathfrak{N}^{cpl}$ over N with $\mathfrak{N} \to \mathfrak{N}^{cpl}$ and $\mathfrak{N}^{cpl} \to \mathfrak{D}$.*

Definition 4 ($\oplus$ Operation). *Let $\mathcal{S} := (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ be a completion system for $(C_0, \mathcal{T})$. For $a \in T \cap O_{\mathsf{a}}$, and either $b \in O_{\mathsf{a}} \setminus T$ and $\ell \in \mathsf{N_R}$, or $b \in O_{\mathsf{c}} \setminus T$ and $\ell \in \mathsf{N_F}$, we denote by $\mathcal{S} \oplus \ell(a,b)$ the completion system $(\mathfrak{T}', \mathfrak{N}', \mathcal{L}')$ obtained from $\mathcal{S}$ as follows. If $\ell \in \mathsf{N_R} \setminus \mathsf{N_{fR}}$, or if $\ell \in \mathsf{N_{fR}} \cup \mathsf{N_F}$ and a has no ℓ-successor, then $T' := T \cup \{b\}$, $E' := E \cup \{(a,b)\}$, and $\mathcal{L}'$ extends $\mathcal{L}$ by $\mathcal{L}'(a,b) := \ell$ plus $\mathcal{L}'(b) := \emptyset$ if $b \in O_{\mathsf{a}}$. If $\ell \in \mathsf{N_{fR}} \cup \mathsf{N_F}$ and a has an ℓ-successor c, then we simply rename c with b. Let $p = (\ell_1, \ldots, \ell_m)$ be a path. With $\mathcal{S} \oplus p(a,b)$ we denote the completion system*

$$\Big(\Big(\cdots\Big(\mathcal{S} \oplus \ell_1(a,b_1)\Big) \oplus \cdots\Big) \oplus \ell_{m-1}(b_{m-2}, b_{m-1})\Big) \oplus \ell_m(b_{m-1}, b),$$

where $b_1, \ldots, b_{m-1}, b \in O_{\mathsf{a}} \cup O_{\mathsf{c}}$ are all fresh w.r.t. $\mathfrak{T}$ and pairwise distinct.

Definition 5 (Blocking). *Let $\mathcal{S} := (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ be a completion system for $(C_0, \mathcal{T})$. We say that $a \in T$ is an* ancestor *of $b \in T$ if b is reachable from a in $(T; E)$. For $a \in T \cap O_{\mathsf{a}}$, we set*

$$\mathrm{Feat}(a) := \{\ell \in \mathsf{N_F} \mid a \textit{ has an } \ell\textit{-successor}\}$$

and define $\mathfrak{N}(a)$ as the substructure of $\mathfrak{N}$ on the set of those $x \in T \cap O_{\mathsf{c}}$ which are an ℓ-successor of a for some $\ell \in \mathsf{N_F}$. We say that a is potentially blocked *by b in $\mathcal{S}$ if b is an ancestor of a, $\mathcal{L}(a) \subseteq \mathcal{L}(b)$, and $\mathrm{Feat}(a) = \mathrm{Feat}(b)$. We say that a is* blocked *by b in $\mathcal{S}$ if a is potentially blocked by b, $\mathfrak{N}(a)$ and $\mathfrak{N}(b)$ are JEPD, and $\mathfrak{N}(a) \cong \mathfrak{N}(b)$. Finally, a is* blocked *in $\mathcal{S}$ if it or one of its ancestors is blocked.*

A.3 A tableau algorithm

Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. For a pair $(C_0, \mathcal{T})$ consisting of an $\mathcal{ALC}^d(\mathfrak{D})$ concept and an $\mathcal{ALC}^d(\mathfrak{D})$ TBox, we denote by $\mathsf{N_C}(C_0, \mathcal{T})$, $\mathsf{N_R}(C_0, \mathcal{T})$ resp. $\mathsf{N_F}(C_0, \mathcal{T})$ the sets of concepts, roles resp. features which appear in C_0 or $C_{\mathcal{T}}$. An *initial completion system* for $(C_0, \mathcal{T})$ is a completion system $\mathcal{S} = (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ where, for some $a_0 \in O_{\mathsf{a}}$, $\mathfrak{T} = (\{a_0\}, \emptyset)$, $\mathfrak{N}$ is the τ-structure over $\{a_0\}$ with empty relations, and $\mathcal{L}$ sends a_0 to $\{C_0\}$.

Note that the $\mathfrak{N}$-rule from Algorithm 2 is always applied with the highest precedence. As usual, all tableau rules in Algorithm 2 can be applied to an arbitrary element of a completion tree.

Algorithm 1: `satisfiable`($\mathcal{S}$).

Input: A completion system $\mathcal{S} = (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ for $(\mathcal{T}, C_0)$.
Output: true or false

if $\mathcal{S}$ *contains a clash* **then**
  return false;
else if *the procedure* $\mathfrak{N}$`-rule` *from Algorithm 2 is applicable to* $\mathcal{S}$ **then**
  return `satisfiable(`$\mathfrak{N}$`-rule(`$\mathcal{S}$`));`
else if *any other procedure* `rule` *from Algorithm 2 is applicable to* $\mathcal{S}$ **then**
  return `satisfiable(rule(`$\mathcal{S}$`));`
else
  return true;

Lemma 5 (Termination). *Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. Furthermore, let C_0 resp. $\mathcal{T}$ be an $\mathcal{ALC}^d_{\vee+}(\mathfrak{D})$ concept resp. TBox in NNF. Then each run of Algorithm 1 starting with an initial completion system for $(C_0, \mathcal{T})$ terminates.*

Proof. Let $\mathcal{S}_0, \mathcal{S}_1, \ldots$ be the sequence of completion systems generated during the run of Algorithm 1, where, for every $i \geq 0$, $\mathcal{S}_i = (\mathfrak{T}_i, \mathfrak{N}_i, \mathcal{L}_i)$. We set $n := |C_0| + |\mathcal{T}|$. Clearly $|\mathrm{Sub}(C_0, \mathcal{T})| \leq n$. We show that for every $i \geq 0$,

a. the out degree of $\mathfrak{T}_i$ is bounded by n, and
b. the depth of $\mathfrak{T}_i$ is bounded by $\ell := 2^{2n} \cdot |\tau|^{d \cdot n^d} + 2$.

