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Institute for Theoretical Computer Science
Chair for Automata Theory

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Franz Baader

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Postal Address:
Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
01062 Dresden

http://lat.inf.tu-dresden.de

Visiting Address:
Nöthnitzer Str. 46
Dresden

An Order-Theoretic View on Optimal Repairs and Complete Sets of Unifiers

Franz Baader[1,2]

[1] Institute of Theoretical Computer Science, TU Dresden, Dresden, Germany
[2] Center for Scalable Data Analytics and AI Dresden/Leipzig (ScaDS.AI), Germany
franz.baader@tu-dresden.de

Abstract. The optimal repair property, which says that there is a finite set of optimal (i.e., entailment-maximal) repairs that covers all repairs, has turned out to be useful both in the context of ontology engineering and in belief change. We provide abstract order-theoretic conditions that guarantee the existence of finite sets of optimal repairs covering all repairs, and illustrate their use with abstract examples as well as with more practical examples from the realm of Description Logic (DL). The order-theoretic view on optimal repairs also reveals that there is a strong similarity between the optimal repair property and the existence of a finite complete set of unifiers for unification modulo equational theories. Applying Siekmann's proposal to divide unification problems into the unification types unitary, finitary, infinitary, and zero to repair problems, we obtain a more fine-grained classification of repair problems. For the DL examples introduced in this paper, we observe that types unitary, finitary and zero can occur, but none of these examples provides us with an infinitary repair problem. However, we also show that unification problems can actually be viewed as repair problems in the abstract framework introduced in our previous work on contractions based on optimal repairs. Thus, within this framework, known results on unification types of certain equational theories provide us with examples of repair problems of these types.

1 Introduction

Representing knowledge in a logic-based knowledge representation language allows one to derive implicit consequences from a given knowledge base (KB). Prominent examples of logic-based KR languages are description logics (DLs) [10]. Modifying a KB such that a certain unwanted consequence no longer follows has been investigated in the area of belief change under the name of contraction [2,21] and in ontology engineering under the name of repair [24,35,15,38]. Belief base contractions [28,21] and classical repair approaches for DL-based knowledge bases [24,35] compute subsets of the given KB. These approaches have been criticised for being syntax-dependent and removing too many consequences [22,15,34,27,6].

On the DL side, optimal repairs have been proposed as a solution to this problem [12]. Optimal repairs maximize the set of consequences of the knowledge base rather than the set of its explicit statements, while still being representable by a finite KB. In general, such optimal repairs need not exist even in cases where there is a repair (see Proposition 2 in [15] and Example 4 in this paper). Even if optimal repairs exist, they may not cover all repairs in the sense that every repair is an instance of an optimal one. If in this case one considers only optimal repairs, then one may lose certain repair options. In the context of repairing DL-based KBs, we were able to determine application-relevant settings were this cannot happen, i.e., where every repair problem has a finite set of optimal repairs that covers all repairs, and where this set of optimal repairs can effectively be computed [12,13,14,25,11].

On the belief change side, we have developed in [19] a new approach for constructing contraction operations, called partial product contractions, which uses optimal repairs in place of classical repairs (called remainders in the belief change literature [23]). However, instead of introducing this approach for a specific instance, we defined a very general framework within which partial product contractions can be constructed. Basically, this framework consider a reflexive and transitive entailment relation between KBs, without making explicit assumptions on the structure of the KBs and their semantics. The repair goal is formulated using repair requests and a repair function that determines which KBs are considered to be repairs of a given KB w.r.t. such a repair goal. The framework makes several other assumptions, but for the purpose of this paper, only the *optimal repair property* is relevant, which says that every repair problem has a finite set of optimal repairs that covers all repairs in the sense that every repair is entailed by an optimal repair (see Definition 2).

In the first part of this paper (Sections 3 and 4), we will investigate the optimal repair property from an order-theoretic point of view, which is possible since the entailment relation can be seen as a quasi-order. We will employ this view to characterize the optimal repair property using well-quasi-orders (wqos) in Section 3. Then we consider several instances of the general framework where KBs are concepts of the DL $\mathcal{EL}$ and show in Section 4 that they satisfy the optimal repair property, though there are also variants that do not.

In the second part of this paper (Section 5), we go beyond the optimal repair property by investigating the possible reasons for the optimal repair property to fail. This is inspired by the use of the order-theoretic view in [4] to characterize unification type zero. In fact, if one replaces repair problems with unification problems and entailment between KBs with the instantiation pre-order between substitutions, then one sees that there is a close relationship between repair problems satisfying the optimal repair property and unification problems that are unitary or finitary. This allows us to transfer the unification hierarchy [31,18] consisting of unification types unitary, finitary, infinitary, and zero to repair problems. Analyzing the repair problems from Section 4 in a more fine-grained way, we see that repair types unitary, finitary, and zero can occur, but type infinitary is not possible in these examples. Whether this has deeper reasons or is just an artefact of the chosen instances is not clear at the moment.

In the third part of the paper (Section 6), we show that (a generalization of) unification modulo equational theories can be seen as an instance of the general repair framework introduced in [19]. We then analyze under what conditions the unification type of the equational theory transfers to the corresponding repair problems.

2 Preliminaries

First, we briefly describe the DL $\mathcal{EL}$ [8,10] since it will later on be used in the instances of the general repair framework presented in the second part of this section.

2.1 The description logic $\mathcal{EL}$

$\mathcal{EL}$ concepts are built inductively, starting with concept names A from a set N_C of such names, and using the concept constructors $\top$ (top concept), $C \sqcap D$ (conjunction), and $\exists r.C$ (existential restriction), where C, D are $\mathcal{EL}$ concepts and r belongs to a set N_R of role names. An atom is a concept name or an existential restriction. A general concept inclusion (GCI) of $\mathcal{EL}$ is of the form $C \sqsubseteq D$ for $\mathcal{EL}$ concepts C, D, and an $\mathcal{EL}$ TBox is a finite set of such GCIs.

The semantics of $\mathcal{EL}$ is defined in a model-theoretic way, using the notion of an *interpretation* $\mathcal{I}$, which is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the domain $\Delta^{\mathcal{I}}$ is a non-empty set and the interpretation function $\cdot^{\mathcal{I}}$ maps each concept name $A \in N_C$ to $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation of an $\mathcal{EL}$ concept is defined inductively as follows: $\top^{\mathcal{I}} := \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$, and $(\exists r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $e \in C^{\mathcal{I}}\}$. A model $\mathcal{I}$ of the $\mathcal{EL}$ TBox $\mathcal{T}$ is an interpretation that satisfies all its GCIs, i.e., $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all $C \sqsubseteq D \in \mathcal{T}$. Given $\mathcal{EL}$ concepts C, D and an $\mathcal{EL}$ TBox $\mathcal{T}$, we say that C is *subsumed by* D *w.r.t.* $\mathcal{T}$ ($C \sqsubseteq^{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in all models $\mathcal{I}$ of $\mathcal{T}$. The $\mathcal{EL}$ concepts C, D are equivalent (written $C \equiv^{\mathcal{T}} D$) if $C \sqsubseteq^{\mathcal{T}} D$ and $D \sqsubseteq^{\mathcal{T}} C$.

The following recursive characterization of subsumption in $\mathcal{EL}$ w.r.t. a TBox has been shown in [7] and will turn out to be useful later on. To formulate this characterization, we must introduce the notion of structural subsumption. Intuitively, a subsumption relationship between two atoms is *structural* if their top-level structure is compatible. To be more precise, structural subsumption between atoms is defined in [7] as follows: the atom C is *structurally subsumed* by the atom D w.r.t. $\mathcal{T}$ ($C \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} D$) if either

- $C = D$ is a concept name, or
- $C = \exists r.C'$, $D = \exists r.D'$, and $C' \sqsubseteq^{\mathcal{T}} D'$.

Lemma 1 ([7]). *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and $C_1, \ldots, C_n$, $D_1, \ldots, D_m$ $\mathcal{EL}$ atoms. Then $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^{\mathcal{T}} D_1 \sqcap \cdots \sqcap D_m$ iff for every $j \in \{1, \ldots, m\}$*

1. *there is an index $i \in \{1, \ldots, n\}$ such that $C_i \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} D_j$, or*

2. there are atoms $A_1, \ldots, A_k, B$ occurring in $\mathcal{T}$ ($k \geq 0$) such that
(i) $A_1 \sqcap \cdots \sqcap A_k \sqsubseteq^{\mathcal{T}} B$,
(ii) for every $\eta \in \{1, \ldots, k\}$ *there is* $i \in \{1, \ldots, n\}$ *with* $C_i \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} A_\eta$, *and*
(iii) $B \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} D_j$.

While this characterization holds for arbitrary $\mathcal{EL}$ TBoxes, we will see in Section 4 that we must restrict the TBox to being cycle-restricted for the optimal repair property to hold.

Definition 1 ([7]). *The $\mathcal{EL}$ TBox $\mathcal{T}$ is* cycle-restricted *if there is no $\mathcal{EL}$ concept C and $m \geq 1$ (not necessarily distinct) role names $r_1, \ldots, r_m$ such that* $C \sqsubseteq^{\mathcal{T}} \exists r_1. \cdots \exists r_m.C$.

