Dresden University of Technology
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Master's Thesis on

The Description Logic ABox Update Problem Revisited

by

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Declaration

Hereby I certify that the thesis has been written by me. Any help that I have received in my research work has been acknowledged. Additionally, I certify that I have not used any auxiliary sources and literature except those cited in the thesis.

Yusri Bong

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Abstract

A description logic ABox is one of the tools to represent a snapshot of a world. Since a world is dynamic, we need a mechanism to update it and also its snapshot. Updates in general can be any kind of information about the change of the world. But here, we consider only the basic updates, i.e., the updates are described using only simple ABoxes. There have been several known results concerning this kind of updates. One of the results is that the existence of nominals and @-constructor is needed to express the updated ABox. More precisely, the simplest DL that is propositionally closed that is able to express the updated ABox is $\mathcal{ALCO}^{@}$.

We try to recover the existence of ABox updates in fragments of $\mathcal{ALCO}^{@}$ that still are propositionally closed, by weakening the definition of ABox updates from semantic updates to syntactic updates. It turns out that this weakening is not enough to recover the existence of ABox updates in these DLs. We then try again to weaken the definition of ABox updates from syntactic updates to extended syntactic updates. With this weakening, we recover the existence of ABox updates in $\mathcal{ALCO}$. Unfortunately, this is not the case for $\mathcal{ALC}$. If we only allow $\mathcal{ALC}$ ABoxes as the result of updates, then this second weakening is still not enough to recover the existence of ABox updates in $\mathcal{ALC}$. We recover the existence of extended syntactic ABox updates in $\mathcal{ALC}$ by allowing KB, i.e., a pair of TBox and ABox, as the result of the update.

Contents

Chapter 1

Introduction

Every intelligent agent must have the ability to acquire new knowledge, reason about it and perform a service based on it. Most of the times, the new information it acquires cannot simply be added to the agent's current knowledge base because it may lead to an inconsistent knowledge base. In [8], Winslett has shown how to update a knowledge base that is represented in propositional logic. The problem of using propositional logic as a knowledge base representation is that, for many applications, it is not expressive enough. Therefore, for such applications, we need a more expressive logic and a mechanism to update it.

Description logics (DLs) are a family of logic-based formalisms for knowledge representation [1]. This logic is more expressive than propositional logic. The knowledge is defined using relevant concepts of the domain. The description logic ABox is one of the tools used to represent a knowledge of a world. It describes a snapshot of the current world. The knowledge represented here tends to be dynamic because we are livin in a dynamic world. In this paper, we will discuss updating description logic ABoxes.

Let us now see an example of how to represent knowledge in DL. Suppose that we are interested in defining knowledge about friends. The following concept describes the class of people that have a strong friend:

$$\mathsf{Person} \sqcap \exists \mathsf{has\_friend}.(\mathsf{Person} \sqcap \mathsf{Strong})$$

This concept is formulated in $\mathcal{ALC}$, the smallest DL that contains all Boolean operators [5]. The following ABox, also formulated in $\mathcal{ALC}$, says that John is a person with at least one strong friend and David is a strong person:

$$\begin{aligned}&\mathsf{john} : \mathsf{Person} \sqcap \exists \mathsf{has\_friend}.(\mathsf{Person} \sqcap \mathsf{Strong})\\ &\mathsf{david} : \mathsf{Person} \sqcap \mathsf{Strong}\end{aligned}$$

The snapshot above does not say anything about the relation between John and David. Since the semantics of ABoxes adopts the open world assumption, we cannot conclude that David is not a friend of John nor that David is a friend of John. Suppose that at the moment, because of aging, David is no longer strong. In order to make sure that the ABox still is a snapshot of the real world, we need to update the ABox. The following ABox, which is formulated in $\mathcal{ALCO}$

(the extension of $\mathcal{ALC}$ with nominals), is the result of updating the above ABox with the new information:

$$\mathsf{john} : \mathsf{Person} \sqcap \exists \mathsf{has\_friend}.(\mathsf{Person} \sqcap (\mathsf{Strong} \sqcup \{\mathsf{david}\}))$$
$$\mathsf{david} : \mathsf{Person} \sqcap \neg \mathsf{Strong}$$

Please note that new information about David influenced the assertion concerning John. The reason is that, there is a possibility of David being a friend of John. Hence, after updating the ABox we have to consider the possibility of David, who is not strong, being a friend of John.

A formal theory of ABox updates has been developed for several DLs. In [4], the authors give a formal definition of ABox updates and present an algorithm to update ABoxes formulated in $\mathcal{ALCQIO}^{@}$ (the extension of $\mathcal{ALCO}$ with number restrictions, inverse roles and @-constructor). The authors also show that under their definition of ABox updates, we are not always able to represent updated ABoxes in some basic DLs such as $\mathcal{ALC}$ and $\mathcal{ALCO}$. They conclude, that the simplest DL which contains $\mathcal{ALC}$ and has ABox updates is $\mathcal{ALCO}^{@}$ (the extension of $\mathcal{ALCO}$ with @-constructor). It has also been shown that for DL-Lite, ABox updates can be computed [2]. There is a huge expressiveness gap between the DLs considered above. The results given in [4] can only be applied to DLs that are very expressive, whereas the result provided in [2] can only be applied to DLs that are not expressive enough for many applications. In this thesis, we try to recover the existence of ABox updates in the basic DLs $\mathcal{ALC}$ and $\mathcal{ALCO}$. We are interested in these DLs because they are more expressive than DL-Lite, less expressive than $\mathcal{ALCO}^{@}$ and there are available standard reasoners for them. Racer [3] and Pellet [6] are examples of standard reasoners that can handle $\mathcal{ALC}$ and $\mathcal{ALCO}$, respectively.

As in [4], we only consider basic updates. The assumption is that new information is formalized in a simple ABox, i.e. an ABox containing only assertions $A(a)$, $r(a, b)$ and their negations, where A is an atomic concept. We do not admit complex assertions in the update because of the same reasons as in [4]. First, there is a single and uncontroversial semantics for updates in this restricted form. Second, we believe that, allowing concepts involving quantifiers nested in a complex way, in the update leads to unintuitive results. We also assume that there is no TBox in the original knowledge base for the same reason.

As stated before, we try to recover the existence of ABox updates in $\mathcal{ALC}$ and $\mathcal{ALCO}$. We do this by weakening the definition of having ABox updates. Let $\mathcal{A}$ be an ABox represented in $\mathcal{L}$, $\mathcal{U}$ an update, and $\mathcal{A}'$ be the result of updating $\mathcal{A}$ with $\mathcal{U}$. In [4], the authors define a DL $\mathcal{L}$ has (semantic) ABox updates if $\mathcal{A}'$ can be represented in $\mathcal{L}$. In this thesis, we define a weaker definition of ABox updates. We call this definition syntactic ABox updates. We say a DL $\mathcal{L}$ has syntactic ABox updates if we can represent an ABox $\mathcal{A}^{\diamond}$ in $\mathcal{L}$ such that $\mathcal{A}^{\diamond}$ has the same $\mathcal{L}$ logical consequences as $\mathcal{A}'$. It turns out that doing this is not enough to recover the existence of ABox updates in both $\mathcal{ALC}$ and $\mathcal{ALCO}$. We show this by introducing a combination of ABox and update in $\mathcal{ALC}$ (respectively $\mathcal{ALCO}$), and then showing that there is no $\mathcal{ALC}$ (respectively $\mathcal{ALCO}$) ABox that has the same $\mathcal{ALC}$ (respectively $\mathcal{ALCO}$) logical consequences as the updated ABox that is represented in $\mathcal{ALCO}^{@}$.

We then again try to weaken the definition of syntactic ABox updates to the extended syntactic ABox updates. Let $\mathcal{A}$ be an ABox represented in $\mathcal{L}$, $\mathcal{U}$ an update, and $\mathcal{A}'$ be the result of updating $\mathcal{A}$ with $\mathcal{U}$. We say a DL $\mathcal{L}$ has extended syntactic ABox updates if we can represent an ABox $\mathcal{A}^\diamond$ in $\mathcal{L}$ such that $\mathcal{A}^\diamond$ has the same $\mathcal{L}$ logical consequences as $\mathcal{A}'$ with respect to assertions that do not use symbols that appear in $\mathcal{A}^\diamond$ but not in $\mathcal{A}'$. The idea here is that $\mathcal{A}^\diamond$ is allowed to use additional symbol that is not used in $\mathcal{A}^\diamond$. By doing this we can recover the existence of extended syntactic ABox updates in $\mathcal{ALCO}$ whereas this does not work for $\mathcal{ALC}$. We can recover the existence of such updates in $\mathcal{ALC}$ by allowing $\mathcal{ALC}$ knowledge bases, i.e. a pairs of general TBox and ABox, as the result of ABox updates. We show that the updated ABox (resp. knowledge base) exists and is expressible in $\mathcal{ALCO}$ (resp. $\mathcal{ALC}$).

This thesis consists of five chapters:

- Chapter 2 contains the basic knowledge that is required to fully understand the thesis. We introduce several description logics that are used throughout this thesis. They are $\mathcal{ALC}$, $\mathcal{ALCO}$, $\mathcal{ALC}^@$ and $\mathcal{ALCO}^@$. We also introduce some known results in this DLs, that will be used in the thesis. We then introduce several definitions of ABox updates and the relations between them. We start with showing how to update interpretations and then introduce three different definitions of ABox updates that we consider: semantic ABox update, syntactic ABox update and extended syntactic ABox update. This chapter ends with the introduction of uniform ABox interpolation and its relation with ABox updates.

- In Chapter 3, we study the existence of syntactic ABox updates in $\mathcal{ALC}$, $\mathcal{ALCO}$ and $\mathcal{ALC}^@$. We show that all of these logics are not expressive enough to represent the syntactically updated ABox. In this chapter, we also introduce the notion of Boolean ABoxes. We then show that Boolean $\mathcal{ALC}$ ABoxes (which are more expressive than standard $\mathcal{ALC}$ and $\mathcal{ALC}^@$ ABoxes) are still not expressive enough to recover the existence of syntactic ABox updates.

- Chapter 4 consists of the results concerning extended syntactic ABox updates. We show that $\mathcal{ALC}$ is still not expressive enough to represent the updated ABox in this weaker definition. To recover the existence, we then allow knowledge bases as the results of updates. This brings a positive result. It turns out that there $\mathcal{ALC}$ KB is expressive enough to represent the updated ABox in this weaker sense. We continue with the DL $\mathcal{ALCO}$. Here, we show that $\mathcal{ALCO}$ has extended syntactic ABox updates.

- In Chapter 5, we give a summary of the results that we have obtained in this thesis and give some suggestions for future works.

Chapter 2

Preliminaries

There are several basic notions and notations that need to be understood before we go into details discussing updates of Description Logic ABoxes. First, we introduce the DLs that we are going to use in this thesis. Then, we introduce several definitions of ABox updates. We also introduce the notion of uniform interpolation at the end of this chapter and its relation to ABox updates.

2.1 Description Logics

In this section, we first introduce the syntax and semantics of description logic $\mathcal{ALCO}^{@}$ and then its fragments: $\mathcal{ALC}$, $\mathcal{ALCO}$, and $\mathcal{ALC}^{@}$. Then we introduce the notion of ABoxes. We also introduce some well known results concerning these DLs, that will be used in this thesis.

2.1.1 Syntax and Semantics

In the DL $\mathcal{ALCO}^{@}$, concepts can be constructed using the constructors listed in Figure 2.1 starting with a set $\mathsf{N_C}$ of concept names, $\mathsf{N_R}$ of role names and $\mathsf{N_I}$ of individual names. There and throughout this paper, we use a, b, and c to denote individual names, r and s to denote role names, A and B to denote concept names, and C and D to denote concepts. As usual, we also use the abbreviation $\top$ for a propositional *tautology*, $\bot$ for $\neg\top$, $\rightarrow$ and $\leftrightarrow$ for the usual Boolean abbreviations.

The most basic fragment of the DL $\mathcal{ALCO}^{@}$ that we consider in this paper is the DL $\mathcal{ALC}$. The DL $\mathcal{ALC}$ is the "smallest" DL that is propositionally closed, i.e. it allows only the following concept constructors: negation, conjunction, disjunction, and both universal and existential restrictions [5]. The availability of the other concept constructors are indicated by concatenating the following letters to $\mathcal{ALC}$: $\mathcal{O}$ stands for nominals and superscript @ for the @ constructor. So, we have the following fragments of $\mathcal{ALCO}^{@}$: $\mathcal{ALC}$, $\mathcal{ALCO}$, and $\mathcal{ALC}^{@}$. A concept is an $\mathcal{L}$ *concept* iff it is constructed using the constructors allowed in the DL $\mathcal{L}$. The following are examples of $\mathcal{ALC}$, $\mathcal{ALCO}$, $\mathcal{ALC}^{@}$ and $\mathcal{ALCO}^{@}$

Name	Syntax	Semantics
Nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
Negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
Conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
Existential restriction	$\exists r.C$	$\{d \in \Delta^{\mathcal{I}} \mid (d,e) \in r^{\mathcal{I}}$ and $e \in C^{\mathcal{I}}\}$
Universal restriction	$\forall r.C$	$\{d \in \Delta^{\mathcal{I}} \mid (d,e) \in r^{\mathcal{I}}$ implies $e \in C^{\mathcal{I}}\}$
@ constructor	$@_a C$	$\Delta^{\mathcal{I}}$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and $\emptyset$ otherwise

Figure 2.1: Syntax and semantics of $\mathcal{ALCO}^{@}$

concepts.