We start with (a). Note that nodes from $T_i \cap O_{\mathsf{c}}$ have no successors and successors of each $a \in T_i \cap O_{\mathsf{a}}$ are created solely by application of $\exists$-rule or $\exists_c$-rule. The $\exists$-rule creates at most one successor $b \in T_i \cap O_{\mathsf{a}}$ and the $\exists_c$-rule generates at most d successors $b_1, \ldots, b_d \in T_i \cap O_{\mathsf{a}}$ of a for every $\exists r.C \in \mathrm{Sub}(C_0, \mathcal{T})$ and $\exists p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathrm{Sub}(C_0, \mathcal{T})$. Furthermore, the $\exists_c$-rule generates at most one successor for every feature from $\mathsf{N_F}(C_0, \mathcal{T})$. Hence, the number of concrete successors is bounded by n.

For (b), suppose that there is an $i \geq 0$ such that the depth of $\mathfrak{T}_i$ exceeds ℓ. Without loss of generality, , we may assume that i is the smallest index with this property. Then $\mathcal{S}_i$ has been obtained from $\mathcal{S}_{i-1}$ by applying the $\exists$-rule or the $\exists_c$-rule to a node on level ℓ, or the $\exists_c$-rule to a node on level $\ell - 1$. Since the $\mathfrak{N}_{i-1}$-rule is applied with highest precedence, it is not applicable to $\mathcal{S}_{i-1}$. Hence, for every $a, b \in T_{i-1} \cap O_{\mathsf{a}}$ such that b is potentially blocked by a, $\mathfrak{N}_{i-1}(a)$ and $\mathfrak{N}_{i-1}(b)$ are both JEPD. For each pair $a, b \in T_{i-1} \cap O_{\mathsf{a}}$, we write $a \sim b$ if and only if $\mathcal{L}_{i-1}(a) = \mathcal{L}_{i-1}(b)$, $\mathrm{Feat}(a) = \mathrm{Feat}(b)$, and $\mathfrak{N}_{i-1}(a) \cong \mathfrak{N}_{i-1}(b)$. Obviously, $\sim$ is an equivalence relation on $T_{i-1} \cap O_{\mathsf{a}}$. Note that if a is an ancestor of b and $a \sim b$, then b is blocked by a in $\mathcal{S}_{i-1}$. Let $T_i \cap O_{\mathsf{a}}/\!\sim$ be the set of all equivalence classes w.r.t. $\sim$ and $\varepsilon := |\mathsf{N_F}(C_0, \mathcal{T})|$. We have $\mathcal{L}(a) \subset \mathrm{Sub}(C_0, \mathcal{T})$, $\mathfrak{N}_{i-1}(a)$ is JEPD, and

Algorithm 2: The tableau rules.

Input: A completion system $\mathcal{S} = (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ for $(\mathcal{T}, C_0)$.

Output: A completion system for $(\mathcal{T}, C_0)$.

$\mathfrak{N}$-`rule`$(\mathcal{S})$
- **if** *1. a is potentially blocked by some $b \in T \cap O_{\mathsf{a}}$ or vice versa*
  2. $\mathfrak{N}(a)$ is not JEPD **then**
  - guess a completion $\mathfrak{N}^{cpl}(a)$ of $\mathfrak{N}(a)$ and extend the relations of $\mathfrak{N}$ by the new tuples from $\mathfrak{N}^{cpl}(a)$;

$\mathcal{T}$-`rule`$(\mathcal{S})$
- **if** $C_{\mathcal{T}} \notin \mathcal{L}(a)$ **then**
  - $\mathcal{L}(a) \longleftarrow \mathcal{L}(a) \cup \{C_{\mathcal{T}}\}$;

$\sqcap$-`rule`$(\mathcal{S})$
- **if** *1. $C_1 \sqcap C_2 \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3. $\{C_1, C_2\} \nsubseteq \mathcal{L}(a)$ **then**
  - $\mathcal{L}(a) \longleftarrow \mathcal{L}(a) \cup \{C_1, C_2\}$;

$\sqcup$-`rule`$(\mathcal{S})$
- **if** *1. $C_1 \sqcup C_2 \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3.$\{C_1, C_2\} \cap \mathcal{L}(a) = \emptyset$ **then**
  - $\mathcal{L}(a) \longleftarrow \mathcal{L}(a) \cup \{C\}$ for some $C \in \{C_1, C_2\}$;

$\exists$-`rule`$(\mathcal{S})$
- **if** *1. $\exists r.C \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3. there is no r-successor b of a such that $C \in \mathcal{L}(b)$ **then**
  - $\mathcal{L} \longleftarrow \mathcal{L} \oplus r(a, b)$ for some fresh $b \in O_{\mathsf{a}}$;
  - $\mathcal{L}(b) \longleftarrow \mathcal{L}(b) \cup \{C\}$;

$\forall$-`rule`$(\mathcal{S})$
- **if** *1. $\forall r.C \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3. b is an r-successor of a such that $C \notin \mathcal{L}(b)$ **then**
  - $\mathcal{L}(b) \longleftarrow \mathcal{L}(b) \cup \{C\}$;

$\exists_c$-`rule`$(\mathcal{S})$
- **if** *1. $\exists p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3. there are no $x_1, \ldots, x_k \in O_{\mathsf{c}}$ such that $(x_1, \ldots, x_k) \in R_j$ for some $j \in [m]$ and x_i is a p_i-successor of a for every $i \in [k]$ **then**
  - $\mathcal{S} \longleftarrow \mathcal{S} \oplus p_1(a, x_1) \oplus \cdots \oplus p_m(a, x_m)$ for some fresh $x_1, \ldots, x_k \in O_{\mathsf{c}}$;
  - $R_j^{\mathfrak{N}} \longleftarrow R_j^{\mathfrak{N}} \cup \{(x_1, \ldots, x_k)\}$ for some $j \in [m]$;