As pointed out in [7], it can be decided in polynomial time whether a given $\mathcal{EL}$ TBox is cycle-restricted or not.

2.2 The optimal repair property

Following [19], we assume that we are given a set of *knowledge bases* (KBs) and an *entailment relation* between knowledge bases. We usually write KBs as $\mathcal{K}$, possibly primed ($\mathcal{K}'$) or with an index ($\mathcal{K}_i$), and entailment as $\models$, i.e., $\mathcal{K} \models \mathcal{K}'$ means that $\mathcal{K}$ *entails* $\mathcal{K}'$, or equivalently that $\mathcal{K}'$ is *entailed by* $\mathcal{K}$. We assume that entailment satisfies the following properties:

- $\mathcal{K} \models \mathcal{K}$ (reflexivity),
- $\mathcal{K} \models \mathcal{K}'$ and $\mathcal{K}' \models \mathcal{K}''$ implies $\mathcal{K} \models \mathcal{K}''$ (transitivity).

From an order-theoretic point of view, this means that entailment (viewed as a $\geq$ relation) is a quasi-order. We call two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$ *equivalent* (and write $\mathcal{K} \equiv \mathcal{K}'$) if $\mathcal{K} \models \mathcal{K}'$ and $\mathcal{K}' \models \mathcal{K}$. We say that $\mathcal{K}$ *strictly entails* $\mathcal{K}'$ if $\mathcal{K} \models \mathcal{K}'$, but $\mathcal{K}' \not\models \mathcal{K}$. In this case we write $\mathcal{K} \models_s \mathcal{K}'$. Note that $\models_s$ is the $>$ relation corresponding to the $\geq$ relation $\models$. The relation $\equiv$ on KBs is indeed an equivalence relation, and we write the equivalence class of a KB $\mathcal{K}$ as $[\mathcal{K}]$, i.e., $[\mathcal{K}] := \{\mathcal{K}' \mid \mathcal{K} \equiv \mathcal{K}'\}$.

When defining repairs, we assume that we have additional syntactic entities called repair requests. Given a KB $\mathcal{K}$, a *repair request* α determines a set of KBs $\mathrm{Rep}(\mathcal{K}, \alpha)$ such that

(a) $\mathcal{K} \models \mathcal{K}'$ holds for every element $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$, and
(b) $\mathcal{K}' \in \mathrm{Rep}(\mathcal{K}, \alpha)$ and $\mathcal{K}' \models \mathcal{K}''$ imply $\mathcal{K}'' \in \mathrm{Rep}(\mathcal{K}, \alpha)$.

We call the elements of $\mathrm{Rep}(\mathcal{K}, \alpha)$ *repairs* of $\mathcal{K}$ for α.

Example 1. In this example, knowledge bases are concepts of the DL $\mathcal{EL}$, and entailment is subsumption w.r.t. the empty TBox, i.e., $C \models D$ iff $C \sqsubseteq^{\emptyset} D$. We use $\mathcal{EL}$ concepts also as repair requests, and define, on the one hand

$$\mathrm{Rep}_{\mathrm{ent}}(C, D) := \{C' \mid C \models C', C' \not\models D\}.$$

This means that we are looking for a subsumer of C that is not subsumed by D. It is easy to see that the function $\mathrm{Rep}_{\mathrm{ent}}$ satisfies conditions (a) and (b).

On the other hand, let $\mathrm{Sig}(C)$ be the set of concept and role names occurring in the concept C. We use finite sets of concept and role names as repair requests, and define

$$\mathrm{Rep}_{\mathrm{for}}(C, \alpha) := \{C' \mid C \models C', \mathrm{Sig}(C') \cap \alpha = \emptyset\}.$$

The goal is here to forget the names from α, i.e., to find a concept that subsumes C and does not contain any of the names in α. Again, condition (a) is satisfied by definition, and (b) holds since it is easy to see that $C' \sqsubseteq^{\emptyset} C''$ implies $\mathrm{Sig}(C') \supseteq \mathrm{Sig}(C'')$ (see Lemma 4 below).

Our second example is more abstract.

Example 2. Assume that KBs are pairs of natural numbers, i.e., elements of $\mathbb{N} \times \mathbb{N}$, and let $\models$ be the lexicographic product order $\geq^{lex}$ induced by the usual order $\geq$ on natural numbers, i.e. $(k, \ell) \geq^{lex} (k', \ell')$ if $k > k'$, or $k = k'$ and $\ell \geq \ell'$. We use natural numbers as repair requests and set

$$\mathrm{Rep}((k, \ell), r) := \{(k', \ell') \mid (k, \ell) \geq^{lex} (k', \ell'), k' \leq r\}.$$

Condition (a) is again satisfied by definition, and (b) holds since it is easy to see that $(k', \ell') \geq^{lex} (k'', \ell'')$ implies $k' \geq k''$.

We are interested in repair settings that satisfy the optimal repair property.

Definition 2. *The repair function* Rep *satisfies the* optimal repair property *if, for every pair $\mathcal{K}, \alpha$ consisting of a KB and a repair request (called a* repair problem*), there exists a finite set of KBs* $\mathrm{Orep}(\mathcal{K}, \alpha)$ *satisfying*

- *every element $\mathcal{K}'$ of* $\mathrm{Orep}(\mathcal{K}, \alpha)$ *is a* repair *of $\mathcal{K}$ for α (repair property),*
- *every element $\mathcal{K}'$ of* $\mathrm{Orep}(\mathcal{K}, \alpha)$ *is* optimal*, i.e., there is no repair of $\mathcal{K}$ for α that strictly entails $\mathcal{K}'$ (optimality),*
- $\mathrm{Orep}(\mathcal{K}, \alpha)$ covers *all repairs, i.e., for every repair $\mathcal{K}''$ of $\mathcal{K}$ for α, there is an element $\mathcal{K}'$ of* $\mathrm{Orep}(\mathcal{K}, \alpha)$ *such that $\mathcal{K}'$ entails $\mathcal{K}''$ (coverage).*

In our first example, the optimal repair property is satisfied for both types of repairs, basically because the set of subsumer of a given $\mathcal{EL}$ concept, i.e., $\mathrm{Subs}(C) := \{D \mid C \sqsubseteq^{\emptyset} D\}$, is finite up to equivalence (see Section 4.1).

In the second example, optimal repairs need not exist although there are repairs, and thus the optimal repair property is not satisfied. For example, $\mathrm{Rep}((1, 0), 0) = \{(0, \ell) \mid \ell \in \mathbb{N}\}$. Clearly, this set is non-empty, but it does not contain an optimal repair. In fact, for every $(0, \ell) \in \mathrm{Rep}((1, 0), 0)$, we have $(0, \ell + 1) \in \mathrm{Rep}((1, 0), 0)$ and $(0, \ell + 1) \models_s (0, \ell)$.

3 An order-theoretic view

Here we use $\geq$ to denote entailment and $>$ to denote strict entailment. Our requirement that entailment be reflexive and transitive then means that $\geq$ is a quasi-order on the set of all knowledge bases. We use A to denote the set of all knowledge bases and write a, b (possibly primed or with index) for its elements. With $\equiv$ we denote the equivalence relation induced by the quasi-order $\geq$, i.e., $a \equiv b$ iff $a \geq b$ and $b \geq a$. We write $[a]$ for the equivalence class of $a \in A$ and $[A]$ for the set of all these equivalence classes. The quasi-order $\geq$ on A induces a partial order on $[A]$, which we write (by a slight abuse of notation) again as $\geq$, i.e., we defined $[a] \geq [b]$ if $a \geq b$.

3.1 Optimal repairs

In our definition of repairs, we require that all repairs are entailed by the original KB. From the order-theoretic point of view, this means that we consider the cone induced by the given KB. For $a \in A$, we define $\mathrm{Cone}(a) := \{b \in A \mid a \geq b\}$. Repairs for a given repair request α are required to be downward-closed subsets of $\mathrm{Cone}(a)$, i.e., the sets $\mathrm{Rep}(a, \alpha)$ must satisfy the following two properties:

1. $\mathrm{Rep}(a, \alpha) \subseteq \mathrm{Cone}(a)$, and
2. if $b \in \mathrm{Rep}(a, \alpha)$, then $b' \in \mathrm{Rep}(a, \alpha)$ for all b' with $b \geq b'$.

Optimal repairs are the maximal elements of $\mathrm{Rep}(a, \alpha)$.

Definition 3. *Let $\geq$ be a quasi-order on the set A, and B a downward-closed subset of A. The subset O of B is* complete *for B if it covers all elements of B, i.e., if for every $b' \in B$ there is $b \in O$ such that $b \geq b'$.*

We can now characterize the optimal repair property as follows.

