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Institute for Theoretical Computer Science
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Franz Baader Renata Wassermann

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Postal Address:
Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
01062 Dresden

http://lat.inf.tu-dresden.de

Visiting Address:
Nöthnitzer Str. 46
Dresden

Contractions Based on Optimal Repairs (Extended Version)

Franz Baader[1] , **Renata Wassermann**[2]

[1]Theoretical Computer Science, Technische Universität Dresden, Germany and
Center for Scalable Data Analytics and Artificial Intelligence (ScaDS.AI), Dresden/Leipzig ,
[2]Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil
franz.baader@tu-dresden.de , renata@ime.usp.br

Abstract

Removing unwanted consequences from a knowledge base has been investigated in belief change under the name contraction and is called repair in ontology engineering. Simple repair and contraction approaches based on removing statements from the knowledge base (respectively called belief base contractions and classical repairs) have the disadvantage that they are syntax-dependent and may remove more consequences than necessary. Belief set contractions do not have these problems, but may result in belief sets that have no finite representation if one works with logics that are not fragments of propositional logic. Similarly, optimal repairs, which are syntax-independent and maximize the retained consequences, may not exist. In this paper, we want to leverage advances in characterizing and computing optimal repairs of ontologies based on the description logics $\mathcal{EL}$ to obtain contraction operations that combine the advantages of belief set and belief base contractions. The basic idea is to employ, in the partial meet contraction approach, optimal repairs instead of optimal classical repairs as remainders. We introduce this new approach in a very general setting, and prove two characterization theorems relating the obtained contractions with well-known postulates. Then, we consider several interesting instances, not only in the standard repair/contraction setting where one wants to get rid of a consequence, but also in other settings such as variants of forgetting in propositional and description logic.

1 Introduction

Representing knowledge in a logic-based knowledge representation language allows one to derive implicit consequences from a given knowledge base (KB), i.e., facts that follow from the statements contained in the KB, but are themselves not explicitly stated there. Modifying a given KB such that a certain unwanted consequence no longer follows is a nontrivial task, which has been investigated in the area of belief change under the name of contraction (Alchourrón, Gärdenfors, and Makinson 1985) and in ontology engineering under the name of repair (Kalyanpur et al. 2006; Schlobach et al. 2007; Baader et al. 2018; Troquard et al. 2018). Whereas research in ontology engineering was mainly concerned with designing, implementing, and testing concrete repair algorithms, research in belief change concentrated on characterizing reasonable classes of contraction operations by formulating certain properties, called postulates, they are supposed to satisfy. Connections between these two areas have, for instance, been investigated in (Flouris, Plexousakis, and Antoniou 2005; Qi and Yang 2008; Ribeiro and Wassermann 2009; Matos et al. 2019; Baader 2023).

The purpose of the present paper is to leverage recent advances in characterizing and computing optimal repairs (Baader, Koopmann, and Kriegel 2023) of ontologies based on Description Logics (DLs) (Baader et al. 2017) to obtain contraction operations that combine the advantages of belief set (Alchourrón, Gärdenfors, and Makinson 1985) and belief base (Hansson 1992) contractions. To be more precise, we will introduce a general framework for constructing contraction operations satisfying certain well-known postulates, which generalizes the partial meet contraction approach. Like base contraction approaches, it has the advantage that (under certain conditions) it can work with finite KBs. However, unlike base contraction, it is syntax independent and loses less consequences.

Partial meet contraction is a well-know approach for constructing contraction operations that satisfy a collection of reasonable postulates. For belief sets, i.e., KBs that are closed under logical consequence, this approach was investigated in the seminal AGM paper (Alchourrón, Gärdenfors, and Makinson 1985). Basically, it considers all maximal subsets of the given belief set that do not *contain* a certain undesired consequence, selects a non-empty collection of these maximal subsets, and then builds their intersection (i.e., the "meet"). This results in a very elegant theory with intuitive postulates, but has the disadvantage that the belief sets obtained by applying this operation may not be representable as the logical closure of a finite KB, even if one starts with belief sets that are finitely representable. To overcome this problem, Nebel (1989) and Hansson (1992) uses finite KBs (called belief bases), takes their maximal subsets that do not *entail* the undesired consequence, and again builds the intersection of a non-empty collection of these maximal subsets. In the belief change literature, these maximal subsets are called remainders, whereas they are called optimal classical repairs in the DL community (Baader et al. 2018). Both partial meet contractions in the belief base setting and optimal classical repairs have the disadvantage that these operations are syntax-dependent and may remove too many consequences (Baader et al. 2018; Santos et al. 2018;

Matos et al. 2019; Baader 2023).

On the DL side, optimal repairs have been introduced, which maximize the set of consequences of the knowledge base rather than the set of its explicit statements (Baader et al. 2018). In cases where such optimal repairs exist (Baader et al. 2021a; Baader et al. 2022; Baader and Kriegel 2022; Baader, Koopmann, and Kriegel 2023), they yield a syntax-independent repair approach that does not lose consequences unnecessarily. The main idea underlying the approach proposed in this paper is to replace, in the partial meet contraction approach, remainders (i.e., optimal classical repairs) with optimal repairs. This approach has been used in (Rienstra, Schon, and Staab 2020; Baader 2023) in the context of designing contraction operations for concepts of the DL $\mathcal{EL}$, though there it was not phrased in this way.

Instead of introducing and applying this new approach in a specific instance, we consider here a very general setup, which clarifies the basic properties needed to apply it. Basically, we consider an entailment relation between KBs, without making explicit assumption on the structure of the KBs and their semantics. For a start, we only require that entailment is reflexive and transitive. In addition, we abstract from non-entailment of a certain consequence as repair goal and only require that the set of repairs is closed under entailment. To apply a variant of the partial meet contraction approach in this setting, we need to make some additional assumptions. First, we assume that operations akin to (but not necessarily equal to) conjunction and disjunction are available, which we will respectively call sum and product. These operations correspond to union and intersection of belief sets, but are performed on (possibly finite) KBs representing them. From a technical point of view, sum is needed to formulate some of the relevant postulates whereas product plays the rôle of meet in the construction of the contraction operation. In addition, we require the existence of remainders, which are optimal repairs in our setting. An important property needed in the proofs of the characterization theorems (i.e., the theorems that state the connections between the constructed contraction operations and the postulates) is that finitely many of these optimal repairs cover all repairs in the sense that every repair is entailed by an optimal one.

In the next section, we describe the general setup and illustrate it with two simple examples, one describing a standard repair/contraction setting, where the repair goal is non-entailment of a certain consequence, and the other one inspired by variable forgetting in propositional logic (Lin and Reiter 1994; Lang, Liberatore, and Marquis 2003). Then, we introduce our new contraction approach (called partial product contractions since the product is used as the meet operation), and prove two characterization theorems. Next, we show that partial meet contraction for belief sets (Alchourrón, Gärdenfors, and Makinson 1985) can be obtained as an instance of our approach, but needing less assumptions on the underlying logic. Finally, we introduce several concrete kinds of knowledge bases, entailment relations, and repair goals that are instances of our general setting, and to which our new partial product contraction approach thus applies. In contrast to the belief set approach, these instances work with finite knowledge bases, and are thus more relevant in practice. For several of these instances, we can use recent results on how to compute optimal repairs for knowledge bases formulated in the DL $\mathcal{EL}$ (Baader et al. 2021a; Baader et al. 2022; Kriegel 2022) to show that the required repair property (existence of a set of optimal repairs that covers all repairs) is satisfied. We also consider an instance where the repair goal is to get rid of certain concept and role names in $\mathcal{EL}$ concepts, and one where KBs are finite representations of formal languages and entailment is induced by language inclusion.

2 The General Setup

We assume that we are given a set of *knowledge bases* (KBs) and an *entailment relation* between knowledge bases. We usually write KBs as $\mathcal{K}$, possibly primed ($\mathcal{K}'$) or with an index ($\mathcal{K}_i$), and entailment as $\models$, i.e., $\mathcal{K} \models \mathcal{K}'$ means that $\mathcal{K}$ *entails* $\mathcal{K}'$, or equivalently that $\mathcal{K}'$ is *entailed by* $\mathcal{K}$. We assume that entailment satisfies the following properties:

- $\mathcal{K} \models \mathcal{K}$ (reflexivity),
- $\mathcal{K} \models \mathcal{K}'$ and $\mathcal{K}' \models \mathcal{K}''$ implies $\mathcal{K} \models \mathcal{K}''$ (transitivity).

We define $\mathrm{Con}(\mathcal{K}) := \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}'\}$, and also call an element of $\mathrm{Con}(\mathcal{K})$ a *consequence* of $\mathcal{K}$. Clearly, reflexivity and transitivity of $\models$ yield the following properties of the Con operator:

- $\mathcal{K} \in \mathrm{Con}(\mathcal{K})$ (inclusion),
- $\mathcal{K} \models \mathcal{K}'$ iff $\mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\mathcal{K})$ (correspondence).

We call two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$ *equivalent* (and write $\mathcal{K} \equiv \mathcal{K}'$) if $\mathrm{Con}(\mathcal{K}) = \mathrm{Con}(\mathcal{K}')$. Obviously, this is the case iff $\mathcal{K} \models \mathcal{K}'$ and $\mathcal{K}' \models \mathcal{K}$. We say that $\mathcal{K}$ *strictly entails* $\mathcal{K}'$ if $\mathcal{K} \models \mathcal{K}'$, but $\mathcal{K}' \not\models \mathcal{K}$. In this case we write $\mathcal{K} \models_s \mathcal{K}'$. The relation $\equiv$ on KBs is indeed an equivalence relation, and we write the equivalence class of a KB $\mathcal{K}$ as $[\mathcal{K}]$, i.e., $[\mathcal{K}] := \{\mathcal{K}' \mid \mathcal{K} \equiv \mathcal{K}'\}$. Note that $\mathrm{Con}(\mathcal{K})$ uniquely determines the equivalence class of $\mathcal{K}$.

To illustrate the notions introduced in this section, we use a very simple example. More practically relevant examples dealing with KBs for the Description Logic $\mathcal{EL}$ are presented in Section 5.

Example 1. *Given a countably infinite set of propositional variables V, a knowledge base is a finite, non-empty conjunction of such variables. Entailment $\models$ between KBs is then classical entailment in propositional logic, which obviously satisfies reflexivity and transitivity. For such a KB $\mathcal{K}$, we denote the set of variables occurring in it with* $\mathrm{Var}(\mathcal{K})$. *Conversely, for a finite, non-empty set $P \subseteq V$, we denote the conjunction of its elements as* $\mathrm{KB}(P)$. *It is easy to see that $\mathcal{K} \models \mathcal{K}'$ iff* $\mathrm{Var}(\mathcal{K}') \subseteq \mathrm{Var}(\mathcal{K})$. *Consequently, $\mathcal{K} \equiv \mathcal{K}'$ iff* $\mathrm{Var}(\mathcal{K}') = \mathrm{Var}(\mathcal{K})$.

In the general case, we make no assumptions on the inner structure of knowledge bases, but we assume that we have operations sum and product available that are akin to conjunction and disjunction.

Definition 2. *We call the operations $\oplus$ and $\otimes$ on finite, non-empty sets of KBs* sum *and* product *operations, respectively, if they satisfy the following properties for each finite, non-empty set of KBs $\mathfrak{K}$:*

- $\mathrm{Con}(\oplus\mathfrak{K}) \supseteq \mathrm{Con}(\mathcal{K})$ *for all* $\mathcal{K} \in \mathfrak{K}$ *and* $\oplus\mathfrak{K}$ *is the least KB satisfying this property, i.e., if* $\mathcal{K}'$ *is a KB satisfying* $\mathrm{Con}(\mathcal{K}') \supseteq \mathrm{Con}(\mathcal{K})$ *for all* $\mathcal{K} \in \mathfrak{K}$*, then* $\mathrm{Con}(\oplus\mathfrak{K}) \subseteq \mathrm{Con}(\mathcal{K}')$.
- $\mathrm{Con}(\otimes\mathfrak{K}) \subseteq \mathrm{Con}(\mathcal{K})$ *for all* $\mathcal{K} \in \mathfrak{K}$ *and* $\otimes\mathfrak{K}$ *is the greatest KB satisfying this property, i.e., if* $\mathcal{K}'$ *is a KB satisfying* $\mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\mathcal{K})$ *for all* $\mathcal{K} \in \mathfrak{K}$*, then* $\mathrm{Con}(\otimes\mathfrak{K}) \supseteq \mathrm{Con}(\mathcal{K}')$.

Note that "least" and "greatest" in the above definition must be read modulo equivalence of KBs. In fact, it is easy to see that the above conditions imply that sum and product of a finite set of KBs are unique up to equivalence. If $\mathfrak{K} = \{\mathcal{K}\}$ is a singleton set, then $\oplus\mathfrak{K} \equiv \mathcal{K} \equiv \otimes\mathfrak{K}$. If $\mathfrak{K} = \{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ for $n \geq 2$, then we will sometimes write its sum as $\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_n$ and its product as $\mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n$.

Lemma 3. *Let* $\mathcal{K}$ *be a KB and* $\mathfrak{K}$ *a finite, non-empty set of KBs. Then the following holds:*

1. $\oplus\mathfrak{K} \models \mathcal{K}'$ *and* $\mathcal{K}' \models \otimes\mathfrak{K}$ *for all* $\mathcal{K}' \in \mathfrak{K}$.
2. $\mathcal{K} \models \oplus\mathfrak{K}$ *iff* $\mathcal{K} \models \mathcal{K}'$ *for all* $\mathcal{K}' \in \mathfrak{K}$.
3. $\otimes\mathfrak{K} \models \mathcal{K}$ *iff* $\mathcal{K}' \models \mathcal{K}$ *for all* $\mathcal{K}' \in \mathfrak{K}$.

Proof. The first part of the lemma is an immediate consequence of monotonicity and the definition of sum and product. Second, assume that $\mathcal{K} \models \oplus\mathfrak{K}$. Then $\mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\oplus\mathfrak{K}) \subseteq \mathrm{Con}(\mathcal{K})$ holds for all $\mathcal{K}' \in \mathfrak{K}$, and thus $\mathcal{K} \models \mathcal{K}'$ for all $\mathcal{K}' \in \mathfrak{K}$. Conversely, assume that $\mathcal{K} \models \mathcal{K}'$ for all $\mathcal{K}' \in \mathfrak{K}$. Then $\mathrm{Con}(\mathcal{K})$ contains the sets $\mathrm{Con}(\mathcal{K}')$ for all $\mathcal{K}' \in \mathfrak{K}$. The definition of the sum thus yields $\mathrm{Con}(\oplus\mathfrak{K}) \subseteq \mathrm{Con}(\mathcal{K})$, which is equivalent to $\mathcal{K} \models \oplus\mathfrak{K}$. The third statement of the lemma can be shown analogously to the second. $\square$

Example 1 (continued). *It is easy to see that sum corresponds to conjunction of KBs, and thus to the union of the corresponding variable sets. Dually, product corresponds to the intersection of the variable sets. Thus, we define*

$$\oplus\mathfrak{K} := \mathrm{KB}\left(\bigcup_{\mathcal{K}\in\mathfrak{K}} \mathrm{Var}(\mathcal{K})\right), \ \otimes\mathfrak{K} := \mathrm{KB}\left(\bigcap_{\mathcal{K}\in\mathfrak{K}} \mathrm{Var}(\mathcal{K})\right).$$

E.g.: $p\wedge q\wedge r \oplus q\wedge s = p\wedge q\wedge r\wedge s$ *and* $p\wedge q\wedge r \otimes q\wedge s = q$.

To see that the sum defined this way satisfies the required properties, first assume that $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ *for some* $\mathcal{K} \in \mathfrak{K}$*. This implies* $\mathrm{Var}(\mathcal{K}') \subseteq \mathrm{Var}(\mathcal{K}) \subseteq \mathrm{Var}(\oplus\mathfrak{K})$*, which yields* $\mathcal{K}' \in \mathrm{Con}(\oplus\mathfrak{K})$*. Thus, we have shown that* $\mathrm{Con}(\oplus\mathfrak{K}) \supseteq \mathrm{Con}(\mathcal{K})$ *holds for all* $\mathcal{K} \in \mathfrak{K}$*. Second, assume that* $\mathcal{K}'$ *is a KB satisfying* $\mathrm{Con}(\mathcal{K}') \supseteq \mathrm{Con}(\mathcal{K})$ *for all* $\mathcal{K} \in \mathfrak{K}$*. Then* $\mathcal{K} \in \mathrm{Con}(\mathcal{K}')$ *for all* $\mathcal{K} \in \mathfrak{K}$*, which yields* $\mathrm{Var}(\mathcal{K}) \subseteq \mathrm{Var}(\mathcal{K}')$ *for all* $\mathcal{K} \in \mathfrak{K}$*, and thus* $\mathrm{Var}(\oplus\mathfrak{K}) \subseteq \mathrm{Var}(\mathcal{K}')$*. Consequently, we obtain* $\mathcal{K}' \models \oplus\mathfrak{K}$*, which is equivalent to* $\mathrm{Con}(\oplus\mathfrak{K}) \subseteq \mathrm{Con}(\mathcal{K}')$.

The proof that, in this example, our definition of the product satisfies the properties required for $\otimes$ *in Definition 2 is similar to our proof for the sum.*

When defining repairs, we assume that we have additional syntactic entities called repair requests, which we usually denote by α.

Definition 4. *Given a KB* $\mathcal{K}$*, a repair request* α *determines a set of KBs* $\mathrm{Rep}(\mathcal{K}, \alpha)$ *such that*

- $\mathcal{K} \models \mathcal{K}'$ *holds for every element* $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$*, and*
- $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ *and* $\mathcal{K}' \models \mathcal{K}''$ *imply* $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$.

We call the elements of $\mathrm{Rep}(\mathcal{K}, \alpha)$ repairs *of* $\mathcal{K}$ *for* α*. Two repair requests* α *and* α' *are* equivalent w.r.t. $\mathcal{K}$ *(*$\alpha \equiv_{\mathcal{K}} \alpha'$*) if they induce the same repairs of* $\mathcal{K}$*, i.e.,* $\mathrm{Rep}(\mathcal{K}, \alpha) = \mathrm{Rep}(\mathcal{K}, \alpha')$.