$\mathcal{ALC}$ concept	:	$\mathsf{Person} \sqcap \forall \mathsf{has\_friend}.(\mathsf{Good} \sqcap \mathsf{Loyal})$
$\mathcal{ALCO}$ concept	:	$\mathsf{Man} \sqcap \exists \mathsf{has\_friend}.(\{\mathsf{david}\} \sqcap (\neg \mathsf{Strong} \sqcup \mathsf{Brave}))$
$\mathcal{ALC}^{@}$ concept	:	$@_{\mathsf{mary}}\mathsf{Happy} \sqcup \exists \mathsf{has\_friend}.\mathsf{Happy}$
$\mathcal{ALCO}^{@}$ concept	:	$\exists \mathsf{has\_friend}.\{\mathsf{john}\} \sqcup @_{\mathsf{john}}\neg \mathsf{Happy}$

The semantics of a DL concept, role, and individual names is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. The *domain* $\Delta^{\mathcal{I}}$ is a set of *individuals* and the function $\cdot^{\mathcal{I}}$ maps each concept name $A \in \mathsf{N_C}$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, each role name $r \in \mathsf{N_R}$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each individual name $a \in \mathsf{N_I}$ to an individual $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The semantics of the concept constructors are listed in Figure 2.1.

A concept C is *satisfiable* if there is an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a *model of* C. A concept C *is subsumed by* D (denoted $C \sqsubseteq D$) if for all interpretations $\mathcal{I}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. The followings are examples of an unsatisfiable concept and a subsumption between two concepts.

- $A \sqcap \neg A$ is unsatisfiable because for all interpretation $\mathcal{I}$, $A^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}) = \emptyset$.
- $A \sqcap B \sqsubseteq B$ because for all interpretations $\mathcal{I}$, $A^{\mathcal{I}} \cap B^{\mathcal{I}} \subseteq B^{\mathcal{I}}$.

Concerning the satisfiability of a concept, the DL $\mathcal{ALC}$ has an interesting property. It turns out that for every satisfiable $\mathcal{ALC}$ concept C, there exists a tree model of it.

Definition 1 (Tree Model)**.** An interpretation $\mathcal{I}$ is a *tree model* of C iff

- $\mathcal{I} = (V, E)$ is a tree where $V = \Delta^{\mathcal{I}}$ and for all $(e, f) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $(e, f) \in E$ iff there is an $r \in \mathsf{N_R}$ such that $(e, f) \in r^{\mathcal{I}}$.
- $d \in C^{\mathcal{I}}$ and d is the root of (V, E).

The proof of the following lemma can be seen in [5].

Lemma 2. *For all $\mathcal{ALC}$ concepts C, C is satisfiable iff C has a tree model.*

Before we introduce TBox and ABox formalisms in DL, we first introduce some basic notions that will often be used in this thesis. A concept C is in *Negation Normal Form* (NNF) if the negation occurs only in front of a concept name or nominal. For example, the concept $\neg(A \sqcup \{a\})$ is not in NNF but the equivalent concept $\neg A \sqcap \neg\{a\}$ is in NNF. Every $\mathcal{ALCO}^@$ concept can be transformed into an equivalent concept in NNF by applying the following rewriting rules exhaustively.

$$
\begin{array}{ll}
\neg\neg C \rightsquigarrow C & \neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D \\
\neg\neg\{a\} \rightsquigarrow \{a\} & \neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D \\
\neg\exists r.C \rightsquigarrow \forall r.\neg C & \neg\forall r.C \rightsquigarrow \exists r.\neg C \\
\neg@_a C \rightsquigarrow @_a\neg C &
\end{array}
$$

The *length* $|C|$ *of a concept* C is the number of symbols needed to write C. It is easy to see that the applications of the rewriting rules are bounded by $|C|$. Hence, it is safe to assume that a concept is given in NNF since we know that the NNF of a concept always exists and can be constructed in linear time.

The set $\mathsf{sub}(C)$ of subconcepts of a concept C is a set of all subconcepts of C. The set $\mathsf{sub}(C)$ is the smallest set satisfying:

$$
\begin{array}{rll}
C \in \mathsf{sub}(C) & & \\
\neg D \in \mathsf{sub}(C) & \text{implies} & D \in \mathsf{sub}(C) \\
D \sqcap E \in \mathsf{sub}(C) & \text{implies} & \{D, E\} \subseteq \mathsf{sub}(C) \\
D \sqcup E \in \mathsf{sub}(C) & \text{implies} & \{D, E\} \subseteq \mathsf{sub}(C) \\
\exists r.D \in \mathsf{sub}(C) & \text{implies} & D \in \mathsf{sub}(C) \\
\forall r.D \in \mathsf{sub}(C) & \text{implies} & D \in \mathsf{sub}(C) \\
@_a D \in \mathsf{sub}(C) & \text{implies} & D \in \mathsf{sub}(C)
\end{array}
$$

The interesting fact about the cardinality of $\mathsf{sub}(C)$ (denoted $|\mathsf{sub}(C)|$) is that $|\mathsf{sub}(C)| \leq |C|$. This fact can be shown using structural induction on C. Please also note that if C is in NNF, then every element of $\mathsf{sub}(C)$ is also in NNF.

A *signature* $\Sigma = \langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ is a pair of a set of concept names $\mathsf{N}'_\mathsf{C} \subseteq \mathsf{N}_\mathsf{C}$ and a set of role names $\mathsf{N}'_\mathsf{R} \subseteq \mathsf{N}_\mathsf{R}$. A concept C is an $\mathcal{L}^\Sigma$ *concept* iff C is an $\mathcal{L}$ concept and for every concept and role names ψ that occurs in C, $\psi \in \mathsf{N}'_\mathsf{C} \cup \mathsf{N}'_\mathsf{R}$. A concept name A belongs to a signature $\Sigma = \langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ (written $A \in \Sigma$) if $A \in \mathsf{N}'_\mathsf{C}$. A role name r belongs to a signature $\Sigma = \langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ (written $r \in \Sigma$) if $r \in \mathsf{N}'_\mathsf{R}$. The signature $\mathsf{sig}(C)$ is a pair $\langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ where N'_C and N'_R are the set of concept and role names used in C. We say that $\Sigma_1 = \langle \mathsf{N}_{\mathsf{C}1}, \mathsf{N}_{\mathsf{R}1} \rangle$ is a sub signature of $\Sigma_2 = \langle \mathsf{N}_{\mathsf{C}2}, \mathsf{N}_{\mathsf{R}2} \rangle$ denoted $\Sigma_1 \subseteq \Sigma_2$ if $\mathsf{N}_{\mathsf{C}1} \subseteq \mathsf{N}_{\mathsf{C}2}$ and $\mathsf{N}_{\mathsf{R}1} \subseteq \mathsf{N}_{\mathsf{R}2}$. The abbreviation $\Sigma_1 \cap \Sigma_2$ denotes the signature that consists of the intersection of the concept names and the intersection role names in Σ_1 and Σ_2. We also use the abbreviation $\Sigma_1 \cup \Sigma_2$ to denote the signature that consists of the union of the concept names and the union role names in Σ_1 and Σ_2.

2.1.2 Description Logic ABox

A description logic *assertional box* (ABox) is a finite set of *concept assertions* $a : C$ (or sometimes written $C(a)$ if C is a short or simple concept), *positive*

$$\begin{array}{lcl}\mathcal{A} & = & \{\mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \{\mathsf{david}\}), \mathsf{david} : \neg\mathsf{Strong} \sqcap \mathsf{Brave}\} \\ \mathcal{A}^{\diamond} & = & \{\mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \mathsf{Brave}), \mathsf{david} : \neg\mathsf{Strong} \sqcap \mathsf{Brave}\}\end{array}$$

Figure 2.2: Examples of ABoxes

role assertions $r(a,b)$, and *negative role assertions* $\neg r(a,b)$. An ABox is used to store an *incomplete* snapshot of a world, i.e., an ABox contains only parts of information of the real world. As we have seen in the introductory chapter, this is the source why updating an ABox is not trivial. An ABox assertion φ is an $\mathcal{L}$ *assertion* iff φ is a role assertion or $\varphi = C(a)$ and C is an $\mathcal{L}$ concept. An ABox is an $\mathcal{L}$ *ABox* if it contains only $\mathcal{L}$ assertions. The ABox $\mathcal{A}$ in Figure 2.2 is an $\mathcal{ALCO}$ ABox while $\mathcal{A}^{\diamond}$ is an $\mathcal{ALC}$ ABox. Let $\Sigma = \langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ be a signature. An ABox assertion φ is an $\mathcal{L}^{\Sigma}$ *assertion* iff

- $\varphi = r(a,b)$ and $r \in \mathsf{N}'_\mathsf{R}$
- $\varphi = \neg r(a,b)$ and $r \in \mathsf{N}'_\mathsf{R}$
- $\varphi = C(a)$ and C is an $\mathcal{L}^{\Sigma}$ concept

If an ABox contains only $\mathcal{L}^{\Sigma}$ assertions, then it is called an $\mathcal{L}^{\Sigma}$ *ABox*. The signature $\mathsf{sig}(\mathcal{A})$ is a pair $\langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ where N'_C and N'_R are the set of concept and role names used in $\mathcal{A}$. The length $|\varphi|$ of an ABox assertion φ is defined as follows.

$$|\varphi| := \begin{cases} |C| & \text{if } \varphi = C(a) \\ 1 & \text{if } \varphi = r(a,b) \\ 1 & \text{if } \varphi = \neg r(a,b) \end{cases}$$

The *size* $|\mathcal{A}|$ of an ABox $\mathcal{A}$ is defined as follows.

$$|\mathcal{A}| := \sum_{\varphi \in \mathcal{A}} |\varphi|$$

An ABox $\mathcal{A}$ is *in NNF* iff for all concept assertions $C(a) \in \mathcal{A}$, C is in NNF. The set $\mathsf{sub}(\mathcal{A})$ is defined as follow.

$$\mathsf{sub}(\mathcal{A}) := \bigcup_{C(a) \in \mathcal{A}} \mathsf{sub}(C)$$

An interpretation $\mathcal{I}$ *satisfies* a concept assertion $C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$, a positive role assertion $r(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$, and a negative role assertion $\neg r(a,b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin r^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *is a model of an ABox* $\mathcal{A}$ (written $\mathcal{I} \models \mathcal{A}$) iff for all assertions $\psi \in \mathcal{A}$, $\mathcal{I}$ satisfies ψ (written $\mathcal{I} \models \psi$). An ABox is *consistent* iff it has a model. An ABox assertion φ is a *logical consequence* of an ABox $\mathcal{A}$ (written $\mathcal{A} \models \psi$) iff every model of $\mathcal{A}$ satisfies φ. An ABox $\mathcal{A}$ *entails* an ABox $\mathcal{B}$ (written $\mathcal{A} \models \mathcal{B}$) if for all $\varphi \in \mathcal{B}$, φ is a logical consequence of $\mathcal{A}$ (sometimes, we also say $\mathcal{A}$ satisfies φ).

Concerning the consistency of ABoxes, the DL $\mathcal{ALC}$ has an interesting property that will be used in this thesis. Since, every satisfiable $\mathcal{ALC}$ concept has a tree model, it is not hard to see that every $\mathcal{ALC}$ ABox has a forest-like model.

Definition 3 (Forest-like Model)**.** An interpretation $\mathcal{I}$ is a *forest-like model* of an ABox $\mathcal{A}$ if

- $\mathcal{I} \models \mathcal{A}$,
- $\mathcal{I}$ is a *forest-like interpretation*, i.e., $\mathcal{I}$ consists of trees with the interpretations of individual names as the roots and every root can only have incoming edge from other roots, and
- if $a^{\mathcal{I}}$ and $b^{\mathcal{I}}$ are roots of the trees in $\mathcal{I}$, then there exists an r edge from $a^{\mathcal{I}}$ to $b^{\mathcal{I}}$ iff $r(a, b) \in \mathcal{A}$ or $r(b, a) \in \mathcal{A}$

Lemma 4. *For all consistent $\mathcal{ALC}$ ABox, there exists a forest-like model of it.*

Proof. (sketch) Let $\mathcal{A}$ be a consistent $\mathcal{ALC}$ ABox. Then $\mathcal{A}$ has a model $\mathcal{I}$. Now, let a be an individual name that occurs in $\mathcal{A}$. We define the concept X_a as follows.

$$\begin{aligned} X_a \quad := \quad & \bigsqcap\{C \in \mathsf{sub}(\mathcal{A}) \mid \mathcal{I} \models C(a)\} \sqcap \\ & \bigsqcap\{\neg C \mid C \in \mathsf{sub}(\mathcal{A}) \text{ and } \mathcal{I} \models \neg C(a)\} \end{aligned}$$

We also define $\mathcal{T}(a)$ as a tree model with $a^{\mathcal{T}(a)}$ as the root and $a^{\mathcal{T}(a)} \in (X_a)^{\mathcal{T}(a)}$. Lemma 2 and the definition of X_a guarantee the existence of these models for all individual names a that occur in $\mathcal{A}$. We collect all these trees and then for all $r(a, b) \in \mathcal{A}$, we connect the root of $\mathcal{T}(a)$ to $\mathcal{T}(b)$. We name this new interpretation $\mathcal{J}$. From the construction, we know that $\mathcal{J}$ is a forest like interpretation. The fact that $\mathcal{J}$ satisfies all role assertions in $\mathcal{A}$ follows directly from the construction. The fact that $\mathcal{J}$ satisfies all concept assertions in $\mathcal{A}$ can be shown by inductively showing the claim for all $a \in \mathsf{N_I}$, $d \in \Delta^{\mathcal{T}(a)}$ and $C \in \mathsf{sub}(A)$, $d \in C^{\mathcal{T}(a)}$ implies $d \in C^{\mathcal{J}}$. □

2.2 ABox Updates

As mentioned in the previous section, an ABox is used to store an incomplete snapshot of the world. The *complete snapshot* itself is represented by an interpretation. So, before going into details about updating an ABox, we need to know how to update an interpretation. In this thesis, we only consider the simplest form of update which only allow simple ABoxes as updates. An ABox $\mathcal{A}$ is *simple* if $C(a) \in \mathcal{A}$ implies that C is a *concept literal*, i.e., a concept name or a negated concept name. We also assume that the update information is consistent since it does not make any sense to update a knowledge base with inconsistent information. To sum up, an *update* is a simple and consistent ABox that contains update information. We first introduce the definition of interpretation update and ABox updates afterwards.