Theorem 1. *Let A be a set of KBs, $\geq$ the quasi-order on A corresponding to the entailment relation between KBs, and* Rep *a repair function that assigns to every tuple (a, α) consisting of an element $a \in A$ and a repair requests α a downward-closed subset $\mathrm{Rep}(a, \alpha)$ of $\mathrm{Cone}(a)$. Then the following are equivalent:*

1. Rep *satisfies the optimal repair property.*
2. *For every input tuple (a, α), the set $\mathrm{Rep}(a, \alpha)$ contains a finite complete set.*
3. *For every input tuple (a, α), the set of all maximal elements of $[\mathrm{Rep}(a, \alpha)] := \{[b] \mid b \in \mathrm{Rep}(a, \alpha)\}$ is finite and complete for $[\mathrm{Rep}(a, \alpha)]$.*

Proof. "$1 \to 2$": If Rep satisfies the optimal repair property, then we can use the set $\mathrm{Orep}(a, \alpha)$ as finite and complete subset of $\mathrm{Rep}(a, \alpha)$.

"$2 \to 3$": Let O be a finite and complete subset of $\mathrm{Rep}(a, \alpha)$. First, we show that, for every maximal element $[b]$ of $[\mathrm{Rep}(a, \alpha)]$, there is a element o of O such that $[o] = [b]$. In fact, since O is complete, there is an element $o \in O$ such that $o \geq b$, and thus $[o] \geq [b]$. Maximality of $[b]$ thus yields $[o] = [b]$. Since O is finite, this implies that the set of maximal elements of $[\mathrm{Rep}(a, \alpha)]$ is finite as well. To show completeness of the set of maximal elements, consider an element $[b]$ of $[\mathrm{Rep}(a,\alpha)]$. Since O is complete, there is an element o_1 of O such that $o_1 \geq b$. If $[o_1]$ is maximal, then we are done since $[o_1] \geq [b]$. Otherwise, there is $[b_1]$ in $[\mathrm{Rep}(a,\alpha)]$ such that $[b_1] > [o_1]$. Since O is complete, there is an element o_2 of O such that $o_2 \geq b_1$. This way, we construct an increasing chain of elements satisfying $o_n \geq b_{n-1} > o_{n-1} \geq \ldots b_1 > o_1 \geq b$ as long as the elements $[o_i]$ are not maximal. Since O is finite, there cannot be an infinite strictly increasing chain in $[O]$, and thus we must reach an n such that $[o_n]$ is maximal. Since $[o_n] \geq [b]$, this shows that the set of maximal elements is complete.

"$3 \to 1$": If the finite set $\{[b_1], \ldots, [b_n]\}$ of maximal elements of $[\mathrm{Rep}(a,\alpha)]$ is complete, then we can define $\mathrm{Orep}(a,\alpha) := \{b_1, \ldots, b_n\}$. It is easy to see that this set satisfies the properties required by Definition 2 □

Note that there are two possible reasons that may be responsible for the third condition of this theorem to fail. On the one hand, the set of maximal elements of $[\mathrm{Rep}(a,\alpha)]$ may be complete, but not finite. On the other hand, the set of maximal elements may not be complete. The repair set $\mathrm{Rep}((1,0),0)$ in Example 2 provides us with an example for this second case.

If the repair sets are always finite up to equivalence, then the second condition of this theorem is always satisfied since we can take a set of representatives for $[\mathrm{Rep}(a,\alpha)]$ as finite complete set.

Corollary 1. *If* $\mathrm{Rep}(a,\alpha)$ *is finite up to equivalence for every input tuple* (a,α)*, then* Rep *satisfies the optimal repair property. This is in particular the case if* $\mathrm{Cone}(a)$ *is finite up to equivalence for every element* a *of* A*.*

3.2 Well-quasi-orders

Well-quasi-orders[3] can be used both to characterize the optimal repair property and finiteness of cones. We write $a \mid b$ if the elements a, b are incomparable w.r.t. $\geq$, i.e., if neither $a \geq b$ nor $b \geq a$ holds. An infinite sequence of elements $a_0, a_1, a_2, \ldots$ is an *infinite descending chain* if $a_i > a_{i+1}$ for all $i \geq 0$, and it is an *infinite anti-chain* if $a_i \mid a_j$ for all distinct $i, j \geq 0$.

Definition 4. *The quasi-order $\geq$ is a well-quasi-order (wqo) if there are neither infinite descending chains nor infinite anti-chains w.r.t. $\geq$.*

Obviously, any quasi-order on a finite set is a wqo. The following lemma provides us with an alternative characterizations of wqos.

Lemma 2. *Let $\geq$ be a quasi-order on A. Then the following are equivalent:*

1. *The quasi-order $\geq$ is a wqo.*
2. *For every subset $[B]$ of $[A]$, the set of its minimal elements is finite and* co-complete*, i.e., for every $[b'] \in [B]$ there is a minimal element $[b]$ of $[B]$ such that $[b'] \geq [b]$.*

[3] see [26,33] for more information on such orders.

Proof. "2 → 1": Assume that, for every subset of $[A]$, the set of its minimal elements is finite and co-complete, but $\geq$ is not a wqo. Then A contains an infinite sequence $a_0, a_1, a_2, \ldots$ that is either an infinite descending chain or an infinite anti-chain. In the former case, the set of minimal elements of $\{[a_0], [a_1], [a_2], \ldots\}$ is empty, and thus cannot be co-complete. In the latter case, all the elements of the set $\{[a_0], [a_1], [a_2], \ldots\}$ are minimal, and thus the set of its minimal elements is not finite. Hence, in both cases, our assumption that $\geq$ is not a wqo leads to a contradiction.

"1 → 2": Assume that $\geq$ is a wqo on A and that $[B]$ is a subset of $[A]$. Since the set $M = \{[m_0], [m_1], [m_2], \ldots\}$ of minimal elements of $[B]$ (enumerated in an arbitrary order[4]) yields an anti-chain $m_0, m_1, m_2, \ldots$, this set must be finite. Now, let $[b]$ be an element of $[B]$. If $[b]$ is minimal, then we are done. Otherwise, there is $[b_1]$ in $[B]$ such that $[b] > [b_1]$. If $[b_1]$ is minimal, then we are again done. Since B (and thus also $[B]$) cannot contain an infinite descending chain, we reach after finitely many steps a minimal element $[b_n]$ such that $[b] > [b_1] > \ldots > [b_{n-1}] > [b_n]$. This completes the proof that M is a finite and co-complete set. □

Comparing the second condition in Lemma 2 with the third condition of Theorem 1, we note that the latter requires the set of maximal elements to be finite and complete, whereas the former requires the set of minimal elements to be finite and co-complete. If we turn around the order, then these two conditions coincide. Given a quasi-order $\geq$, we denote the corresponding inverse quasi-order by $\geq^-$, i.e., $a \geq^- b$ iff $b \geq a$. Theorem 1 together with Lemma 2 thus yields the following characterization of the optimal repair property.

Theorem 2. *Let A be a set of KBs, $\geq$ the quasi-order on A corresponding to the entailment relation between KBs, and* Rep *a repair function that assigns to every tuple (a, α) consisting of an element $a \in A$ and a repair requests α a downward-closed subset* $\mathrm{Rep}(a, \alpha)$ *of* $\mathrm{Cone}(a)$*. Then the following are equivalent:*

1. Rep *satisfies the optimal repair property.*
2. *The inverse quasi-order $\geq^-$ is a wqo on* $\mathrm{Rep}(a, \alpha)$ *for every input tuple* (a, α).

Since the restriction of a wqo to a subset of its domain is again a wqo, requiring that $\geq^-$ is a wqo on $\mathrm{Cone}(a)$ for every $a \in A$ is a sufficient condition for the optimal repair property to be satisfied.

Requiring that $\geq$ itself is a wqo (on $\mathrm{Rep}(a, \alpha)$ or $\mathrm{Cone}(a)$) does not guarantee the optimal repair property, as demonstrated by Example 2. In fact, it is easy to see that the order $\geq^{lex}$ used there is a wqo since it is linear (and thus excluding anti-chains of size more than 1) and well-founded (and thus excluding infinite descending chains). Nevertheless, showing that $\geq$ is a wqo may be a useful step towards proving that $\mathrm{Cone}(a)$ is finite, which then yields the optimal repair property by Corollary 1. For this, we consider the one-step relation induced by $\geq$, as introduced in [15].

[4] If this set is uncountable, we just enumerate a countable subset.

Definition 5. *The* one-step relation[5] *induced by the quasi-order* $\geq$ *is defined as*

$$>_1 := \{(a,b) \in > \mid \text{ there is no } c \text{ such that } a > c > b\}.$$

We say that $>_1$ generates $\geq$ *if its reflexive-transitive closure is again* $\geq$*. In this case we also say that* $\geq$ *is* one-step generated.

If $\geq$ is one-step generated, then every smaller element can be reached by a finite one-step sequence, i.e., if $a > b$, then there are finitely many elements $c_0, \ldots, c_n$ ($n \geq 1$) such that $a = c_0 >_1 c_1 >_1 \ldots >_1 c_n = b$. This yields the following characterization of quasi-orders that are *not* one-step generated stated in [15].

Lemma 3 ([15]). *The quasi-order* $\geq$ *is* not *one-step generated iff there is a pair of strictly comparable elements* $a > b$ *such that every finite chain* $a = c_0 > c_1 > \ldots > c_n = b$ *can be* refined *in the sense that there is an* $i, 0 \leq i < n$*, and an element* c *such that* $c_i > c > c_{i+1}$.