Example 1 (continued). *In this example, we consider a standard repair setting, where each KB can also be used as a repair request. Given a KB* $\mathcal{K}$ *and a repair request* α*, the goal then is to find a KB entailed by* $\mathcal{K}$ *that does not entail* α*, i.e., the induced set of repairs is defined as* $\mathrm{Rep}(\mathcal{K}, \alpha) := \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}', \mathcal{K}' \not\models \alpha\}$*, where* $\mathcal{K}'$ *range over KBs. The first condition on repair sets of Definition 4 is satisfied by definition and the second by transitivity of* $\models$*. Clearly, two repair requests are equivalent w.r.t.* $\mathcal{K}$ *if they are equivalent as KBs.*

Continuing with presenting our general setup, we additionally assume the *optimal repair property*, which says that, for every pair $\mathcal{K}, \alpha$ consisting of a KB and a repair request (called a *repair problem*), there exists a finite set of KBs $\mathrm{Orep}(\mathcal{K}, \alpha)$ satisfying

- every element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K}, \alpha)$ is a *repair* of $\mathcal{K}$ for α (repair property),
- every element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K}, \alpha)$ is *optimal*, i.e., there is no repair of $\mathcal{K}$ for α that strictly entails $\mathcal{K}'$ (optimality),
- $\mathrm{Orep}(\mathcal{K}, \alpha)$ *covers* all repairs, i.e., for every repair $\mathcal{K}''$ of $\mathcal{K}$ for α, there is an element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K}, \alpha)$ such that $\mathcal{K}'' \in \mathrm{Con}(\mathcal{K}')$ (coverage).

Example 1 (continued). *In this example, the optimal repair property is satisfied. Let* $\mathcal{K}$ *and* α *be KBs. If* $\mathcal{K} \not\models \alpha$*, then we set* $\mathrm{Orep}(\mathcal{K}, \alpha) := \{\mathcal{K}\}$*, which in this case clearly is a set of optimal repairs that covers all repairs.*

Thus, assume that $\mathcal{K} \models \alpha$*, which means that* $\mathrm{Var}(\alpha) \subseteq \mathrm{Var}(\mathcal{K})$*. For every* $p \in \mathrm{Var}(\alpha)$ *we define* $\mathcal{K}^{-p} := \mathrm{KB}(\mathrm{Var}(\mathcal{K}) \setminus \{p\})$*. It is easy to see that each such KB* $\mathcal{K}^{-p}$ *is a repair of* $\mathcal{K}$ *for* α*, i.e., is entailed by* $\mathcal{K}$ *and does not entail* α*. We claim that* $\mathrm{Orep}(\mathcal{K}, \alpha) := \{\mathcal{K}^{-p} \mid p \in \mathrm{Var}(\alpha)\}$ *is a set of optimal repairs of* $\mathcal{K}$ *for* α *that covers all repairs.*

To show optimality, assume that $\mathcal{K}'$ *is a repair of* $\mathcal{K}$ *for* α *that entails* $\mathcal{K}^{-p}$*, which implies that* $\mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}') \supseteq \mathrm{Var}(\mathcal{K}^{-p})$*. Since* $\mathrm{Var}(\mathcal{K}^{-p})$ *is obtained from* $\mathrm{Var}(\mathcal{K})$ *by removing a single element, this chain of inclusions implies* $\mathrm{Var}(\mathcal{K}') = \mathrm{Var}(\mathcal{K})$ *or* $\mathrm{Var}(\mathcal{K}') = \mathrm{Var}(\mathcal{K}^{-p})$*. Since* $\mathcal{K}'$ *does not entail* α*, but* $\mathcal{K}$ *does, the former identity cannot hold. Thus, the latter identity holds, which shows that* $\mathcal{K}'$ *is equivalent to* $\mathcal{K}^{-p}$.

To show coverage, assume that $\mathcal{K}''$ *is a repair of* $\mathcal{K}$ *for* α*. This implies* $\mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}'')$ *and* $\mathrm{Var}(\mathcal{K}'') \not\supseteq \mathrm{Var}(\alpha)$*. The latter non-inclusion yields a variable* $p \in \mathrm{Var}(\alpha)$ *such that* $p \notin \mathrm{Var}(\mathcal{K}'')$*. Together with the former inclusion, we obtain* $\mathrm{Var}(\mathcal{K}^{-p}) \supseteq \mathrm{Var}(\mathcal{K}'')$*, and thus* $\mathcal{K}'' \in \mathrm{Con}(\mathcal{K}^{-p})$.

We conclude this section with a simple example that considers repair requests that do not require non-entailment.

It is inspired by variable forgetting in propositional logic (Lang, Liberatore, and Marquis 2003).

Example 5. *Given a countably infinite set of propositional variables V, a knowledge base is a formula of propositional logic (built using the connectives $\wedge$, $\vee$, $\neg$, and the truth constants $\top$ and $\bot$). Entailment $\models$ between KBs is the following restriction of classical entailment $\models_{\mathsf{PL}}$ in propositional logic: $\mathcal{K} \models \mathcal{K}'$ if $\mathcal{K} \models_{\mathsf{PL}} \mathcal{K}'$ and additionally $\mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}')$ is satisfied. This entailment relation is clearly reflexive and transitive. As* repair requests*, we consider finite subsets of the set of propositional variables V. Given a KB $\mathcal{K}$ and a repair request α, the induced set of repairs is defined as $\mathrm{Rep}(\mathcal{K},\alpha) := \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}', \mathrm{Var}(\mathcal{K}') \cap \alpha = \emptyset\}$.*

The sum *operation again corresponds to conjunction, i.e., $\mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_n := \mathcal{K}_1 \wedge \ldots \wedge \mathcal{K}_n$. To see that the sum defined this way satisfies the required properties, first assume that $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ for some $\mathcal{K} \in \mathfrak{K}$. We must show that $\mathcal{K}' \in \mathrm{Con}(\oplus\mathfrak{K})$, i.e., that $\oplus\mathfrak{K} \models \mathcal{K}'$. Now, $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ means that $\mathcal{K} \models_{\mathsf{PL}} \mathcal{K}'$ and $\mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}')$, which together with $\mathcal{K} \in \mathfrak{K}$ imply $\oplus\mathfrak{K} \models_{\mathsf{PL}} \mathcal{K}'$ and $\mathrm{Var}(\oplus\mathfrak{K}) \supseteq \mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}')$. Thus, we have shown that $\oplus\mathfrak{K} \models \mathcal{K}'$ holds, as required. Second, assume that $\mathcal{K}'$ is a KB satisfying $\mathrm{Con}(\mathcal{K}') \supseteq \mathrm{Con}(\mathcal{K})$ for all $\mathcal{K} \in \mathfrak{K}$. This means that $\mathcal{K}' \models_{\mathsf{PL}} \mathcal{K}$ and $\mathrm{Var}(\mathcal{K}') \supseteq \mathrm{Var}(\mathcal{K})$ hold for all $\mathcal{K} \in \mathfrak{K}$. Consequently, $\mathcal{K}' \models_{\mathsf{PL}} \oplus\mathfrak{K}$ and $\mathrm{Var}(\mathcal{K}') \supseteq \bigcup_{\mathcal{K}\in\mathfrak{K}} \mathrm{Var}(\mathcal{K}) = \mathrm{Var}(\oplus\mathfrak{K})$. This shows $\mathcal{K}' \models \oplus\mathfrak{K}$, as required.*

For the product, one could be tempted to use the disjunction operation of propositional logic. While disjunction behaves correctly w.r.t. $\models_{\mathsf{PL}}$, there is a problem with the containment condition for the variables. The set of variables occurring in a disjunction is again the union of the set of variables occurring in its disjuncts, but we would need it to be the intersection. For this reason, we defer defining the product, and first consider the optimal repair property.

Consider a repair problem *$\mathcal{K}, \alpha$, i.e., a propositional formula $\mathcal{K}$ and a finite set of propositional variables α. For every mapping $\tau : \alpha \to \{\top, \bot\}$, let $\mathcal{K}^\tau$ be the propositional formula obtained from $\mathcal{K}$ by replacing every variable $p \in \alpha$ with $\tau(p)$. We set $\mathrm{Orep}(\mathcal{K},\alpha) := \{\mathcal{K}^{-\alpha}\}$, where $\mathcal{K}^{-\alpha}$ is the disjunction of the formulas $\mathcal{K}^\tau$ with τ ranging over all mappings from α to $\{\top, \bot\}$. Clearly, the formulas $\mathcal{K}^\tau$ do not contain any of the variables of α, and thus the same is true for $\mathcal{K}^{-\alpha}$. To prove that $\mathcal{K}^{-\alpha}$ is a repair of $\mathcal{K}$ for α, it is thus sufficient to show $\mathcal{K} \models_{\mathsf{PL}} \mathcal{K}^{-\alpha}$ since $\mathrm{Var}(\mathcal{K}) \supseteq \mathrm{Var}(\mathcal{K}^{-\alpha})$ obviously holds. Hence, let $v : V \to \{0, 1\}$ be a propositional valuation that makes $\mathcal{K}$ true. We define the mapping τ_v from α to $\{\top, \bot\}$ as follows: $\tau_v(p) = \top$ if $v(p) = 1$ and $\tau_v(p) = \bot$ if $v(p) = 0$. Obviously, v then also makes $\mathcal{K}^{\tau_v}$ true, and thus also $\mathcal{K}^{-\alpha}$. This show $\mathcal{K} \models_{\mathsf{PL}} \mathcal{K}^{-\alpha}$, and thus also $\mathcal{K} \models \mathcal{K}^{-\alpha}$.*

To show optimality and coverage, it is sufficient to prove that every repair $\mathcal{K}'$ of $\mathcal{K}$ for α is entailed by $\mathcal{K}^{-\alpha}$. We know that $\mathcal{K} \models \mathcal{K}'$ and $\mathcal{K}'$ contains none of the variables of α. This implies $\mathrm{Var}(\mathcal{K}') \subseteq \mathrm{Var}(\mathcal{K}) \setminus \alpha = \mathrm{Var}(\mathcal{K}^{-\alpha})$. Thus, it remains to show $\mathcal{K}^{-\alpha} \models_{\mathsf{PL}} \mathcal{K}'$. Let v be a propositional valuation that makes $\mathcal{K}^{-\alpha}$ true. Then there is a disjunct $\mathcal{K}^\tau$ of $\mathcal{K}^{-\alpha}$ such that v makes $\mathcal{K}^\tau$ true. We modify v to v_τ by setting $v_\tau(p) = 1$ if $\tau(p) = \top$ and $v_\tau(p) = 0$ if $\tau(p) = \bot$, for all $p \in \alpha$, and leaving the value unchanged for all other propositional variables. Then the fact that v makes $\mathcal{K}^\tau$ true implies that v_τ makes $\mathcal{K}$ true, which in turn yields that v_τ makes $\mathcal{K}'$ true. Since $\mathcal{K}'$ does not contain any element of α, the latter implies that also v makes $\mathcal{K}'$ true. Thus, we have shown $\mathcal{K}^{-\alpha} \models_{\mathsf{PL}} \mathcal{K}'$.

To come back to the product*, consider KBs $\mathcal{K}_1, \ldots, \mathcal{K}_n$, and set $\beta := \bigcup_{1\leq i\leq n} \mathrm{Var}(\mathcal{K}_i) \setminus \bigcap_{1\leq i\leq n} \mathrm{Var}(\mathcal{K}_i)$. We define $\mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n := (\mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n)^{-\beta}$. It is easy to see that $\mathrm{Var}(\mathcal{K}_i) \supseteq \bigcap_{1\leq j\leq n} \mathrm{Var}(\mathcal{K}_j) = \mathrm{Var}(\mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n)$. In addition, $\mathcal{K}_i \models_{\mathsf{PL}} \mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n \models_{\mathsf{PL}} (\mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n)^{-\beta}$, where the former is obvious and the latter was shown above. Thus, we have shown that $\mathcal{K}_i \models \mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n$.*

Now, assume that $\mathcal{K}_i \models \mathcal{K}'$ for all $i, 1 \leq i \leq n$. This implies that $\mathrm{Var}(\mathcal{K}') \subseteq \bigcap_{1\leq i\leq n} \mathrm{Var}(\mathcal{K}_i) = \mathrm{Var}(\mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n)$ and $\mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n \models_{\mathsf{PL}} \mathcal{K}'$. Since $\bigcap_{1\leq i\leq n} \mathrm{Var}(\mathcal{K}_i) \cap \beta = \emptyset$, the KB $\mathcal{K}'$ is a repair of $\mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n$ for β. As shown above, this implies $(\mathcal{K}_1 \vee \ldots \vee \mathcal{K}_n)^{-\beta} \models_{\mathsf{PL}} \mathcal{K}'$. Overall, we have thus shown $\mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n \models \mathcal{K}'$, as required.

3 Partial Product Contractions

In this section, we assume that we are given a set of KBs, a set of repair requests inducing repair sets that satisfy the conditions in Definition 4, and an entailment relation $\models$ with the associated consequence operator Con such that all the properties introduced in the previous section are satisfied. In the following, we adapt the partial meet contraction approach to this setting, but call the resulting approach the partial product contraction approach since intersection (meet) is replaced with the product. Since the properties of entailment relations introduced in the previous section are needed for this contraction approach to work, we call such entailment relations partial product contraction enabling.

Definition 6. *Given a set of knowledge bases (KBs), a set of repair requests inducing repair sets, and a binary relation $\models$ between KBs, we call $\models$* partial product contraction enabling *if it is reflexive and transitive, has sum and product operations $\oplus$ and $\otimes$ satisfying the properties stated in Definition 2, and for every repair problem $\mathcal{K}, \alpha$ the induced set of repairs $\mathrm{Rep}(\mathcal{K},\alpha)$ satisfies the conditions in Definition 4 and has a finite subset $\mathrm{Orep}(\mathcal{K},\alpha)$ that consists of optimal repairs and covers all repairs.*

Let $\mathcal{K}$ be a KB and $\mathrm{Orep}(\mathcal{K},\alpha)$ for each repair request α the corresponding set of optimal repairs, which covers all repairs of $\mathcal{K}$ for α. A *selection function* γ for $\mathcal{K}$ takes such sets of optimal repairs as input and satisfies the following properties, for each repair request α:

- If $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$, then the selected set $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ satisfies $\emptyset \neq \gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \subseteq \mathrm{Orep}(\mathcal{K},\alpha)$.
- If $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$, then $\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) = \{\mathcal{K}\}$.

Note that coverage of $\mathrm{Orep}(\mathcal{K},\alpha)$ implies that this set is empty iff $\mathrm{Rep}(\mathcal{K},\alpha) = \emptyset$. In this case, the selection function returns the singleton set consisting of $\mathcal{K}$. Otherwise, it returns a non-empty set consisting of some of the optimal repairs.

In addition, we require that selection functions are *invariant under equivalence* of their input sets, where we say that two sets $\mathfrak{K}$ and $\mathfrak{K}'$ of knowledge bases are *equivalent* (written $\mathfrak{K} \equiv \mathfrak{K}'$) if they induce the same sets of equivalence classes, i.e., $\{[\mathcal{K}] \mid \mathcal{K} \in \mathfrak{K}\} = \{[\mathcal{K}'] \mid \mathcal{K}' \in \mathfrak{K}'\}$. More formally, the third condition on selection functions requires that, for all repair requests α and α', the following property is satisfied:

- If $\mathrm{Orep}(\mathcal{K}, \alpha) \equiv \mathrm{Orep}(\mathcal{K}, \alpha')$, then $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha)) \equiv \gamma(\mathrm{Orep}(\mathcal{K}, \alpha'))$.

Each selection function γ induces a *partial product contraction* operation ctr_γ as follows:

$$\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) := \otimes\gamma(\mathrm{Orep}(\mathcal{K}, \alpha)).$$

A partial product contraction operation defined using a selection function γ satisfying $|\gamma(\mathrm{Orep}(\mathcal{K}, \alpha))| = 1$ for all repair requests α is called a *MaxiChoice* partial product contraction operation. In this setting, the selection function returns a singleton set consisting of $\mathcal{K}$ (if there is no repair) or an optimal repair (otherwise). In the latter case, $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ is then this optimal repair. This is the way (optimal) repairs are normally used in ontology engineering: the KB to be repaired is replaced by a single repair chosen by the ontology engineer.

Postulates

We show that each partial product contraction operation ctr satisfies the following postulates:

- $\mathrm{ctr}(\mathcal{K}, \alpha) \in \mathrm{Con}(\mathcal{K})$ (logical inclusion),
- if $\mathrm{Rep}(\mathcal{K}, \alpha) \neq \emptyset$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \in \mathrm{Rep}(\mathcal{K}, \alpha)$ (success),
- if $\mathrm{Rep}(\mathcal{K}, \alpha) = \emptyset$, then $\mathrm{ctr}(\mathcal{K}, \alpha) = \mathcal{K}$ (failure),
- if $\mathcal{K} \in \mathrm{Rep}(\mathcal{K}, \alpha)$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \equiv \mathcal{K}$ (vacuity),
- if $\alpha \equiv_\mathcal{K} \alpha'$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \equiv \mathrm{ctr}(\mathcal{K}, \alpha')$ (preservation),
- if $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ and $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}(\mathcal{K}, \alpha))$, then there is $\mathcal{K}''$ such that $\mathcal{K} \models_s \mathcal{K}'' \models \mathrm{ctr}(\mathcal{K}, \alpha)$, $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$, and $\mathcal{K}'' \oplus \mathcal{K}' \notin \mathrm{Rep}(\mathcal{K}, \alpha)$ (relevance).

MaxiChoice partial product contraction operations also satisfy the following postulate, which is stronger than *relevance*:

- if $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ and $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}(\mathcal{K}, \alpha))$, then $\mathrm{ctr}(\mathcal{K}, \alpha) \oplus \mathcal{K}' \notin \mathrm{Rep}(\mathcal{K}, \alpha)$ (fullness).