Definition 5 (Interpretation Update)**.** Let $\mathcal{U}$ be an update, $\mathcal{I}, \mathcal{I}'$ interpretations such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and for all $a \in \mathsf{N_I}$, $a^{\mathcal{I}} = a^{\mathcal{I}'}$. Then $\mathcal{I}'$ is the *result of updating $\mathcal{I}$ with $\mathcal{U}$* (written $\mathcal{I} \Longrightarrow_{\mathcal{U}} \mathcal{I}'$) if

- for all $A \in \mathsf{N_C}$, $A^{\mathcal{I}'} = (A^{\mathcal{I}} \cup \{a^{\mathcal{I}} \mid A(a) \in \mathcal{U}\}) \setminus \{a^{\mathcal{I}} \mid \neg A(a) \in \mathcal{U}\}$;
- for all $r \in \mathsf{N_R}$, $r^{\mathcal{I}'} = (r^{\mathcal{I}} \cup \{(a,b) \mid r(a,b) \in \mathcal{U}\}) \setminus \{(a,b) \mid \neg r(a,b) \in \mathcal{U}\}$.

Now, we define the ABox updates problem. Given an ABox $\mathcal{A}$ that describes the current knowledge and an update $\mathcal{U}$ containing update information, we want to construct a new ABox $\mathcal{A}'$ that is the snapshot of the updated world. In this thesis, we introduce three types of ABox updates: semantic updates, syntactic updates and extended syntactic updates.

From now on, we often treat sets of models similar to ABoxes. In particular, we write $\mathbf{I} \equiv \mathcal{A}$ if $\mathbf{I}$ is the set of models of $\mathcal{A}$ and say that an assertion φ is a *logical consequence of a set of models* $\mathbf{I}$ (written $\mathbf{I} \models \varphi$) iff for all $\mathcal{I} \in \mathbf{I}$, $\mathcal{I} \models \varphi$.

Definition 6 (Semantic ABox Update)**.** Let $\mathcal{A}$ be an ABox and $\mathcal{U}$ an update. We denote $\mathcal{A} * \mathcal{U}$ as the (unique) set of models such that

(U1) $\forall \mathcal{I}, \mathcal{I}' : ((\mathcal{I} \models \mathcal{A} \wedge \mathcal{I} \Longrightarrow_{\mathcal{U}} \mathcal{I}') \rightarrow \mathcal{I}' \in \mathcal{A} * \mathcal{U})$ and

(U2) $\forall \mathcal{I}' : (\mathcal{I}' \in \mathcal{A} * \mathcal{U} \rightarrow \exists \mathcal{I} : (\mathcal{I} \models \mathcal{A} \wedge \mathcal{I} \Longrightarrow_{\mathcal{U}} \mathcal{I}'))$.

An ABox $\mathcal{A}'$ is the *result of semantically updating* $\mathcal{A}$ *with* $\mathcal{U}$ if $\mathcal{A}' \equiv \mathcal{A} * \mathcal{U}$. We call $\mathcal{A}$ the *original ABox*, $\mathcal{A} * \mathcal{U}$ the *updated models* and $\mathcal{A}'$ the *semantically updated ABox*.

We know that given an original ABox $\mathcal{A}$ and update $\mathcal{U}$, up to equivalence, there exists at most one semantically updated ABox $\mathcal{A}'$. We say a DL $\mathcal{L}$ *has semantic ABox updates* if for every original $\mathcal{L}$ ABox and update $\mathcal{U}$, there exists an $\mathcal{L}$ ABox $\mathcal{A}'$ such that $\mathcal{A}'$ and the updated models $\mathcal{A} * \mathcal{U}$ are equivalent ($\mathcal{A}' \equiv \mathcal{A} * \mathcal{U}$). There is an interesting fact concerning the signature of the semantically updated ABox $\mathcal{A}'$. We know that $\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U}) \subseteq \mathsf{sig}(\mathcal{A}')$. As stated in [4], the DLs $\mathcal{ALC}$, $\mathcal{ALCO}$, and $\mathcal{ALC}^{@}$ do not have ABox semantic updates. The simplest DL that contains $\mathcal{ALC}$ and has semantic ABox updates is the DL $\mathcal{ALCO}^{@}$.

Consider the following scenario. We are working with an ABox $\mathcal{A}$ that is expressed in $\mathcal{L}$ where $\mathcal{L} \in \{\mathcal{ALC}, \mathcal{ALCO}, \mathcal{ALC}^{@}\}$. For some reason, our ABox $\mathcal{A}$ needs to be updated with $\mathcal{U}$. In general, we get an $\mathcal{ALCO}^{@}$ ABox $\mathcal{A}'$ as the semantically updated ABox that cannot be represented in $\mathcal{L}$. Since we are working on an $\mathcal{L}$-ABox, we are most like interested only in $\mathcal{L}$ assertions. Then, we do not need to have an updated ABox that is equivalent to $\mathcal{A}'$. We just need an ABox that has the same $\mathcal{L}$ logical consequences as $\mathcal{A}'$.

Definition 7 ($\mathcal{L}$-indistinguishable)**.**

- φ is an $\mathcal{L}$ *logical consequence* of $\mathcal{A}$ if $\mathcal{A} \models \varphi$ and φ is an $\mathcal{L}$ assertion.
- ABoxes $\mathcal{A}$ and $\mathcal{A}^{\diamond}$ are $\mathcal{L}$*-indistinguishable* if for all $\mathcal{L}$ assertions φ, $\mathcal{A} \models \varphi$ iff $\mathcal{A}^{\diamond} \models \varphi$.
- A set of models $\mathbf{I}$ and an ABox $\mathcal{A}^{\diamond}$ are $\mathcal{L}$*-indistinguishable* if for all $\mathcal{L}$ assertions φ, $\mathbf{I} \models \varphi$ iff $\mathcal{A}^{\diamond} \models \varphi$.

This scenario leads us to the idea of introducing a weaker definition of ABox updates. We hope by weakening the definition of semantic updates to syntactic updates, we are able to recover the existence of ABox updates in some basic

DLs $\mathcal{ALC}$, $\mathcal{ALCO}$ and $\mathcal{ALC}^{@}$. We say a DL $\mathcal{L}$ *has syntactic ABox updates* if for every original $\mathcal{L}$ ABox $\mathcal{A}$ and update $\mathcal{U}$, there exists an $\mathcal{L}$ ABox $\mathcal{A}^{\diamond}$ such that the updated models $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^{\diamond}$ are $\mathcal{L}$-indistinguishable.

Definition 8 (Syntactic ABox Update)**.** Let $\mathcal{A}$ be an original $\mathcal{L}$ ABox, $\mathcal{U}$ an update and $\mathcal{A}*\mathcal{U}$ the updated models. An ABox $\mathcal{A}^{\diamond}$ is the *result of syntactically updating $\mathcal{A}$ with $\mathcal{U}$ in DL $\mathcal{L}$* if $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^{\diamond}$ are $\mathcal{L}$-indistinguishable. We call $\mathcal{A}^{\diamond}$ the *syntactically updated ABox in DL $\mathcal{L}$*.

We will give an example of this kind of update. Before going into details of the example, we introduce a new notation that will be used. Let $\mathcal{A}$ be an ABox. We use the notation $[\![\mathcal{A}]\!] = \{\mathcal{I} \mid \mathcal{I} \models \mathcal{A}\}$ for the set of all models of $\mathcal{A}$. The following $\mathcal{ALC}$ ABox $\mathcal{A}$ expresses that John has at least one strong friend. We also know that David is strong and brave.

$$\begin{array}{l}\mathsf{john} : \exists \mathsf{has\_friend}.\mathsf{Strong}\\ \mathsf{david} : \mathsf{Strong} \sqcap \mathsf{Brave}\end{array}$$

Suppose that, because of aging, David is no longer strong but he is still a brave man. Hence, we need to update our knowledge by the following update $\mathcal{U}$.

$$\neg\mathsf{Strong}(\mathsf{david})$$

Let $[\![\mathcal{A}]\!] = \{\mathcal{I} \mid \mathcal{I} \models \mathcal{A}\}$, we divide these interpretations into two groups such that $[\![\mathcal{A}_1]\!] \cup [\![\mathcal{A}_2]\!] = [\![\mathcal{A}]\!]$ as follows:

1. $[\![\mathcal{A}_1]\!] = \{\mathcal{I} \mid \mathcal{I} \in [\![\mathcal{A}]\!] \text{ and } \mathcal{I} \models \mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\}$
2. $[\![\mathcal{A}_2]\!] = \{\mathcal{I} \mid \mathcal{I} \in [\![\mathcal{A}]\!] \text{ and } \mathcal{I} \models \neg\mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\}$

From the definition, it is clear that for all $\mathcal{I} \in [\![\mathcal{A}_1]\!] \cup [\![\mathcal{A}_2]\!]$,

$$\begin{array}{lcl}\mathcal{I} & \models & \mathsf{david} : \mathsf{Strong} \sqcap \mathsf{Brave}\\ \mathcal{I} & \models & \mathsf{john} : \exists \mathsf{has\_friend}.\mathsf{Strong}\end{array}$$

Abusing notation, let $[\![\mathcal{A}_1^{\mathcal{U}}]\!]$ and $[\![\mathcal{A}_2^{\mathcal{U}}]\!]$ be the results of updating the interpretations in $[\![\mathcal{A}_1]\!]$ and $[\![\mathcal{A}_2]\!]$ respectively. Then we have:

$$\begin{array}{rcl}[\![\mathcal{A}_1^{\mathcal{U}}]\!] & = & \{\mathcal{I} \mid \mathcal{I} \models \mathsf{david} : \neg\mathsf{Strong} \sqcap \mathsf{Brave}\} \cap\\ & & \{\mathcal{I} \mid \mathcal{I} \models \mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\} \cap\\ & & \{\mathcal{I} \mid \mathcal{I} \models \mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \{\mathsf{david}\}\}\\ [\![\mathcal{A}_2^{\mathcal{U}}]\!] & = & \{\mathcal{I} \mid \mathcal{I} \models \mathsf{david} : \neg\mathsf{Strong} \sqcap \mathsf{Brave}\} \cap\\ & & \{\mathcal{I} \mid \mathcal{I} \models \neg\mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\} \cap\\ & & \{\mathcal{I} \mid \mathcal{I} \models \mathsf{john} : \exists \mathsf{has\_friend}.\mathsf{Strong}\}\end{array}$$

Due to (U1) and (U2), we have that the update models $\mathcal{A} * \mathcal{U}$ is the union of these two sets. Hence, the semantically updated ABox $\mathcal{A}'$ can be expressed in $\mathcal{ALCO}$ as follows:

$$\begin{array}{l}\mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \{\mathsf{david}\})\\ \mathsf{david} : \neg\mathsf{Strong} \sqcap \mathsf{Brave}\end{array}$$

This ABox is exactly the ABox $\mathcal{A}$ described in Figure 2.2.

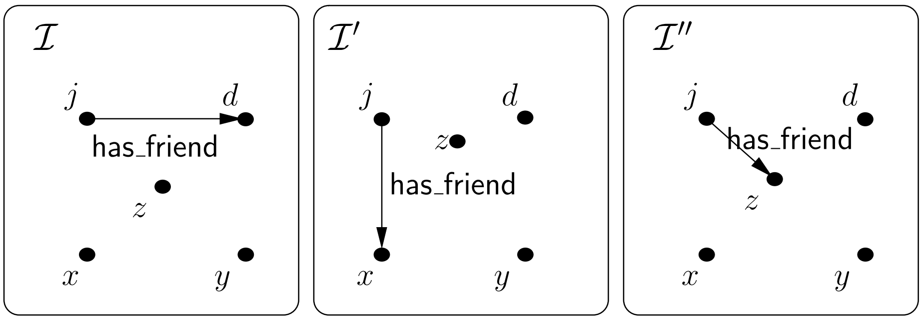

Figure 2.3: $\mathcal{I}, \mathcal{I}'$ and $\mathcal{I}''$

Lemma 9. *There exists no $\mathcal{ALC}$ ABox that is equivalent to the $\mathcal{ALCO}$ ABox $\mathcal{A}$ as described in Figure 2.2.*

Proof. (sketch) Let $\mathcal{I}, \mathcal{I}'$ and $\mathcal{I}''$ be the interpretations shown in Figure 2.3. The individual names john and david are mapped to j and d respectively. Moreover, all other individual names are mapped to y. The concept name Brave is mapped to $\{d, x\}$, Strong to $\{z\}$ and the other concept names are interpreted as empty set. It is easy to see that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{I}'' \models \mathcal{A}$, but $\mathcal{I}' \not\models \mathcal{A}$.

We claim that there is no $\mathcal{ALC}$ concept that can differentiate the interpretations $\mathcal{I}$ and $\mathcal{I}'$. This means that for all $\mathcal{ALC}$ concepts C and all individual names α, $\mathcal{I} \models C(\alpha)$ iff $\mathcal{I}' \models C(\alpha)$. The proof of the claim can be done using a very similar argument as in [4].