Note that satisfaction of this condition does not imply that there is an infinite descending sequence starting with a. It only implies that there are descending sequences of arbitrary length between a and b. For example, the partial order $\geq^{lex}$ of Example 2 satisfies this condition. In fact, $(1,0) >^{lex} (0,0)$ and any finite $>^{lex}$-sequence between these two tuples is of the form $(1,0) >^{lex} (0,k_1) >^{lex} \ldots (0,k_{n-1}) >^{lex} (0,0)$ with $k_1 > \ldots > k_{n-1} > 0$. Choosing $c := (0, k_1 + 1)$, we obtain $(1,0) >^{lex} c >^{lex} (0,k_1)$.

Following [15], we say that the one-step generated quasi-order $\geq$ is *finitely branching* if for every element a the cardinality of the set $\{b \mid a >_1 b\}$ is finite up to equivalence. We are now ready to characterize quasi-orders for which all cones are finite up to equivalence.

Proposition 1. *The quasi-order* $\geq$ *is one-step generated, finitely branching, and well-founded iff* Cone(a) *is finite up to equivalence for every element* a.

Proof. "$\Leftarrow$" First, assume that $\geq$ is not one-step generated. By Lemma 3, this implies that there are elements $a > b$ such that there are descending sequences $a = c_0 > c_1 > \ldots > c_n = b$ of arbitrary length between a and b. However, since all the elements of such a sequence belong to Cone(a), the maximal length of such a sequence is bounded by the cardinality of a set of representatives for the equivalence classes in Cone(a), which is finite. Thus, $\geq$ is one-step generated. For every element a, the set $\{b \mid a >_1 b\}$ is a subset of Cone(a), and thus finite up to equivalence, which shows that $\geq$ is finitely branching. Similarly, any descending chain issuing from a is contained in Cone(a), and thus cannot be infinite.

"$\Rightarrow$" If $\geq$ is one-step generated, then every element of Cone(a) can be reached by a finite $>_1$-sequence from a. Thus, Cone(a) can be seen as a tree with root a and edges corresponding to $>_1$-relationships, where we merge equivalent elements into a single node. By König's lemma, finitely branching and well-founded imply that this tree is finite. □

[5] This is sometimes also called the transitive reduction of $\geq$ [1].

Obviously, if $\geq$ is a wqo, then it is also well-founded. In addition, since the representatives of the equivalence classes in the sets $\{b \mid a >_1 b\}$ are $>$-incomparable, these sets must be finite up to equivalence. Together with Corollary 1, the above proposition thus yields the following result.

Corollary 2. *Let A be a set of KBs, $\geq$ the quasi-order on A corresponding to the entailment relation between KBs, and* Rep *a repair function that assigns to every tuple (a, α) consisting of an element $a \in A$ and a repair requests α a downward-closed subset* $\mathrm{Rep}(a, \alpha)$ *of* $\mathrm{Cone}(a)$*. Then* Rep *satisfies the optimal repair property if $\geq$ is a one-step generated wqo.*

4 Instances satisfying the optimal repair property

As instances that satisfy the optimal repair property, we consider concepts of the DL $\mathcal{EL}$ as KBs and the subsumption relation (both without TBox and w.r.t. a cycle-restricted TBox) as entailment. In both cases, we will see that the optimal repair property holds since the cone of a given KB is finite up to equivalence. Since, for an $\mathcal{EL}$ concept C, its cone consist of the subsumers of C, we will denote it as $\mathrm{Subs}(C)$ (or $\mathrm{Subs}^{\mathcal{T}}(C)$ in the presence of a TBox $\mathcal{T}$). In these two cases, it turns out that proving finiteness of these subsumer sets directly is easier than resorting to Proposition 1. We will also see that, for non-cycle-restricted TBoxes, the optimal repair property need not hold.

4.1 Concepts of the DL $\mathcal{EL}$

When using subsumption $\sqsubseteq^{\emptyset}$ between concepts of the description logic $\mathcal{EL}$ as entailment relation, then we must consider the order-theoretic properties of the sets $\mathrm{Subs}(C)$ for $\mathcal{EL}$ concepts C. As already mentioned, $C \sqsubseteq^{\emptyset} D$ implies $\mathrm{Sig}(C) \supseteq \mathrm{Sig}(D)$, and thus all elements of $\mathrm{Subs}(C)$ are built over a fixed finite signature.

Lemma 4. *Let C, D be $\mathcal{EL}$ concepts. If $C \sqsubseteq^{\emptyset} D$, then* $\mathrm{Sig}(C) \supseteq \mathrm{Sig}(D)$*.*

Proof. We prove the contrapositive. Thus, assume that $\mathrm{Sig}(C) \not\supseteq \mathrm{Sig}(D)$. Define the interpretation $\mathcal{I}$ such that $\Delta^{\mathcal{I}} = \{d\}$, $A^{\mathcal{I}} = \{d\}$ or $A^{\mathcal{I}} = \emptyset$ depending on whether the concept name A belongs to $\mathrm{Sig}(C)$ or not, and $r^{\mathcal{I}} = \{(d, d)\}$ or $r^{\mathcal{I}} = \emptyset$ depending on whether the role name r belongs to $\mathrm{Sig}(C)$ or not. It is easy to see that $C^{\mathcal{I}} = \{d\}$ and $D^{\mathcal{I}} = \emptyset$, the latter holds since D contains a symbol (concept name A or role name r) not belonging to $\mathrm{Sig}(C)$, and thus A or any existential restriction for r is interpreted as the empty set. This shows that the subsumption $C \sqsubseteq^{\emptyset} D$ cannot hold. □

In addition, the role depth $\mathrm{rd}(D)$ of the elements $D \in \mathrm{Subs}(C)$ is bounded by the role depth of C, where the role depth of an $\mathcal{EL}$ concept C is the maximal nesting of existential restrictions in C.

Lemma 5. *Let C, D be $\mathcal{EL}$ concepts. If $C \sqsubseteq^{\emptyset} D$, then* $\mathrm{rd}(D) \leq \mathrm{rd}(C)$*.*

Proof. This is a consequence of the following observations, which are easy to see:

1. If $\mathrm{rd}(D) = n$, then there are role names $r_1, \ldots, r_n$ s.t. $D \sqsubseteq^{\emptyset} \exists r_1. \cdots \exists r_n.\top$.
2. If $\mathrm{rd}(C) < n$, then there is a tree-shaped interpretation $\mathcal{I}$ of depth smaller than n with root d s.t. $d \in C^{\mathcal{I}}$, and thus $C \not\sqsubseteq^{\emptyset} \exists r_1. \cdots \exists r_n.\top$ for any sequence of role names $r_1, \ldots, r_n$.

We can now show the contrapositive of the statement of the lemma. Assume that $n := \mathrm{rd}(D) > \mathrm{rd}(C)$. If $C \sqsubseteq^{\emptyset} D$, then the first observation implies $C \sqsubseteq^{\emptyset} D \sqsubseteq^{\emptyset} \exists r_1. \cdots \exists r_n.\top$ for an appropriate sequence $r_1, \ldots, r_n$ of role names, which contradicts the second observation since $\sqsubseteq^{\emptyset}$ is transitive. Thus, $C \sqsubseteq^{\emptyset} D$ cannot be the case. □

It is well-known that, for a finite signature and variable set, there are up to equivalence only finitely many first-order formulas of a given fixed bound on the quantifier depth. In the context of DLs, this results was, e.g., formulated in [17] for the DL $\mathcal{ALE}$, which contains $\mathcal{EL}$: for given finite sets of concept and role names, there are up to equivalence only finitely many concepts for a given fixed upper-bound on the role depth of the concepts. The two lemmas shown above thus provide us with the following result.

Proposition 2. *If C is an $\mathcal{EL}$ concept, then* $\mathrm{Subs}(C)$ *is finite up to equivalence.*

By Corollary 1, this implies that the optimal repair property is satisfied both for $\mathrm{Rep}_{\mathrm{ent}}$ and $\mathrm{Rep}_{\mathrm{for}}$. However, since these results are special cases of the results shown in the next subsection (see Corollaries 3 and 4, and note that the empty TBox is cycle-restricted), we do not formulate them as separate corollaries here.

4.2 Concepts of the DL $\mathcal{EL}$ w.r.t. a cycle-restricted TBox

Given an $\mathcal{EL}$ TBox $\mathcal{T}$, we define $\mathrm{Subs}^{\mathcal{T}}(C) := \{D \mid C \sqsubseteq^{\mathcal{T}} D\}$, and extend the notion of entailment repairs to $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(C, D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$. For arbitrary TBoxes, finiteness up to equivalence of the set $\mathrm{Subs}^{\mathcal{T}}(C)$ cannot be guaranteed.

Example 3. For the TBox $\mathcal{T} := \{A \sqsubseteq \exists r.A\}$, the concepts $C_n := \exists r. \cdots \exists r.\top$, which are nestings of $n \geq 0$ existential restrictions with the top concept $\top$ at the end, are infinitely many $\mathcal{EL}$ concepts in $\mathrm{Subs}^{\mathcal{T}}(A)$ that are pairwise non-equivalent w.r.t. $\mathcal{T}$. In fact, we have $C_{n+1} \sqsubseteq^{\mathcal{T}} C_n$, but $C_n \not\sqsubseteq^{\mathcal{T}} C_{n+1}$.