In the AGM setting, MaxiChoice operations have been criticized for producing belief sets that are too large (Alchourrón, Gärdenfors, and Makinson 1985). However, this only happens when dealing with logics that contain full propositional logic. In some cases, it is the most appropriate way to define contractions (Makinson 1987; Wassermann 2000).

It is easy to see that, in the presence of *logical inclusion*, *success*, and *failure*, the postulate *fullness* implies *relevance* since one can simply set $\mathcal{K}'' := \mathrm{ctr}(\mathcal{K}, \alpha)$. In fact, $\mathcal{K} \models_s \mathrm{ctr}(\mathcal{K}, \alpha)$ due to *logical inclusion* and the fact that $\mathcal{K}$ entails $\mathcal{K}'$, whereas $\mathrm{ctr}(\mathcal{K}, \alpha)$ does not. This strict entailment also implies that $\mathrm{Rep}(\mathcal{K}, \alpha) \neq \emptyset$ since otherwise $\mathrm{ctr}(\mathcal{K}, \alpha)$ would be equivalent to $\mathcal{K}$ due to the postulate *failure*. Thus, *success* yields $\mathrm{ctr}(\mathcal{K}, \alpha) \in \mathrm{Rep}(\mathcal{K}, \alpha)$.

Proposition 7. *Let γ be a selection function. Then the partial product contraction operation ctr_γ induced by γ satisfies the postulates* logical inclusion*,* success*,* failure*,* vacuity*,* preservation*, and* relevance*. If γ is such that $|\gamma(\mathrm{Orep}(\mathcal{K}, \alpha))| = 1$ for all repair requests α, then ctr_γ additionally satisfies* fullness*.*

Proof. By definition, $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) = \mathcal{K}$ or $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ is a non-empty product of optimal repairs of $\mathcal{K}$ for α. In the former case, $\mathcal{K}$ entails $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ by reflexivity. In the latter, we know, for all $\mathcal{K}' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha)) \subseteq \mathrm{Orep}(\mathcal{K}, \alpha)$, that $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$, and thus $\mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\mathcal{K})$. Since in this case $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha)) \neq \emptyset$, there exists a KB $\mathcal{K}' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$. Then we know that the product $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ of the elements of $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$ satisfies $\mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)) \subseteq \mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\mathcal{K})$, which shows that the postulate *logical inclusion* is satisfied.

If $\mathrm{Rep}(\mathcal{K}, \alpha) \neq \emptyset$, then $\mathrm{Orep}(\mathcal{K}, \alpha) \neq \emptyset$ by coverage, and thus $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ is a non-empty product of repairs of $\mathcal{K}$ for α. Let $\mathcal{K}'$ be one of the repairs occurring in this product. Then $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ and $\mathcal{K}' \models \mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ yield $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) \in \mathrm{Rep}(\mathcal{K}, \alpha)$ by Definition 4. This establishes the postulate *success*.

If $\mathrm{Rep}(\mathcal{K}, \alpha) = \emptyset$, then $\mathrm{Orep}(\mathcal{K}, \alpha) = \emptyset$, and thus $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) = \mathcal{K}$ by the definition of selection functions, which yields the postulate *failure*.

If $\mathcal{K} \in \mathrm{Rep}(\mathcal{K}, \alpha)$, then $\mathcal{K}$ is an optimal repair. This implies that every element of $\mathrm{Orep}(\mathcal{K}, \alpha)$ is equivalent to $\mathcal{K}$, and thus $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) \equiv \mathcal{K}$, which shows that the postulate *vacuity* is satisfied.

Now, assume that $\alpha \equiv_\mathcal{K} \alpha'$. We claim that this implies that $\mathrm{Orep}(\mathcal{K}, \alpha) \equiv \mathrm{Orep}(\mathcal{K}, \alpha')$. Thus, consider an element $\mathcal{K}'$ of $\mathrm{Orep}(\mathcal{K}, \alpha)$. We must show that there is an element $\mathcal{K}''$ of $\mathrm{Orep}(\mathcal{K}, \alpha')$ such that $\mathcal{K}' \equiv \mathcal{K}''$. Since $\alpha \equiv_\mathcal{K} \alpha'$, every repair of $\mathcal{K}$ for α is also a repair of $\mathcal{K}$ for α' and vice versa. Thus, coverage of $\mathrm{Orep}(\mathcal{K}, \alpha')$ implies that there is $\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K}, \alpha')$ such that $\mathcal{K}'' \models \mathcal{K}'$. But then coverage of $\mathrm{Orep}(\mathcal{K}, \alpha)$ yields an element $\mathcal{K}''' \in \mathrm{Orep}(\mathcal{K}, \alpha)$ with $\mathcal{K}''' \models \mathcal{K}''$, and thus $\mathcal{K}''' \models \mathcal{K}'$. Optimality of $\mathcal{K}'$ implies that $\mathcal{K}'''$ and $\mathcal{K}'$ are equivalent. Since $\mathcal{K}''$ lies between these two KBs w.r.t. entailment, this shows that $\mathcal{K}' \equiv \mathcal{K}''$. Since the other direction can be shown in the same way, we have thus established that $\mathrm{Orep}(\mathcal{K}, \alpha) \equiv \mathrm{Orep}(\mathcal{K}, \alpha')$. Consequently, the third condition on selection functions yields $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha)) \equiv \gamma(\mathrm{Orep}(\mathcal{K}, \alpha'))$.

It remains to show that this implies $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha) \equiv \mathrm{ctr}_\gamma(\mathcal{K}, \alpha')$. Thus, assume that $\mathcal{L} \in \mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K}, \alpha))$. Then, by Lemma 3, $\mathcal{L}$ belongs to $\mathrm{Con}(\mathcal{K}')$ for all $\mathcal{K}' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$. Assume that $\mathcal{L} \notin \mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K}, \alpha'))$. Then, again by Lemma 3, there is $\mathcal{K}'' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha'))$ such that $\mathcal{L} \notin \mathrm{Con}(\mathcal{K}'')$. This yields a contradiction since there is $\mathcal{K}' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$ such that $\mathrm{Con}(\mathcal{K}') = \mathrm{Con}(\mathcal{K}'')$. The other direction can be shown in the same way. Thus, we have proved that *preservation* holds.

To show *relevance*, assume that $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ and $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K}, \alpha))$. Since $\mathrm{ctr}_\gamma(\mathcal{K}, \alpha)$ is the product of the elements of $\gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$, Lemma 3 implies that there must be an element $\mathcal{K}'' \in \gamma(\mathrm{Orep}(\mathcal{K}, \alpha))$ such that $\mathcal{K}' \notin \mathrm{Con}(\mathcal{K}'')$. Thus, we have $\mathrm{Con}(\mathcal{K}) \supset \mathrm{Con}(\mathcal{K}'') \supseteq$

$\mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K},\alpha))$, which yields $\mathcal{K} \models_s \mathcal{K}'' \models \mathrm{ctr}_\gamma(\mathcal{K},\alpha)$. In addition, since $\mathcal{K}''$ is an element of $\mathrm{Orep}(\mathcal{K},\alpha)$, it is a repair of $\mathcal{K}$ for α. Now assume that $\mathcal{K}'' \oplus \mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$. We know that $\mathcal{K}''$ is entailed by $\mathcal{K}'' \oplus \mathcal{K}'$ by Lemma 3. However, since $\mathcal{K}''$ is an optimal repair, the repair $\mathcal{K}'' \oplus \mathcal{K}'$ entailing it must be equivalent to $\mathcal{K}''$. This contradicts the fact that $\mathcal{K}''$ was chosen such that $\mathcal{K}' \notin \mathrm{Con}(\mathcal{K}'')$ since clearly also $\mathcal{K}'$ is entailed by $\mathcal{K}'' \oplus \mathcal{K}'$. Thus, our assumption that $\mathcal{K}'' \oplus \mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$ was wrong, which means that we have shown that *relevance* is satisfied.

Finally, we prove *fullness* under the assumption that the selection function γ is such that $|\gamma(\mathrm{Orep}(\mathcal{K},\alpha))| = 1$ for all repair requests α. Thus, assume again that $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ and $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K},\alpha))$, but $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \oplus \mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$. The MaxiChoice assumption implies that $\mathrm{ctr}_\gamma(\mathcal{K},\alpha)$ is actually an optimal repair of $\mathcal{K}$ for α. Since this optimal repair is entailed by the repair $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \oplus \mathcal{K}'$, $\mathrm{ctr}_\gamma(\mathcal{K},\alpha)$ must be equivalent to $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \oplus \mathcal{K}'$. However, this contradicts the assumption that $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}_\gamma(\mathcal{K},\alpha))$. Thus, our assumption that $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \oplus \mathcal{K}' \in \mathrm{Rep}(\mathcal{K},\alpha)$ must have been wrong, which shows that *fullness* holds. $\square$

The postulates *logical inclusion*, *success*, *vacuity*, and *preservation* are variants of the original AGM postulates for belief set contraction (Alchourrón, Gärdenfors, and Makinson 1985), but adapted to a setting where the belief set is represented by a KB $\mathcal{K}$ and the goal of the contraction may be different from getting rid of an unwanted consequence (see Example 5). In case the repair request α is itself a knowledge base, and $\mathrm{Rep}(\mathcal{K},\alpha)$ consists of the KBs entailed by $\mathcal{K}$, but not entailing α, the AGM *recovery* postulate can be formulated in our setting as

- $\mathrm{Con}(\mathcal{K}) \subseteq \mathrm{Con}(ctr(\mathcal{K},\alpha) \oplus \alpha)$ (recovery).

However, even in this restricted setting, it need not hold. It is replaced by *failure* and *relevance* (or *fullness* for the MaxiChoice case), which are adaptations of postulates employed in the belief base setting (Hansson 1992). For the simple instance of our setup introduced in Example 1, *recovery* does actually hold. In the setting of Example 5, writing $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \oplus \alpha$ does not even make sense since α is not a KB. An instance where formulating *recovery* make sense, but it nevertheless fails, can be found in Section 5.1.

Characterization theorems

We now show that, modulo equivalence, the converse of Proposition 7 holds as well. We say that two contraction operations ctr and ctr' are equivalent if $\mathrm{ctr}(\mathcal{K},\alpha) \equiv \mathrm{ctr}'(\mathcal{K},\alpha)$ holds for all KBs $\mathcal{K}$ and repair requests α. Obviously, equivalent contraction operations behave the same w.r.t. satisfaction of the postulates introduced above.

We start with the MaxiChoice setting. The following lemma is needed in the proof of the characterization theorem for this cases.

Lemma 8. *If $\mathcal{K}$ and $\mathcal{K}'$ are KBs such that $\mathcal{K} \models \mathcal{K}'$, then $\mathcal{K} \oplus \mathcal{K}' \equiv \mathcal{K}$.*

Proof. We know that $\mathcal{K} \models \mathcal{K}'$ implies $\mathrm{Con}(\mathcal{K}') \subseteq \mathrm{Con}(\mathcal{K})$, and thus $\mathrm{Con}(\mathcal{K}) \supseteq \mathrm{Con}(\mathcal{K}) \cup \mathrm{Con}(\mathcal{K}')$. In addition, any KB $\mathcal{K}''$ satisfying $\mathrm{Con}(\mathcal{K}'') \supseteq \mathrm{Con}(\mathcal{K}) \cup \mathrm{Con}(\mathcal{K}')$ also satisfies $\mathrm{Con}(\mathcal{K}) \subseteq \mathrm{Con}(\mathcal{K}'')$. Consequently, $\mathcal{K}$ satisfies the properties required for the sum of $\mathcal{K}$ and $\mathcal{K}'$, and thus is equivalent to $\mathcal{K} \oplus \mathcal{K}'$. $\square$

Theorem 9. *Assume that $\models$ is partial product contraction enabling, and let ctr be an operation that receives as input a KB and a repair requests, and returns as output a KB. Then the following are equivalent:*

1. *The operation ctr satisfies* logical inclusion*,* success*,* failure*,* vacuity*,* preservation*, and* fullness.
2. *The operation ctr is equivalent to a MaxiChoice partial product contraction operation.*

Proof. The implication "$2 \Rightarrow 1$" is just the statement of Proposition 7, which we have already proved.

To prove "$1 \Rightarrow 2$," we assume that ctr satisfies the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *fullness*. To show that ctr is a MaxiChoice partial product contraction operation, we define an appropriate selection function. For a KB $\mathcal{K}$ and repair request α, we set

$$\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) := \begin{cases} \{\mathcal{K}'\} & \text{if there is } \mathcal{K}' \in \mathrm{Orep}(\mathcal{K},\alpha) \\ & \text{such that } \mathcal{K}' \equiv \mathrm{ctr}(\mathcal{K},\alpha), \\ \{\mathcal{K}\} & \text{otherwise.} \end{cases}$$

We claim that this definition yields a well-defined selection function γ satisfying $|\gamma(\mathrm{Orep}(\mathcal{K},\alpha))| = 1$ and $\mathrm{ctr} \equiv \mathrm{ctr}_\gamma$.

To prove this claim, first assume that $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$. Then $\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) = \{K\}$, and thus $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) = \mathcal{K}$. In addition, *failure* implies that $\mathrm{ctr}(\mathcal{K},\alpha) = \mathcal{K}$. Thus, in this case γ satisfies the required properties.

Second, assume that $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$. We must show that $\mathrm{Orep}(\mathcal{K},\alpha)$ contains an element that is equivalent to $\mathrm{ctr}(\mathcal{K},\alpha)$. Since $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$, *success* implies that $\mathrm{ctr}(\mathcal{K},\alpha)$ is a repair of $\mathcal{K}$ for α. It is sufficient to prove that $\mathrm{ctr}(\mathcal{K},\alpha)$ is optimal since then coverage of $\mathrm{Orep}(\mathcal{K},\alpha)$ implies the existence of an element of $\mathrm{Orep}(\mathcal{K},\alpha)$ that is equivalent to it. Assume to the contrary that $\mathrm{ctr}(\mathcal{K},\alpha)$ is not optimal, i.e., there is a repair $\mathcal{K}'$ of $\mathcal{K}$ for α that strictly entails $\mathrm{ctr}(\mathcal{K},\alpha)$. This repair satisfies $\mathcal{K}' \in \mathrm{Con}(\mathcal{K})$ and $\mathcal{K}' \notin \mathrm{Con}(\mathrm{ctr}(\mathcal{K},\alpha))$. Thus, *fullness* yields $\mathrm{ctr}(\mathcal{K},\alpha) \oplus \mathcal{K}' \notin \mathrm{Rep}(\mathcal{K},\alpha)$. However, we also know that $\mathcal{K}' \models \mathrm{ctr}(\mathcal{K},\alpha)$, which implies that $\mathrm{ctr}(\mathcal{K},\alpha) \oplus \mathcal{K}' \equiv \mathcal{K}'$ by Lemma 8. This contradicts our assumption that $\mathcal{K}'$ is repair.

Summing up, we have shown that, in both cases, $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ is a singleton set whose element is equivalent to $\mathrm{ctr}(\mathcal{K},\alpha)$. Since $\mathrm{ctr}_\gamma(\mathcal{K},\alpha)$ is equal to this element, we have shown that $\mathrm{ctr} \equiv \mathrm{ctr}_\gamma$.

It remains to prove that our third condition on selection functions is also satisfied by γ. Thus, assume that $\mathrm{Orep}(\mathcal{K},\alpha) \equiv \mathrm{Orep}(\mathcal{K},\alpha')$. We claim that in this case $\alpha \equiv_{\mathcal{K}} \alpha'$. To show $\mathrm{Rep}(\mathcal{K},\alpha) \subseteq \mathrm{Rep}(\mathcal{K},\alpha')$, assume that $\mathcal{L} \in \mathrm{Rep}(\mathcal{K},\alpha)$. Coverage of $\mathrm{Orep}(\mathcal{K},\alpha)$ yields an element $\mathcal{K}' \in \mathrm{Orep}(\mathcal{K},\alpha)$ such that such that $\mathcal{K}' \models \mathcal{L}$. But them the assumed equivalence of $\mathrm{Orep}(\mathcal{K},\alpha)$ and $\mathrm{Orep}(\mathcal{K},\alpha')$ yields an element $\mathcal{K}'' \in \mathrm{Orep}(\mathcal{K},\alpha')$ such that $\mathcal{K}'' \equiv \mathcal{K}'$, and thus $\mathcal{K}'' \models \mathcal{L}$. Since $\mathcal{K}''$ is a repair of $\mathcal{K}$ for α', this shows $\mathcal{L} \in \mathrm{Rep}(\mathcal{K},\alpha')$. The inclusion in the other direction can be shown symmetrically. We can now apply *preservation* to conclude that $\mathrm{ctr}(\mathcal{K},\alpha) \equiv \mathrm{ctr}(\mathcal{K},\alpha')$, which shows that $\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \equiv \gamma(\mathrm{Orep}(\mathcal{K},\alpha'))$. ☐

Next, we prove a characterization of arbitrary partial product contraction operations. The following lemma is used in this proof.

Lemma 10. *If $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{K}''$ are KBs such that $\mathcal{K} \models \mathcal{K}'$, then $\mathcal{K} \oplus \mathcal{K}'' \models \mathcal{K}' \oplus \mathcal{K}''$.*

Proof. By Lemma 3, it is sufficient to show that $\mathcal{K} \oplus \mathcal{K}'' \models \mathcal{K}'$ and $\mathcal{K} \oplus \mathcal{K}'' \models \mathcal{K}''$. The second entailment is an immediate consequence of the first part of Lemma 3. This part also yields $\mathcal{K} \oplus \mathcal{K}'' \models \mathcal{K}$, and thus with $\mathcal{K} \models \mathcal{K}'$ also $\mathcal{K} \oplus \mathcal{K}'' \models \mathcal{K}'$. ☐

Theorem 11. *Assume that $\models$ is partial product contraction enabling, and let* ctr *be an operation that receives as input a KB and a repair requests, and returns as output a KB. Then the following are equivalent:*

1. *The operation* ctr *satisfies* logical inclusion*,* success*,* failure*,* vacuity*,* preservation*, and* relevance*.*
2. *The operation* ctr *is equivalent to a partial product contraction operation.*

Proof. The implication "2 $\Rightarrow$ 1" is an immediate consequence of Proposition 7.