$\forall_c$-`rule`$(\mathcal{S})$
- **if** *1. $\forall p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathcal{L}(a)$,*
  2. a is not blocked, and
  3. there are $x_1, \ldots, x_k \in O_{\mathsf{c}}$ such that x_i is a p_i-successor of a for every $i \in [k]$ and $(x_1, \ldots, x_k) \notin R_j$ for all $j \in [m]$ **then**
  - $R_j^{\mathfrak{N}} \longleftarrow R_j^{\mathfrak{N}} \cup \{(x_1, \ldots, x_k)\}$ for some $j \in [m]$;

$|N_{i-1}(a)| \leq \varepsilon$. Then

$$|T_{i-1} \cap O_{\mathsf{a}}/{\sim}| \leq 2^{|\mathrm{Sub}(C_0,\mathcal{T})|} \cdot \sum_{i=0}^{m} \binom{m}{i} |\tau|^{\sum_{j=0}^{d} i^j}.$$

Since $m \leq n$, we obtain $|T_{i-1} \cap O_{\mathsf{a}}/{\sim}| \leq 2^n \cdot 2^n \cdot |\tau^{\leqslant d}|^{d \cdot n^d}$. Let $a \in T_{i-1} \cap O_{\mathsf{a}}$ be the node to which a rule is applied in $\mathcal{S}_{i-1}$ in order to obtain $\mathcal{S}_i$. As noted before, the level k of a in $\mathfrak{T}_{i-1}$ is at least $\ell - 1 \geq |T_{i-1} \cap O_{\mathsf{a}}/{\sim}| + 1$. Let $a_0, \ldots, a_k$ be the path in $\mathfrak{T}_{i-1}$ leading from the root to a. Since $k > |T_{i-1} \cap O_{\mathsf{a}}/{\sim}|$, we have $a_j \sim a_i$ for some $0 \leq i < j \leq k$. But then a is blocked, which contradicts that a rule was applied to a.

Thus the statement of Lemma 5 holds due to to the following reasons:

- Algorithm 1 constructs a completion system $\mathcal{S} = (\mathfrak{T}, \mathfrak{N}, \mathcal{L})$ whose underlying completion tree $\mathfrak{T}$ has bounded out-degree and depth. No nodes, concepts or tuples of relations are removed from $\mathcal{S}$ in the process.
- Every tableau rule from Algorithm 2 adds a new node or a concept to $\mathcal{S}$ or a new tuple to a relation of $\mathfrak{N}$.
- The cardinality of node labels is bounded by $|\mathrm{Sub}(C_0, \mathcal{T})|$ and the number of relations in the constraint system of $\mathcal{S}$ is bounded by $|\tau| \cdot \sum_{j=0}^{d} |T \cap O_{\mathsf{c}}|^j$. $\square$

Lemma 6 (Soundness). *Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. Furthermore, let C_0 resp. $\mathcal{T}$ be an $\mathcal{ALC}^d_{\vee^+}(\mathfrak{D})$ concept resp. TBox in NNF. If there is a run of Algorithm 1 starting with an initial completion system for $(C_0, \mathcal{T})$ that returns* `true`*, then C_0 is satisfiable w.r.t. $\mathcal{T}$.*

Proof. By the assumption, there exists a complete and clash-free completion system $\mathcal{S}$ for $(C_0, \mathcal{T})$. We use it to construct a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $(C_0, \mathcal{T})$. Let *root* be the root of $\mathfrak{T}$, and Blocks: $T \cap O_{\mathsf{a}} \to T \cap O_{\mathsf{a}}$ a function that, for every blocked b, returns an unblocked a that blocks b in $\mathcal{S}$.

A *path* in $\mathcal{S}$ is a (possibly empty) sequence $((a_i, b_i))_{i=1}^{n}$ of pairs of nodes from $T \cap O_{\mathsf{a}}$ such that for each $1 \leq i < n$, one of the following holds:

- $a_{i+1} = b_{i+1}$ is an unblocked successor of a_i in $\mathfrak{T}$, or
- $a_{i+1} = \mathrm{Blocks}(b_{i+1})$ and b_{i+1} is a blocked successor of a_i in $\mathfrak{T}$.

Intuitively, the first components in a path represent a sequence of individuals and the second components provide justification for the existence of this sequence in case of blocking situations. Observe that b_{i+1} is always a successor of a_{i+1}. We use Paths to denote the set of all paths in S. For $p, q, q' \in \mathrm{Paths}$ we say that p *extends* q by q' and write $p = q \cdot q'$ (or $p = qq'$ if clear from context), if

$$p = ((a_i, b_i))_{i \in [n]}, q = ((a_i, b_i))_{i \in [m]} \text{ and } q' = ((a_i, b_i))_{i \in [n] \setminus [m]},$$

such that $n > m$. For $p \in \mathrm{Paths}$, the *tail* of p, denoted by $\mathrm{Tail}(p)$, is the last pair of p. We first define the abstract part of $\mathcal{I}$. The domain of $\mathcal{I}$ is

$$\Delta^{\mathcal{I}} := \{p \in \mathrm{Paths} \mid p \text{ is non-empty with first pair being } (root, root)\}.$$

For every $C \in \mathsf{N_C}(C_0, \mathcal{T})$, we set

$$C^{\mathcal{I}} := \{p \in \Delta^{\mathcal{I}} \mid \mathrm{Tail}(p) = (a, b) \text{ with } C \in \mathcal{L}(a)\},$$

and for every $r \in \mathsf{N_R}(C_0, \mathcal{T})$, we set

$$r^{\mathcal{I}} := \{(p, p') \in (\Delta^{\mathcal{I}})^2 \mid p' \text{ extends } p \text{ with } \mathrm{Tail}(p) = (a, b) \text{ by one pair } \mathrm{Tail}(p') = (a', b') \text{ such that } b' \text{ is an } r\text{-successor of } a \text{ in } \mathfrak{T}\}.$$

Note that $\Delta^{\mathcal{I}}$ is non-empty, since $(root, root) \in \Delta^{\mathcal{I}}$. Also note that $f^{\mathcal{I}}$ is a function for every $f \in \mathsf{N_{fR}}(C_0, \mathcal{T})$, which is ensured by $\oplus$ generating at most one f-successor per abstract node, and by the definition of Paths in which we choose only a single blocking node to be put into a path.