We then assume that such $\mathcal{ALC}$ ABox $\mathcal{B}$ exists. Then $\mathcal{I} \models \mathcal{B}$ and $\mathcal{I}'' \models \mathcal{B}$, but $\mathcal{I}' \not\models \mathcal{B}$. We know that for all role assertions $\varphi \in \mathcal{B}$, $\mathcal{I}'' \models \varphi$ iff $\mathcal{I}' \models \varphi$. We also know that for all concept assertions $\psi \in \mathcal{B}$, $\mathcal{I} \models \psi$ iff $\mathcal{I}' \models \psi$. Hence, this shows that $\mathcal{I} \models \mathcal{B}$ and $\mathcal{I}'' \models \mathcal{B}$ then $\mathcal{I}' \models \mathcal{B}$ (contradiction). □

Lemma 9 implies the fact that $\mathcal{ALC}$ does not have semantic ABox updates. But, if we are interested only in the logical consequences in the DL $\mathcal{ALC}$, then there exists an $\mathcal{ALC}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}'$ and $\mathcal{A}^\diamond$ are $\mathcal{ALC}$-indistinguishable. This is shown in the following lemma. Moreover, we show that the $\mathcal{ALC}$ ABoxes $\mathcal{A}$ and $\mathcal{A}^\diamond$ in Figure 2.2 are $\mathcal{ALC}$-indistinguishable.

Lemma 10. *Let $\mathcal{A}$ and $\mathcal{A}^\diamond$ be ABoxes in Figure 2.2. Then, $\mathcal{A}$ and $\mathcal{A}^\diamond$ are $\mathcal{ALC}$-indistinguishable.*

Proof. We need to show that for all $\mathcal{ALC}$ assertions φ, $\mathcal{A} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$. Since $\mathcal{A}^\diamond \subseteq \mathcal{A}$, we know that $\mathcal{A} \models \mathcal{A}^\diamond$. Hence, we can conclude that for all $\mathcal{ALC}$ assertions φ, $\mathcal{A}^\diamond \models \varphi$ implies $\mathcal{A} \models \varphi$. To conclude the proof, we still need to show for all $\mathcal{ALC}$ assertions φ, $\mathcal{A} \models \varphi$ implies $\mathcal{A}^\diamond \models \varphi$. We do a case analysis on φ. If φ is a role assertion, then the case is trivial because we know both of the ABoxes above do not entail any role assertions. We now come to a more interesting case, which is, φ is a concept assertion.

We prove this by showing the following claim. For all $\mathcal{ALC}$ concept assertions φ, if $\mathcal{A}^\diamond \cup \{\neg\varphi\}$ is consistent, then $\mathcal{A} \cup \{\neg\varphi\}$ is also consistent. Assume that $\mathcal{A}^\diamond \cup \{\neg\varphi\}$ is consistent. Then there exists a model $\mathcal{I}$ of $\mathcal{A}^\diamond \cup \{\neg\varphi\}$. From Lemma 4, we know that without loss of generality, we can assume that $\mathcal{I}$ is a forest-like model. We show that we can construct another interpretation $\mathcal{J}$ by extending $\mathcal{I}$ such that $\mathcal{J} \models \mathcal{A} \cup \{\neg\varphi\}$. This then concludes the proof.

We construct the interpretation $\mathcal{J}$ as an extension of $\mathcal{I}$. The construction of $\mathcal{J}$ depends on the individual name in φ. If the individual name is david, then we add $(\mathsf{john}^\mathcal{I}, \mathsf{david}^\mathcal{I})$ to the interpretation $\mathsf{has\_friend}^\mathcal{I}$. Otherwise, we change the interpretation of $\mathsf{david}^\mathcal{I}$ to the individual who is a brave friend of John. This individual is guaranteed to exists due to the fact that

$$\begin{aligned} \mathcal{I} &\models \mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \mathsf{Brave}) \text{ and} \\ \mathcal{I} &\not\models \mathsf{john} : \exists \mathsf{has\_friend}.(\mathsf{Strong} \sqcup \{\mathsf{david}\}). \end{aligned}$$

Formally, $\mathcal{J}$ is constructed as follows. Let $d \in \Delta^\mathcal{I}$ such that $(\mathsf{john}^\mathcal{I}, d) \in \mathsf{has\_friend}^\mathcal{I}$ and $d \in \mathsf{Brave}^\mathcal{I}$.

$$\begin{aligned} \mathsf{has\_friend}^\mathcal{J} &:= \begin{cases} \mathsf{has\_friend}^\mathcal{I} & \text{if } \neg\varphi \neq C(\mathsf{david}) \\ \mathsf{has\_friend}^\mathcal{I} \cup \{(\mathsf{john}^\mathcal{I}, \mathsf{david}^\mathcal{I})\} & \text{otherwise} \end{cases} \\ \mathsf{david}^\mathcal{J} &:= \begin{cases} d & \text{if } \neg\varphi \neq C(\mathsf{david}) \\ \mathsf{david}^\mathcal{I} & \text{otherwise} \end{cases} \end{aligned}$$

As a direct result of our construction, we have $\mathcal{J} \models \mathcal{A}$. It remains to show that $\mathcal{J} \models \neg\varphi$. Case analysis:

- $\neg\varphi = C(\mathsf{david})$. This case is trivial because from the construction and the fact that $\mathcal{I}$ is a forest-like interpretation, we know that for all $\mathcal{ALC}$ concepts C, $\mathsf{david}^\mathcal{J} \in C^\mathcal{J}$ iff $\mathsf{david}^\mathcal{I} \in C^\mathcal{I}$.

- $\neg\varphi \neq C(\mathsf{david})$. This case is also trivial because from the construction we have for all $d \in \Delta^\mathcal{I}$ and $\mathcal{ALC}$ concepts C, $d \in C^\mathcal{I}$ iff $d \in C^\mathcal{J}$ and for all $a \in \mathsf{N_I}$, $a \neq david$ implies $a^\mathcal{I} = a^\mathcal{J}$.

Hence, $\mathcal{J} \models \mathcal{A}$ and $\mathcal{J} \models \neg\varphi$.

□

We have seen that there exists an $\mathcal{ALC}$ ABox that is $\mathcal{ALC}$-indistinguishable with a semantically updated ABox which cannot be expressed in $\mathcal{ALC}$. From this, one may be tempted to conjecture that if we started with an original $\mathcal{ALC}$ ABox, the syntactically updated ABox can always be expressed in $\mathcal{ALC}$. Unfortunately, this is not the case. We will show this in the next chapter.

In some applications, we are often interested only in a certain set of concept and role names (signatures) and we do not care about the others. So going back to the previous scenario, instead of being interested only in $\mathcal{L}$ assertions, we are interested only in $\mathcal{L}$ assertions which are build using a particular signature. These particular signatures usually contain only finite sets of concept and role names. This means we can use additional concept and role names in our syntactically updated ABox and not care about the assertions that use the additional symbols.

Definition 11 ($\mathcal{L}^\Sigma$-indistinguishable)**.**

- φ is an $\mathcal{L}^\Sigma$ *logical consequence* of $\mathcal{A}$ if $\mathcal{A} \models \varphi$ and φ is an $\mathcal{L}^\Sigma$ assertion.

- ABoxes $\mathcal{A}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$*-indistinguishable* if for all $\mathcal{L}^\Sigma$ assertions φ, $\mathcal{A} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$.

- A set of models $\mathbf{I}$ and an ABox $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$-*indistinguishable* if for all $\mathcal{L}^\Sigma$ assertions φ, $\mathbf{I} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$.

The scenario above and the fact that weakening semantic updates to syntactic updates is not enough to recover the existence of ABox updates in the DLs we are interested in, are the reasons why we introduce the ABox extended syntactic update. We will show that using this definition, we can partially recover the existence of ABox updates in the DL $\mathcal{ALC}$ and fully in the DL $\mathcal{ALCO}$. We say a DL $\mathcal{L}$ *has extended syntactic ABox updates* if for every original $\mathcal{L}$ ABox $\mathcal{A}$, update $\mathcal{U}$, there exists an $\mathcal{L}$ ABox $\mathcal{A}^\diamond$ such that the updated models $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_I} \rangle \setminus (\mathsf{sig}(\mathcal{A}^\diamond) \setminus (\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})))$.

It is not hard to see that $\mathcal{A}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_I} \rangle \setminus (\mathsf{sig}(\mathcal{A}^\diamond) \setminus \mathsf{sig}(\mathcal{A}))$ iff for all $\mathcal{L}$ assertion φ where $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, we have $\mathcal{A} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$. We will be using these sentences interchangeably throughout the thesis.

Definition 12 (Extended Syntactic ABox Update)**.** Let $\mathcal{A}$ be an $\mathcal{L}$ ABox, $\mathcal{U}$ an update and $\mathcal{A} * \mathcal{U}$ the updated models. An ABox $\mathcal{A}^\diamond$ is *the result of extended syntactically updating $\mathcal{A}$ with $\mathcal{U}$ in DL $\mathcal{L}$* if $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_I} \rangle \setminus (\mathsf{sig}(\mathcal{A}^\diamond) \setminus (\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})))$. We call $\mathcal{A}^\diamond$ the *extended syntactically updated ABox* in $\mathcal{L}$.

Intuitively, the definition allows us to use additional helping symbols, that will not be considered when we check the logical consequences, in expressing $\mathcal{A}^\diamond$. For this kind of update, consider the following example. Assume that we know that John has at least one strong friend. We then see that John and David are fighting with each other in a bar. This means we have to update our knowledge with the information that David is not John's friend. The following $\mathcal{ALCO}$ ABox $\mathcal{A}$ describes the snapshot.

$$\mathsf{john} : \exists \mathsf{has\_friend}.\mathsf{Strong}$$

Let $[\![\mathcal{A}]\!] = \{\mathcal{I} \mid \mathcal{I} \models \mathcal{A}\}$, we divide these interpretations into three groups such that $[\![\mathcal{A}_1]\!] \cup [\![\mathcal{A}_2]\!] \cup [\![\mathcal{A}_3]\!] = [\![\mathcal{A}]\!]$ as follows:

1. $[\![\mathcal{A}_1]\!] = \{\mathcal{I} \mid \mathcal{I} \in [\![\mathcal{A}]\!] \wedge \mathcal{I} \models \mathsf{Strong}(\mathsf{david}) \wedge \mathcal{I} \models \mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\}$
2. $[\![\mathcal{A}_2]\!] = \{\mathcal{I} \mid \mathcal{I} \in [\![\mathcal{A}]\!] \wedge \mathcal{I} \models \mathsf{Strong}(\mathsf{david}) \wedge \mathcal{I} \models \neg\mathsf{has\_friend}(\mathsf{john}, \mathsf{david})\}$
3. $[\![\mathcal{A}_3]\!] = \{\mathcal{I} \mid \mathcal{I} \in [\![\mathcal{A}]\!] \wedge \mathcal{I} \models \neg\mathsf{Strong}(\mathsf{david})\}$

From the definition, it is clear that for all $\mathcal{I} \in [\![\mathcal{A}_1]\!] \cup [\![\mathcal{A}_2]\!] \cup [\![\mathcal{A}_3]\!]$,

$$\mathcal{I} \quad \models \quad \mathsf{john} : \exists \mathsf{has\_friend}.\mathsf{Strong}$$

Abusing notation, let $[\![\mathcal{A}_1^\mathcal{U}]\!]$, $[\![\mathcal{A}_2^\mathcal{U}]\!]$ and $[\![\mathcal{A}_3^\mathcal{U}]\!]$ be the results of updating the interpretations in $[\![\mathcal{A}_1]\!]$, $[\![\mathcal{A}_2]\!]$ and $[\![\mathcal{A}_3]\!]$ respectively. Then we have:

$$\begin{aligned}
[\![\mathcal{A}_1^{\mathcal{U}}]\!] &= \{\mathcal{I} \mid \mathcal{I} \models \mathsf{Strong}(\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \neg\mathsf{has_friend}(\mathsf{john},\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \mathsf{john} : \exists\mathsf{has_friend}.(\mathsf{Strong} \sqcap \neg\{\mathsf{david}\}\} \\
[\![\mathcal{A}_2^{\mathcal{U}}]\!] &= \{\mathcal{I} \mid \mathcal{I} \models \mathsf{Strong}(\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \neg\mathsf{has_friend}(\mathsf{john},\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \mathsf{john} : \exists\mathsf{has_friend}.\mathsf{Strong}\} \\
[\![\mathcal{A}_3^{\mathcal{U}}]\!] &= \{\mathcal{I} \mid \mathcal{I} \models \neg\mathsf{Strong}(\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \neg\mathsf{has_friend}(\mathsf{john},\mathsf{david})\} \cap \\
&\quad \{\mathcal{I} \mid \mathcal{I} \models \mathsf{john} : \exists\mathsf{has_friend}.\mathsf{Strong}\}
\end{aligned}$$

Due to (U1) and (U2), we have the updated models $\mathcal{A} * \mathcal{U}$ is the union of these three sets. Hence, the semantically updated ABox $\mathcal{A}'$ can be expressed in $\mathcal{ALC}^{@}$ as follows:

$$\begin{aligned}
&\mathsf{john} : \exists\mathsf{has_friend}.\mathsf{Strong} \sqcup @_{\mathsf{david}}\mathsf{Strong} \\
&\neg\mathsf{has_friend}(\mathsf{john},\mathsf{david})
\end{aligned}$$

It has been shown in [4] that there is no $\mathcal{ALCO}$ ABox that is equivalent to the above $\mathcal{ALC}^{@}$ ABox. Moreover, we will show in Chapter 3 that there is no $\mathcal{ALCO}$ ABox $\mathcal{B}$ such that $\mathcal{A}'$ and $\mathcal{B}$ has the same $\mathcal{ALCO}$ logical consequences. But, if we consider only $\mathcal{ALCO}^{\Sigma}$ logical consequences where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_I} \rangle \setminus (\mathsf{sig}(\mathcal{A}^{\diamond}) \setminus \mathsf{sig}(\mathcal{A}'))$, then there exists an $\mathcal{ALCO}$ ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}'$ and $\mathcal{A}^{\diamond}$ are $\mathcal{ALCO}^{\Sigma}$-indistinguishable.