To prove "1 $\Rightarrow$ 2," we assume that ctr satisfies the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance*. To show that ctr is a partial product contraction operation, we again define an appropriate selection function. For a KB $\mathcal{K}$ and repair request α, we set

$$\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) := \begin{cases} \{\mathcal{K}' \in \mathrm{Orep}(\mathcal{K},\alpha) \mid \mathcal{K}' \models \mathrm{ctr}(\mathcal{K},\alpha)\} \\ \qquad \text{if } \mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset, \\ \{\mathcal{K}\} \quad \text{otherwise.} \end{cases}$$

We claim that this yields a well-defined selection function γ satisfying $\mathrm{ctr} \equiv \mathrm{ctr}_\gamma$. The case where $\mathrm{Orep}(\mathcal{K},\alpha) = \emptyset$ can be handled as in the proof of Theorem 9.

Assuming $\mathrm{Orep}(\mathcal{K},\alpha) \neq \emptyset$, we now show $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) \equiv \mathrm{ctr}(\mathcal{K},\alpha)$. Because of our definition of γ, we know that the inclusion $\mathrm{Con}(\mathrm{ctr}(\mathcal{K},\alpha)) \subseteq \mathrm{Con}(\mathcal{K}')$ holds for all $\mathcal{K}' \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha))$. The definition of the product thus yields $\mathrm{Con}(\otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha))) \supseteq \mathrm{Con}(\mathrm{ctr}(\mathcal{K},\alpha))$, and thus $\mathrm{ctr}_\gamma(\mathcal{K},\alpha) = \otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \models \mathrm{ctr}(\mathcal{K},\alpha)$.

To show $\mathrm{ctr}(\mathcal{K},\alpha) \models \otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$, we assume to the contrary that there is $\mathcal{K}' \in \mathrm{Con}(\otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ not belonging to $\mathrm{Con}(\mathrm{ctr}(\mathcal{K},\alpha))$, i.e, $\otimes\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \models \mathcal{K}'$, but $\mathrm{ctr}(\mathcal{K},\alpha) \not\models \mathcal{K}'$. Since $\mathcal{K}$ entails every element $\mathcal{L}$ of $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$, and each such element entails the product of these elements, we know that $\mathcal{K} \models \mathcal{K}'$ holds. Consequently, *relevance* yields the existence of a KB $\mathcal{K}''$ such that $\mathcal{K} \models_s \mathcal{K}'' \models \mathrm{ctr}(\mathcal{K},\alpha)$, $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K},\alpha)$, and $\mathcal{K}'' \oplus \mathcal{K}' \notin \mathrm{Rep}(\mathcal{K},\alpha)$. Coverage of $\mathrm{Orep}(\mathcal{K},\alpha)$ implies that it contains an element $\mathcal{K}'''$ such that $\mathcal{K}''' \models \mathcal{K}''$. Our definition of γ thus yields $\mathcal{K}''' \in \gamma(\mathrm{Orep}(\mathcal{K},\alpha))$, and thus $\mathcal{K}''' \models \mathcal{K}'$. This holds by Lemma 3 since we have assumed that the product of the elements of $\gamma(\mathrm{Orep}(\mathcal{K},\alpha))$ entails $\mathcal{K}'$. Consequently, $\mathcal{K}''' \models \mathcal{K}'' \oplus \mathcal{K}'$ by Lemma 3. Since $\mathcal{K}''' \in \mathrm{Rep}(\mathcal{K},\alpha)$, this yields a contradiction to the fact that $\mathcal{K}'' \oplus \mathcal{K}' \notin \mathrm{Rep}(\mathcal{K},\alpha)$.

Finally, the third condition on selection functions can be shown as in the proof of Theorem 9. In fact, we have shown there that $\mathrm{Orep}(\mathcal{K},\alpha) \equiv \mathrm{Orep}(\mathcal{K},\alpha')$ implies $\alpha \equiv_\mathcal{K} \alpha'$, and thus $\mathrm{ctr}(\mathcal{K},\alpha) \equiv \mathrm{ctr}(\mathcal{K},\alpha')$ due to *preservation*. It is easy to see that, together with $\mathrm{Orep}(\mathcal{K},\alpha) \equiv \mathrm{Orep}(\mathcal{K},\alpha')$, this implies $\gamma(\mathrm{Orep}(\mathcal{K},\alpha)) \equiv \gamma(\mathrm{Orep}(\mathcal{K},\alpha'))$. ☐

4 Belief Set Contraction as Instance

Contraction operations and in particular partial meet contractions were introduced in the seminal AGM paper (Alchourrón, Gärdenfors, and Makinson 1985) for belief sets, i.e., sets of formulas that are closed under the inference relation of an underlying logic. We show that this can be seen as an instance of the approach introduced in this paper. However, the instance we investigate here is more general than the original AGM setting (Alchourrón, Gärdenfors, and Makinson 1985) since we make less assumptions on the underlying logic.

We assume that we are given a set of formulas $\mathcal{F}$ (without any assumptions on their syntactic form) and a *closure operator* Cl mapping sets of formulas to sets of formulas (which generalizes inference closure w.r.t. some logic). A *belief set* $\mathcal{B}$ is a *closed* subset of $\mathcal{F}$, i.e. $\mathrm{Cl}(\mathcal{B}) = \mathcal{B} \subseteq \mathcal{F}$. The closure operator is assumed to satisfy the following properties (for all $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{F}$):

- $\mathcal{A} \subseteq \mathrm{Cl}(\mathcal{A})$ (inclusion),
- $\mathcal{A} \subseteq \mathcal{A}'$ implies $\mathrm{Cl}(\mathcal{A}) \subseteq \mathrm{Cl}(\mathcal{A}')$ (monotonicity),
- $\mathrm{Cl}(\mathrm{Cl}(\mathcal{A})) = \mathrm{Cl}(\mathcal{A})$ (idempotency),
- $\varphi \in \mathrm{Cl}(\mathcal{A})$ implies that there is a *finite* set $\mathcal{E} \subseteq \mathcal{A}$ such that $\varphi \in \mathrm{Cl}(\mathcal{E})$ (compactness).

Note that the first three properties imply that, for every set of formulas $\mathcal{A}$, its closure $\mathrm{Cl}(\mathcal{A})$ is the least belief set containing $\mathcal{A}$. These are exactly the conditions needed for a closure operator to be compliant with the relevance postulate (Ribeiro et al. 2013).

We use $\mathcal{F}$ and a closure operator Cl satisfying inclusion, monotonicity, idempotency, and compactness to define the following instance of our general framework:

- *Knowledge bases* are belief sets, i.e., subsets of $\mathcal{F}$ that are closed under Cl.
- *Entailment* is the superset relation between belief sets, i.e., $\mathcal{B}_1$ entails $\mathcal{B}_2$ (written $\mathcal{B}_1 \models_\supseteq \mathcal{B}_2$) iff $\mathcal{B}_1 \supseteq \mathcal{B}_2$.
- *Repair requests* are of the form φ for $\varphi \in \mathcal{F}$, and they induce the repair sets $\mathrm{Rep}(\mathcal{B},\alpha) := \{\mathcal{B}' \mid \mathcal{B} \supseteq \mathcal{B}' \text{ and } \varphi \notin \mathcal{B}'\}$.

Note that the consequence operator $\mathrm{Con}^\supseteq$ induced by $\models_\supseteq$ does not coincide with Cl. The operator Cl applies to arbitrary sets of formulas and defines what we consider to be KBs (i.e., sets that are closed under Cl). The operator $\mathrm{Con}^\supseteq$ applies to KBs and yields all KBs that are subsets of its input KB. Since the superset relation is *reflexive* and *transitive*, the entailment relation $\models_\supseteq$ satisfies these two properties required by our framework. The repair operator Rep satisfies the first condition of Definition 4 by definition and the second one since $\varphi \notin \mathcal{B}' \supseteq \mathcal{B}''$ clearly implies $\varphi \notin \mathcal{B}''$.

As sum operation on belief sets we define $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_n := \mathrm{Cl}(\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n)$.

Lemma 12. *The operation $\oplus$ on belief sets satisfies the properties of sum for the entailment relation $\models_\supseteq$.*

Proof. First, note that $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_n$ is a belief set due to idempotency of Cl.

Second, we must show that $\mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_n \models_\supseteq \mathcal{B}_i$ for $i = 1, \ldots, n$, i.e., that $\mathrm{Cl}(\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n) \supseteq \mathcal{B}_i$ holds for $i = 1, \ldots, n$. Since $\mathcal{B}_i \subseteq \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$, monotonicity of Cl and the fact that $\mathcal{B}_i$ is closed w.r.t. Cl yield $\mathcal{B}_i = \mathrm{Cl}(\mathcal{B}_i) \subseteq \mathrm{Cl}(\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n)$, as required.

Third, assume that $\mathcal{B}$ is a belief set that satisfies $\mathcal{B} \models_\supseteq \mathcal{B}_i$ for $i = 1, \ldots, n$. Then $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n \subseteq \mathcal{B}$, and thus $\mathrm{Cl}(\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n) \subseteq \mathrm{Cl}(\mathcal{B}) = \mathcal{B}$ by monotonicity of Cl and the fact that $\mathcal{B}$ is closed w.r.t. Cl. This yields $\mathcal{B} \models_\supseteq \mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_n$, as required. □

As product operation on belief sets we take intersection, i.e., $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n := \mathcal{B}_1 \cap \ldots \cap \mathcal{B}_n$.

Lemma 13. *The operation $\otimes$ on belief sets satisfies the properties of product for the entailment relation $\models_\supseteq$.*

Proof. It is sufficient to shows that the intersection of belief sets is again a belief set since intersection clearly satisfies the properties required for the product w.r.t. $\models_\supseteq$.

By monotonicity of Cl and the fact that the belief sets $\mathcal{B}_i$ are closed, the inclusion $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n = \mathcal{B}_1 \cap \ldots \cap \mathcal{B}_n \subseteq \mathcal{B}_i$ yields $\mathrm{Cl}(\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n) \subseteq \mathrm{Cl}(\mathcal{B}_i) = \mathcal{B}_i$ for $i = 1, \ldots, n$, and thus $\mathrm{Cl}(\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n) \subseteq \mathcal{B}_1 \cap \ldots \cap \mathcal{B}_n = \mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$. The inclusion in the other direction follows from the fact that Cl satisfies inclusion. □

Regarding repairs, given a belief set $\mathcal{B}$ and a repair request φ, we define $\mathrm{Orep}(\mathcal{B}, \varphi)$ to consist of the maximal subsets of $\mathcal{B}$ whose closure does not contain φ.

Lemma 14. *The elements of $\mathrm{Orep}(\mathcal{B}, \varphi)$ are belief sets and optimal repairs of $\mathcal{B}$ for φ.*

Proof. Let $\mathcal{A}$ be an element of $\mathrm{Orep}(\mathcal{B}, \varphi$. Then $\varphi \notin \mathrm{Cl}(\mathcal{A})$, and thus idempotency of Cl yields $\varphi \notin \mathrm{Cl}(\mathrm{Cl}(\mathcal{A}))$. Monotonicity of Cl, $\mathcal{A} \subseteq \mathcal{B}$, and the fact that $\mathcal{B}$ is closed imply that $\mathrm{Cl}(\mathcal{A})$ is a subset of $\mathcal{B}$ whose closure does not contain φ. In addition, inclusion yields $\mathcal{A} \subseteq \mathrm{Cl}(\mathcal{A})$, which in turn implies $\mathcal{A} = \mathrm{Cl}(\mathcal{A})$ due to the assumed maximality of $\mathcal{A}$. Thus, we have shown that the elements of $\mathrm{Orep}(\mathcal{B}, \mathrm{Cl}(\varphi)$ are belief sets.

Every element $\mathcal{A}$ of $\mathrm{Orep}(\mathcal{B}, \varphi)$ is a belief set contained in $\mathcal{B}$ and satisfying $\varphi \notin \mathcal{A}$, which shows that $\mathcal{A}$ is a repair of $\mathcal{B}$ for φ. Optimality of these repairs is an immediate consequence of the fact that they were chosen to be maximal. □

Lemma 15. *The set $\mathrm{Orep}(\mathcal{B}, \varphi)$ covers all repairs of $\mathcal{B}$ for φ. In particular, it contains all optimal repairs.*

Proof. The second statement is an immediate consequence of the first. In fact, assume that $\mathcal{B}'$ is an optimal repair of $\mathcal{B}$ for φ. Then coverage of all repairs by the set $\mathrm{Orep}(\mathcal{B}, \varphi)$ implies that there is an element $\mathcal{B}''$ of $\mathrm{Orep}(\mathcal{B}, \varphi)$ such that $\mathcal{B}'' \supseteq \mathcal{B}'$. Since $\mathcal{B}''$ is a repair, this inclusion cannot be strict since this would contradict the optimality of $\mathcal{B}'$. Consequently, $\mathcal{B}'' = \mathcal{B}'$, and thus $\mathcal{B}' \in \mathrm{Orep}(\mathcal{B}, \varphi)$.

Let $\mathcal{B}'$ be a repair of $\mathcal{B}$ for φ. We use transfinite induction to extend $\mathcal{B}'$ to a maximal subset of $\mathcal{B}$ whose closure does not contain φ. To this purpose, we assume[1] that $\mathcal{B}$ consist of the formulas φ_α where α ranges over the ordinals $\prec \beta$ for an appropriate ordinal β. We define sets $\mathcal{B}_\alpha$ for all ordinals $\alpha \preceq \beta$ as follows:

- $\mathcal{B}_0 := \mathcal{B}'$,
- $\mathcal{B}_{\alpha+1} := \mathcal{B}_\alpha \cup \{\varphi_\alpha\}$ if $\varphi \notin \mathrm{Cl}(\mathcal{B}_\alpha \cup \{\varphi_\alpha\})$,
- $\mathcal{B}_{\alpha+1} := \mathcal{B}_\alpha$ if $\varphi \in \mathrm{Cl}(\mathcal{B}_\alpha \cup \{\varphi_\alpha\})$,
- $\mathcal{B}_\alpha := \bigcup_{\alpha' \prec \alpha} \mathcal{B}_{\alpha'}$ if α is a limit ordinal.

By definition, $\mathcal{B}' \subseteq \mathcal{B}_\beta$. In addition, one can easily prove by transfinite induction that all sets $\mathcal{B}_\alpha$ for $\alpha \preceq \beta$ satisfy $\varphi \notin \mathrm{Cl}(\mathcal{B}_\alpha)$. This proof uses compactness of Cl to deal with the limit case.

It remains to show maximality of $\mathcal{B}_\beta$. Assume to the contrary that there is a strict superset $\mathcal{B}'$ of $\mathcal{B}_\beta$ whose closure does not entail φ, i.e., $\mathcal{B}_\beta \subset \mathcal{B}' \subseteq \mathcal{B}$ and $\varphi \notin \mathrm{Cl}(\mathcal{B}')$. Monotonicity of Cl implies that there is an element ψ in $\mathcal{B} \setminus \mathcal{B}_\beta$ such that $\varphi \notin \mathrm{Cl}(\mathcal{B}_\beta \cup \{\psi\})$. Let $\alpha \prec \beta$ be the ordinal such that $\psi = \varphi_\alpha$. Since $\varphi_\alpha \notin \mathcal{B}_\beta$, it cannot belong to $\mathcal{B}_{\alpha+1} \subseteq \mathcal{B}_\beta$. However, this means that $\varphi \in \mathrm{Cl}(\mathcal{B}_\alpha \cup \{\varphi_\alpha\})$ since otherwise φ_α would have been added to $\mathcal{B}_{\alpha+1}$. Since $\mathcal{B}_\alpha \cup \{\varphi_\alpha\} \subseteq \mathcal{B}_\beta \cup \{\psi\}$, monotonicity of Cl yields $\varphi \in \mathrm{Cl}(\mathcal{B}_\beta \cup \{\psi\})$, which contradicts our assumption that ψ is such that $\varphi \notin \mathrm{Cl}(\mathcal{B}_\beta \cup \{\psi\})$. Consequently, we have shown that $\mathcal{B}_\beta$ is an element of $\mathrm{Orep}(\mathcal{B}, \mathrm{Cl}(\{\varphi\}))$ that covers $\mathcal{B}'$. □

Summing up, we have thus shown that the entailment relation $\models_\supseteq$ on belief sets induced by a closure operator Cl that satisfies the conditions introduced above fulfills all the properties introduced in Section 2.

Theorem 16. *Consider as knowledge bases belief sets that are closed w.r.t. a closure operator* Cl *that satisfies inclusion, monotonicity, idempotency, and compactness, and as repair requests single formulas with associated repair sets of the form* $\mathrm{Rep}(\mathcal{B}, \varphi) := \{\mathcal{B}' \mid \mathcal{B} \supseteq \mathcal{B}' \text{ and } \varphi \notin \mathcal{B}'\}$. *Then the entailment relation $\models_\supseteq$ corresponding to the superset relation between belief sets is partial product contraction enabling.*

As a consequence, we can use the partial product contraction approach introduced in Section 3 to obtain contraction operations for belief sets that satisfy the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance* (and additionally *fullness* in the MaxiChoice case).

Since in this case product is intersection and optimal repairs are obtained as maximal sets that do not have the consequence, the construction of partial product contractions as described in Section 3 coincides with the construction of the partial meet contractions for belief sets introduced in the

[1]This assumption is based on the well ordering theorem, whose validity is equivalent to the axiom of choice (Halmos 1960).

seminal AGM paper. Nevertheless, our postulates do not coincide with the ones given in (Alchourrón, Gärdenfors, and Makinson 1985). In particular, instead of *recovery* we have *relevance* or *fullness*. The reason is that Alchourrón, Gärdenfors, and Makinson make additional assumptions on the formulas and the closure operator. Their proof of *recovery* actually employs the fact that their closure operator corresponds to logical consequence for a logic that has negation and disjunction.