If we consider the repair set $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$, then we see that all the concepts C_n are contained in this set. Thus, we have a situation that is similar to the one encountered in Example 2. Nevertheless, the optimal repair property is still satisfied since the set of repairs also contains $D := \exists r.A$, which satisfies $D \sqsubseteq^{\mathcal{T}} C_n$ for all $n \geq 0$. In fact, $\{D\}$ is a set of repairs that covers all repairs. To show this, it is enough to prove that $D \sqsubseteq^{\mathcal{T}} E$ holds for every element E of $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$.

To prove the claim "$D \sqsubseteq^{\mathcal{T}} E$" made in Example 3, we make use of the recursive characterization of subsumption in $\mathcal{EL}$ w.r.t. TBoxes given in Lemma 1.

Lemma 6. *Let $\mathcal{T} := \{A \sqsubseteq \exists r.A\}$, and $D := \exists r.A$. Then $D \sqsubseteq^{\mathcal{T}} E$ holds for all $E \in \mathrm{Rep}^{\mathcal{T}}_{ent}(A, A)$.*

Proof. Note that the only atoms of $\mathcal{T}$ are A and $\exists r.A$. From $E \in \mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$ we obtain that $A \sqsubseteq^{\mathcal{T}} E$ holds. Thus, according to Lemma 1, every atom F in the top-level conjunction of E satisfies $A \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} F$ or $\exists r.A \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} F$. The former structural subsumption implies $A = F$, and thus $D \sqsubseteq^{\mathcal{T}} A$, which contradicts our assumption that $E \in \mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$. Thus, $\exists r.A \sqsubseteq^{\mathcal{T}}_{\mathsf{s}} F$ holds for all atoms F in the top-level conjunction of E, which yields $D = \exists r.A \sqsubseteq^{\mathcal{T}} E$, again by Lemma 1. □

Lemma 6 shows that repair sets that contain infinite increasing[6] chains, as in Example 3, may nevertheless satisfy the optimal repair property. However, in general this property may be violated in the presence of general $\mathcal{EL}$ TBoxes, as illustrated by the following variant of Example 3, which is similar to the example used in the proof of Proposition 1 in [15].

Example 4. Let $\mathcal{T} := \{A \sqsubseteq \exists r.A, \exists r.A \sqsubseteq A\}$ and consider $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$. We claim that this repair set is non-empty, but does not contain an optimal repair. Obviously, all the concepts C_n introduced in Example 3 belong to $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$. In addition, it is easy to see that no concept in $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$ can contain the concept name A. In fact, concepts subsuming A w.r.t. $\mathcal{T}$ must be built using only A, r, and $\top$, and if such a concept contains A then it is also subsumed by A w.r.t. $\mathcal{T}$. Thus, the concepts in $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$ must be built using only $\top$ and r. For this reason, $\sqsubseteq^{\mathcal{T}}$ coincides with $\sqsubseteq^{\emptyset}$ on $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$. Now assume that C is an optimal repair in $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$, and let m be its role depth. It is easy to see that the concept $C \sqcap C_{m+1}$ satisfies $A \sqsubseteq^{\mathcal{T}} C \sqcap C_{m+1} \sqsubseteq^{\mathcal{T}} C$ and $C \sqcap C_{m+1} \not\sqsubseteq^{\mathcal{T}} A$. The latter non-subsumption follows from the fact that neither C nor C_{m+1} contains A. Consequently, $C \sqcap C_{m+1} \in \mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$, and thus optimality of C yields $C \sqcap C_{m+1} \equiv^{\mathcal{T}} C$, which implies $C \sqsubseteq^{\mathcal{T}} C_{m+1}$ and thus $C \sqsubseteq^{\emptyset} C_{m+1}$. Lemma 5 shows that the latter subsumption is not possible since the role depth of C_{m+1} is larger than that of C. Thus, our assumption that $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(A, A)$ contains an optimal repair has been refuted.

Finiteness of $\mathrm{Subs}^{\mathcal{T}}(C)$ holds, however, if we restrict the attention to cycle-restricted TBoxes (see Definition 1). The following lemma can easily be shown by adapting results for entailment between qABoxes from [12].

Lemma 7. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept. Then one can compute in at most exponential time an $\mathcal{EL}$ concept $\mathrm{sat}^{\mathcal{T}}(C)$ such that $C \sqsubseteq^{\mathcal{T}} D$ iff $\mathrm{sat}^{\mathcal{T}}(C) \sqsubseteq^{\emptyset} D$ holds for all $\mathcal{EL}$ concepts D.*

This lemma implies that $\mathrm{Subs}^{\mathcal{T}}(C) = \mathrm{Subs}(\mathrm{sat}^{\mathcal{T}}(C))$. By Proposition 2, $\mathrm{Subs}(\mathrm{sat}^{\mathcal{T}}(C))$ is finite even up to equivalence w.r.t. the empty TBox, and thus all the more up to equivalence w.r.t. $\mathcal{T}$.

[6] Recall that in this setting the quasi-order $\geq$ induced by the entailment relation corresponds to $\sqsubseteq^{\mathcal{T}}$, and thus a chain that is decreasing w.r.t. subsumption is increasing w.r.t. this quasi-order.

Proposition 3. *If $\mathcal{T}$ is a cycle-restricted $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept, then* $\mathrm{Subs}^{\mathcal{T}}(C)$ *is finite up to equivalence.*

As an immediate consequence of Corollary 1 we thus obtain that $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}$ has the optimal repair property.

Corollary 3. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}^{\mathcal{T}}_{ent}$ the repair function that is defined as $\mathrm{Rep}^{\mathcal{T}}_{ent}(C,D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$. Then $\mathrm{Rep}^{\mathcal{T}}_{ent}$ satisfies the optimal repair property.*

When trying to extend the definition of $\mathrm{Rep}_{\mathrm{for}}$ to the setting with a TBox $\mathcal{T}$, one must be careful since, in contrast to the case of $\sqsubseteq^{\emptyset}$, subsumption w.r.t. $\mathcal{T}$ may not preserve the signature, i.e., Lemma 4 does no hold if we replace $\sqsubseteq^{\emptyset}$ with $\sqsubseteq^{\mathcal{T}}$. Thus, the condition that repair sets must be closed under entailment would not be satisfied if we used a straightforward extension of $\mathrm{Rep}_{\mathrm{for}}$ to $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}$ in which $\sqsubseteq^{\mathcal{T}}$ replaces $\sqsubseteq^{\emptyset}$.

In [19], this problem is addressed by imposing a compatibility condition on the TBox and the signature used as repair request, which ensures that Lemma 4 holds also w.r.t. $\sqsubseteq^{\mathcal{T}}$. Here, we restrict the entailment relation instead, i.e., we define $C \sqsubseteq^{\mathcal{T}}_{\supseteq} D$ if $C \sqsubseteq^{\mathcal{T}} D$ and $\mathrm{Sig}(C) \supseteq \mathrm{Sig}(D)$, set $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}(C,\alpha) := \{C' \mid C \sqsubseteq^{\mathcal{T}}_{\supseteq} C', \mathrm{Sig}(C') \cap \alpha = \emptyset\}$, and use $\sqsubseteq^{\mathcal{T}}_{\supseteq}$ rather than $\sqsubseteq^{\mathcal{T}}$ when comparing repairs in the definition of the optimal repair property. Since the set $\mathrm{Subs}^{\mathcal{T}}(C)_{\supseteq} := \{D \mid C \sqsubseteq^{\mathcal{T}}_{\supseteq} D\}$ is a subset of $\mathrm{Subs}^{\mathcal{T}}(C)$, it is clearly also finite up to equivalence.[7] Thus, Corollary 1 yields the following result.

Corollary 4. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}^{\mathcal{T}}_{for}$ the repair function that is defined as $\mathrm{Rep}^{\mathcal{T}}_{for}(C,\alpha) := \{C' \mid C \sqsubseteq^{\mathcal{T}}_{\supseteq} C', \mathrm{Sig}(C') \cap \alpha = \emptyset\}$. Then $\mathrm{Rep}^{\mathcal{T}}_{for}$ satisfies the optimal repair property.*

5 Beyond the optimal repair property

Considering Item 3 of Theorem 1, we see that there are two possible reasons for the optimal repair property to fail: the set of all maximal elements of $[\mathrm{Rep}(a,\alpha)]$ may still be complete, but not finite, or this set may not even be complete. In case the optimal repair property holds, we can also distinguish two cases, depending on whether $[\mathrm{Rep}(a,\alpha)]$ has a unique greatest element or more than one maximal element.

Following the terminology employed in unification theory [18], we distinguish between the following types of repair problems.

Definition 6. *Let A be a set of KBs, $\geq$ the quasi-order on A corresponding to the entailment relation between KBs, and* Rep *a repair function that assigns to every repair problem (a,α) consisting of an element $a \in A$ and a repair requests α a downward-closed subset $\mathrm{Rep}(a,\alpha)$ of $\mathrm{Cone}(a)$. The repair problem (a,α) is*

[7] Note that, due to Lemma 4, equivalence w.r.t. the empty TBox is contained in the equivalence relation induced by $\sqsubseteq^{\mathcal{T}}_{\supseteq}$.