A path p is called a *hook* if $p = (root, root)$, or $\mathrm{Tail}(p) = (a, b)$ with $a \neq b$, that is, b is blocked by a. We denote the set of all hooks by Hooks. For every $q \in \mathrm{Hooks}$ we define

$$\mathrm{Patched}(q) := \{p \in \Delta^{\mathcal{I}} \mid \exists q' \in \mathrm{Paths} \text{ such that } p = qq' \text{ and all pairs } (a', b') \text{ in } q' \text{ satisfy } a' = b' \text{ with the possible exception of } \mathrm{Tail}(q')\}$$

For a pair $p, q \in \mathrm{Hooks}$, we call p a *successor* of q and write $p \succ q$, if $p \in \mathrm{Patched}(q)$ and $p \neq q$.

Since $\mathcal{S}$ is clash-free, we have $\mathfrak{N} \to \mathfrak{D}$. Let $\mathfrak{N}^{cpl}$ be any completion of $\mathfrak{N}$. Now we rename every concrete node x that is an g-successor of an abstract node a to the pair (a, g). This renaming procedure is well-defined since, by the definition of $\oplus$, every abstract node has at most one g-successor. We now define the τ-structure $\tilde{\mathfrak{N}}$ that represents the τ-constraints put on the concrete part of $\mathcal{I}$. For every $q \in \mathrm{Hooks}$ with $p \in \mathrm{Patched}(q)$ and $\mathrm{Tail}(p) = (a, b)$, we set

$$\mathrm{Rep}_q(p) := \begin{cases} b \text{ if } p \neq q \text{ and } a \neq b, \\ a \text{ otherwise.} \end{cases}$$

Let $\mathrm{Unf}(\mathfrak{N})$ be the structure over $\Delta^{\mathcal{I}} \times \mathsf{N_F}(C_0, \mathcal{T})$ such that, for every k-ary $R \in \tau$,

$$R^{\mathrm{Unf}(\mathfrak{N})} = \{((p_1, g_1), \ldots, (p_k, g_k)) \in (\Delta^{\mathcal{I}} \times \mathsf{N_F}(C_0, \mathcal{T}))^k \mid \exists q \in \mathrm{Hooks} \text{ such that } p_1, \ldots, p_k \in \mathrm{Patched}(q) \text{ and } ((\mathrm{Rep}_q(p_1), g_1), \ldots, (\mathrm{Rep}_q(p_k), g_k)) \in R^{\mathfrak{N}^{cpl}}\}.$$

We define $\tilde{\mathfrak{N}}$ as the substructure of $\mathrm{Unf}(\mathfrak{N})$ on the set of all elements from N which are contained in a tuple from a relation of $\mathrm{Unf}(\mathfrak{N})$. Our goal now is to show $\tilde{\mathfrak{N}} \to \mathfrak{D}$. For every $p \in \mathrm{Hooks}$, we set $\tilde{\mathfrak{N}}(p)$ to be the substructure of $\tilde{\mathfrak{N}}$ on $\tilde{N}(p) := \{(q, g) \in \tilde{N} \mid q \in \mathrm{Patched}(p)\}$.

Claim 1

a. $\tilde{\mathfrak{N}} = \bigcup_{p \in \mathrm{Hooks}} \tilde{\mathfrak{N}}(p)$.
b. *If* $p, q \in \mathrm{Hooks}$, $p \neq q$, *and neither* $q \succ p$, *nor* $p \succ q$, *then* $\tilde{N}(p) \cap \tilde{N}(q) = \emptyset$.

Proof. For (a), let $((p_1,g_1),\dots,(p_k,g_k))\in R^{\tilde{\mathfrak{N}}}$ for some k-ary $R\in\tau$. Then there is $q\in\mathrm{Hooks}$ such that $p_1,\dots,p_k\in\mathrm{Patched}(q)$ and the statement follows directly from the definition of $\tilde{\mathfrak{N}}(p)$.

We prove (b) by contradiction. Let $(q''',g)\in\tilde{N}(p)\cap\tilde{N}(q)$. Then $q'''\in\mathrm{Patched}(p)\cap\mathrm{Patched}(q)$, that is, there are $q',q''\in\mathrm{Paths}$ such that

i. $q'''=pq'$ and $q'''=qq''$, and
ii. every pair (a,b) in q' or q'' satisfies $a=b$ with the possible exception of $\mathrm{Tail}(q')$ and $\mathrm{Tail}(q'')$.

Due to (i), either $p=q$, or q properly extends p, or vice versa. In the first case, we are done. In the second case, since $q\in\mathrm{Hooks}$, we have $\mathrm{Tail}(q)=(a,b)$ for some $a\neq b$. Because (i) and (ii) hold and q properly extends p, we have $q=q'''$. Thus $q=pq'$. A subsequent application of (ii) shows that q is a successor of p. The third case is analogous to the second. $\square$

The properties P1 and P2 presented in the following claim are essential for the rest of the proof.

Claim 2

(P1): *If* $q,q'\in\mathrm{Hooks}$ *are such that* $q'\succ q$*, then* $\mathrm{Patched}(q)\cap\mathrm{Patched}(q')=\{q'\}$.
(P2): *If* $((q_1,g_1),\dots,(q_k,g_k))\in R^{\tilde{\mathfrak{N}}(p)}$*, then* $(\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k))\in R^{\mathfrak{N}^{cpl}}$.