The setting introduced in this subsection does not make any assumptions on the formulas, and only requires the closure operator to satisfy inclusion, monotonicity, idempotency, and compactness. For example, we could use as formulas Horn implications or more generally concepts of the Description Logic $\mathcal{EL}$, and as closure operator logical consequence for Horn formulas or subsumption between $\mathcal{EL}$ concepts. In these setting, recovery does not hold (Delgrande and Wassermann 2013; Zhuang and Pagnucco 2009). Intuitionistic Logic (Heyting 1956) is another example where recovery does not hold (Ribeiro et al. 2013). A detailed study of the postulates recovery and relevance for logics that do not satisfy all the assumptions of the original AGM paper can be found in (Ribeiro et al. 2013)

Considering belief sets as knowledge bases has the disadvantage that, for logics that are more powerful than propositional logic, the optimal repairs, and thus also the belief sets produced by applying the contraction operator, may become infinite without appropriate finite representation, even if one starts with finitely generated belief sets. A practical example where this problem occurs are ABoxes of the description logic (DL) $\mathcal{EL}$ (Baader et al. 2017) as KBs and inference w.r.t. an $\mathcal{EL}$ TBox as entailment relation. Note that ABoxes are assumed to be finite in the DL community. As shown in the proof of Proposition 2 in (Baader et al. 2018), in this setting there are repair problems that have repairs, but no optimal repairs. Basically, the reason is that one would need an infinite ABox to represent such an optimal repair. The approach for belief set contraction introduced in the present section applies to the setting of $\mathcal{EL}$ ABoxes w.r.t. an $\mathcal{EL}$ TBox if one allows ABoxes to be infinite. However, the obtained contraction operation may then return infinite ABoxes, which makes this approach useless in practice unless one finds an appropriate finite representation for the infinite ABoxes.[2] In the next section, we consider instances of our general setup where knowledge bases are finite.

5 Instances with Finite KBs

As practically relevant instances of the general setup for which KBs are finite, we consider KBs and entailment relations connected with the DL $\mathcal{EL}$ (Baader et al. 2017). As a consequence of the results shown in Section 3, we can then use the partial product contraction approach to obtain contraction operations for these instances satisfying the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance* (and additionally *fullness* in the MaxiChoice case).

[2]First steps in this direction are described in Section 5 of (Baader, Koopmann, and Kriegel 2023).

In this setting, when showing that a set of KBs, repair requests, and an entailment relation satisfy the properties required for the entailment relation to be partial product contraction enabling, the most challenging task will be to prove that the properties related to optimal repairs are satisfied. Fortunately, in most of the cases considered below, this task has already been solved by recent work on optimal repairs in $\mathcal{EL}$. Nevertheless, the overall task of showing that the considered entailment relations are partial product contraction enabling remains non-trivial since we must prove the existence of appropriate product and sum operations.

In most of the cases, we consider a standard repair setting where the repair request is a KB that is supposed to be no longer entailed, i.e., $\mathrm{Rep}(\mathcal{K},\alpha) = \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}', \mathcal{K}' \not\models \alpha\}$. It is easy to see that in this case the conditions of Definition 4 are satisfied. In addition, we will consider a modified version of the standard repair setting where non-entailment of the repair request is demanded for a different entailment relation, i.e., $\mathrm{Rep}(\mathcal{K},\alpha) = \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}', \mathcal{K}' \not\models_r \alpha\}$. If $\models_r$ is transitive and a stronger entailment relation than $\models$ (i.e., $\models \subseteq \models_r$), then the conditions of Definition 4 are still satisfied in this extended setting.

In case KBs are finite sets of formulas and repair requests α are not assumed to be just singleton sets, there are actually (at least) two possibilities for how to define repairs, corresponding to choice and package contraction in the belief change literature (Fuhrmann and Hansson 1994; Fermé, Saez, and Sanz 2003; Resina, Ribeiro, and Wassermann 2014). What we have defined above corresponds to choice contraction since $\mathcal{K}' \not\models \alpha$ means that at least one of the elements of α should not be entailed. For package contraction, none of the elements of α is allowed to be entailed, i.e., repairs are defined as $\mathrm{Rep}(\mathcal{K},\alpha) = \{\mathcal{K}' \mid \mathcal{K} \models \mathcal{K}', \mathcal{K}' \not\models \varphi \text{ for all } \varphi \in \alpha\}$. We will call these two forms of repairs choice and package repairs, respectively. Package repair is actually the notion of repair employed in our previous work on optimal repairs (Baader et al. 2021a). Thus, at first sight, one might think that these results show the optimal repair property required by our framework only for the package setting. However, it is easy to see that satisfaction of this property in the package setting implies that it is also satisfied in the choice setting since we consider finite KBs (Fuhrmann and Hansson 1994).

In addition to such standard repair settings, we will also introduce an instance that is akin to variable forgetting (see Example 5), but considered in the context of concepts of the DL $\mathcal{EL}$, where concept and role names may be forgotten. Finally, we introduce an instances of the general setup that has nothing to do with logic, but considers automata or grammars as KBs, and uses language inclusion to define entailment. The main reason for introducing this instance is to demonstrate the generality of our approach. To show the partial product contraction enabling property in this setting, one can use results on the closure properties for the language classes of the Chomsky hierarchy (Chomsky 1959).

5.1 $\mathcal{EL}$ Concept Contraction

In this setting, knowledge bases and repair requests are $\mathcal{EL}$ concepts and entailment is subsumption w.r.t. an $\mathcal{EL}$

TBox (Baader et al. 2017).

$\mathcal{EL}$ concepts are built inductively, starting with concept names A from a set N_C of such names, and using the concept constructors $\top$ (top concept), $C \sqcap D$ (conjunction), and $\exists r.C$ (existential restriction), where C, D are $\mathcal{EL}$ concepts and r belongs to a set N_R of role names. A general concept inclusion (GCI) of $\mathcal{EL}$ is of the form $C \sqsubseteq D$ for $\mathcal{EL}$ concepts C, D, and an $\mathcal{EL}$ TBox is a finite set of such GCIs.

The semantics of $\mathcal{EL}$ is defined in a model-theoretic way, using the notion of an *interpretation* $\mathcal{I}$, which is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the domain $\Delta^{\mathcal{I}}$ is a non-empty set and the interpretation function $\cdot^{\mathcal{I}}$ maps each concept name $A \in N_C$ to $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation of an $\mathcal{EL}$ concept is defined inductively as follows: $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$, and $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}$ such that $(d,e) \in r^{\mathcal{I}}$ and $e \in C^{\mathcal{I}}\}$. A model $\mathcal{I}$ of the $\mathcal{EL}$ TBox $\mathcal{T}$ is an interpretation that satisfies all its GCIs, i.e., $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all $C \sqsubseteq D \in \mathcal{T}$. Given $\mathcal{EL}$ concepts C, D and an $\mathcal{EL}$ TBox $\mathcal{T}$, we say that C is *subsumed by* D *w.r.t.* $\mathcal{T}$ (and write $C \sqsubseteq^{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in all models $\mathcal{I}$ of $\mathcal{T}$. The $\mathcal{EL}$ concepts C, D are equivalent (written $C \equiv^{\mathcal{T}} D$) if $C \sqsubseteq^{\mathcal{T}} D$ and $D \sqsubseteq^{\mathcal{T}} C$.

For a given $\mathcal{EL}$ TBox $\mathcal{T}$, we obtain the following instance of our general framework:

- *Knowledge bases* are $\mathcal{EL}$ concepts.
- *Entailment* is given by the subsumption relation w.r.t. $\mathcal{T}$, i.e., C entails D (written $C \models_{\sqsubseteq^{\mathcal{T}}} D$) iff $C \sqsubseteq^{\mathcal{T}} D$.
- *Repair requests* are $\mathcal{EL}$ concepts, and repairs are defined as $\mathrm{Rep}^{\mathcal{T}}(C, D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$.

This instance has first been considered in (Rienstra, Schon, and Staab 2020) for subsumption w.r.t. the empty TBox ($\sqsubseteq^{\emptyset}$) and was then extended to subsumption w.r.t. a cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$ ($\sqsubseteq^{\mathcal{T}}$) in (Baader 2023).

It is easy to see that the sum operation for the entailment relation $\models_{\sqsubseteq^{\mathcal{T}}}$ is conjunction of concepts, and the product is the least common subsumer (lcs) w.r.t. the TBox $\mathcal{T}$:

- the $\mathcal{EL}$ concept C is a *least common subsumer* of the $\mathcal{EL}$ concepts $C_1, \ldots, C_n$ w.r.t. $\mathcal{T}$ if $C_i \sqsubseteq^{\mathcal{T}} C$ for all $i = 1, \ldots, n$, and C is the least $\mathcal{EL}$ concept (for $\sqsubseteq^{\mathcal{T}}$) with this property, i.e., if D is an $\mathcal{EL}$ concept satisfying $C_i \sqsubseteq^{\mathcal{T}} D$ for all $i = 1, \ldots, n$, then $C \sqsubseteq^{\mathcal{T}} D$.

Obviously, if it exists, then such an lcs is unique up to equivalence $\equiv^{\mathcal{T}}$. For the case of the empty TBox, the lcs in $\mathcal{EL}$ always exists (Baader, Küsters, and Molitor 1999), but this is not the case w.r.t. an arbitrary $\mathcal{EL}$ TBox. The characterization of the existence of the lcs w.r.t. an $\mathcal{EL}$ TBox given in (Zarrieß and Turhan 2013) implies that the lcs always exists for cycle-restricted TBoxes:

- The $\mathcal{EL}$ TBox $\mathcal{T}$ is *cycle-restricted* if there is no $\mathcal{EL}$ concept C and $m \geq 1$ (not necessarily distinct) role names $r_1, \ldots, r_m$ such that $C \sqsubseteq^{\mathcal{T}} \exists r_1. \cdots \exists r_m.C$.

As stated in (Baader, Borgwardt, and Morawska 2012), it can be decided in polynomial time whether a given $\mathcal{EL}$ TBox is cycle-restricted or not.

Cycle-restrictedness is also required to obtain the necessary repair properties. As explained in more detail in (Baader 2023), satisfaction of these properties is an easy consequence of the results on optimal ABox repairs shown in (Baader et al. 2022).

Theorem 17. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and $\models_{\sqsubseteq^{\mathcal{T}}}$ subsumption w.r.t. $\mathcal{T}$ between $\mathcal{EL}$ concepts, and consider $\mathcal{EL}$ concepts repair requests as inducing repair sets defined as* $\mathrm{Rep}^{\mathcal{T}}(C, D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$. *Then $\models_{\sqsubseteq^{\mathcal{T}}}$ is partial product contraction enabling.*

The following example, which is due to (Rienstra, Schon, and Staab 2020), demonstrates that in this setting no contraction operation that satisfies *success* and *logical inclusion* can also satisfy the *recovery* postulate.

Example 18. *Let $\mathcal{T} = \emptyset$ and $C = \exists r.(A \sqcap B)$, and consider the repair request $D = \exists r.A$. Thus, any contraction operation satisfying* success *and* logical inclusion *must return an $\mathcal{EL}$ concepts C' such that $C \sqsubseteq^{\emptyset} C'$ and $C' \not\sqsubseteq^{\emptyset} D$. It is easy to see that the only $\mathcal{EL}$ concepts satisfying these two (non-)subsumption relationships are $\exists r.B$, $\exists r.\top$, and the top concept $\top$. Conjoining $\exists r.A$ with any of these concepts does no yield a concept that is subsumed by C, which implies that* recovery *does not hold.*

5.2 Contractions for Quantified ABoxes: Classical Entailment

ABoxes of $\mathcal{EL}$ are finite sets of concept assertions $C(a)$ and role assertions $r(a, b)$, where C is an $\mathcal{EL}$ concept, r a role name, and a, b are individuals from a set N_I. In the presence of an ABox, an interpretation $\mathcal{I}$ additionally interprets individuals a as elements $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is a model of the $\mathcal{EL}$ ABox $\mathcal{A}$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ respectively holds for all concept and role assertions $C(a)$ and $r(a, b)$ in $\mathcal{A}$.

Quantified ABoxes were first introduced in (Baader et al. 2020) since they allow for the existence of optimal repairs in situations where this would not be the case if only ABoxes were used. Basically, they are variants of ABoxes where some of the individual names are assumed to be anonymous, which we express by writing them as existentially quantified variables. More formally, a *quantified ABox (qABox)* $\exists X.\mathcal{A}$ consists of a finite set X of variables, which is disjoint with N_I, and a matrix $\mathcal{A}$, which is a finite set of concept assertions $A(u)$ and role assertions $r(u, v)$, where $A \in N_C$, $r \in N_R$ and $u, v \in N_I \cup X$. Thus, the matrix is an ABox built using the extended set of individuals $N_I \cup X$, but cannot contain complex concept descriptions. Semantically, the latter is not a restriction since it is easy to see that a concept assertions $C(a)$ for a complex $\mathcal{EL}$ concept C ican be expressed by a qABox.

The interpretation $\mathcal{I}$ is a model of the qABox $\exists X.\mathcal{A}$ if there is a variable assignment $\mathcal{Z} : X \to \Delta^{\mathcal{I}}$ such that the augmented interpretation $\mathcal{I}[\mathcal{Z}]$ that additionally maps each variable x to $\mathcal{Z}(x)$ is a model of the matrix $\mathcal{A}$, i.e, $u^{\mathcal{I}[\mathcal{Z}]} \in A^{\mathcal{I}}$ for each $A(u) \in \mathcal{A}$ and $(u^{\mathcal{I}[\mathcal{Z}]}, v^{\mathcal{I},\mathcal{Z}}) \in r^{\mathcal{I}}$ for each $r(u, v) \in \mathcal{A}$. The qABox $\exists X.\mathcal{A}$ entails the qABox $\exists Y.\mathcal{B}$ w.r.t. the $\mathcal{EL}$ TBox $\mathcal{T}$ (written $\exists X.\mathcal{A} \models^{\mathcal{T}} \exists Y.\mathcal{B}$) if every model of $\exists X.\mathcal{A}$ and $\mathcal{T}$ is also a model of $\exists Y.\mathcal{B}$. Note that this also defines entailment of a concept assertion $C(a)$ by a qABox w.r.t. an $\mathcal{EL}$ TBox since $C(a)$ can be expressed by a qABox. For the empty TBox, we write the entailment relation as $\models$ rather than $\models^{\emptyset}$.

The entailment relation $\models$ between qABoxes can be characterized using the notion of a homomorphism. Given qABoxes $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$, a *homomorphism* from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$ is a mapping h from the objects (i.e., variables or individuals) of $\mathcal{A}$ to the objects of $\mathcal{B}$ such that

- $h(a) = a$ for all individuals a,
- $A(u) \in \mathcal{A}$ implies $A(h(u)) \in \mathcal{B}$ for all objects u and concept names A,
- $r(u, v) \in \mathcal{A}$ implies $r(h(u), h(v)) \in \mathcal{B}$ for all objects u, v and role names r.

The following characterization of entailment was shown in (Baader et al. 2020): $\exists Y.\mathcal{B} \models \exists X.\mathcal{A}$ iff there is a homomorphism from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$.[3] This characterization also works in the setting with a background TBox $\mathcal{T}$ if one first saturates the qABox $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$. However, a finite saturation only exists if the TBox is *cycle-restricted*. Given a qABox $\exists Y.\mathcal{B}$ and a cycle-restricted TBox $\mathcal{T}$, one can compute the saturation $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B})$ of $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$ in exponential time, and this saturation satisfies $\exists Y.\mathcal{B} \models^{\mathcal{T}} \exists X.\mathcal{A}$ iff $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B}) \models \exists X.\mathcal{A}$ for each qABox $\exists X.\mathcal{A}$ (Baader et al. 2021a). Thus, we have the following characterization of entailment w.r.t. a cycle-restricted TBox.

Lemma 19. *Let $\exists X.\mathcal{A}, \exists Y.\mathcal{B}$ be qABoxes, and $\mathcal{T}$ a cycle-restricted $\mathcal{EL}$ TBox. Then the following are equivalent:*

- $\exists Y.\mathcal{B} \models^{\mathcal{T}} \exists X.\mathcal{A}$,
- $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B}) \models \exists X.\mathcal{A}$,
- *there is a homomorphism from $\exists X.\mathcal{A}$ to $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B})$.*

The saturation of a qABox is of at most exponential size, and there are examples showing that this size-bound is tight (see Example III in (Baader et al. 2021b)). Nevertheless, as pointed out in (Baader et al. 2021a), deciding the entailment relation $\models^{\mathcal{T}}$ is an NP-complete problem (where hardness already holds without TBox).

In the following, we use qABoxes as KBs, $\models^{\mathcal{T}}$ for a cycle-restricted TBox $\mathcal{T}$ as entailment, and finite sets of $\mathcal{EL}$ concept assertions as repair requests. Repairs are defined as package repairs, i.e., $\mathrm{Rep}^{\mathcal{T}}(\exists X.\mathcal{A}, \alpha) := \{\exists Y.\mathcal{B} \mid \exists X.\mathcal{A} \models^{\mathcal{T}} \exists Y.\mathcal{B}, \exists Y.\mathcal{B} \not\models^{\mathcal{T}} C(a) \text{ for all } C(a) \in \alpha\}$. We show that this yields an entailment relation such that all the properties introduced in Section 2 are satisfied, i.e., we show that $\models^{\mathcal{T}}$ is then partial product contraction enabling.

Reflexivity and transitivity of $\models^{\mathcal{T}}$ are obvious. Next, we introduce an appropriate sum operation. For a singleton set $\mathfrak{K} = \{\exists X.\mathcal{A}\}$, its sum is simply $\exists X.\mathcal{A}$ itself. Given a set of $n \geq 2$ qABoxes $\mathfrak{K} = \{\exists X_1.\mathcal{A}_1, \ldots, \exists X_n.\mathcal{A}_n\}$, we construct its disjoint union as follows: we first rename the qABoxes in $\mathfrak{K}$ into equivalent ones $\exists X'_1.\mathcal{A}'_1, \ldots, \exists X'_n.\mathcal{A}'_n$ with pairwise disjoint sets of variables $X'_1, \ldots, X'_n$, and then set $\uplus\mathfrak{K} := \exists(X'_1 \cup \ldots \cup X'_n).(\mathcal{A}'_1 \cup \ldots \cup \mathcal{A}'_n)$.