Lemma 13. *If*

$$\mathcal{A}' = \{\mathsf{john} : \exists\mathsf{has\_friend}.\mathsf{Strong} \sqcup @_{\mathsf{david}}\mathsf{Strong}, \neg\mathsf{has\_friend}(\mathsf{john},\mathsf{david})$$

is the semantically updated $\mathcal{ALCO}^{@}$ ABox. Then

$$\begin{aligned}
\mathcal{A}^{\diamond} &= \{\mathsf{john} : \exists\mathsf{has_friend}.\mathsf{Strong} \sqcup (\exists\mathsf{u}.\{\mathsf{david}\} \sqcap \mathsf{Strong}), \\
&\quad \neg\mathsf{has_friend}(\mathsf{john},\mathsf{david})\}
\end{aligned}$$

is the extended syntactically updated ABox.

We will proof this in a more general manner in Chapter 4. We now want to see the relation between the different definitions of ABox updates. This explains why we say one definition is weaker than the other.

Lemma 14. *Let $\mathcal{A}$ be an ABox.*

1. *If $\mathcal{A}$ is the semantically updated ABox, then $\mathcal{A}$ is also the syntactically updated ABox in $\mathcal{L}$ for all DLs $\mathcal{L}$.*

2. *If $\mathcal{A}$ is the syntactically updated ABox in $\mathcal{L}$, then $\mathcal{A}$ is also the extended syntactically updated ABox in $\mathcal{L}$.*

Proof.

1. From the Definition 7, we know that for all ABoxes $\mathcal{A}$, $\mathcal{B}$ and DLs $\mathcal{L}$, we have $\mathcal{A} \equiv \mathcal{B}$ implies $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-indistinguishable. Hence, the fact that $\mathcal{A} \equiv \mathcal{A}$ concludes the case.

2. Let $\mathcal{A}'$ be the semantically updated ABox. Then from the assumption that $\mathcal{A}$ is the syntactically updated ABox in $\mathcal{L}$, we get $\mathcal{A}$ and $\mathcal{A}'$ are $\mathcal{L}$-indistinguishable. From the Definition 11, we know that for all ABoxes $\mathcal{A}$, $\mathcal{B}$ and signatures Σ, we have $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-indistinguishable implies $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}^{\Sigma}$-indistinguishable. Hence, we have $\mathcal{A}$ and $\mathcal{A}'$ are $\mathcal{L}^{\Sigma}$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_R} \rangle \setminus (\mathsf{sig}(\mathcal{A}) \setminus \mathsf{sig}(\mathcal{A}'))$.

□

Corollary 15.

1. *If a DL $\mathcal{L}$ has semantic ABox updates, then $\mathcal{L}$ has syntactic ABox updates.*

2. *If a DL $\mathcal{L}$ has syntactic ABox updates, then $\mathcal{L}$ has extended syntactic ABox updates.*

Proof.

1. Assume that $\mathcal{L}$ has semantic ABox updates. Then for every original ABox $\mathcal{A}$ and update $\mathcal{U}$, there exists a semantically updated $\mathcal{L}$ ABox $\mathcal{A}'$. From Point 1 in Lemma 14, we know that $\mathcal{A}'$ is also the syntactically updated ABox.

2. Assume that $\mathcal{L}$ has syntactic ABox updates. Then for every original ABox $\mathcal{A}$ and update $\mathcal{U}$, there exists a syntactically updated $\mathcal{L}$ ABox $\mathcal{A}'$. From Point 2 in Lemma 14, we know that $\mathcal{A}'$ is also the extended syntactically updated ABox.

□

There are some cases where the existence of syntactic ABox updates coincides with the existence of extended syntactic ABox updates. One result that we will see in the next subsection is that if a description logic has uniform ABox interpolation, the existence of syntactic ABox updates and the existence of extended syntactic ABox updates coincides.

2.3 Uniform Interpolation

In this subsection, we introduce the notion of uniform concept interpolation and then the notion of uniform ABox interpolation. After that we show the relation between uniform interpolation and ABox updates.

Definition 16 (Uniform Concept Interpolation)**.**

1. An $\mathcal{L}$ concept C^{Σ} is the *uniform concept interpolant* of an $\mathcal{L}$ concept C w.r.t. $\Sigma \subseteq \mathsf{sig}(C)$ if

   (a) $\mathsf{sig}(C^{\Sigma}) \subseteq \Sigma$

   (b) $C \sqsubseteq C^{\Sigma}$

(c) for all $\mathcal{L}$ concepts D with $\mathsf{sig}(D) \cap \mathsf{sig}(C) \subseteq \Sigma$, $C \sqsubseteq D$ implies $C^\Sigma \sqsubseteq D$.

2. A description logic $\mathcal{L}$ *has uniform concept interpolation* if for all $\mathcal{L}$ concept C and $\Sigma \subseteq \mathsf{sig}(C)$, there exists an $\mathcal{L}$ concept C^Σ that is the uniform concept interpolant of C with respect to Σ.

Theorem 17 ([7])**.** *$\mathcal{ALC}$ has uniform concept interpolation.*

We now extend the notion of uniform concept interpolation to uniform ABox interpolation. It turns out that the DL $\mathcal{ALC}$ does not have uniform ABox interpolation even if it has uniform concept interpolation.

Definition 18 (Uniform ABox Interpolation)**.**

1. An $\mathcal{L}$ ABox $\mathcal{A}^\Sigma$ is the uniform ABox interpolant of an $\mathcal{L}$ ABox $\mathcal{A}$ w.r.t. $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$ if
   (a) $\mathsf{sig}(\mathcal{A}^\Sigma) \subseteq \Sigma$
   (b) $\mathcal{A} \models \mathcal{A}^\Sigma$
   (c) for all $\mathcal{L}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, $\mathcal{A} \models \varphi$ implies $\mathcal{A}^\Sigma \models \varphi$.
2. A description logic $\mathcal{L}$ has uniform ABox interpolation if for all $\mathcal{L}$ ABox $\mathcal{A}$ and $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$, there exists an $\mathcal{L}$ ABox $\mathcal{A}^\Sigma$ that is the uniform ABox interpolant of $\mathcal{A}$ with respect to Σ.

Intuitively, Definition 18 says adding auxiliary variables does not increase the expressive power of ABoxes for description logics that have uniform ABox interpolation. The following lemma shows the relation between uniform ABox interpolation and ABox updates definition.

Lemma 19. *If a DL $\mathcal{L}$ has uniform ABox interpolation, then $\mathcal{L}$ has syntactic ABox updates iff $\mathcal{L}$ has extended syntactic updates.*

Proof. This ($\Rightarrow$) is exactly what is stated in Point 2 in Corollary 15. We now show the other direction.

($\Leftarrow$) Assume that $\mathcal{L}$ does not have syntactic ABox updates but has extended syntactic ABox updates. Then there exists an original $\mathcal{L}$ ABox $\mathcal{A}$ and update $\mathcal{U}$ such that there is no $\mathcal{L}$ ABox that is $\mathcal{L}$-indistinguishable with the updated ABox $\mathcal{A} * \mathcal{U}$.

Since $\mathcal{L}$ has extended syntactic ABox updates, we know that there exists an $\mathcal{L}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}^\Sigma$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_R} \rangle \backslash (\mathsf{sig}(\mathcal{A}^\diamond) \backslash \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U}))$. Equivalently, we can write for all $\mathcal{L}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$, we have $\mathcal{A}^\diamond \models \varphi$ iff $\mathcal{A} * \mathcal{U} \models \varphi$.

It is not hard to see that $\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U}) \subseteq \mathsf{sig}(\mathcal{A}^\diamond)$. The idea now is to show that the uniform interpolant of $\mathcal{A}^\diamond$ w.r.t. $\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$ is $\mathcal{L}$-indistinguishable with $\mathcal{A} * \mathcal{U}$. Let $\mathcal{A}_\diamond^\Sigma$ be the uniform ABox interpolant of $\mathcal{A}^\diamond$ w.r.t. $\Sigma = \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$. Then we know that

- $\mathsf{sig}(\mathcal{A}_\diamond^\Sigma) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$ and
- for all $\mathcal{L}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$, $\mathcal{A}^\diamond \models \varphi$ iff $\mathcal{A}_\diamond^\Sigma \models \varphi$ iff $\mathcal{A} * \mathcal{U} \models \varphi$.

We can write the second item equivalenty as follows. $\mathcal{A} * \mathcal{U}$, $\mathcal{A}^{\diamond}$, and $\mathcal{A}^{\Sigma}_{\diamond}$ are $\mathcal{L}^{\Sigma'}$-indistinguishable where $\Sigma' = \langle \mathsf{N_C}, \mathsf{N_R} \rangle \setminus (\mathsf{sig}(\mathcal{A}^{\Sigma}_{\diamond}) \setminus (\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})))$. Since $\mathsf{sig}(\mathcal{A}^{\Sigma}_{\diamond}) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$, we have $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^{\Sigma}_{\diamond}$ are $\mathcal{L}^{\langle \mathsf{N_C}, \mathsf{N_R} \rangle}$-indistinguishable. Thus, $\mathcal{A}^{\Sigma}_{\diamond}$ and $\mathcal{A} * \mathcal{U}$ are $\mathcal{L}$-indistinguishable (contradiction).

□

We have seen in Lemma 19 that if a DL has uniform ABox interpolation, the existence of syntactic and the extended syntactic updates coincides. Now, to make sure that it is worthy to weaken the definition of syntactic ABox updates to extended syntactic ABox updates, we need to show that there is a description logic that does not have uniform ABox interpolation. Moreover, we show that $\mathcal{ALC}$ does not have ABox interpolation in the following lemma.

Lemma 20. *$\mathcal{ALC}$ does not have uniform ABox interpolation.*

Proof. Let $\mathcal{A} = \{a : A \sqcup \forall r.A, r(a,b), s(a,b), t(a,b)\}$ be an $\mathcal{ALC}$ ABox and $\Sigma = \langle \{A\}, \{s,t\} \rangle$. We want to show there is no uniform ABox interpolant of $\mathcal{A}$ w.r.t. Σ.

Assume that there exists an $\mathcal{ALC}$ ABox $\mathcal{A}^{\Sigma}$ that is the uniform ABox interpolant of $\mathcal{A}$ w.r.t. Σ. We will first show that there exists a model $\mathcal{I}$ of $\mathcal{A}^{\Sigma}$ such that:

1. $\mathcal{I} \models \neg A(a)$,
2. $\mathcal{I} \models \neg A(b)$ and
3. $\{d \mid (a^{\mathcal{I}}, d) \in s^{\mathcal{I}}\} \cap \{d \mid (a^{\mathcal{I}}, d) \in t^{\mathcal{I}}\} = \{b^{\mathcal{I}}\}$.

Let B be a concept name that does not occur in both $\mathcal{A}$ and $\mathcal{A}'$. Now, using the fact that we have a model $\mathcal{I}$ of $\mathcal{A}^{\Sigma}$ satisfying the three properties above, we show that $\mathcal{A}^{\Sigma} \not\models a : \forall s.B \rightarrow (A \sqcup \exists t.(A \sqcap B))$ eventhough $\mathcal{A} \models a : \forall s.B \rightarrow (A \sqcup \exists t.(A \sqcap B))$. Let us construct a new interpretation $\mathcal{I}'$ by only modifying the interpretation of the concept name B as follows

$$B^{\mathcal{I}'} := \{d \mid (a^{\mathcal{I}'}, d) \in s^{\mathcal{I}}\}.$$

Since the concept name B does not appear in the ABox $\mathcal{A}^{\Sigma}$, we know for sure that $\mathcal{I}' \models \mathcal{A}^{\Sigma}$. It is easy to see that $\mathcal{I}' \models a : \forall s.B$ but $\mathcal{I}' \not\models a : A \sqcup \exists t.(A \sqcap B)$. This contradicts the fact that $\mathcal{A}^{\Sigma}$ is the uniform ABox interpolant of $\mathcal{A}$ w.r.t. signature Σ.

To conclude the proof, we need to show that such interpretation $\mathcal{I}$ exists. Let $C_1 = \exists s.\top \sqcup \exists t.\top$ and $C_2 = \exists ss.\top \sqcup \exists st.\top \sqcup \exists ts.\top \sqcup \exists tt.\top$. The concepts C_1 and C_2 are used as a technical trick. We will see the usage of these concepts later on in the proof. From the assumption that $\mathcal{A}^{\Sigma}$ is the uniform ABox interpolant of $\mathcal{A}$ w.r.t. Σ, we know that:

- $\mathcal{A}^{\Sigma} \not\models a : A \sqcup \forall s.A \sqcup \forall t.A \sqcup C_2$ since $\mathcal{A} \not\models a : A \sqcup \forall s.A \sqcup \forall t.A \sqcup C_2$ and
- $\mathcal{A}^{\Sigma} \not\models b : A \sqcup C_1$ since $\mathcal{A} \not\models b : A \sqcup C_1$.

Now, let $\mathcal{J}$ and $\mathcal{K}$ be models of $\mathcal{A}^{\Sigma}$ such that $\mathcal{J} \models a : \neg A \sqcap \exists s.\neg A \sqcap \exists t.\neg A \sqcap \neg C_2$ and $\mathcal{K} \models b : \neg A \sqcap \neg C_1$. Without loss of generality, we assume $\Delta^{\mathcal{J}} \cap \Delta^{\mathcal{K}} = \emptyset$.