Proof. The property P1 follows directly from the definition of hooks and successors. For P2, let $((q_1,g_1),\dots,(q_k,g_k))\in R^{\tilde{\mathfrak{N}}(p)}$. Then $q_1,\dots,q_k\in\mathrm{Patched}(p)$ and there exists $p'\in\mathrm{Hooks}$ such that $(\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k))\in R^{\mathfrak{N}^{cpl}}$ and $q_1,\dots,q_k\in\mathrm{Patched}(p')$. If $p=p'$, then we are done. Suppose that $p\neq p'$. By (b) of Claim 1 and P1, $q_1,\dots,q_k\in\mathrm{Patched}(p)\cap\mathrm{Patched}(p')$ implies that either $q_1=\dots=q_k=p$ and $p\succ p'$, or $q_1=\dots=q_k=p'$ and $p'\succ p$. Without loss of generality, we assume that the former is the case. Let $\mathrm{Tail}(p)=(a,b)$. Since $q_1=\dots=q_k=p$ and p is a hook, we have $a\neq b$ and that b is blocked by a in $\mathcal{S}$. Thus, for every $i\in[k]$, $\mathrm{Rep}_{p'}(q_i)=b$ and $\mathrm{Rep}_p(q_i)=a$. Then $(\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k))\in R^{\mathfrak{N}^{cpl}}$ yields $(b,g_1),\dots,(b,g_k))\in R^{\mathfrak{N}^{cpl}}$. Since b is blocked by a, this means that $((a,g_1),\dots,(a,g_k))\in R^{\mathfrak{N}^{cpl}}$. $\square$

Claim 3 *For every* $p\in\mathrm{Hooks}$*,* $\tilde{\mathfrak{N}}(p)$ *is finite, JEPD, and* $\tilde{\mathfrak{N}}(p)\to\mathfrak{D}$.

Proof. Let $p\in\mathrm{Hooks}$. Since $\mathfrak{T}$ is finite and acyclic, it follows from the definition of blocking that $\mathrm{Patched}(p)$ is finite, which implies the finiteness of $\tilde{N}(p)$. We show that $\tilde{\mathfrak{N}}$ is JEPD. Let $(q_1,g_1),\dots,(q_k,g_k)\in\tilde{N}(p)$ for some $k\leq d$. From the definitions of $\mathrm{Unf}(\mathfrak{N})$, $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{N}}(p)$, it follows directly that $(\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k)\in N^{cpl}$. We have $((\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k))\in R^{\mathfrak{N}^{cpl}}$ for some $R\in\tau$ because $\mathfrak{N}^{cpl}$ is JE, which implies $((q_1,g_1),\dots,(q_k,g_k))\in R^{\tilde{\mathfrak{N}}(p)}$ by the definition of $\tilde{\mathfrak{N}}(p)$. Now if $((q_1,g_1),\dots,(q_k,g_k))\in R^{\tilde{\mathfrak{N}}(p)}\cap R'^{\tilde{\mathfrak{N}}(p)}$ for some $R,R'\in\tau$, then, by P2, we have $(\mathrm{Rep}_p(q_1),g_1),\dots,(\mathrm{Rep}_p(q_k),g_k))\in R^{\mathfrak{N}^{cpl}}\cap R'^{\mathfrak{N}^{cpl}}$. But then $R=R'$, since $\mathfrak{N}^{cpl}$ is PD. Since $\tilde{\mathfrak{N}}(p)$ is JEPD, we get $\tilde{\mathfrak{N}}(p)\to\mathfrak{D}$ due to P2, because $\mathfrak{N}^{cpl}\to\mathfrak{D}$. $\square$

Claim 4 $\tilde{\mathfrak{N}} \to \mathfrak{D}$.

Proof. First consider the case where there are no blocked nodes in $\mathcal{S}$. Then $\text{Hooks} = \{(root, root)\}$. By (a) of Claim 1, we have $\tilde{\mathfrak{N}} = \tilde{\mathfrak{N}}(root, root)$. Then the statement follow directly from Claim 3. Now suppose that $\mathcal{S}$ contains some blocked nodes. Since $T \cap O_{\mathsf{a}}$ is finite (see the proof of Lemma 5), Hooks is countably infinite. Moreover, the successor relation on Hooks describes an infinite tree whose out-degree is bounded by $|T \cap O_{\mathsf{a}}|$. Fix any enumeration of $\{p_0, p_1, \dots\}$ of Hooks such that $p_0 = (root, root)$, and, if $p_i \succ p_j$, then $i > j$. By (a) of Claim 1, we have $\tilde{\mathfrak{N}} = \bigcup_{i \in \mathbb{N}} \tilde{\mathfrak{N}}(p_i)$. We show by induction on k that for every $k \in \mathbb{N}$ we have $\tilde{\mathfrak{N}}_k \to \mathfrak{D}$, where $\tilde{\mathfrak{N}}_k := \bigcup_{0 \leq i \leq k} \tilde{\mathfrak{N}}(p_i)$. The base case $k = 0$ follows directly from Claim 3. For the induction step, suppose that $\tilde{\mathfrak{N}}_{k-1} \to \mathfrak{D}$. Let $\tilde{\mathfrak{N}}_{k-1}^{cpl}$ be any completion of $\tilde{\mathfrak{N}}_{k-1}$. There exists a unique $p_n \in \text{Hooks}$ with $k > n$ such that $p_k \succ p_n$. By (b) of Claim 1, we have $\tilde{N}_{k-1}^{cpl} \cap \tilde{N}(p_k) = \tilde{N}(p_n) \cap \tilde{N}(p_k)$. Let $\mathfrak{M}_{k-1}^{cpl}, \mathfrak{M}(p_n)$, and $\mathfrak{M}(p_k)$ be the substructures of $\tilde{\mathfrak{N}}_{k-1}^{cpl}, \tilde{\mathfrak{N}}(p_n)$ and $\tilde{\mathfrak{N}}(p_k)$ on $\tilde{N}_{k-1} \cap \tilde{N}(p_k)$, respectively. By Claim 3, $\tilde{\mathfrak{N}}(p_n)$ is JEPD, and thus $\mathfrak{M}_{k-1}^{cpl} = \mathfrak{M}(p_n)$. We also clearly have $\mathfrak{M}(p_n) = \mathfrak{M}(p_k)$ by the definition of $\tilde{\mathfrak{N}}(p_n)$ and $\tilde{\mathfrak{N}}(p_k)$. By Claim 3, $\tilde{\mathfrak{N}}(p_k)$ is finite, JEPD, and $\tilde{\mathfrak{N}}(p_k) \to \mathfrak{D}$. Since $\mathfrak{D}$ is a patchwork, we have $\tilde{\mathfrak{N}}_k = \tilde{\mathfrak{N}}_{k-1}^{cpl} \cup \tilde{\mathfrak{N}}(p_k) \to \mathfrak{D}$ by choosing $\mathfrak{A} := \mathfrak{M}_{k-1}^{cpl}$, $\mathfrak{B}_1 := \tilde{\mathfrak{N}}_{k-1}^{cpl}$ and $\mathfrak{B}_2 := \tilde{\mathfrak{N}}(p_k)$ in the definition of a patchwork. Now let $\mathfrak{A}$ be any finite structure that embeds to $\tilde{\mathfrak{N}}$. Then there exists $k \in \mathbb{N}$ such that $\mathfrak{A} \hookrightarrow \tilde{\mathfrak{N}}_k \to \mathfrak{D}$. Since $\mathfrak{D}$ has homomorphism compactness, we have $\tilde{\mathfrak{N}} \to \mathfrak{D}$. □