[3]Note that checking for the existence of homomorphism between qABoxes is an NP-complete problem (Baader et al. 2020).

Lemma 20. *Disjoint union $\uplus$ of qABoxes satisfies the properties of sum for $\models^{\mathcal{T}}$.*

Proof. First, we must show that $\uplus\mathfrak{K} \models^{\mathcal{T}} \exists X_i.\mathcal{A}_i$ for $i = 1, \ldots, n$. In fact, this entailment even holds without TBox since we can define a homomorphism from $\exists X_i.\mathcal{A}_i$ to $\uplus\mathfrak{K}$ by mapping individuals to individuals and the variables in X_i to their renamings in X'_i.

Second, assume that $\exists Y.\mathcal{B}$ satisfies $\exists Y.\mathcal{B} \models^{\mathcal{T}} \exists X_i.\mathcal{A}_i$ for $i = 1, \ldots, n$. By Lemma 19, this implies that there are homomorphisms h_i (for $i = 1, \ldots, n$) from $\exists X_i.\mathcal{A}_i$ to $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B})$. These homomorphisms can be turned into a single homomorphism h from $\uplus\mathfrak{K}$ to $\mathrm{sat}^{\mathcal{T}}(\exists Y.\mathcal{B})$ by mapping $x' \in X'_i$ to $h_i(x)$ where x' is the renaming of $x \in X_i$, and of course a to a for all individuals a. □

The product of a set of qABoxes $\mathfrak{K} = \{\exists X_1.\mathcal{A}_1, \ldots, \exists X_n.\mathcal{A}_n\}$ is $\exists X_1.\mathcal{A}_1$ if $n = 1$. For $n \geq 2$, we consider the saturations $\exists Y_1.\mathcal{B}_1 := \mathrm{sat}^{\mathcal{T}}(\exists X_1.\mathcal{A}_1), \ldots, \exists Y_n.\mathcal{B}_n := \mathrm{sat}^{\mathcal{T}}(\exists X_n.\mathcal{A}_n)$ of $\exists X_1.\mathcal{A}_1, \ldots, \exists X_n.\mathcal{A}_n$. Let Ind be the set of individuals occurring in at least one of the ABoxes $\mathcal{B}_1, \ldots, \mathcal{B}_n$ and $\mathrm{Obj}_i := Y_i \cup \mathrm{Ind}$ for $i = 1, \ldots, n$. We set $\mathrm{Ind}^{\times} := \{(a, \ldots, a) \mid a \in \mathrm{Ind}\}$ and $Y := \mathrm{Obj}_1 \times \ldots \times \mathrm{Obj}_n \setminus \mathrm{Ind}^{\times}$, and define $\otimes\mathfrak{K} := \exists Y.\mathcal{B}$ where

$$\begin{aligned}\mathcal{B} := \{&A(u_1, \ldots, u_n) \mid A(u_i) \in \mathcal{B}_i && \text{for } i = 1, \ldots, n\} \cup \\ \{&r((u_1, \ldots, u_n), (v_1, \ldots, v_n)) \mid && r(u_i, v_i) \in \mathcal{B}_i \\ & && \text{for } i = 1, \ldots, n\}.\end{aligned}$$

In this qABox, each tuple $(a, \ldots, a) \in \mathrm{Ind}^{\times}$ is viewed as representing the individual $a \in \mathrm{Ind}$.

Lemma 21. *The product $\otimes$ of qABoxes satisfies the properties of product for $\models^{\mathcal{T}}$.*

Proof. First, we must show that $\exists X_i.\mathcal{A}_i \models^{\mathcal{T}} \otimes\mathfrak{K}$ for $i = 1, \ldots, n$. By Lemma 19, it is sufficient to show that, for all $i = 1, \ldots, n$, there is a homomorphisms from $\otimes\mathfrak{K}$ to $\exists Y_i.\mathcal{B}_i$. Obviously, the projection to the i-th component yields such a homomorphism.

Second, assume that $\exists Z.\mathcal{C}$ satisfies $\exists X_i.\mathcal{A}_i \models^{\mathcal{T}} \exists Z.\mathcal{C}$ for $i = 1, \ldots, n$, which means that there are homomorphisms h_i from $\exists Z.\mathcal{C}$ to $\exists Y_i.\mathcal{B}_i$ for $i = 1, \ldots, n$. These homomorphisms can be turned into a single homomorphism h from $\exists Z.\mathcal{C}$ to $\otimes\mathfrak{K}$ by mapping each object u of $\exists Z.\mathcal{C}$ to $(h_1(u), \ldots, h_n(u))$. It is easy to see that the function h defined this way really yields a homomorphism. □

Repairs of qABoxes w.r.t. cycle-restricted TBoxes for repair requests given as finite sets of $\mathcal{EL}$ concept assertions have been investigated in (Baader et al. 2021a). It is shown there that, up to equivalence, the set of all optimal repairs of a qABox for a repair request w.r.t. a cycle-restricted TBox can be computed in exponential time using an NP oracle (Theorem 9 in (Baader et al. 2021a)). To be more precise, the paper introduces the notion of canonical repairs induced by repair seed functions. There are at most exponentially many such canonical repairs, each of which is of at most exponential size. These canonical repairs are indeed repairs, and the set of canonical repairs covers all repairs (Proposition 8 in (Baader et al. 2021a)). As a consequence, up to equivalence, this set contains all optimal repairs, which can be obtained by removing elements that are strictly entailed by another element.[4] The coverage property for the obtained set of optimal repairs $\mathrm{Orep}^{\mathcal{T}}(\exists X.\mathcal{A}, \alpha)$ is then an easy consequence of the coverage property for the set of canonical repairs. Summing up, we have thus shown that $\models^{\mathcal{T}}$ for a cycle-restricted TBox $\mathcal{T}$ as entailment satisfies all the properties introduced in Section 2.

Theorem 22. *Let $\mathcal{T}$ be a cycle-restricted TBox and $\models^{\mathcal{T}}$ entailment w.r.t. $\mathcal{T}$ between qABoxes, and consider as repair requests finite sets of $\mathcal{EL}$ concept assertions inducing repair sets according to the package approach. Then $\models^{\mathcal{T}}$ is partial product contraction enabling.*

The same result holds if we use the choice approach for defining repairs, i.e., if we define the induced repair sets as $\mathrm{Rep}_c^{\mathcal{T}}(\exists X.\mathcal{A}, \alpha) := \{\exists Y.\mathcal{B} \mid \exists X.\mathcal{A} \models^{\mathcal{T}} \exists Y.\mathcal{B}, \exists Y.\mathcal{B} \not\models^{\mathcal{T}} C(a) \text{ for some } C(a) \in \alpha\}$. To show this we must demonstrate that the optimal repair property is satisfied in this setting. Given a qABox $\exists X.\mathcal{A}$ and a repair request $\alpha = \{C_1(a_1), \ldots, C_n(a_n)\}$, we consider the union of the sets $\mathrm{Orep}^{\mathcal{T}}(\exists X.\mathcal{A}, \{C_i(a_i)\})$. It is easy to see that this set covers $\mathrm{Rep}_c^{\mathcal{T}}(\exists X.\mathcal{A}, \alpha)$. Thus, the set of all optimal repairs in the choice setting is obtained by removing elements that are strictly entailed by another elements.

Corollary 23. *Let $\mathcal{T}$ be a cycle-restricted TBox and $\models^{\mathcal{T}}$ entailment w.r.t. $\mathcal{T}$ between qABoxes, and consider as repair requests finite sets of $\mathcal{EL}$ concept assertions inducing repair sets according to the choice approach. Then $\models^{\mathcal{T}}$ is partial product contraction enabling.*

As a consequence, in both the package and the choice setting, we can use the partial product contraction approach to obtain contraction operations for qABoxes w.r.t. cycle-restricted TBoxes that satisfy the postulates *logical inclusion*, *success*, *failure*, *vacuity*, *preservation*, and *relevance* (and additionally *fullness* in the MaxiChoice case).

5.3 Contractions for Quantified ABoxes: IQ-Entailment

If one is only interested in answering instance queries (i.e., checking which concept assertions a qABox entails), then it makes sense to compare qABoxes w.r.t. the instance relationships they entail rather than w.r.t. the models they have or (equivalently) w.r.t. the conjunctive queries they entail (as classical entailment does) (Baader, Koopmann, and Kriegel 2023). We say that $\exists X.\mathcal{A}$ IQ-*entails* $\exists Y.\mathcal{B}$ w.r.t. the TBox $\mathcal{T}$ (written $\exists X.\mathcal{A} \models_{\mathsf{IQ}}^{\mathcal{T}} \exists Y.\mathcal{B}$) if every concept assertion entailed by $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$ is also entailed by $\exists X.\mathcal{A}$ w.r.t. $\mathcal{T}$. Two qABoxes are called IQ-*equivalent* w.r.t. $\mathcal{T}$ if they IQ-entail each other w.r.t. $\mathcal{T}$, which is the case iff they entail the same concept assertions w.r.t. $\mathcal{T}$.

Using IQ-entailment rather than classical entailment has several practical advantages. First, IQ-entailment between qABoxes can be characterized using the notion of a simulation, which has the advantage that the existence of a simulation can be decided in polynomial time (Henzinger, Henzinger, and Kopke 1995). Given qABoxes $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$, a *simulation* from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$ is a binary relation $\mathfrak{S}$ between the objects of $\mathcal{A}$ and the objects of $\mathcal{B}$ such that

- $(a, a) \in \mathfrak{S}$ for all individuals a,
- $A(u) \in \mathcal{A}$ and $(u, u') \in \mathfrak{S}$ implies $A(u') \in \mathcal{B}$ for all objects u, u' and concept names A,
- $r(u, v) \in \mathcal{A}$ and $(u, u') \in \mathfrak{S}$ implies the existence of an object v' with $r(u', v') \in \mathcal{B}$ and $(v, v') \in \mathfrak{S}$ for all objects u, v, u' and role names r.

As shown in (Baader et al. 2020), IQ-entailment without TBox can be characterized as follows: $\exists Y.\mathcal{B} \models_{\mathsf{IQ}} \exists X.\mathcal{A}$ iff there is a simulation from $\exists X.\mathcal{A}$ to $\exists Y.\mathcal{B}$. Again, this characterization also works in the setting with a background TBox $\mathcal{T}$ if one first IQ-saturates the qABox $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$. However, in the IQ case, a finite saturation (of polynomial size) exists for all TBoxes. Given a qABox $\exists Y.\mathcal{B}$ and an arbitrary $\mathcal{EL}$ TBox $\mathcal{T}$, one can compute the IQ-saturation $\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B})$ of $\exists Y.\mathcal{B}$ w.r.t. $\mathcal{T}$ in polynomial time, and this saturation satisfies $\exists Y.\mathcal{B} \models_{\mathsf{IQ}}^{\mathcal{T}} \exists X.\mathcal{A}$ iff $\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B}) \models_{\mathsf{IQ}} \exists X.\mathcal{A}$ for each qABox $\exists X.\mathcal{A}$ (Baader et al. 2021a). Thus, we have the following characterization of IQ-entailment w.r.t. an $\mathcal{EL}$ TBox.

Lemma 24. *Let $\exists X.\mathcal{A}$ and $\exists Y.\mathcal{B}$ be qABoxes, and $\mathcal{T}$ an $\mathcal{EL}$ TBox. Then the following are equivalent:*

- *$\exists Y.\mathcal{B} \models_{\mathsf{IQ}}^{\mathcal{T}} \exists X.\mathcal{A}$,*
- *$\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B}) \models_{\mathsf{IQ}} \exists X.\mathcal{A}$,*
- *there is a simulation from $\exists X.\mathcal{A}$ to $\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B})$.*

Since the IQ-saturation of a given qABox is of polynomial size and the existence of a simulation can be decided in polynomial time, this shows that IQ-entailment w.r.t. an $\mathcal{EL}$ TBox can be checked in polynomial time.

We now show that, just as classical entailment $\models^{\mathcal{T}}$, IQ-entailment $\models_{\mathsf{IQ}}^{\mathcal{T}}$ is partial product contraction enabling. However, unlike the case of classical entailment, the TBox $\mathcal{T}$ need not be required to be cycle-restricted. Reflexivity and transitivity of $\models_{\mathsf{IQ}}^{\mathcal{T}}$ are again obvious. For the product, we can use the same construction as in Section 5.2.

Lemma 25. *The product $\otimes$ of qABoxes satisfies the properties of product for $\models_{\mathsf{IQ}}^{\mathcal{T}}$.*

Proof. First, recall that we have already shown the entailments $\exists X_i.\mathcal{A}_i \models^{\mathcal{T}} \otimes\mathfrak{K}$ for $i = 1, \ldots, n$. This obviously implies $\exists X_i.\mathcal{A}_i \models_{\mathsf{IQ}}^{\mathcal{T}} \otimes\mathfrak{K}$ for $i = 1, \ldots, n$.

Second, assume that $\exists Z.\mathcal{C}$ satisfies $\exists X_i.\mathcal{A}_i \models^{\mathcal{T}} \exists Z.\mathcal{C}$ for $i = 1, \ldots, n$, which means that there are simulations $\mathfrak{S}_i$ from $\exists Z.\mathcal{C}$ to $\exists Y_i.\mathcal{B}_i := \mathrm{sat}^{\mathcal{T}}(\exists X_n.\mathcal{A}_n)$ for $i = 1, \ldots, n$. These simulations can be turned into a single simulation $\mathfrak{S}$ from $\exists Z.\mathcal{C}$ to $\otimes\mathfrak{K}$ by setting

$$\mathfrak{S} := \{(u, (v_1, \ldots, v_n) \mid (u, v_i) \in \mathfrak{S}_i \text{ for } i = 1, \ldots, n\}.$$

It is easy to see that the relation $\mathfrak{S}$ defined this way yields a simulation. $\square$

For the sum, we cannot simply take the disjoint union as defined in Section 5.2. This is illustrated by the next example, where we assume that the TBox is empty.

[4] The NP oracle is used to realize these entailment tests.

Example 26. *Consider the qABoxes $\exists X_1.\mathcal{A}_1 := \exists\emptyset.\{r(a,a)\}$, $\exists X_2.\mathcal{A}_2 := \exists\emptyset.\{A(a)\}$, and $\exists Y.\mathcal{B} := \exists\{y\}.\{A(a), r(a,y), r(y,y)\}$. Then $\exists Y.\mathcal{B} \models_{\mathsf{IQ}} \exists X_i.\mathcal{A}_i$ for $i = 1,2$, which can be certified by the simulations $\mathfrak{S}_1 := \{(a,a),(a,y)\}$ and $\mathfrak{S}_2 := \{(a,a)\}$. The disjoint union of $\exists X_1.\mathcal{A}_1$ and $\exists X_2.\mathcal{A}_2$, as defined in Section 5.2, is $\exists\emptyset.\{r(a,a), A(a)\}$. However, $\exists Y.\mathcal{B}$ does not IQ-entail this qABox. In fact, $\exists\emptyset.\{r(a,a), A(a)\}$ entails the concept assertion $(A \sqcap \exists r.A)(a)$, whereas $\exists Y.\mathcal{B}$ does not.*

In order to overcome this problem, we first observe that any qABox is IQ-equivalent to one that does not contain role assertions of the form $r(u,a)$ for an individual a.

Lemma 27. *Let $\exists X.\mathcal{A}$ be a qABox. Then there exists a qABox $\exists Y.\mathcal{B}$ that is IQ-equivalent to $\exists X.\mathcal{A}$ such that $r(u,v) \in \mathcal{B}$ implies $v \in Y$. This qABox can be computed from $\exists X.\mathcal{A}$ in polynomial time.*

Proof. We define the qABox $\exists Y.\mathcal{B}$ as follows. The quantifier-prefix Y consists of copies y_u of all objects (variables and individuals) of $\exists X.\mathcal{A}$. The matrix $\mathcal{B}$ consists of the following assertions:

$$\begin{aligned}\mathcal{B} := \{&A(y_u) \mid A(u) \in \mathcal{A}\} \cup \{r(y_u, y_v)) \mid r(u,v) \in \mathcal{A}\} \cup \\ \{&A(a) \mid A(a) \in \mathcal{A} \text{ where } a \text{ is an individual}\} \cup \\ \{&r(a, y_v)) \mid r(a,v) \in \mathcal{A} \text{ where } a \text{ is an individual}\}.\end{aligned}$$

By definition of $\mathcal{B}$, $r(u,v) \in \mathcal{B}$ implies $v \in Y$. To show that $\exists X.\mathcal{A} \models_{\mathsf{IQ}} \exists Y.\mathcal{B}$, we define the following simulation from $\exists Y.\mathcal{B}$ to $\exists X.\mathcal{A}$:

$$\begin{aligned}\mathfrak{S} \;\; := \;\; &\{(a,a) \mid \text{where } a \text{ is an individual}\} \cup \\ &\{(y_u, u) \mid \text{where } u \text{ is any object of } \exists X.\mathcal{A}\}.\end{aligned}$$

To see that $\mathfrak{S}$ is indeed a simulation, first note that $(a,a) \in \mathfrak{S}$ for all individuals a holds by the definition of $\mathfrak{S}$.

Second, assume that $A(v) \in \mathcal{B}$ and $(v,\hat{v}) \in \mathfrak{S}$. If $v = a$ is an individual, then $\hat{v} = a$ and $A(a) \in \mathcal{B}$ can only be the case if $A(a) \in \mathcal{A}$. If $v = y_u$ for an object u of $\exists X.\mathcal{A}$, then $\hat{v} = u$ and $A(y_u) \in \mathcal{B}$ can only be the case if $A(u) \in \mathcal{A}$. Thus, we have shown that in both cases $A(\hat{v}) \in \mathcal{A}$ holds.