We now construct $\mathcal{I}$ as follows. For all concept names A, role names r and individual names $b \neq a$,

$$\begin{aligned}
\Delta^{\mathcal{I}} &:= \Delta^{\mathcal{J}} \cup \Delta^{\mathcal{K}}, \\
A^{\mathcal{I}} &:= A^{\mathcal{J}} \cup A^{\mathcal{K}}, \\
a^{\mathcal{I}} &:= a^{\mathcal{J}}, \\
b^{\mathcal{I}} &:= b^{\mathcal{K}}, \\
r^{\mathcal{I}} &:= r^{\mathcal{J}} \cup r^{\mathcal{K}} \cup \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid r \in \{s, t\}\}.
\end{aligned}$$

It is easy to see that $\mathcal{I}$ satisfies all three conditions above. To conclude the proof, it remains to show that $\mathcal{I}$ is a model of $\mathcal{A}^{\Sigma}$. Let $\varphi \in \mathcal{A}$ be an ABox assertion. We do a case analysis on φ.

- φ is a role assertion. Then φ is either $s(a, b)$ or $t(a, b)$. It is easy to see that $\mathcal{I} \models \varphi$.
- φ is a concept assertion. We show $\mathcal{I} \models \mathcal{A}^{\Sigma}$ by showing the following claims. For all $\mathcal{ALC}$ concepts C built using the signature Σ, we have

  1. for all $d \in \Delta^{\mathcal{K}}$, $d \in C^{\mathcal{K}}$ implies $d \in C^{\mathcal{I}}$ and
  2. for all $d \in \Delta^{\mathcal{J}}$, $d \in C^{\mathcal{J}}$ implies $d \in C^{\mathcal{I}}$.

  The first claim can be shown easily using structural induction on C. We will only show the more interesting structural induction for the second claim. Without loss of generality, we assume that C is in NNF.

  – The cases $C = A$ and $C = \neg A$ are trivial.
  – The cases $C = D \sqcup E$ and $C = D \sqcap E$ follow directly from the induction hypothesis.
  – $C = \exists r.D$, where $r \in \{s, t\}$. This case is trivial since we do not remove any outgoing edge from d during the construction.
  – $C = \forall r.D$, where $r \in \{s, t\}$. If $d \neq a^{\mathcal{I}}$ then the case is trivial because we did not add any outgoing edge from d during the construction of $\mathcal{I}$. Assume that $d = a^{\mathcal{I}} = a^{\mathcal{J}}$. Then during the construction, we added the edge $(a^{\mathcal{I}}, b^{\mathcal{I}})$ to $r^{\mathcal{I}}$. To conclude the case, we need to show that $b^{\mathcal{I}} \in D^{\mathcal{I}}$.
    Since $\mathcal{J} \models a : \exists s.\neg A \sqcap \exists t.\neg A$, we know that there exist $e, e' \in \Delta^{\mathcal{J}}$ such that $(d, e) \in s^{\mathcal{J}}$ and $(d, e') \in t^{\mathcal{J}}$ and $\{e, e'\} \cap A^{\mathcal{J}} = \emptyset$. This implies $\{e, e'\} \subseteq D^{\mathcal{I}}$. We claim that for all $\mathcal{ALC}$ concepts E built using signature Σ, $e \in E^{\mathcal{J}}$ implies $b^{\mathcal{I}} \in E^{\mathcal{I}}$ and $e' \in E^{\mathcal{J}}$ implies $b^{\mathcal{I}} \in E^{\mathcal{I}}$. This is where the concepts C_1 and C_2 come in handy. The facts $\mathcal{J} \models \neg C_2(a)$, $(a^{\mathcal{J}}, e) \in s^{\mathcal{J}}$ and $(a^{\mathcal{J}}, e') \in t^{\mathcal{J}}$ guarantee that both e and e' have no r-successor. The fact that $\mathcal{K} \models \neg C_1(b)$ guarantees that $b^{\mathcal{K}} = b^{\mathcal{I}}$ has no r-successor. The fact that $\mathcal{K} \models \neg A(b)$ guarantees that $b^{\mathcal{K}} = b^{\mathcal{I}} \notin A^{\mathcal{K}}$. This implies $b^{\mathcal{I}} \notin A^{\mathcal{I}}$ Using the facts mentioned above, the claim can easily be shown easily using structural induction on E.

□

Chapter 3

Syntactic Updates

We say a description logic $\mathcal{L}$ has syntactic ABox updates iff for every ABox $\mathcal{A}$, formulated in $\mathcal{L}$, and every update $\mathcal{U}$, there exists an ABox $\mathcal{A}^\diamond$, formulated in $\mathcal{L}$, such that $\mathcal{A} * \mathcal{U}$ and $\mathcal{A}^\diamond$ are $\mathcal{L}$-indistinguishable. In this chapter, we study the existence of syntactic ABox updates in the DLs $\mathcal{ALC}$, $\mathcal{ALCO}$ and $\mathcal{ALC}^@$.

3.1 Syntactic Updates in $\mathcal{ALC}$

We analyze the basic description logic $\mathcal{ALC}$ and show that it does not have syntactic ABox updates. In particular, we consider the combination of ABoxes given in the following lemma. Please note that the original ABox $\mathcal{A}$ is formulated in $\mathcal{ALC}$, but the updated ABox $\mathcal{A}'$ is formulated in $\mathcal{ALCO}$.

Lemma 21. *Let* $\mathcal{A} = \{a : \exists r.A, A(b), r(b,b)\}$, $\mathcal{U} = \{\neg A(b)\}$ *and*

$$\mathcal{A}' = \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\}.$$

Then $\mathcal{A} * \mathcal{U} \equiv \mathcal{A}'$.

To show that the description logic $\mathcal{ALC}$ does not have syntactic ABox updates, it is enough to show that there is no $\mathcal{ALC}$ ABox that has the same $\mathcal{ALC}$ logical consequences as $\mathcal{A}'$. We first introduce the abbreviation $\exists r^n.C$ for the concept $\underbrace{\exists r \ldots \exists r}_{n}.C$. We will use this abbreviation throughout the thesis. Now, consider the concept assertion $C_n(a)$ where $C_n = \exists r.(A \sqcup \exists r^n.\neg A)$. Clearly, for all $n \in \mathbb{N}$, $\mathcal{A}' \models C_n(a)$, but $\mathcal{A}' \not\models a : \exists r.A$. We can also see that $r(b,b)$ is the only role assertion that is a logical consequence of $\mathcal{A}'$. To show that there is no $\mathcal{ALC}$ ABox which has the same $\mathcal{ALC}$ logical consequences as $\mathcal{A}'$, we show that there is no $\mathcal{ALC}$ ABox $\mathcal{A}^\diamond$ with a finite size that has the following properties:

(i) $r(b,b)$ is the only role assertion that is a logical consequence of $\mathcal{A}^\diamond$,

(ii) $\mathcal{A}^\diamond$ entails the concept assertion $C_n(a)$, for all $n \in \mathbb{N}$ and

(iii) $a : \exists r.A$ is not a logical consequence of $\mathcal{A}^\diamond$.

Now, let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox. Then, we define $\mathcal{A}|_a$ as follows.

$$\mathcal{A}|_a = \{C(a) \mid C(a) \in \mathcal{A}\}$$

Intuitively, $\mathcal{A}|_a$ is a set of concept assertions in $\mathcal{A}$ that describes the individual name a. First, we show that if a consistent $\mathcal{ALC}$ ABox $\mathcal{A}$ does not entail any role assertion going in or out from an individual name a, then $\mathcal{A}$ entails a concept assertion $C(a)$ iff $\mathcal{A}|_a$ entails $C(a)$. Without loss of generality, we can always assume that $\mathcal{A}|_a$ contains only a single concept assertion $D(a)$. Hence, we have $\mathcal{A}$ entails $C(a)$ iff $\mathcal{A}|_a$ entails $C(a)$ iff $D \sqsubseteq C$. We then show that if D is an $\mathcal{ALC}$ concept that is subsumed by $C_n(a)$ (Property (ii)) but not by $\exists r.A$ (Property (iii)) where $n \in \mathbb{N}$, then $|C| > n$. This implies $|\mathcal{A}|_a| > n$ which then implies $|\mathcal{A}| > n$. Hence, we have that $|\mathcal{A}|$ cannot be finite because $|\mathcal{A}|$ has to be greater than n for all $n \in \mathbb{N}$.

Lemma 22. *Let $\mathcal{A}$ be a consistent $\mathcal{ALC}$ ABox and a an individual name. If for all $b \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{A} \not\models r(a, b)$ and $\mathcal{A} \not\models r(b, a)$, then*

$$\mathcal{A} \models C(a) \text{ iff } \mathcal{A}|_a \models C(a).$$

Proof. ($\Leftarrow$) Assume $\mathcal{A}|_a \models C(a)$. Since $\mathcal{A}|_a \subseteq \mathcal{A}$, we know that $\mathcal{A} \models \mathcal{A}|_a$. Hence, $\mathcal{A} \models C(a)$.

($\Rightarrow$) Assume $\mathcal{A}|_a \not\models C(a)$. Then there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A}|_a$ but $\mathcal{I} \not\models C(a)$. Now, let $\mathcal{K}$ be a model of $\mathcal{A}$. Without loss of generality, we assume $\Delta^{\mathcal{K}} \cap \Delta^{\mathcal{I}} = \emptyset$. From this we can construct an interpretation $\mathcal{J}$ such that the following conditions hold. Let D be an arbitrary $\mathcal{ALC}$ concept in NNF.

(i) for all $d \in \Delta^{\mathcal{I}}$, $d \in D^{\mathcal{I}}$ implies $d \in D^{\mathcal{J}}$,

(ii) for all $d \in \Delta^{\mathcal{K}}$, $d \in D^{\mathcal{K}}$ implies $d \in D^{\mathcal{J}}$,

(iii) for all role assertions $\varphi \in \mathcal{A}$, $\mathcal{J} \models \varphi$ iff $\mathcal{K} \models \varphi$ and

(iv) $a^{\mathcal{J}} = a^{\mathcal{I}}$

From Point (ii), (iii) and the fact that $\mathcal{K}$ is a model of $\mathcal{A}$, we know that $\mathcal{J} \models \mathcal{A} \backslash \mathcal{A}|_a$. And then from Point (i), (iv) and the assumption $\mathcal{I} \models \mathcal{A}|_a$ but $\mathcal{I} \not\models C(a)$ we have $\mathcal{J} \models \mathcal{A}|_a$ but $\mathcal{J} \not\models C(a)$. Hence, we have $\mathcal{A} \not\models C(a)$.

Now, to conclude the proof, we show that there exists an interpretation $\mathcal{J}$ that satisfies all of the conditions mentioned above. Such $\mathcal{J}$ can be constructed as follows.

$$\begin{aligned}
\Delta^{\mathcal{J}} &:= \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{K}} \\
a^{\mathcal{J}} &:= a^{\mathcal{I}} \\
b^{\mathcal{J}} &:= b^{\mathcal{K}} \text{ for all } b \in \mathsf{N_I} \setminus \{a\} \\
A^{\mathcal{J}} &:= A^{\mathcal{I}} \cup A^{\mathcal{K}} \text{ for all } A \in \mathsf{N_C} \\
r^{\mathcal{J}} &:= r^{\mathcal{I}} \cup r^{\mathcal{K}} \text{ for all } r \in \mathsf{N_R}
\end{aligned}$$

Point (iv) follows directly from the construction of $\mathcal{J}$. Let φ be a role assertion in $\mathcal{A}$. To show Point (iii), we do a case analysis on φ.

- $\varphi = \neg r(\alpha, \beta)$. This case is trivial because we do not add any edge to the interpretation of any role name during the construction of $\mathcal{J}$.
- $\varphi = r(\alpha, \beta)$. From the assumption $\mathcal{A}$ is consistent and for all $b \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{A} \not\models r(a, b)$ and $\mathcal{A} \not\models r(b, a)$, we know that $\alpha \neq a$ and $\beta \neq a$. This means $\alpha^{\mathcal{J}} \notin \Delta^{\mathcal{I}}$ and $\beta^{\mathcal{J}} \notin \Delta^{\mathcal{I}}$ which then implies $\alpha^{\mathcal{J}} \in \Delta^{\mathcal{K}}$ and $\beta^{\mathcal{J}} \in \Delta^{\mathcal{K}}$. Since during the construction of $\mathcal{J}$ we do not remove any edge from the interpretation of role names, we have $\mathcal{J} \models r(\alpha, \beta)$ iff $\mathcal{K} \models r(\alpha, \beta)$.

We show Point (i) using structural induction on D. Point (ii) can be proven analogously.

- $D = A$ or $D = \neg A$, where $A \in \mathsf{N_C}$. Since $d \in \Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{I}} \cap \Delta^{\mathcal{K}} = \emptyset$, we know that $d \notin A^{\mathcal{K}}$. Hence, $d \in A^{\mathcal{J}}$ iff $d \in A^{\mathcal{I}}$.

- $D = E \sqcap F$ or $D = E \sqcup F$. These cases follow directly from the semantics and induction hypothesis.

- $D = \exists r.E$ or $D = \forall r.E$. These cases are trivial because we do not add or remove any edge from $r^{\mathcal{I}}$ during the construction of $\mathcal{J}$.

□

Next, we introduce the notion of role depth of an $\mathcal{ALC}$ concept. Using this notion, we show in the next lemma that if C is an $\mathcal{ALC}$ concept that is subsumed by C_n but not by $\exists r.A$ then $|C| > n$ where $n \in \mathbb{N}$. The idea is to exploit the fact that C cannot "see" deeper than its own role depth and $|C|$ is always greater than its role depth.

Definition 23 (Role depth of $\mathcal{ALC}$ concepts)**.** Let A be a concept name and C an $\mathcal{ALC}$ concept. Then we define the *role depth* of a concept C (written $\mathsf{rd}(C)$) as follows.