We now define the concrete part of $\mathcal{I}$. Let $h\colon \tilde{\mathfrak{N}} \to \mathfrak{D}$ be any homomorphism. For every $g \in \mathsf{N_F}(C_0, \mathcal{T})$,

$$g^{\mathcal{I}} := \{(p, h(p, g)) \in \Delta^{\mathcal{I}} \times D \mid \text{Tail}(p) = (a, b) \text{ and } g \in \text{Feat}(a)\}.$$

By definition, $g^{\mathcal{I}}$ is functional for every $g \in \mathsf{N_F}$. We need one additional claim to show that $\mathcal{I}$ is a model of $(C_0, \mathcal{T})$.

Claim 5 *For every $s \in \Delta^{\mathcal{I}}$ and $C \in \text{Sub}(C_0, \mathcal{T})$, if $\text{Tail}(s) = (a, b)$ and $C \in \mathcal{L}(a)$, then $s \in C^{\mathcal{I}}$.*

Proof. We proceed using induction on the subconcepts of C. Let $s \in \Delta^{\mathcal{I}}$ be arbitrary such that $\text{Tail}(s) = (a, b)$, and $C \in \mathcal{L}(a)$. Suppose that the statements holds for every proper subconcept of C. By definition of Paths, a is not blocked in $\mathcal{S}$. We make case distinction w.r.t. the topmost operator in C.

If C is a concept name, then by definition of $\mathcal{I}$ we have $s \in C^{\mathcal{I}}$.

Suppose that $C = \neg D$. Since C is in NNF, D must be a concept name. Since $\mathcal{S}$ is clash-free, $D \notin \mathcal{L}(a)$. Then by definition of $\mathcal{I}$ we have $s \notin D^{\mathcal{I}}$, which yields $s \in C^{\mathcal{I}}$.

Suppose that $C = D \sqcap E$. Since $\mathcal{S}$ is complete, $\{D, E\} \subseteq \mathcal{L}(a)$. Then by the induction hypothesis we have $s \in D^{\mathcal{I}}$ and $s \in D^{\mathcal{I}}$. Thus $s \in (D \sqcap E)^{\mathcal{I}}$.

Suppose that $C = D \sqcup E$. Since $\mathcal{S}$ is complete, $\{D, E\} \cap \mathcal{L}(a) \neq \emptyset$. Then by the induction hypothesis we have $s \in D^{\mathcal{I}}$ or $s \in D^{\mathcal{I}}$. Thus $a \in (D \sqcup E)^{\mathcal{I}}$.

Suppose that $C = \exists r.\, D$. Since $\mathcal{S}$ is complete, the $\exists$-rule is not applicable. Thus a has an r-successor c such that $D \in \mathcal{L}(c)$. By definition of $\mathcal{I}$, there is a path $t = s(d, c) \in \Delta^{\mathcal{I}}$ such that either $c = d$ or c is blocked by d in $\mathcal{S}$. Since $\mathcal{L}(c) \subseteq \mathcal{L}(d)$ in both cases, we have $D \in \mathcal{L}(d)$. By the induction hypothesis, we have $t \in D^{\mathcal{I}}$. By definition of $\mathcal{I}$, we have $(s, t) \in r^{\mathcal{I}}$, which implies $s \in C^{\mathcal{I}}$.

Suppose that $C = \forall r.\, D$. Consider any $(s, t) \in r^{\mathcal{I}}$. By definition of $\mathcal{I}$, we have $t = s \cdot (d, c)$ such that c is an r-successor of a. Since $\mathcal{S}$ is complete, the $\forall$-rule is not applicable. Thus we have $D \in \mathcal{L}(c)$. Since $\mathcal{L}(c) \subseteq \mathcal{L}(d)$ as in the previous case, we have $D \in \mathcal{L}(d)$. By the induction hypothesis, we have $t \in D^{\mathcal{I}}$. Since this holds independently of the choice of t, we get $s \in C^{\mathcal{I}}$.

Suppose that $C = \exists p_1, \ldots, p_k.\, [R_1 \vee \cdots \vee R_m]$ for k-ary symbols $R_1, \ldots, R_m \in \tau$. Since C is in NNF, each p_i is either a feature or a path of length two. We consider only the case where $p_i = (r_i, g_i)$ for each $i \in [k]$. The argumentation for the remaining cases is similar but easier. Since $\mathcal{S}$ is complete, the $\exists_c$-rule cannot be applied. Thus there exists an r_i-successor c_i of a and a g_i-successor x_i of c_i for each $i \in [k]$ such that $(x_1, \ldots, x_k) \in R_j^{\mathfrak{N}}$ for some $j \in [m]$. Then $((c_1, g_1), \ldots, (c_k, g_k)) \in R_j^{\mathfrak{N}^{cpl}}$. Moreover, for each $i \in [k]$ there is a $t_i = s \cdot (d_i, c_i) \in \Delta^{\mathcal{I}}$ such that $c_i = d_i$ or c_i is blocked by d_i. By definition of $r_i^{\mathcal{I}}$, we have $(s, t_i) \in r_i^{\mathcal{I}}$ for every $i \in [k]$. Since a is not blocked and $c_1, \ldots, c_k$ are its successors, there exists $p \in \mathrm{Hooks}$ such that $t_1, \ldots, t_k \in P(p)$ and $\mathrm{Rep}_p(t_i) = c_i$ for every $i \in [k]$. Thus, by the definition of $\tilde{\mathfrak{N}}$ we have $((t_1, g_1), \ldots, (t_k, g_k)) \in R_j^{\tilde{\mathfrak{N}}}$, which implies $(g(t_1, g_1), \ldots, g(t_k, g_k)) \in R_j^{\mathfrak{D}}$. Since $g_i^{\mathcal{I}}(t_i) = g(t_i, g_i)$ for every $i \in [k]$, we get $s \in C^{\mathcal{I}}$.