Third, assume that $r(v,w) \in \mathcal{B}$ and $(v,\hat{v}) \in \mathfrak{S}$. If $v = a$ is an individual, then $\hat{v} = a$ and $r(a,w) \in \mathcal{B}$ implies that $w = y_u$ for an object u with $r(a,u) \in \mathcal{A}$. Since $(y_u, u) \in \mathfrak{S}$, this finishes the proof that $\mathfrak{S}$ satisfies the required property for the case $v = a$. If v is not an individual, then $v = y_u$ for an object u of $\exists X.\mathcal{A}$, and $\hat{v} = u$. In addition, $r(v,w) \in \mathcal{B}$ implies that $w = y_z$ for an object z of $\exists X.\mathcal{A}$. Thus, $r(u,z) \in \mathcal{A}$ and $(w,z) \in \mathfrak{S}$, which finishes the proof that $\mathfrak{S}$ is a simulation.

The IQ-entailment in the other direction can be shown by proving that the inverse $\mathfrak{S}^{-1}$ of $\mathfrak{S}$ is also a simulation. □

In our example, $\exists X_2.\mathcal{A}_2$ already satisfies the restriction that role assertions must not have an individual in the second position, but $\exists X_1.\mathcal{A}_1$ does not. The qABox $\exists Y_1.\mathcal{B}_1 := \exists\{y_a\}.\{r(a,y_a), r(y_a,y_a)\}$ is IQ-equivalent to $\exists X_1.\mathcal{A}_1$ and satisfies this restriction.

Lemma 28. *Let $\exists Y.\mathcal{B}$ be a qABox such that $r(u,v) \in \mathcal{B}$ implies $v \in Y$. If $\exists Z.\mathcal{C} \models_{\mathsf{IQ}} \exists Y.\mathcal{B}$, then there is a simulation $\mathfrak{S}$ from $\exists Y.\mathcal{B}$ to $\exists Z.\mathcal{C}$ such that $(a,u) \in \mathfrak{S}$ implies $u = a$ for all individuals a.*

Proof. If $\exists Z.\mathcal{C} \models_{\mathsf{IQ}} \exists Y.\mathcal{B}$, then there is a simulation $\mathfrak{S}$ from $\exists Y.\mathcal{B}$ to $\exists Z.\mathcal{C}$. To ensure the additional condition required by the lemma, we modify $\mathfrak{S}$ to $\mathfrak{S}'$ by removing all pair (a,u) where a is an individual and $u \neq a$. We claim that $\mathfrak{S}'$ is also a simulation. The only condition in the definition of a simulation where the removal of such pairs could lead to a problem is the one dealing with role assertions. Thus, assume that $r(u,v) \in \mathcal{B}$ and $(u,u') \in \mathfrak{S}'$. Then $(u,u') \in \mathfrak{S}$, and thus there exists of an object v' with $r(u',v') \in \mathcal{C}$ and $(v,v') \in \mathfrak{S}$. Since v is a role successor in $\mathcal{B}$, it cannot be an individual. This implies the (v,v') also belongs to $\mathfrak{S}'$. □

We now define the sum operation $\oplus$ on finite, non-empty sets of qABoxes $\mathfrak{K} = \{\exists X_1.\mathcal{A}_1, \ldots, \exists X_n.\mathcal{A}_n\}$ as follows. If $n = 1$, then $\oplus\mathfrak{K} := \exists X_1.\mathcal{A}_1$. If $n \geq 2$, then we construct $\mathfrak{K}' := \{\exists Y_1.\mathcal{B}_1, \ldots, \exists Y_n.\mathcal{B}_n\}$, where (for $i = 1,\ldots,n$) $\exists Y_i.\mathcal{B}_i$ is the qABox obtained from $\exists X_i.\mathcal{A}_i$ by applying the construction in the proof of Lemma 27, and set $\oplus\mathfrak{K} := \uplus\mathfrak{K}'$.

Lemma 29. *The operation $\oplus$ on qABoxes satisfies the properties of sum for $\models^{\mathcal{T}}_{\mathsf{IQ}}$.*

Proof. First, recall that we have shown in the proof of Lemma 20 that $\oplus\mathfrak{K} = \uplus\mathfrak{K}' \models^{\mathcal{T}} \exists Y_i.\mathcal{B}_i$ for all i, $1 \leq i \leq n$. Since $\exists Y_i.\mathcal{B}_i$ is IQ-equivalent to $\exists X_i.\mathcal{A}_i$ by Lemma 27, this implies $\oplus\mathfrak{K} \models^{\mathcal{T}}_{\mathsf{IQ}} \exists X_i.\mathcal{A}_i$ for $i = 1,\ldots,n$.

Second, assume that $\exists Y.\mathcal{B}$ satisfies $\exists Y.\mathcal{B} \models^{\mathcal{T}}_{\mathsf{IQ}} \exists X_i.\mathcal{A}_i$ for $i = 1,\ldots,n$. The IQ-equivalence of $\exists X_i.\mathcal{A}_i$ and $\exists Y_i.\mathcal{B}_i$ yields $\exists Y.\mathcal{B} \models^{\mathcal{T}}_{\mathsf{IQ}} \exists Y_i.\mathcal{B}_i$ for $i = 1,\ldots,n$. By Lemma 24 and Lemma 28, this implies that there are simulations $\mathfrak{S}_i$ (for $i = 1,\ldots,n$) from $\exists Y_i.\mathcal{B}_i$ to $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$ such that $(a,u) \in \mathfrak{S}_i$ implies $u = a$ for all individuals a and all $i = 1,\ldots,n$. These simulations can be turned into a single simulation $\mathfrak{S}$ from $\uplus\mathfrak{K}$ to $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$ by setting

$$\begin{aligned}\mathfrak{S} := \{&(y',v) \mid (y,v) \in \mathfrak{S}_i,\ y' \text{ renaming of } y \in Y_i\} \cup \\ \{&(a,a) \mid a \text{ is an individual}\}.\end{aligned}$$

It remains to show that $\mathfrak{S}$ is indeed a simulation from $\oplus\mathfrak{K} = \uplus\mathfrak{K}'$ to $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$. First, note that $(a,a) \in \mathfrak{S}$ for all individuals a holds by the definition of $\mathfrak{S}$.

Second, assume that $A(u)$ belongs to the matrix of $\oplus\mathfrak{K}$ and $(u,v) \in \mathfrak{S}$. If u is an individual a, then $u = a = v$. Since $A(a)$ belongs to the matrix of $\oplus\mathfrak{K}$, we know that there is an i such that $A(a)$ belongs to $\mathcal{B}_i$. Thus, $(a,a) \in \mathfrak{S}_i$ yields that $A(a)$ belongs to the matrix of $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$. If u is a variable, then $u = y'$ where y' is the renaming of $y \in Y_i$ for some $i, 1 \leq i \leq n$, and thus $(y,v) \in \mathfrak{S}_i$ and $A(y)$ belongs to $\mathcal{B}_i$. Since $\mathfrak{S}_i$ is a simulation, this implies that $A(v)$ belongs to the matrix of $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$.

Third, assume that $r(u,v)$ belongs to the matrix of $\oplus\mathfrak{K}$ and $(u,\hat{u}) \in \mathfrak{S}$. We must show that this implies the existence of an object $\hat{v}$ with $r(\hat{u},\hat{v})$ in the matrix of $\mathrm{sat}^{\mathcal{T}}_{\mathsf{IQ}}(\exists Y.\mathcal{B})$ and $(v,\hat{v}) \in \mathfrak{S}$. Since $r(u,v)$ belongs to the matrix of $\oplus\mathfrak{K}$, there is an index i such that $r(u,v) \in \mathcal{B}'_i$, where $\mathcal{B}'_i$ is the renamed version of $\mathcal{B}_i$ that was created when constructing the disjoint union. Due to our construction of the qABoxes $\mathcal{B}_i$, we know that v cannot be an individual. Thus, $v = y'$ is the renaming of a variable $y \in Y_i$.

If $u = a$ is an individual, then $\hat{u} = a$, and $r(a,y) \in \mathcal{B}_i$. Thus, $(a,a) \in \mathfrak{S}_i$ yields an object $\hat{v}$ such that $r(a,\hat{v})$

belongs to the matrix of $\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B})$ and $(y,\hat{v}) \in \mathfrak{S}_i$. The definition of $\mathfrak{S}$, together with the fact that $v = y'$, yields $(v,\hat{v}) \in \mathfrak{S}$.

Finally, assume $u = z'$ is a variable, where z' is the renaming of $z \in Y_i$ for some $i, 1 \leq i \leq n$. Then we know that $(z,\hat{u}) \in \mathfrak{S}_i$ and $r(z,y) \in \mathcal{B}_i$, which implies the existence of an object $\hat{v}$ such that $r(\hat{u},\hat{v})$ belongs to the matrix of $\mathrm{sat}_{\mathsf{IQ}}^{\mathcal{T}}(\exists Y.\mathcal{B})$ and $(y,\hat{v}) \in \mathfrak{S}_i$. The definition of $\mathfrak{S}$, together with the fact that $v = y'$, again yields $(v,\hat{v}) \in \mathfrak{S}$. □

IQ-repairs of qABoxes w.r.t. $\mathcal{EL}$ TBoxes for repair requests formulated as $\mathcal{EL}$ instance assertions have also been investigated in (Baader et al. 2021a), again in the package setting. It is shown there that, up to IQ-equivalence, the set of all optimal IQ-repairs of a qABox for a repair request w.r.t. an $\mathcal{EL}$ TBox can be computed in exponential time (Theorem 9 in (Baader et al. 2021a)). As in the case of classical entailment, the paper introduces the notion of canonical IQ-repairs induced by repair seed functions. There are again at most exponentially many such canonical IQ-repairs, each of which is of at most exponential size. These canonical IQ-repairs are indeed IQ-repairs, and the set of canonical IQ-repairs IQ-covers all IQ-repairs (Proposition 8 in (Baader et al. 2021a)). As a consequence, up to IQ-equivalence, this set contains all optimal IQ-repairs, which can be obtained by removing elements that are strictly IQ-entailed by another elements.[5] The coverage property for the set of optimal IQ-repairs is then an easy consequence of the coverage property for the set of canonical IQ-repairs. As in the case of classical entailment, this also yields satisfaction of the optimal repair property in the choice setting. Summing up, we have thus shown that $\models_{\mathsf{IQ}}^{\mathcal{T}}$ for an $\mathcal{EL}$ TBox $\mathcal{T}$ as entailment satisfies all the properties introduced in Section 2 both for the package and the choice setting.

Theorem 30. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $\models_{\mathsf{IQ}}^{\mathcal{T}}$ IQ-entailment w.r.t. $\mathcal{T}$ between qABoxes, and consider as repair requests finite sets of $\mathcal{EL}$ concept assertions inducing repair sets according to the package (choice) approach. Then $\models_{\mathsf{IQ}}^{\mathcal{T}}$ is partial product contraction enabling.*

5.4 Contractions for $\mathcal{EL}$ TBoxes

In the context of repairing $\mathcal{EL}$ TBoxes, the following entailment relation between such TBoxes was introduced in (Kriegel 2022).

Definition 31. *Let $\mathcal{T}$ and $\mathcal{T}'$ be $\mathcal{EL}$ TBoxes. Then $\mathcal{T}'$ is a* generalized-conclusion weakening (GC-weakening) *of $\mathcal{T}$ (written $\mathcal{T} \models_{\mathsf{GC}} \mathcal{T}'$) if for each GCI $C \sqsubseteq D$ in $\mathcal{T}'$ there is a GCI $C \sqsubseteq E$ in $\mathcal{T}$ such that $E \sqsubseteq^{\emptyset} D$.*

Obviously, generalized-conclusion weakening implies classical entailment, i.e., $\mathcal{T} \models_{\mathsf{GC}} \mathcal{T}'$ implies $\mathcal{T} \models \mathcal{T}'$. Since the subsumption relation $\sqsubseteq^{\emptyset}$ between $\mathcal{EL}$ concepts is decidable in polynomial time, the same is true for the entailment relation $\models_{\mathsf{GC}}$ between $\mathcal{EL}$ TBoxes. The idea underlying generalized-conclusion weakening is that one wants to repair $\mathcal{EL}$ TBoxes, but preserve their structure as much as possible. Thus, one only allows to remove GCIs or weaken them by weakening their conclusion. This way, every GCI in the repair is obtained in a transparent way from a GCI in the original TBox. However, classical entailment is used for the non-entailment demanded for the repair request. To be more precise, following (Kriegel 2022), we consider GCIs (or equivalently, TBoxes consisting of a single GCI) as *repair requests*, and define

$$\mathrm{Rep}_{\mathsf{GC}}(\mathcal{T}, \{C \sqsubseteq D\}) := \{\mathcal{T}' \mid \mathcal{T} \models_{\mathsf{GC}} \mathcal{T}', C \not\sqsubseteq^{\mathcal{T}'} D\}.$$

Due to the fact that GC-weakening implies classical entailment, it is easy to see that this definition of repairs satisfies the conditions of Definition 4 for the entailment relation $\models_{\mathsf{GC}}$.

In the following, we show that $\models_{\mathsf{GC}}$ is partial product contraction enabling. First, note that, as sum, we can just use union of TBoxes.

Lemma 32. *The operation $\cup$ (i.e., set union) on $\mathcal{EL}$ TBoxes satisfies the properties of sum for $\models_{\mathsf{GC}}$.*

Proof. Obviously $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n$ for $\mathcal{EL}$ TBoxes $\mathcal{T}_1, \ldots, \mathcal{T}_n$ satisfies $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n \models_{\mathsf{GC}} \mathcal{T}_i$ for $i = 1, \ldots, n$. Now, assume that $\mathcal{T}' \models_{\mathsf{GC}} \mathcal{T}_i$ for $i = 1, \ldots, n$. We must show that $\mathcal{T}' \models_{\mathsf{GC}} \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n$. Thus, let $C \sqsubseteq D$ be a GCI in the union. This means that there is an index i such that $C \sqsubseteq D \in \mathcal{T}_i$. Then $\mathcal{T}' \models_{\mathsf{GC}} \mathcal{T}_i$ yields a GCI $C \sqsubseteq E$ in $\mathcal{T}'$ such that $E \sqsubseteq^{\emptyset} D$. Since $C \sqsubseteq D$ was chosen as an arbitrary element of the union, this shows $\mathcal{T}' \models_{\mathsf{GC}} \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_n$. □

To construct the product for $\models_{\mathsf{GC}}$, we use the lcs w.r.t. the empty TBox. As shown in (Baader, Küsters, and Molitor 1999), the lcs w.r.t. the empty TBox always exists in $\mathcal{EL}$, and it is unique up to equivalence. We write $\mathrm{lcs}_{\emptyset}(C_1, \ldots, C_m)$ to denote (an arbitrary element of the equivalence class of) the lcs of $C_1, \ldots, C_m$. If the number m of concepts to which the lcs operation is applied is assumed to be constant, then it can be computed in polynomial time. However, the size of the lcs may be exponential in m, and thus computing it may take exponential time if m is assumed to be part of the input (Baader and Turhan 2002).

Given $\mathcal{EL}$ TBoxes $\mathcal{T}_1, \ldots, \mathcal{T}_n$, we denote with $\mathrm{Pre}(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ the set of all $\mathcal{EL}$ concepts C such that each of the TBoxes $\mathcal{T}_i$ contains a GCI with premise C. For each $C \in \mathrm{Pre}(\mathcal{T}_1, \ldots, \mathcal{T}_n)$, we define $\mathrm{Pos}(C, \mathcal{T}_1, \ldots, \mathcal{T}_n) := \{\mathrm{lcs}(D_1, \ldots, D_n) \mid C \sqsubseteq D_i \in \mathcal{T}_i \text{ for } i = 1, \ldots, n\}$, and set $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n := \{C \sqsubseteq D \mid C \in \mathrm{Pre}(\mathcal{T}_1, \ldots, \mathcal{T}_n) \text{ and } D \in \mathrm{Pos}(C, \mathcal{T}_1, \ldots, \mathcal{T}_n)\}$.

Lemma 33. *The operation $\otimes$ on $\mathcal{EL}$ TBoxes satisfies the properties of product for $\models_{\mathsf{GC}}$.*

Proof. First, we show that $\mathcal{T}_i \models_{\mathsf{GC}} \mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n$ holds for all $i = 1, \ldots, n$. Thus, let $C \sqsubseteq D \in \mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n$. Then $C \in \mathrm{Pre}(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ and $D = \mathrm{lcs}(D_1, \ldots, D_n)$ where $C \sqsubseteq D_i \in \mathcal{T}_i$ for $i = 1, \ldots, n$. Since $D_i \sqsubseteq^{\emptyset} \mathrm{lcs}(D_1, \ldots, D_n)$ holds for all $i = 1, \ldots, n$ and $C \sqsubseteq D_i \in \mathcal{T}_i$, this shows $\mathcal{T}_i \models_{\mathsf{GC}} \mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n$.

Second, assume that $\mathcal{T}$ is such that $\mathcal{T}_i \models_{\mathsf{GC}} \mathcal{T}$ for $i = 1, \ldots, n$. We must show that $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n \models_{\mathsf{GC}} \mathcal{T}$. Thus, let $C \sqsubseteq D$ be an element of $\mathcal{T}$. Then, for each $i, 1 \leq i \leq n$,

[5] Since IQ-entailment can be decided in polynomial time, no NP-oracle is needed.

there is a GCI $C \sqsubseteq D_i$ in $\mathcal{T}_i$ such that $D_i \sqsubseteq^\emptyset D$. Consequently, $C \in \mathrm{Pre}(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ and $\mathrm{lcs}(D_1, \ldots, D_n) \in \mathrm{Pos}(C, \mathcal{T}_1, \ldots, \mathcal{T}_n)$. In addition, $D_i \sqsubseteq^\emptyset D$ for $i = 1, \ldots, n$ yields $\mathrm{lcs}(D_1, \ldots, D_n) \sqsubseteq^\emptyset D$. Since $C \sqsubseteq \mathrm{lcs}(D_1, \ldots, D_n)$ belongs to $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n$, this completes the proof that $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_n \models_{\mathsf{GC}} \mathcal{T}$. □

Regarding repairs, it was shown in (Kriegel 2022) that, for a given repair problem consisting of an $\mathcal{EL}$ TBox and a GCI as repair request, a finite set of optimal *generalized conclusion repairs (*GC*-repairs)* can be computed in exponential time, and this set covers all repairs in the sense that every GC-repair is a GC-weakening of an element of this set. Note that the notion of (optimal) GC-repairs employed in (Kriegel 2022) coincides with our notion of (optimal) repairs if one uses $\models_{\mathsf{GC}}$ as entailment relation and our above definition of $\mathrm{Rep}_{\mathsf{GC}}$ as repairs.