- $\mathsf{rd}(A) := 0$,
- $\mathsf{rd}(\neg C) := \mathsf{rd}(C)$,
- $\mathsf{rd}(C \sqcap D) := \mathsf{rd}(C \sqcup D) := \max\{\mathsf{rd}(C), \mathsf{rd}(D)\}$,
- $\mathsf{rd}(\exists r.C) := \mathsf{rd}(\forall r.C) := 1 + \mathsf{rd}(C)$.

Let C be an $\mathcal{ALC}$ concept. Suppose that we have two different tree models $\mathcal{I}$ and $\mathcal{I}'$ of C that have depth $n > \mathsf{rd}(C)$ and if we cut these trees at depth $\mathsf{rd}(C)$ we end up with two identical trees. We show that C will not be able to differentiate these two models, i.e. C is satisfied at the root of $\mathcal{I}$ iff C is satisfied at the root of $\mathcal{I}'$. So in order for the concept C to be able to differentiate these two trees, the role depth of C has to be at least n which then implies $|C| > n$.

Before showing the lemma, we first introduce the notion of depth in an interpretation tree. Let $\mathcal{I}$ be a tree interpretation and $d \in \Delta^{\mathcal{I}}$. We define the notion of the *depth* of an individual in a finite tree.

$$\mathtt{depth}(d) := \begin{cases} 0 & \text{if } d \text{ is a root,} \\ n+1 & \text{if there exists } e \in \Delta^{\mathcal{I}} \text{ and } r \in \mathsf{N_R} \text{ such that} \\ & \mathtt{depth}(e) = n \text{ and } (e,d) \in r^{\mathcal{I}} \end{cases}$$

Lemma 24. *For all $n \in \mathbb{N}$, $C \not\sqsubseteq \exists r.A$ and $C \sqsubseteq C_n$ implies $|C| > n$*

Proof. Assume $C \not\sqsubseteq \exists r.A$ and $C \sqsubseteq C_n$. Then we know that $C \sqcap \neg\exists r.A$ is satisfiable. Hence, from Lemma 2, we get $C \sqcap \neg\exists r.A$ has a tree model. Now, assume $\mathsf{rd}(C) < n$ and let $\mathcal{I}$ be a tree model of $C \sqcap \neg\exists r.A$ such that $d \in \Delta^{\mathcal{I}}$ is the root of $\mathcal{I}$ where $d \in (\exists r^{n+1}.A)^{\mathcal{I}} \setminus (\exists r.A)^{\mathcal{I}}$.

Now, let us construct a new interpretation $\mathcal{I}'$ by cutting all outgoing edges from all nodes at depth $\mathsf{rd}(C)$. Formally, $\mathcal{I}'$ is constructed by extending $\mathcal{I}$ as follows.

$$r^{\mathcal{I}'} := r^{\mathcal{I}} \setminus \{(e, f) \mid \mathtt{depth}(e) = \mathsf{rd}(C)\}.$$

We claim that the following properties hold.

(i) $d \notin (\exists r^{n+1}.A)^{\mathcal{I}'}$

(ii) $d \in C^{\mathcal{I}'}$

With the properties above, we can see that $\mathcal{I}'$ shows that $C \not\sqsubseteq C_n$ (contradiction). Therefore, we have $\mathsf{rd}(C) \geq n$ which implies $|C| > n$.

To conclude the proof, we need to show that $\mathcal{I}'$ satisfies the properties above. Point (i) is a direct consequence of the construction of $\mathcal{I}'$. We show Point (ii) by claiming that for all subconcepts D of C and all $e \in \Delta^{\mathcal{I}'}$ with $\mathtt{depth}(e) \leq \mathsf{rd}(C) - \mathsf{rd}(D)$, $e \in D^{\mathcal{I}}$ implies $e \in D^{\mathcal{I}'}$. We prove the claim using structural induction on D. Without loss of generality, we assume that D is in NNF.

- $D = A$ or $D = \neg A$, where $A \in \mathsf{N_C}$. This case is trivial, since during the construction of $\mathcal{I}'$ we do not change the interpretation of concept names.
- $D = E \sqcap F$ or $D = E \sqcup F$. From Definition 23, we know that $\mathsf{rd}(E) \leq \mathsf{rd}(D)$ and $\mathsf{rd}(F) \leq \mathsf{rd}(D)$. Hence, from the semantics and induction hypothesis we have $e \in D^{\mathcal{I}'}$.
- $D = \exists r.E$. Assume that $\mathtt{depth}(e) \geq \mathsf{rd}(C)$, then $\mathsf{rd}(D) \leq 0$. But then this contradicts the fact that $\mathsf{rd}(D) = \mathsf{rd}(\exists r.E) \geq 1$. Hence, we know that $\mathtt{depth}(e) < \mathsf{rd}(C)$. This case then becomes trivial because we do not remove any outgoing edge from e for all individuals e with $\mathtt{depth}(e) < \mathsf{rd}(C)$.
- $D = \forall r.E$. This case is trivial because we do not add any edge to $r^{\mathcal{I}}$ during the construction of $\mathcal{I}'$.

□

We are now ready to show that there is no $\mathcal{ALC}$ ABox which does not entail any role assertion except $r(b, b)$ and entails the assertions $C_n(a)$ for all $n \in \mathbb{N}$, but does not entail $a : \exists r.A$.

Lemma 25. *There is no $\mathcal{ALC}$ ABox $\mathcal{A}$ such that $\mathcal{A}$ has the following properties:*

- *for all $n \in \mathbb{N}$, $\mathcal{A} \models C_n(a)$,*
- *$\mathcal{A} \not\models a : \exists r.A$ and*
- *for all $c \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{A} \not\models r(a, c)$ and $\mathcal{A} \not\models r(c, a)$.*

Proof. Assume that such an ABox $\mathcal{A}$ exists. Lemma 22 tells us that $\mathcal{A}|_a \models C_n(a)$ for all $n \in \mathbb{N}$ and $\mathcal{A} \not\models a : \exists r.A$. This means that $\mathcal{A}|_a \neq \emptyset$. Without loss of generality, we assume that $C(a)$ is the only assertion in $\mathcal{A}|_a$. Hence, we have $\{C(a)\} \models C_n(a)$ but $\{C(a)\} \not\models a : \exists r.A$ for all $n \in \mathbb{N}$. This is equivalent to $C \sqsubseteq C_n$ and $C \not\sqsubseteq \exists r.A$ for all $n \in \mathbb{N}$. Hence, by Lemma 24, we know that $|C| > n$ for all $n \in \mathbb{N}$. This gives a contradiction to the fact that $|C|$ is finite. □

Combining the results that we have so far, we are now ready to show that there is no $\mathcal{ALC}$-ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}'$ and $\mathcal{A}^\diamond$ are $\mathcal{ALC}$-indistinguishable.

Lemma 26. *Let $\mathcal{A}'$ be the semantically updated ABox described in Lemma 21. Then there is no $\mathcal{ALC}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}'$ and $\mathcal{A}^\diamond$ are $\mathcal{ALC}$-indistinguishable.*

Proof. It is easy to see that $\mathcal{A}'$ has all properties from Lemma 25. □

Theorem 27. *$\mathcal{ALC}$ does not have syntactic ABox updates.*

Recall that the updated ABox $\mathcal{A}'$ described in Lemma 21 is an $\mathcal{ALCO}$ ABox. From this, one may be tempted to conjecture that adding nominals is enough to recover the existence of syntactic ABox updates. Unfortunately, this is not the case. We show this in the following section.

3.2 Syntactic Updates in $\mathcal{ALCO}$

We now consider the description logic $\mathcal{ALCO}$. We show that extending $\mathcal{ALC}$ by adding nominals is not enough to recover the existence of syntactic ABox updates.

It has been shown in [4] that the DL $\mathcal{ALCO}$ does not have semantic ABox updates. In this section, we will see that this is also the case even if we only consider syntactic ABox updates. In particular, we consider the combination of ABoxes in the following lemma (which is taken from [4]). Please note that the original ABox $\mathcal{A}$ is formulated in $\mathcal{ALC}$ (which is also an $\mathcal{ALCO}$ ABox), but the semantically updated ABox $\mathcal{A}'$ is formulated in $\mathcal{ALC}^{@}$. We show that there is no $\mathcal{ALCO}$ ABox that has the same $\mathcal{ALCO}$ logical consequences as the semantically updated ABox $\mathcal{A}'$.

Lemma 28. *Let $\mathcal{A} = \{a : \exists r.A\}$, $\mathcal{U} = \{\neg r(a, b)\}$ and*

$$\mathcal{A}' = \{a : \exists r.A \sqcup @_b A, \neg r(a, b)\}.$$

Then $\mathcal{A} * \mathcal{U} \equiv \mathcal{A}'$.

Let us describe the models of $\mathcal{A}'$. We know $\mathcal{A}' \models \neg r(a, b)$ so for all models $\mathcal{I}$, we have $\mathcal{I} \not\models r(a, b)$. We also know $\mathcal{A}' \models a : \exists r.A \sqcup @_b A$. From this, we can derive the fact that for all models $\mathcal{I}$ of $\mathcal{A}'$, we have $\mathcal{I} \models a : \exists r.A$ or $\mathcal{I} \models A(b)$. In summary we have, for all models $\mathcal{I}$ of $\mathcal{A}'$,

(i) $\mathcal{I} \models \neg r(a, b)$ and

(ii) $\mathcal{I} \models a : \exists r.A$ or $\mathcal{I} \models A(b)$.

The idea now is to find $\mathcal{ALCO}$ concept assertions that are always satisfied by the interpretations that satisfies both properties above, but cannot be entailed by a single $\mathcal{ALCO}$ ABox. There are not so many interesting $\mathcal{ALCO}$ ABox assertions that we can derive from Property (i). That is why we will concentrate on Property (ii). We choose to consider an assertion for individual name b (one can also try to find for individual name a). The negation of Property (ii) is $\mathcal{I} \models a : \neg\exists r.A$ and $\mathcal{I} \models \neg A(b)$. Let $\mathbf{I}$ be the set of interpretations that satisfies the negation of Property (ii). We now use the expressiveness power of concept description in the DL $\mathcal{ALCO}$ to express concept assertions as concepts. For example, we can use the concept $\{a\} \sqcap \neg\exists r.A$ to express the concept assertion $a : \neg\exists r.A$. Now, it is easy to see that for all $s \in \mathsf{N_R}$, $\mathcal{I} \models b : (\neg A \sqcap \exists s.(\{a\} \sqcap \neg(\exists r.A)))$ implies $\mathcal{I} \in \mathbf{I}$. Since we know that $\mathbf{I}$ contains only interpretations that satisfy the negation of Property (ii), for all $\mathcal{I} \notin \mathbf{I}$, $\mathcal{I}$ satisfies Property (ii). So we can conclude that for all interpretation $\mathcal{I}$, if $\mathcal{I}$ satisfies Property (ii), then for all $s \in \mathsf{N_R}$, $\mathcal{I} \not\models b : (\neg A \sqcap \exists s.(\{a\} \sqcap \neg(\exists r.A)))$ or written in a nicer way $\mathcal{I} \models b : (A \sqcup \forall s.(\{a\} \rightarrow (\exists r.A)))$ for all $s \in \mathsf{N_R}$. From now on, we will be using the abbreviation $C_s = A \sqcup \forall s.(\{a\} \rightarrow (\exists r.A))$ throughout this section. We also use the following abbreviations. Let R be a finite subset of $\mathsf{N_R}$. We define concept abbreviation $\exists\mathsf{path}(R, n).C$ as follows. For $n \in \mathbb{N}$,

$$\exists\mathsf{path}(R, n).C := \bigsqcup_{\substack{p \in (R)^* \\ \mathrm{len}(p) \leq n}} \exists p.C$$

where * is the kleene-* operator.

Lemma 29. *For all finite subsets R of $\mathsf{N_R}$, role names $u \notin R$, finite subsets I of $\mathsf{N_I} \setminus \{b\}$ and $n \in \mathbb{N}$,*

$$\mathcal{A}' \not\models a : \exists r.A \sqcup \bigsqcup_{c \in I} \forall u.(\{c\} \rightarrow \exists\mathsf{path}(R, n).\{b\}) \text{ and} \quad (3.1)$$

$$\mathcal{A}' \not\models b : A \sqcup \bigsqcup_{c \in I} \exists\mathsf{path}(R, n).\{c\}. \quad (3.2)$$

Proof. Consider the interpretations $\mathcal{I}$ and $\mathcal{I}'$ depicted in Figure 3.1. We assume that the individual names a and b are mapped to the objects of the same name, and all other individual names are mapped to the individual c. Moreover, the concept name A and role names r and u are interpreted as shown in the figure and all other concept and role names are interpreted as empty sets. It can easily be checked that $\mathcal{I}$ and $\mathcal{I}'$ are models of $\mathcal{A}'$ where (3.1) and (3.2) holds respectively. □

To show $\mathcal{ALCO}$ does not have syntactic ABox updates, it suffices to show that there is no $\mathcal{ALCO}$ ABox which has the same $\mathcal{ALCO}$ consequences as $\mathcal{A}'$. We show this by showing there is no $\mathcal{ALCO}$ ABox that entails $C_s(b)$ for all $s \in \mathsf{N_R}$ and also satisfies (3.1) and (3.2) at the same time.

We now introduce a property of $\mathcal{ALCO}$ concept assertions. Let us view an interpretation $\mathcal{I}$ as a graph $G^{\mathcal{I}} = (V^{\mathcal{I}}, E^{\mathcal{I}})$ where $V^{\mathcal{I}} = \Delta^{\mathcal{I}}$ and $E^{\mathcal{I}} = \{(d, e) \mid \text{there exists } r \in \mathsf{N_R} \text{ such that } (d, e) \in r^{\mathcal{I}}\}$. Then, the truth value of an $\mathcal{ALCO}$ concept assertion $C(a)$ in an interpretation $\mathcal{I}$ only depends on the set of individuals that are *reachable* from $a^{\mathcal{I}}$ in the graph $G^{\mathcal{I}}$ using less than $|C|$ steps. Formally, this property can be defined as in the following lemma.