Suppose that $C = \forall p_1, \ldots, p_k.\, [R_1 \vee \cdots \vee R_m]$ for k-ary symbols $R_1, \ldots, R_m \in \tau$. As in the previous case, we assume that each p_i is of the form (r_i, g_i). For every $i \in [k]$, let t_i be such that $(s, t_i) \in r_i^{\mathcal{I}}$ and $g_i^{\mathcal{I}}(t_i)$ is defined. By the definition of $\mathcal{I}$, we have $t_i = s \cdot (d_i, c_i) \in \Delta^{\mathcal{I}}$ such that c_i is an r_i-successor of a for every $i \in [k]$. Moreover, there is exists a g_i-successor y_i of c_i for every $i \in [k]$. Since the $\exists$-rule is not applicable, $\forall p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathcal{L}(a)$ implies that $(y_1, \ldots, y_k) \in R_j^{\mathfrak{N}}$ for some $j \in [m]$. Thus, $((c_1, g_1), \ldots, (c_k, g_k)) \in R_j^{\mathfrak{N}^{cpl}}$. Moreover, since a is not blocked, there is a $p \in \mathrm{Hooks}$ such that $t_1, \ldots, t_k \in P(p)$ and $\mathrm{Rep}_p(t_i) = c_i$ for every $i \in [k]$. Thus, by the definition of $\tilde{\mathfrak{N}}$, we have $((t_1, g_1), \ldots, (t_k, g_k)) \in R_j^{\tilde{\mathfrak{N}}}$, which implies $(g(t_1, g_1), \ldots, g(t_k, g_k)) \in R_j^{\mathfrak{D}}$. Thus $s \in C^{\mathcal{I}}$. □

Since $C_0 \in \mathcal{L}(root)$ and $(root, root) \in \Delta^{\mathcal{I}}$, Claim 5 implies that $\mathcal{I}$ is a model of C_0. Finally, let us show that $\mathcal{I}$ is a model of the input TBox $\mathcal{T} = \{\top \sqsubseteq C_{\mathcal{T}}\}$. Choose an $s \in \Delta^{\mathcal{I}}$ and let $\mathrm{Tail}(s) = (a, b)$. Since $\mathcal{S}$ is complete, the $\mathcal{T}$-rule is not applicable, and thus $C_{\mathcal{T}} \in \mathcal{L}(a)$. By Claim 5 we have that $s \in C_{\mathcal{T}}^{\mathcal{I}}$. Since this holds independently of the choice of s, we have $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$. □

Lemma 7 (Completeness). *Let $\mathfrak{D}$ be an ω-admissible τ-structure with at most d-ary relations. Furthermore, let C_0 resp. $\mathcal{T}$ be an $\mathcal{ALC}^d_{\vee+}(\mathfrak{D})$ concept resp. TBox in NNF. If C_0 is satisfiable w.r.t. $\mathcal{T}$, then there is a run of Algorithm 1 starting with an initial completion system for $(C_0, \mathcal{T})$ that returns* `true`.

Proof. Suppose that C_0 is satisfiable w.r.t. $\mathcal{T}$, $\mathcal{I}$ is a model for $(C_0, \mathcal{T})$, and $a_0 \in \Delta^{\mathcal{I}}$ such that $a_0 \in C_0^{\mathcal{I}}$. We say that an completion system $\mathcal{S}$ is *$\mathcal{I}$-compatible* if there exist mappings $\pi\colon T \cap O_{\mathsf{a}} \to \Delta^{\mathcal{I}}$ and $\tau\colon T \cap O_{\mathsf{c}} \to D$ such that

a. if $C \in \mathcal{L}(a)$, then $\pi(a) \in C^{\mathcal{I}}$,
b. if b is an r-successor of a, then $(\pi(a), \pi(b)) \in r^{\mathcal{I}}$,
c. if x is a g-successor of a, then $g^{\mathcal{I}}(\pi(a)) = \tau(x)$, and
d. if $(x_1, \ldots, x_k) \in R^{\mathfrak{N}}$, then $(\tau(x_1), \ldots, \tau(x_k)) \in R^{\mathfrak{D}}$.

Claim 6 *If $\mathcal{S}$ is $\mathcal{I}$-compatible and a rule is applicable to $\mathcal{S}$, then this rule can be applied so that the resulting completion system $\mathcal{S}'$ is $\mathcal{I}$-compatible as well.*

Proof. We make a case distinction according to the choice of a rule from Algorithm 2.

The $\sqcap$-rule is applied to a concept $C_1 \sqcap C_2 \in \mathcal{L}(a)$. By (a), $C_1 \sqcap C_2 \in \mathcal{L}(a)$ implies $\pi(a) \in (C_1 \sqcap C_2)^{\mathcal{I}}$ and hence $\pi(a) \in C_1^{\mathcal{I}}$ and $\pi(a) \in C_2^{\mathcal{I}}$. Since the rule adds C_1 and C_2 to $\mathcal{L}(a)$, it yields a completion system that is $\mathcal{I}$-compatible via π and τ.

The $\sqcup$-rule is applied to a concept $C_1 \sqcup C_2 \in \mathcal{L}(a)$. This implies $\pi(a) \in C_1^{\mathcal{I}}$ or $\pi(a) \in C_2^{\mathcal{I}}$. Since the rule adds either C_1 or C_2 to $\mathcal{L}(a)$, it can be applied such that it yields as completion system that is $\mathcal{I}$-compatible via π and τ.