– unitary *if* $[\mathrm{Rep}(a,\alpha)]$ *has a greatest element (i.e., the set of its maximal elements has cardinality* 1*), and this greatest element covers* $[\mathrm{Rep}(a,\alpha)]$*,*
– finitary *if the set of maximal elements of* $[\mathrm{Rep}(a,\alpha)]$ *is finite, but not a singleton set, and this set covers* $[\mathrm{Rep}(a,\alpha)]$*,*
– infinitary *if the set of maximal elements of* $[\mathrm{Rep}(a,\alpha)]$ *is infinite, and this set covers* $[\mathrm{Rep}(a,\alpha)]$*,*
– of type zero *if the set of maximal elements of* $[\mathrm{Rep}(a,\alpha)]$ *does not cover* $[\mathrm{Rep}(a,\alpha)]$.

Unitary and finitary repair problems satisfy the optimal repair property, whereas infinitary and type zero repair problems do not.

We call a repair problem *solvable* if it has at least one repair, and *unsolvable* otherwise. Obviously, unsolvable repair problems satisfy the optimal repair property since the empty set of maximal repairs covers the empty set of repairs. However, such problems do not have any of the types introduced in the above definition. For solvable repair problems, it is easy to see that they must have one of the four types introduced there.

Using the distinction between unitary and finitary repair problems, we can now give a more fine-grained version of Corollary 3.

Proposition 4. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}^{\mathcal{T}}_{ent}$ the repair function that is defined as $\mathrm{Rep}^{\mathcal{T}}_{ent}(C,D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$. Then every solvable repair problem for $\mathrm{Rep}^{\mathcal{T}}_{ent}$ is either unitary or finitary, and both types can occur. The latter is already true for the empty TBox.*

Proof. The first part of the proposition is an immediate consequence of Corollary 3. To show the second part, we present both a unitary and a finitary repair problem for $\mathcal{T} = \emptyset$. First, consider $C = A \sqcap B = D$. It is easy to see that, up to equivalence, $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(C,D)$ consists of A, B, and $\top$, where A and B are maximal. Thus, this repair problem is finitary. Second, consider $C = A = D$. It is easy to see that, up to equivalence, $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{ent}}(C,D)$ consists of $\top$, and thus this repair problem is unitary. □

For $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}$, we can strengthen Corollary 4 by showing that all solvable repair problems are unitary if $\mathcal{T}$ is cycle-restricted.

Proposition 5. *Let $\mathcal{T}$ be a cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}^{\mathcal{T}}_{for}$ the repair function that is defined as $\mathrm{Rep}^{\mathcal{T}}_{for}(C,\alpha) := \{C' \mid C \sqsubseteq^{\mathcal{T}}_{\supseteq} C', \mathrm{Sig}(C') \cap \alpha = \emptyset\}$. Then every solvable repair problem for $\mathrm{Rep}^{\mathcal{T}}_{for}$ is unitary.*

Proof. Let (C,α) by a solvable repair problem for $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}$, i.e., $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}(C,\alpha) \neq \emptyset$. By Corollary 4, we know that this repair problem is unitary or finitary. Assume that it is finitary. Then there are two elements C_1, C_2 in $\mathrm{Rep}^{\mathcal{T}}_{\mathrm{for}}(C,\alpha)$ that are maximal w.r.t. $\sqsubseteq^{\mathcal{T}}_{\supseteq}$ and incomparable w.r.t. $\sqsubseteq^{\mathcal{T}}_{\supseteq}$. Since both C_1 and C_2 subsume C w.r.t. $\mathcal{T}$, we know that $C \sqsubseteq^{\mathcal{T}} C_1 \sqcap C_2$. In addition, $\mathrm{Sig}(C) \supseteq \mathrm{Sig}(C_i)$ for $i = 1,2$ yields $\mathrm{Sig}(C) \supseteq \mathrm{Sig}(C_1) \cup \mathrm{Sig}(C_2) = \mathrm{Sig}(C_1 \sqcap C_2)$. Thus, we have shown that $C \sqsubseteq_{\supseteq}^{\mathcal{T}} C_1 \sqcap C_2$. Finally, since neither C_1 nor C_2 contains an element of α, $C_1 \sqcap C_2$ also does not. This shows that $C_1 \sqcap C_2 \in \mathrm{Rep}_{\mathrm{for}}^{\mathcal{T}}(C, \alpha)$.

It is also easy to see that $C_1 \sqcap C_2 \sqsubseteq_{\supseteq}^{\mathcal{T}} C_i$ for $i = 1, 2$. Maximality of C_1 and C_2 implies that these entailments cannot be strict. Thus, C_1, $C_1 \sqcap C_2$, and C_2 must be equivalent w.r.t. $\sqsubseteq_{\supseteq}^{\mathcal{T}}$, which contradicts our assumption that C_1 and C_2 are incomparable. Thus, two such incomparable maximal elements cannot exist in $\mathrm{Rep}_{\mathrm{for}}^{\mathcal{T}}(C, \alpha)$, which shows that the repair problem (C, α) is unitary. □

For a non-cycle-restricted TBox, repair problems may also be of type unitary or finitary. In fact, Example 3 provides us with a repair problem (A, A) for the non-cycle-restricted TBox $\mathcal{T} = \{A \sqsubseteq \exists r.A\}$ that is unitary, although $\mathrm{Rep}_{\mathrm{ent}}^{\mathcal{T}}(A, A)$ is not finite. However, we have shown in Example 4 that the same repair problem has type zero if one uses the non-cycle-restricted TBox $\{A \sqsubseteq \exists r.A, \exists r.A \sqsubseteq A\}$ instead. We will show now that in this setting (i.e., $\mathcal{EL}$ concepts w.r.t. an arbitrary TBox and $\mathrm{Rep}_{\mathrm{ent}}$) type infinitary cannot occur.

Proposition 6. *Let $\mathcal{T}$ be a non-cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}_{ent}^{\mathcal{T}}$ the repair function that is defined as $\mathrm{Rep}_{ent}^{\mathcal{T}}(C, D) := \{C' \mid C \sqsubseteq^{\mathcal{T}} C', C' \not\sqsubseteq^{\mathcal{T}} D\}$. Then every solvable repair problem for $\mathrm{Rep}_{ent}^{\mathcal{T}}$ is unitary, finitary, or of type zero, and all three types can occur.*

Proof. We have already seen in Example 3 and Example 4, respectively, that type unitary and type zero may occur for non-cycle-restricted TBoxes. It is easy to adapt the example of a finitary repair problem from the proof of Proposition 4 to work also w.r.t. a non-cycle-restricted TBox by replacing the empty TBox with $\mathcal{T} = \{A' \sqsubseteq \exists r.A'\}$, which does not influence the repair problem $\mathrm{Rep}_{\mathrm{ent}}^{\mathcal{T}}(A \sqcap B, A \sqcap B)$.

It remains to show that type infinitary cannot occur. Thus, assume to the contrary that $\mathcal{T}$ is a non-cycle-restricted $\mathcal{EL}$ TBox and (C, D) an infinitary repair problem for $\mathrm{Rep}_{\mathrm{ent}}^{\mathcal{T}}$. One can view this repair problem as a repair problem for quantified ABoxes (qABoxes) [12], where the input qABox is the qABox representation of the $\mathcal{EL}$ ABox $\{C(a)\}$ and the repair request is $D(a)$. If repairs are required to be finite qABoxes, then the optimal repair property need not be satisfied. However, if one allows for rational qABoxes (which is a restricted class of infinite qABoxes), then it is satisfied again [11]. Thus, there is a finite set of rational qABoxes $\{\exists X_1.\mathcal{A}_1, \ldots, \exists X_n.\mathcal{A}_n\}$ that are entailed by (the qABox representation of) $\{C(a)\}$, do not entail $D(a)$, and cover all repairs. Since the repair problem (C, D) is infinitary, there are two $\sqsubseteq^{\mathcal{T}}$-incomparable maximal elements C_1, C_2 of $\mathrm{Rep}_{\mathrm{ent}}^{\mathcal{T}}(C, D)$ such that $\{C_1(a)\}$ and $\{C_2(a)\}$ are entailed by the same rational qABox $\exists X_i.\mathcal{A}_i$. Consequently, $\exists X_i.\mathcal{A}_i$ also entails $\{(C_1 \sqcap C_2)(a)\}$. Since $\exists X_i.\mathcal{A}_i$ does not entail $D(a)$, neither can $\{(C_1 \sqcap C_2)(a)\}$, which shows that $C_1 \sqcap C_2 \in \mathrm{Rep}_{\mathrm{ent}}^{\mathcal{T}}(C, D)$. Maximality of C_1 and C_2 implies that C_1, $C_1 \sqcap C_2$, and C_2 must be equivalent w.r.t. $\sqsubseteq^{\mathcal{T}}$, which contradicts our assumption that C_1 and C_2 are incomparable. □

For $\mathrm{Rep}_{\mathrm{for}}^{\mathcal{T}}$, neither type finitary nor type infinitary is possible.