Summing up, we have thus shown that the entailment relation $\models_{\mathsf{GC}}$ satisfies all the properties introduced in Section 2.

Theorem 34. *Let $\models_{\mathsf{GC}}$ be generalized conclusion weakening between $\mathcal{EL}$ TBoxes, and consider $\mathcal{EL}$ GCIs as repair requests inducing repair sets defined as $\mathrm{Rep}_{\mathsf{GC}}(\mathcal{T}, \{C \sqsubseteq D\}) := \{\mathcal{T}' \mid \mathcal{T} \models_{\mathsf{GC}} \mathcal{T}', C \not\sqsubseteq^{\mathcal{T}'} D\}$. Then $\models_{\mathsf{GC}}$ is partial product contraction enabling.*

5.5 Forgetting for $\mathcal{EL}$ Concepts

In the DL literature, different versions of forgetting concept and role names have been investigated (see, e.g., (Konev, Walther, and Wolter 2009; Lutz and Wolter 2011; Ludwig and Konev 2014; Koopmann and Schmidt 2015; Sakr and Schmidt 2021). Here, we consider a variant of forgetting that is akin to the $\mathcal{EL}$ concept contraction considered in Section 5.1, but now the goal is to remove concepts or role names rather than to remove subsuming concepts.

As in Section 5.1, *knowledge bases* are $\mathcal{EL}$ concepts and *entailment* $\models_{\sqsubseteq^\mathcal{T}}$ is subsumption $\sqsubseteq^\mathcal{T}$ w.r.t. a fixed cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$. Given an $\mathcal{EL}$ concept C, its signature $\mathrm{Sig}(C)$ consists of the concept and role names occurring in C. *Repair requests* are finite sets of concept and role names satisfying an additional restriction. Given an $\mathcal{EL}$ concept C, such a repair request α induces the following set of repairs:

$$\mathrm{Rep}(C, \alpha) := \{D \mid C \sqsubseteq^\mathcal{T} D \text{ and } \mathrm{Sig}(D) \cap \alpha = \emptyset\}.$$

To ensure that the second condition of Definition 4 is satisfied, we must impose an additional restriction on repair requests: α must be compatible with $\mathcal{T}$. A finite set α of concept and role names is *compatible with* $\mathcal{T}$ if $\mathrm{Sig}(E) \cap \alpha = \emptyset$ implies $\mathrm{Sig}(F) \cap \alpha = \emptyset$ for all GCIs $E \sqsubseteq F$ in $\mathcal{T}$.

Lemma 35. *Let α be a repair request and D an $\mathcal{EL}$ concept with $\mathrm{Sig}(D) \cap \alpha = \emptyset$. If $D \sqsubseteq^\mathcal{T} D'$, then $\mathrm{Sig}(D') \cap \alpha = \emptyset$.*

Proof. Assume to the contrary that $D \sqsubseteq^\mathcal{T} D'$, but $\mathrm{Sig}(D') \cap \alpha \neq \emptyset$. Let $\mathcal{I}$ be the interpretation with $\Delta^\mathcal{I} = \{d\}$ and

- $r^\mathcal{I} = \emptyset$ and $A^\mathcal{I} = \emptyset$ for all role names r and concept names A in α,
- $r^\mathcal{I} = \{(d, d)\}$ and $A^\mathcal{I} = \{d\}$ for all role names r and concept names A not belonging to α.

It is easy to see that the following is satisfied for all $\mathcal{EL}$ concepts C: if $\mathrm{Sig}(C) \cap \alpha = \emptyset$, then $C^\mathcal{I} = \{d\}$; and $C^\mathcal{I} = \emptyset$ otherwise. Due to compatibility of α with $\mathcal{T}$, this implies that $\mathcal{I}$ is a model of $\mathcal{T}$. In fact, if $E \sqsubseteq F$ is a GCI in $\mathcal{T}$ with $\mathrm{Sig}(E) \cap \alpha \neq \emptyset$, then $E^\mathcal{I} = \emptyset$, and thus $E^\mathcal{I} \subseteq F^\mathcal{I}$ clearly holds. If $\mathrm{Sig}(E) \cap \alpha = \emptyset$, then also $\mathrm{Sig}(F) \cap \alpha = \emptyset$, and thus $E^\mathcal{I} \subseteq F^\mathcal{I}$ since both are equal to $\{d\}$. Our assumptions that $\mathrm{Sig}(D) \cap \alpha = \emptyset$ and $\mathrm{Sig}(D') \cap \alpha \neq \emptyset$ yield $D^\mathcal{I} = \{d\} \not\subseteq \emptyset = D'^\mathcal{I}$. This contradicts the assumed subsumption $D \sqsubseteq^\mathcal{T} D'$. □

By adapting Lemma 19 of Section 5.2, we obtain the following characterization of subsumption w.r.t. a cycle-restricted $\mathcal{EL}$ TBox.

Lemma 36. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept. Then one can compute in at most exponential time an $\mathcal{EL}$ concept $\mathrm{sat}^\mathcal{T}(C)$ such that the following are equivalent for all $\mathcal{EL}$ concepts D:*

- *$C \sqsubseteq^\mathcal{T} D$,*
- *$\mathrm{sat}^\mathcal{T}(C) \sqsubseteq^\emptyset D$,*
- *there is a homomorphism from D to $\mathrm{sat}^\mathcal{T}(C)$.*

The notion of homomorphism between $\mathcal{EL}$ concepts E and F employed in this lemma is the one introduced in (Baader, Küsters, and Molitor 1999) as homomorphism between $\mathcal{EL}$ description trees. It is easy to see that it coincides with the notion of homomorphism between the qABox representations of the ABoxes $\{E(a)\}$ and $\{F(a)\}$.

We have already seen in Section 5.1 that $\models_{\sqsubseteq^\mathcal{T}}$ has products and sums. Thus, it remains to prove that the optimal repair property is satisfied as well. Given a cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$, an $\mathcal{EL}$ concept C, and a finite set α of concept and role names, we first saturate C w.r.t. $\mathcal{T}$, i.e., compute the concept $\mathrm{sat}^\mathcal{T}(C)$. Then we remove from $\mathrm{sat}^\mathcal{T}(C)$ all concept names occurring in α and all existential restrictions of the form $\exists r.E$ for $r \in \alpha$. We denote the resulting concept as $\mathrm{sat}^\mathcal{T}(C)^{-\alpha}$ and set $\mathrm{Orep}(C, \alpha) := \{\mathrm{sat}^\mathcal{T}(C)^{-\alpha}\}$.

Example 37. *Let $\mathcal{T} := \{A \sqsubseteq B \sqcap \exists r.B\}$, $C := A$, and $\alpha := \{A, r\}$. Then α is admissible as repair request for $\mathcal{T}$, and $\mathrm{sat}^\mathcal{T}(C) = A \sqcap B \sqcap \exists r.B$. Removing A and $\exists r.B$ from this concept yields $\mathrm{sat}^\mathcal{T}(C)^{-\alpha} = B$, and thus $\mathrm{Orep}(C, \alpha) = \{B\}$.*

To shows that $\mathrm{Orep}(C, \alpha)$ consists of optimal repairs and covers all repairs, it is sufficient to prove the following lemma.

Lemma 38. *The concept $\mathrm{sat}^\mathcal{T}(C)^{-\alpha}$ is a repair of C for α that entails every repair of C for α.*

Proof. Since all concept names in α and all existential restrictions for roles in α are removed by our construction of $\mathrm{sat}^\mathcal{T}(C)^{-\alpha}$ from $\mathrm{sat}^\mathcal{T}(C)$, we know that $\mathrm{Sig}(\mathrm{sat}^\mathcal{T}(C)^{-\alpha}) \cap \alpha = \emptyset$. In addition, this construction also implies that $\mathrm{sat}^\mathcal{T}(C) \sqsubseteq^\mathcal{T} \mathrm{sat}^\mathcal{T}(C)^{-\alpha}$. Since $C \sqsubseteq^\mathcal{T} \mathrm{sat}^\mathcal{T}(C)$ by Lemma 36, transitivity of subsumption yields $C \sqsubseteq^\mathcal{T} \mathrm{sat}^\mathcal{T}(C)^{-\alpha}$. Thus, we have shown that $\mathrm{sat}^\mathcal{T}(C)^{-\alpha} \in \mathrm{Rep}(C, \alpha)$. Optimality of this repair follows from the fact that it entails every repair.

To show this coverage property, assume that $D \in \mathrm{Rep}(C, \alpha)$, i.e., $C \sqsubseteq^{\mathcal{T}} D$ and $\mathrm{Sig}(C) \cap \alpha = \emptyset$. By Lemma 36, the former subsumption implies that there is a homomorphism from D to $\mathrm{sat}^{\mathcal{T}}(C)$. Since D does not contain any of the concept and role names from α, this also yields a homomorphism from D to $\mathrm{sat}^{\mathcal{T}}(C)^{-\alpha}$. This shows $\mathrm{sat}^{\mathcal{T}}(C)^{-\alpha} \sqsubseteq^{\mathcal{T}} D$. □

Summing up, we have thus shown that, in the setting introduced in this subsection, the entailment relation $\models_{\sqsubseteq^{\mathcal{T}}}$ satisfies all the properties introduced in Section 2.

Theorem 39. *Let $\models_{\sqsubseteq^{\mathcal{T}}}$ be subsumption w.r.t. a cycle-restricted $\mathcal{EL}$ TBox $\mathcal{T}$, and consider as repair requests finite sets of concept and role names that are compatible with $\mathcal{T}$ and induce repair sets defined as $\mathrm{Rep}(C, \alpha) := \{D \mid C \sqsubseteq^{\mathcal{T}} D$ and $\mathrm{Sig}(D) \cap \alpha = \emptyset\}$. Then $\models_{\sqsubseteq^{\mathcal{T}}}$ is partial product contraction enabling.*

5.6 Contractions for Automata, Grammars, and Turing Machines

To illustrate the generality of our approach, we consider a setting where KBs define formal languages and entailment corresponds to language inclusion. We start with the simple case of finite automata. Given a finite automaton $\mathcal{A}$ over a finite alphabet Σ, we denote the set of words over Σ accepted by $\mathcal{A}$ as $L(\mathcal{A})$. We say that $\mathcal{A}$ L-entails $\mathcal{B}$ (written $\mathcal{A} \models_{\mathsf{L}} \mathcal{B}$) if every word accepted by $\mathcal{B}$ is also accepted by $\mathcal{A}$, i.e., if $L(\mathcal{A}) \supseteq L(\mathcal{B})$.

It is easy to see that, in this setting, *sum* corresponds to *union* and *product* to *intersection* of the corresponding languages. In addition, it is well-known that the class of recognizable languages (i.e., languages accepted by finite automata) is closed under finite union and intersection (Hopcroft, Motwani, and Ullman 2007). Thus, given finite automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, their sum $\mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_n$ is a finite automaton accepting $L(\mathcal{A}_1) \cup \ldots \cup L(\mathcal{A}_n)$ and their product is a finite automaton accepting $L(\mathcal{A}_1) \cap \ldots \cap L(\mathcal{A}_n)$. These automata can be obtained using the constructions employed to show closure under union and intersection for the class of recognizable languages in standard textbooks such as (Hopcroft, Motwani, and Ullman 2007).

As repair requests, we consider finite sets of words. Note that, given such a set $R = \{w_1, \ldots, w_m\}$, there is a finite automaton $\mathcal{R}$ with $L(\mathcal{R}) = \{w_1, \ldots, w_m\}$ that has at most $|w_1| + \ldots + |w_m| + 1$ states. We use the choice approach to define repairs. This means that, given a repair problem $\mathcal{A}$ and R, a *repair* is an automaton $\mathcal{B}$ such that $L(\mathcal{B}) \subseteq L(\mathcal{A})$ and $L(\mathcal{B}) \not\supseteq R$, i.e., there is a $w \in R$ such that $w \notin L(\mathcal{B})$. In case $L(\mathcal{A}) \not\supseteq R$, then up to equivalence (which coincides with the usual notion of equivalence for finite automata), $\mathcal{A}$ is the only optimal repair, which clearly covers all repairs. If $L(\mathcal{A}) \supseteq R$, then

$$\{\mathcal{A}^{-w} \mid w \in R\}$$

is (up to equivalence) the set of all optimal repairs, where $\mathcal{A}^{-w}$ is a finite automaton accepting the language $L(\mathcal{A}) \setminus \{w\}$. Since $\{w\}$ can be accepted by a deterministic finite automaton whose size is linear in $|w|$, and the class of recognizable languages is closed under intersection and complement, the finite automaton $\mathcal{A}^{-w}$ can be constructed in time polynomial in the size of the repair problem, using standard textbook constructions.

Theorem 40. *Let $\models_{\mathsf{L}}$ be the superset relation for the induced languages for finite automata, and consider finite sets of words as repair requests inducing repair sets according to the choice approach. Then $\models_{\mathsf{L}}$ is partial product contraction enabling.*

Our proof of this theorem uses the fact that the class of recognizable languages is closed under union, intersection, and complement. The same is true for the class of context-sensitive languages. Thus, if we replace finite automata by context-sensitive grammars (or equivalently, linear bounded automata), the above theorem still holds. However, in this case, the entailment relation (i.e., language inclusion) is not decidable.

The class of context-free (cf) languages is not closed under intersection and complement. The latter is not a problem since removing the word w from a cf-language can be achieved by intersecting it with a recognizable language (the complement of the recognizable language $\{w\}$), and the intersection of a cf language with a recognizable language is again cf. However, failure of closure under intersection of the class of cf languages implies that there is no appropriate product operation. In fact, assume that G_1, G_2 are cf grammars such that $L(G_1) \cap L(G_2)$ is not a cf language. Now assume that $G_1 \otimes G_2$ is a cf grammar that is the product of G_1, G_2, i.e., $L(G_1 \otimes G_2) \subseteq L(G_1) \cap L(G_2)$ and there is no cf language L such that $L(G_1 \otimes G_2) \subset L \subseteq L(G_1) \cap L(G_2)$. Since $L(G_1) \cap L(G_2)$ is not cf, $L(G_1 \otimes G_2) \subset L(G_1) \cap L(G_2)$, and thus there is a word w such that $w \in (L(G_1) \cap L(G_2)) \setminus L(G_1 \otimes G_2)$. Since cf languages are closed under union and $\{w\}$ is cf, $L := L(G_1 \otimes G_2) \cup \{w\}$ is a cf language satisfying $L(G_1 \otimes G_2) \subset L \subseteq L(G_1) \cap L(G_2)$, which contradicts our assumption that $G_1 \otimes G_2$ is the product of G_1, G_2.

The class of Turing recognizable languages (aka languages generated by a general Chomsky grammar) is closed under union and intersection, but not under complement. The latter is, however, again not a problem. In fact, given a Turing machine accepting the language L and a word w, one can easily construct one that accepts $L \setminus \{w\}$. Note, however, that entailment (i.e., language inclusion) is again undecidable.

Corollary 41. *If we replace in Theorem 40 finite automata with Turing machines (linear bounded automata), then $\models_{\mathsf{L}}$ is partial product contraction enabling. However, if we use cf grammars instead, then $\models_{\mathsf{L}}$ is not partial product contraction enabling since the product need not exist.*

The definition of repairs used until now in this subsections follows the choice approach. If we employ the package approach, then a repair of a finite automaton $\mathcal{A}$ for the repair request R is a finite automaton $\mathcal{B}$ such that $L(\mathcal{B}) \subseteq L(\mathcal{A})$ and $L(\mathcal{B}) \cap R = \emptyset$. It is easy to see that then, up to equivalence, the set $\{\mathcal{A}^{-R}\}$ where $\mathcal{A}^{-R}$ is a finite automaton accepting the language $L(\mathcal{A}) \setminus R$, is the set of optimal repair, and this set covers all repairs. Similar arguments can be used to show that $\models_{\mathsf{L}}$ is partial product contraction enabling not only for the case of finite automata, but also for Turing machines and linear bounded automata.

6 Conclusion

We have shown that the partial meet contraction approach can be generalized to the setting of a reflexive and transitive entailment relation between KBs with associated sum and product operations generalizing conjunction and disjunction. The main novelty of the approach is that we employ optimal repairs in place of remainders. Under the additional assumption that the optimal repairs cover all repairs, we were able to prove characterization theorems linking the obtained contraction operations, called partial product contractions, with reasonable postulates, both for the MaxiChoice and the general case. In contrast to belief base contractions, our partial product contractions are syntax-independent and usually preserve more consequences. Though partial product contractions can express belief set contractions, they also work in settings where finite KBs generating the belief sets are required. In these settings, the main challenge is usually to show that the required repair properties are satisfied. In Sections 5.1 to 5.4 we were able to use recent results on optimal repairs for the DL $\mathcal{EL}$ to obtain instances of our approach that are relevant for ontology engineering.

A second important novelty of our approach is that it generalizes the notion of contraction and repair towards repair goals different from non-entailment of a certain formula or knowledge base. This allows us, for instance, to treat different approaches to multiple contraction, such a choice and package contraction, in a uniform way. Additionally, we have shown in Example 5 and Section 5.5 that certain notions of variable forgetting in propositional logic and concept and role forgetting in DLs can be seen as instances of our approach, and thus satisfy the same postulates as the more standard contraction approaches that have non-entailment as a goal.

One interesting direction for future research is to identify instances of our approach also for other logics, or for repair goals other than non-entailment or signature forgetting. Another is to determine whether other contraction approaches, such as kernel contractions (Hansson 1994), can be generalized in a similar way. Finally, the relationship to previous work on forgetting, both in the DL community (Konev, Walther, and Wolter 2009; Lutz and Wolter 2011; Ludwig and Konev 2014; Koopmann and Schmidt 2015; Sakr and Schmidt 2021) and in the belief change community (Lang and Marquis 2010; Delgrande 2017; Kern-Isberner et al. 2019a; Kern-Isberner et al. 2019b) needs to be investigated in more detail.

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