The $\exists$-rule is applied to $\exists r.C \in \mathcal{L}(a)$. By (a), we have $\pi(a) \in (\exists r.C)^{\mathcal{I}}$ and hence there exists a $d \in \Delta^{\mathcal{I}}$ such that $(\pi(a), d) \in r^{\mathcal{I}}$. By the definition of the $\exists$-rule and $\oplus$, the rule application either adds a new r-successor b of a and sets $\mathcal{L}(b) = \{C\}$, or reuses an existing r-successor, renames it to b in $\mathcal{S}$ and sets $\mathcal{L}(b) = \mathcal{L}(b) \cup \{C\}$. Extend π by setting $\pi(b) := d$. The resulting completion system is $\mathcal{I}$-compatible via the extension of π and the original τ.

The $\forall$-rule is applied to $\forall r.C \in \mathcal{L}(a)$ and it adds C to the label $\mathcal{L}(b)$ of every existing r-successor of a. By (a), $\pi(a) \in (\forall r.C)^{\mathcal{I}}$ and by (b), $(\pi(a), \pi(b)) \in R^{\mathcal{I}}$. Therefore, $\pi(b) \in C^{\mathcal{I}}$ and the resulting completion system in $\mathcal{I}$-compatible via π and τ.

The $\exists_c$-rule is applied to a concept $\exists p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathcal{L}(a)$. We assume that $p_i = (r_i, g_i)$ for every $i \in [k]$. The argumentation for the case where some paths are of length one is similar. The rule application generates new abstract nodes $b_1, \ldots, b_k$ and concrete nodes $x_1, \ldots, x_k$ (or reuses existing ones while renaming them) such that for every $i \in [k]$, b_i is an r_i-successor of a and x_i is a g_i-successor of b_i. By (a), we have $\pi(a) \in (\exists p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m])^{\mathcal{I}}$. Thus, there exist $d_1, \ldots, d_k \in \Delta^{\mathcal{I}}$, $v_1, \ldots, v_k \in D$ and a $j \in [m]$ such that for every $i \in [k]$: $(\pi(a), d_i) \in r_i^{\mathcal{I}}$, $g_j^{\mathcal{I}}(d_j) = v_i$, and $(v_1, \ldots, v_k) \in R_j^{\mathfrak{D}}$. Thus, the rule can be guided such that it adds $(x_1, \ldots, x_k)$ to $R_j^{\mathfrak{N}}$. We extend π by setting $\pi(b_i) := d_i$, and τ by setting $\tau(x_i) := v_i$ for every $i \in [k]$. It is easy to see that the resulting completion system is $\mathcal{I}$-compatible via the extensions of π and τ.

The $\forall_c$-rule is applied to an abstract node a with $\forall p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m] \in \mathcal{L}(a)$ such that there are $x_1, \ldots, x_k \in T \cap O_{\mathsf{c}}$ with x_i a p_i-successor of a for every $i \in [k]$. By (a), $\pi(a) \in (\forall p_1, \ldots, p_k.[R_1 \vee \cdots \vee R_m])^{\mathcal{I}}$. By (b) and (c), we have $(\pi(a), \tau(x_i)) \in p_i^{\mathcal{I}}$ for every $i \in [k]$. It follows that there exists $j \in [m]$ such that $(\tau(x_1), \ldots, \tau(x_k)) \in R_j^{\mathfrak{D}}$.

Thus the application of the rule can be guided such that it adds $(x_1, \dots, x_k)$ to $R_j^{\mathfrak{N}}$. Thus the resulting completion system is $\mathcal{I}$-compatible via π and τ.

The $\mathfrak{N}$-rule is applied to an abstract node a which is potentially blocked by an abstract node b and $\mathfrak{N}(a)$ is not JEPD (the other case is analogous). The rule application guesses a completion $\mathfrak{N}^{cpl}(a)$ of $\mathfrak{N}(a)$ and extends the relations of $\mathfrak{N}$ by the new tuples from $\mathfrak{N}^{cpl}(a)$. We define $\mathfrak{N}^{cpl}(a)$ to be the τ-structure over $N(a)$ such that for every $R \in \tau$,

$$\begin{aligned} R^{\mathfrak{N}^{cpl}(a)} := \{(x_1, \dots, x_k) \in N(a)^k \mid & x_i \text{ is a } g_i\text{-successor of } a \\ & \text{for every } i \in [k] \text{ and } (\tau(x_1), \dots, \tau(x_k)) \in R_j^{\mathfrak{D}}\} \end{aligned}$$

By (d), we have $R^{\mathfrak{N}(a)} \subseteq R^{\mathfrak{N}^{cpl}(a)}$ for every $R \in \tau$. Since $\mathfrak{D}$ is JEPD, $\mathfrak{N}'(a)$ is JEPD as well. Finally, $\tau|_{N(a)} \colon \mathfrak{N}^{cpl}(a) \to \mathfrak{D}$, which means that $\mathfrak{N}^{cpl}(a)$ is a completion of $\mathfrak{N}(a)$. We apply the $\mathfrak{N}$-rule so that $\mathfrak{N}^{cpl}(a)$ is guessed. Then the resulting system is $\mathcal{I}$-compatible via π and τ.

The $\mathcal{T}$-rule adds $C_{\mathcal{T}}$ to $\mathcal{L}(a)$ for some $a \in T \cap O_{\mathsf{a}}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we have $\pi(a) \in C_{\mathcal{T}}^{\mathcal{I}}$. Thus, the resulting completion system is $\mathcal{I}$-compatible via π and τ. $\square$

Claim 7 *Every $\mathcal{I}$-compatible completion system is clash-free.*

Proof. Let $\mathcal{S}$ be an $\mathcal{I}$-compatible completion system. Consider the two kinds of clashes. If $\{A, \neg A\} \in \mathcal{L}(a)$, we get a contradiction to (a). If $\mathfrak{N} \not\to \mathfrak{D}$, we get a contradiction to (d). $\square$

We can now describe the guidance of the rules from Algorithm 1 by the model $\mathcal{I}$. We ensure that, at all times, the considered completion systems are $\mathcal{I}$-compatible. This clearly holds for the initial system. By Claim 6, we can guide the rule applications such that a $\mathcal{I}$-compatible completion system is obtained in each step. By Lemma 5, Algorithm 1 terminates on the input of an initial completion system for $(C_0, \mathcal{T})$. Since it does not find a clash by Claim 7, it returns true. $\square$