Proposition 7. *Let $\mathcal{T}$ be a non-cycle-restricted $\mathcal{EL}$ TBox and $\mathrm{Rep}^{\mathcal{T}}_{for}$ the repair function that is defined as $\mathrm{Rep}^{\mathcal{T}}_{for}(C,\alpha) := \{C' \mid C \sqsubseteq^{\mathcal{T}}_{\sqsupseteq} C', \mathrm{Sig}(C') \cap \alpha = \emptyset\}$. Then every solvable repair problem for $\mathrm{Rep}^{\mathcal{T}}_{for}$ is either unitary or of type zero, and both types can occur.*

Proof. To obtain an example of a unitary repair problem, one just takes one for the empty TBox, and adds a cyclic GCI that does not influence the repair problem. For type zero, one can re-use Example 4, but now employ $\alpha := \{A\}$ as repair request. The proof that neither type finitary nor type infinitary is possible is identical to the one of Proposition 5. □

6 Unification as an instance

Given a *signature* Σ consisting of a finite set of function symbols (with associated arities) and a countably infinite set of *variables* V, the set $T(\Sigma, V)$ of *terms* over Σ with variables in V is defined in the usual way [18]. An equational theory E is given by a finite set of identities $s \approx t$ between terms, which are (implicitly) assumed to be universally quantified. Such a set of identities E induces the congruence relation $\approx_E$ on terms, which can either be defined syntactically through rewriting or semantically through first-order interpretations of Σ, with $\approx$ as identity relation [16].

A *substitution* σ is a mapping from V to $T(\Sigma, V)$ that has a finite *domain* $\mathrm{Dom}(\sigma) := \{x \in V \mid \sigma(x) \neq x\}$. It can be homomorphically extended to a mapping from $T(\Sigma, V)$ to $T(\Sigma, V)$ by defining $\sigma(f(t_1, \ldots, t_n)) := f(\sigma(t_1), \ldots, \sigma(t_n))$. The *variable range* $\mathrm{VRan}(\sigma)$ of σ consists of the set of variables occurring in the terms $\sigma(x)$ for $x \in \mathrm{Dom}(\sigma)$. Substitutions can be compared using the instantiation quasi-order: given an equational theory E, a finite set of variables $X \subseteq V$, and two substitutions σ, τ, we say that σ is *more general* than τ (or τ is an *instance* of σ) w.r.t. E and X (written $\sigma \geq^X_E \tau$)[8] if there is a substitution λ such that $\lambda(\sigma(x)) \approx_E \tau(x)$ holds for all $x \in X$.

We use substitutions as knowledge bases and the instantiation quasi-order as entailment. Repair requests are unification problems. Let E be an equational theory and $X \subseteq V$ a finite set of variables. A *unification problem* for E and X is a finite set of equations $\Gamma = \{s_1 \approx^?_E t_1, \ldots, s_n \approx^?_E t_n\}$ such that $s_1, t_1, \ldots, s_n, t_n$ are terms containing only variables from X. An *E-unifier* of Γ is a substitution θ that solves all the equations in Γ, i.e., satisfies $\theta(s_i) \approx_E \theta(t_i)$ for all $i, 1 \leq i \leq n$. The unification problem Γ is *solvable* if it has an E-unifier. Given a substitution σ with $\mathrm{VRan}(\sigma) \subseteq X$ and a unification problem Γ for E and X, we define

$$\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma) := \{\theta \mid \sigma \geq^X_E \theta \text{ and } \theta \text{ is an } E\text{-unifier of } \Gamma\}.$$

[8] In unification theory [18], the order is usually written the other way round, i.e., more general substitutions are the smaller ones, but here it is more convenient to consider instances to be the smaller substitutions since this is more in line with our representation of entailment as a quasi-order $\geq$ in Section 3.

In unification theory, one usually considers only a unification problem Γ (without an additional substitution σ), and looks for E-unifiers of Γ. This is the special case of the repair problem defined above where σ is the identity substitution σ_{id} with empty domain.

Lemma 8. $\mathrm{Rep}^E_{uni}(\sigma_{id}, \Gamma) = \{\theta \mid \theta$ *is an* E*-unifier of* $\Gamma\}$.

Proof. This is an immediate consequence of the fact that every substitution is an instance of the identity substitution w.r.t. any equational theory E and finite set of variables X. □

In the case of a substitution σ with non-empty domain, we can find elements of $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ by considering the unification problem

$$\sigma(\Gamma) := \{\sigma(s_1) \approx^?_E \sigma(t_1), \ldots, \sigma(s_n) \approx^?_E \sigma(t_n)\}$$

and then use the E-unifiers of this problem to construct the instances of σ that are unifiers of Γ. Before we can make this idea more formal, we need to introduce some notation. Given two substitutions σ and λ, their *composition* $\lambda\sigma$ is the substitution obtained by first applying σ and then λ. We write $\sigma \approx^X_E \theta$ if $\sigma(x) \approx_E \theta(x)$ for all $x \in X$. The relation $\approx^X_E$ can be extended to sets of substitutions $\mathcal{S}_1$ and $\mathcal{S}_2$ by setting $\mathcal{S}_1 \approx^X_E \mathcal{S}_2$ if these sets yield the same $\approx^X_E$ equivalence classes, i.e., for every element of $\mathcal{S}_1$ there is an $\approx^X_E$-equivalent element in $\mathcal{S}_2$ and vice versa.

Lemma 9. $\mathrm{Rep}^E_{uni}(\sigma, \Gamma) \approx^X_E \{\lambda\sigma \mid \lambda$ *is an* E*-unifier of* $\sigma(\Gamma)\}$.

Proof. First, assume that θ is an element of $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$. Then θ is an E-unifier of Γ and there exists a substitution λ such that $\theta(x) \approx_E \lambda(\sigma(x))$ holds for all $x \in X$. Note that this means that $\theta \approx^X_E \lambda\sigma$. Thus, to show that the right-hand side of the identity in the formulation of the lemma contains a substitution that is $\approx^X_E$-equivalent to θ, it is sufficient to prove that λ is an E-unifier of $\sigma(\Gamma)$. Assume that $\Gamma = \{s_1 \approx^?_E t_1, \ldots, s_n \approx^?_E t_n\}$. Since the terms in Γ contain only variables from X, we thus have for all $i, 1 \leq i \leq n$:

$$\lambda(\sigma(s_i)) \approx_E \theta(s_i) \approx_E \theta(t_i) \approx_E \lambda(\sigma(t_i)),$$

which shows that λ is an E-unifier of $\sigma(\Gamma)$.

Second, assume that λ is an E-unifier of $\sigma(\Gamma)$. Then $\lambda(\sigma(s_i)) \approx_E \lambda(\sigma(t_i))$ holds for all $i, 1 \leq i \leq n$, which shows that $\lambda\sigma$ is an E-unifier of Γ. Since $\sigma \geq^X_E \lambda\sigma$ obviously holds, we thus obtain that $\lambda\sigma$ is an element of $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$. □

Unification types are defined analogously to repair types. Let X be a finite set of variables and E an equational theory. Given a unification problem Γ for E and X, we say that a set $\mathcal{S}$ of substitutions is a *complete set* of E-unifiers of Γ if it consists of E-unifiers, and every E-unifier of Γ is an instance of an element of the complete set, i.e., for every E-unifier θ of Γ there exists $\tau \in \mathcal{S}$ such that $\tau \geq^X_E \theta$. Such a set is called *minimal* if it does not contain two distinct elements that are comparable w.r.t. $\geq_E^X$. It is easy to see that minimal complete sets of E-unifiers of a given unification problem Γ are unique up to the equivalence relation induced by the quasi-order $\geq_E^X$ (see, e.g., Corollary 3.13 in [18]), and thus all have the same cardinality.

Definition 7. *Let Γ be a solvable unification problem for E and X. Then Γ has* unification type

– unitary *if it has a minimal complete set of E-unifiers of cardinality one,*
– finitary *if it has a finite minimal complete set of E-unifiers of cardinality greater than one,*
– infinitary *if it has an infinite minimal complete set of E-unifiers,*
– zero *if it does not have a minimal complete set of E-unifiers, i.e., every complete set of E-unifiers is redundant in the sense that it must contain two distinct elements that are comparable w.r.t. $\geq_E^X$.*

Using the order-theoretic characterization of unification types given in Section 3.3.1 of [18], it is easy to show the following connection between repair and unification types. Basically, this characterization states a correspondence between minimal complete sets of unifiers and sets of representatives of the maximal[9] elements of the set of unifiers, which is in line with our Definition 6 of repair types.

Proposition 8. *Let Γ be a unification problem for the equational theory E and X a finite set of variables containing all the variables in Γ. Then the repair type of (σ_{id}, Γ) for Rep_{uni}^E coincides with the unification type of Γ if this unification problem is solvable. Otherwise, both the repair problem and the unification problem Γ are unsolvable.*

In the general case with arbitrary substitution σ, we must consider the unification type of $\sigma(\Gamma)$, but the correspondence need not be exact and does not hold for all types. We order unification types w.r.t. how "bad" they are (larger is worse) by setting

$$\text{type zero} > \text{infinitary} > \text{finitary} > \text{unitary}.$$

Proposition 9. *Let σ be a substitution with $\mathrm{VRan}(\sigma) \subseteq X$ and Γ a unification problem for the equational theory E and the finite set of variables X. Then the repair type of (σ, Γ) for Rep_{uni}^E is not worse than the unification type of $\sigma(\Gamma)$ if this unification problem is solvable and of type unitary or finitary. Unsolvability of the repair problem (σ, Γ) implies unsolvability of the unification problem $\sigma(\Gamma)$ and vice versa.*

Proof. The main observation on which this result is based is the following: if λ_1, λ_2 are substitutions such that $\lambda_1 \geq_E^X \lambda_2$, then $\lambda_1\sigma \geq_E^X \lambda_2\sigma$. To see this

[9] Note that in [18] these are the minimal elements since the instantiation quasi-order is written the other way round.

assume that $x \in X$ and that $\lambda_2 \approx_E^X \lambda\lambda_1$. Since $\mathrm{VRan}(\sigma) \subseteq X$, this implies $\lambda_2(\sigma(x)) \approx_E \lambda(\lambda_1(\sigma(x)))$, and since this holds for all $x \in X$, we obtain $\lambda_2\sigma \approx_E^X \lambda(\lambda_1\sigma)$ as required.

Now, assume that $\sigma(\Gamma)$ is solvable and unitary. Then there is a complete set $\{\theta\}$ of E-unifiers of $\sigma(\Gamma)$. We claim that $\theta\sigma$ is an optimal repair for $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ and it covers all repairs. It is sufficient to show the latter since optimality of $\theta\sigma$ is an immediate consequence of this. Thus, assume that $\tau \in \mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$. Then there exists a substitution λ such that $\lambda\sigma \approx_E^X \tau$ and τ is an E-unifier of Γ. It is easy to see that this implies that λ is an E-unifier of $\sigma(\Gamma)$ (see the proof of Lemma 9), and thus $\theta \geq_E^X \lambda$. By the observation made above, this yields $\theta\sigma \geq_E^X \lambda\sigma \approx_E^X \tau$, and thus τ is an instance of $\theta\sigma$. This shows that the repair type of (σ, Γ) is unitary, and thus not worse than the unification type of $\sigma(\Gamma)$.

If $\sigma(\Gamma)$ is solvable and finitary, then there is a finite complete set $\{\theta_1, \ldots, \theta_n\}$ of E-unifiers of $\sigma(\Gamma)$. As in the unitary case, we can show that $\{\theta_1\sigma, \ldots, \theta_n\sigma\}$ is a set of repairs of $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ that covers all repairs. By removing repairs that are not optimal from this set, we obtain a finite set of optimal repairs that covers all repairs, and thus the repair type of (σ, Γ) is unitary or finitary.

Finally, we have observed above that an element τ of $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ yields an E-unifier λ of $\sigma(\Gamma)$. Consequently, if $\sigma(\Gamma)$ is not solvable, then there cannot be a solution of the repair problem, i.e., $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ must be empty. Conversely, any solution λ of $\sigma(\Gamma)$ clearly yields a repair $\lambda\sigma$ in $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$. Thus, if $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma) = \emptyset$, then $\sigma(\Gamma)$ cannot be solvable. □

One could ask why we cannot show that the types are exactly preserved when going from the unification problem $\sigma(\Gamma)$ to the repair problem (σ, Γ) for $\mathrm{Rep}^E_{\mathrm{uni}}$. The reason is that in general $\lambda_1\sigma \geq_E^X \lambda_2\sigma$ need not imply $\lambda_1 \geq_E^X \lambda_2$. Thus, it could be the case that two repairs in $\mathrm{Rep}^E_{\mathrm{uni}}(\sigma, \Gamma)$ are comparable, whereas the corresponding unifiers of $\sigma(\Gamma)$ are not. This opens the possibility that unification type finitary becomes repair type unitary, but also that unification type infinitary (zero) becomes repair type zero (infinitary), though concrete examples where this happens still need to be found. However, the take-home message of the above proposition is that positive results (unitary, finitary) transfer from the unification to the repair problem.

Research in unification theory has produced many results on the unification types of specific equational theories, where the *unification type of an equational theory* is the worst type of a unification problem for this theory (see [18] for an overview). For example, the empty theory is unitary [32], commutativity of a binary function symbol is finitary [37], associativity of a binary function symbol is infinitary [30], and associativity and idempotency of a binary function symbol is of type zero [3,36]. In particular, associativity yields an example of a repair problem whose type is infinitary.

Example 5. Let f be a binary function symbol and a a constant symbol, define $A := \{f(x, f(y, z)) \approx f(f(x, y), z)\}$, and consider the unification problem $\Gamma := \{f(x, a) \approx^?_A f(a, x)\}$ for A and $X := \{x\}$. Then all unifiers of Γ map x to a term built from f and a only. Let θ_n be the unifier with domain $\{x\}$ that maps x to the term $f(a, f(a, \ldots, f(a, a) \ldots))$ with n occurrences of a. Then it is easy to see that the set $\{\theta_1, \theta_2, \theta_3, \ldots\}$ is an infinite minimal complete set of A-unifiers of Γ. Consequently, by Proposition 8, the repair problem (σ_{id}, Γ) is infinitary, which is a repair type that we did not encounter in the context of $\mathcal{EL}$ concepts.

The results on the unification type of equational theories in the literature are shown under the assumption that the instantiation quasi-order on substitutions is defined w.r.t. the finite set of variables occurring in the unification problem (restricted instantiation quasi-order). If one were to compare unifiers w.r.t. the set V of all variables (unrestricted instantiation quasi-order), then a different unification type would be obtained in some cases. In fact, in [9] it is shown that the unification type for the equational theories ACUI, ACU, and AC switches from respectively unitary, unitary, and finitary for the restricted instantiation quasi-order to infinitary in all three cases for the unrestricted instantiation quasi-order. For the equational theory axiomatizing equivalence of $\mathcal{EL}$ concepts, the unification type improves from type zero to infinitary when replacing the restricted instantiation quasi-order with the unrestricted one [9].

In the present paper, we use comparison w.r.t. a finite set of variables X that contains the set of variables occurring in the unification problem, but may be larger than this set. However, as shown in [9], this difference does not impact the unification type.

Lemma 10 ([9]). *Let E be an equational theory, Γ a unification problem, and $X_0 \subseteq X$ finite sets of variables such that X_0 consists of all the variables occurring in Γ. If Γ has a minimal complete set of unifiers w.r.t. $\geq_E^X$, then it has a minimal complete set of unifiers w.r.t. $\geq_E^{X_0}$ of the same cardinality, and vice versa.*

In [9], this result is actually shown in the more general setting where X may be infinite, but $V \setminus X$ is required to be infinite as well. For finite sets of variables X, this condition is clearly satisfied since V is assumed to be infinite.

This lemma shows that the unification types shown in unification theory for the setting where unifiers are compared w.r.t. the variable occurring in the unification problem also apply to the setting introduced in this paper, where unifiers are compared w.r.t. a finite set of variables X that contains all the variables occurring in the unification problem. The following is now an immediate consequence of Proposition 9.

Proposition 10. *Let E be an equational theory whose unification type is unitary or finitary. Then every repair problem (σ, Γ) for Rep_{uni}^E satisfies the optimal repair property.*

7 Conclusion

We have used the order-theoretic view on optimal repairs to find characterizations and sufficient criteria for the optimal repair property to hold. When considering the instantiation quasi-order between substitutions rather than a reflexive and transitive entailment relation, these results can also be used to show that a unification problem or equational theory has unification type unary or finitary. In practice, one then still needs to develop appropriate algorithms for computing optimal repairs or complete sets of unifiers. In fact, the optimal repair property and unification types unitary/finitary were until now often shown by exhibiting algorithm for computing the appropriate finite sets of repairs or unifiers. On the repair side, this was, e.g., done in [12,13,14,25,11] (for equational unification, see [18] for references to algorithms computing minimal complete sets of unifiers). Also note that, in [19], we have shown the optimal repair property for $\mathrm{Rep}_{\mathrm{ent}}$ and $\mathrm{Rep}_{\mathrm{for}}$ by describing algorithms for computing sets of optimal repairs, and then showing that these sets cover all repairs. Nevertheless, we believe that applying such abstract criteria may turn out to be useful as a first step when investigating a new kind of repair or unification problem. The characterizations and necessary/sufficient conditions for unification type zero presented in [4] were recently employed in [20,9] to prove unification type zero for equational theories induced by certain modal and description logics. They may also turn out to be useful for showing that a given repair problem has type zero.

The fact that unification problems can be seen as repair problems in the repair framework of [19] shows how general this framework is. It also provides us with examples of repair problems that have type infinitary. It is currently unclear whether this type can also occur in the more traditional repair setting where entailment is the entailment relation of some logic and the repair goal is non-entail of an unwanted consequence. The overall framework of [19] for constructing contraction operations has as additional requirements the existence of product and sum operations between KBs. In general, such products and sums need not exist for the unification instance. However, if one considers a so-called commutative/monoidal theory [5,29] as underlying equational theory, then it is easy to see that products and sums w.r.t. the instantiation quasi-order exist.

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