We first define several abbreviations that are used throughout this thesis. Let C be a concept. We will be using $\mathsf{concs}(C)$, $\mathsf{roles}(C)$ and $\mathsf{inds}(C)$ for the set of concept, role and individual names that occur in C respectively. This abbreviation is also extended to ABoxes.

Lemma 30. *Let $C(a)$ be an $\mathcal{ALCO}$ concept assertion. Then, for all interpretations $\mathcal{I}$ and $\mathcal{I}'$, if for all $d \in \Delta^{\mathcal{I}}$, d is reachable from $a^{\mathcal{I}}$ in $G^{\mathcal{I}}$ using less than $|C|$ steps implies*

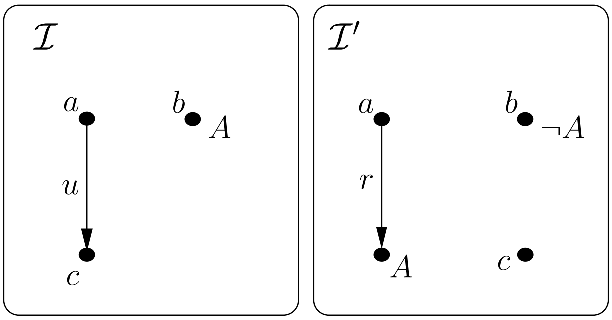

Figure 3.1: $\mathcal{I}$ and $\mathcal{I}'$

- $d \in \Delta^{\mathcal{I}'}$,
- *for all* $A \in \mathsf{concs}(C)$, $d \in A^{\mathcal{I}}$ *iff* $d \in A^{\mathcal{I}'}$,
- *for all* $r \in \mathsf{roles}(C)$, $(d, e) \in r^{\mathcal{I}}$ *iff* $(d, e) \in r^{\mathcal{I}'}$ *and*
- *for all* $a \in \mathsf{inds}(\{C(a)\})$, $a^{\mathcal{I}} = d$ *iff* $a^{\mathcal{I}'} = d$

then

$$\mathcal{I} \models C(a) \textit{ iff } \mathcal{I}' \models C(a).$$

Intuitively, Lemma 30 says that the assertion $C(a)$ cannot differentiate two interpretations $\mathcal{I}$, $\mathcal{I}'$ if the two interpretations have differences only:

- at the individuals that is not reachable from $a^{\mathcal{I}}$ in $G^{\mathcal{I}}$ using less than $|C|$ steps or
- on the interpretations of concept, role and individual names that do not occur in $C(a)$.

Lemma 31. *Let $\mathcal{A}$ be an $\mathcal{ALCO}$ ABox, u a role name such that $u \notin \mathsf{roles}(\mathcal{A})$ and $|\mathcal{A}| = n$. If*

$$\begin{aligned} \mathcal{A} &\not\models a : \exists r.A \sqcup \bigsqcup_{c \in \mathsf{inds}(\mathcal{A}) \setminus \{b\}} \forall u.(\{c\} \rightarrow \exists \mathsf{path}(\mathsf{roles}(\mathcal{A}), n).\{b\}) \textit{ and} \\ \mathcal{A} &\not\models b : A \sqcup \bigsqcup_{c \in \mathsf{inds}(\mathcal{A}) \setminus \{b\}} \exists \mathsf{path}(\mathsf{roles}(\mathcal{A}), n).\{c\} \end{aligned}$$

then there exists a model $\mathcal{I}$ of $\mathcal{A}$ such that

$$\mathcal{I} \not\models a : \exists r.A \textit{ and } \mathcal{I} \not\models A(b).$$

Proof. From the assumptions, we know that there are models $\mathcal{J}$ and $\mathcal{K}$ of $\mathcal{A}$ such that

(i) $\mathcal{J} \not\models a : \exists r.A$,

(ii) $\mathcal{J} \models a : \exists u.(\{c\} \sqcap \neg\exists \mathsf{path}(\mathsf{roles}(\mathcal{A}), n).\{b\})$ for all $c \in \mathsf{inds}(\mathcal{A}) \setminus \{b\}$,

(iii) $\mathcal{K} \not\models A(b)$,

(iv) $\mathcal{K} \models b : \neg\exists \mathsf{path}(\mathsf{roles}(\mathcal{A}), n).\{c\}$ for all $c \in \mathsf{inds}(\mathcal{A}) \setminus \{b\}$,

Now we will construct an interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models a : \exists r.A$ and $\mathcal{I} \not\models b : A$ by combining $\mathcal{J}$ and $\mathcal{K}$. And later, we show that $\mathcal{I}$ is still a model of $\mathcal{A}$. Without loss of generality, we assume $\Delta^{\mathcal{J}} \cap \Delta^{\mathcal{K}} = \emptyset$. Then $\mathcal{K}$ is constructed as follows. For all concept names A, role names r, and individuals $c \in \mathsf{N_I} \setminus \{a, b\}$,

$$\begin{array}{rcl} \Delta^{\mathcal{I}} & := & \Delta^{\mathcal{J}} \cup \Delta^{\mathcal{K}} \\ A^{\mathcal{I}} & := & A^{\mathcal{J}} \cup A^{\mathcal{K}} \\ r^{\mathcal{I}} & := & r^{\mathcal{J}} \cup r^{\mathcal{K}} \\ a^{\mathcal{I}} & := & a^{\mathcal{J}} \\ b^{\mathcal{I}} & := & b^{\mathcal{K}} \\ c^{\mathcal{I}} & := & c^{\mathcal{J}} \end{array}$$

From the construction, we can derive the following facts: $\mathcal{I} \not\models a : \exists r.A$ since $\mathcal{J} \not\models a : \exists r.A$ and $\mathcal{I} \not\models A(b)$ since $\mathcal{K} \not\models A(b)$. It remains to show that $\mathcal{I} \models \mathcal{A}$. We make a case distinction according to the type of assertion $\varphi \in \mathcal{A}$:

- φ is a positive role assertion. The only possible positive role assertions in $\mathcal{A}$ that can be violated by $\mathcal{I}$ are role assertions that involve individual b and some individual $\alpha \neq b$. But Point (ii) and (iv) guarantee that these role assertions do not exists in $\mathcal{A}$.
- φ is a negative role assertion. Since we did not add any new pair to the interpretation of any role, it is easy to see that if $\mathcal{K} \models \varphi$ or $\mathcal{J} \models \varphi$ then $\mathcal{I} \models \varphi$.
- φ is a concept assertion $C(\alpha)$. From Lemma 30,we know that the truth value of an assertion $C(a) \in \mathcal{A}$ in the interpretation $\mathcal{I}$ only depends on the set of individuals that are reachable from $\alpha^{\mathcal{I}}$ in $G^{\mathcal{I}}$ using role names that occurs in C with less than $|C|$ steps. We make a case distinction on α:
  - $\alpha \neq b$. Then we have to pay attention to the individuals in $\Delta^{\mathcal{J}}$ because $\alpha^{\mathcal{I}} = \alpha^{\mathcal{J}}$. The only individual in $\Delta^{\mathcal{J}}$ that is involved during the construction of $\mathcal{J}$ is $b^{\mathcal{J}}$. But then, from Point (ii) we know that $\alpha^{\mathcal{J}}$ cannot reach $b^{\mathcal{J}}$ using role names that occurs in C with role depth at most $|C|$. Hence, it is clear that $\alpha^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $\alpha^{\mathcal{J}} \in C^{\mathcal{J}}$.
  - $\alpha = b$. For this case, we must pay attention on the individuals in $\Delta^{\mathcal{K}}$ because $\alpha^{\mathcal{I}} = \alpha^{\mathcal{K}}$. During the constructions, we changed the interpretations of individuals names in $\mathsf{N_I} \setminus \{b\}$. But then, from point (iv) we know that $\alpha^{\mathcal{K}}$ cannot reach any other individual names using role names that occurs in C with role depth at most $|C|$. Hence, it is clear that $\alpha^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $\alpha^{\mathcal{K}} \in C^{\mathcal{K}}$.

□

Combining the results that we have so far, we are now ready to show that there is no $\mathcal{ALCO}$-ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}'$ and $\mathcal{A}^{\diamond}$ are $\mathcal{ALCO}$-indistinguishable.

Lemma 32. *Let $\mathcal{A}'$ be the semantically updated ABox described in Lemma 28. Then there exists no $\mathcal{ALCO}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}'$ and $\mathcal{A}^\diamond$ have the same $\mathcal{ALCO}$ logical consequences.*

Proof. Assume that there is an $\mathcal{ALCO}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}'$ and $\mathcal{A}^\diamond$ are $\mathcal{ALCO}$-indistinguishable. Let $C_s = A \sqcup \forall s.(\{a\} \rightarrow \exists r.A)$. It is easy to see that $\mathcal{A}' \models C_s(b)$ for all $s \in \mathsf{N_R}$. Then from the assumption, we have $\mathcal{A}^\diamond \models C_s(b)$. Lemma 29 and the assumption implies $\mathcal{A}^\diamond$ has the following properties.

$$\begin{aligned}\mathcal{A}^\diamond &\not\models a : \exists r.A \sqcup \bigsqcup_{c \in \mathsf{inds}(\mathcal{A}^\diamond)\setminus\{b\}} \forall u.(\{c\} \rightarrow \exists \mathsf{path}(\mathsf{roles}(\mathcal{A}^\diamond), n).\{b\}) \text{ and}\\ \mathcal{A}^\diamond &\not\models b : A \sqcup \bigsqcup_{c \in \mathsf{inds}(\mathcal{A}^\diamond)\setminus\{b\}} \exists \mathsf{path}(\mathsf{roles}(\mathcal{A}^\diamond), n).\{c\}.\end{aligned}$$

From Lemma 31, we know that there exists a model $\mathcal{I}$ of $\mathcal{A}^\diamond$ such that $\mathcal{I} \not\models a : \exists r.A$ and $\mathcal{I} \not\models A(b)$. Since $\mathcal{I} \not\models A(b)$ and for all $s \in \mathsf{N_R}$, $\mathcal{A}^\diamond \models C_s(b)$, it has to be the case that for all $s \in \mathsf{N_R}$, $\mathcal{I} \models b : \forall s.(\{a\} \rightarrow \exists r.A)$ (otherwise $\mathcal{I}$ is already a counter model). The idea now is to show that there exists an interpretation $\mathcal{I}'$ and a role name t such that

(i) $\mathcal{I}' \models \mathcal{A}^\diamond$,

(ii) $\mathcal{I}' \not\models A(b)$ and

(iii) $\mathcal{I}' \not\models b : \forall t.(\{a\} \rightarrow \exists r.A)$.

This implies that there exists a $t \in \mathsf{N_R}$ such that $\mathcal{A}^\diamond \not\models C_t(b)$ (contradiction).

To conclude the proof, we show how to construct $\mathcal{I}'$. Let t be a role name such that $t \notin \mathsf{roles}(\mathcal{A}^\diamond)$. Then $\mathcal{I}'$ can be obtained by extending $\mathcal{I}$ as follows.

$$t^{\mathcal{I}'} := t^{\mathcal{I}} \cup \{(b^{\mathcal{I}}, a^{\mathcal{I}})\}$$

It is easy to see that the Point (i) hold because during the construction of $\mathcal{I}'$, we do not change the interpretation of concept, role and individual names that occur in $\mathcal{A}^\diamond$. Point (ii) holds because $\mathcal{I} \not\models A(b)$ and during the construction of $\mathcal{I}'$, we do not change the interpretation of any concept name. From the fact that $\mathcal{I} \not\models a : \exists r.A$ and we do not change the interpretation of $r^{\mathcal{I}}$ and any concept name, we have $\mathcal{I}' \not\models a : \exists r.A$. Point (iii) holds because $(b^{\mathcal{I}'}, a^{\mathcal{I}'}) \in t^{\mathcal{I}'}$ and $\mathcal{I}' \not\models a : \exists r.A$. □

Theorem 33. *$\mathcal{ALCO}$ does not have syntactic ABox updates.*

Please recall that the semantically updated ABox in Lemma 28 is given in the DL $\mathcal{ALC}^{@}$. From this, one might again be tempted to conjecture that adding the @ constructor to the DL $\mathcal{ALC}$ will recover the ABox syntactic update. We will see in the next subsection that this is not the case.

3.3 Syntactic Updates in $\mathcal{ALC}^{@}$ and Boolean ABoxes

As we have seen in the previous section, extending $\mathcal{ALC}$ with only nominals is not enough to recover the existence of syntactic ABox updates. We now study the existence of syntactic ABox updates in $\mathcal{ALC}^@$. It has been shown in [4] that $\mathcal{ALC}^@$ does not have semantic ABox update. In this section, we study the existence of syntactic ABox update in $\mathcal{ALC}^@$. It turns out that $\mathcal{ALC}^@$ ABoxes are not expressive enough to express the syntactically updated $\mathcal{ALC}^@$ ABoxes. We also show some results related to Boolean ABox in this section. First, we introduce the definition of restricted Boolean ABox and then study the relation between it and $\mathcal{ALC}^@$ ABox.

A *restricted Boolean $\mathcal{L}$ ABox* is a finite set of role and *restricted Boolean $\mathcal{L}$ assertions*, i.e., Boolean combinations of $\mathcal{L}$ concept assertions expressed in terms of the connectives $\wedge$ and $\vee$. We do not need to introduce negation since concept negation is available in every DL considered in this thesis. For example, $\mathcal{A}$ is *not* a restricted Boolean $\mathcal{ALC}$ ABox, but $\mathcal{B}$ is.

- $\mathcal{A} = \{A(a), r(a,b) \wedge (B(a) \vee B(b))\}$
- $\mathcal{B} = \{A(a), b : \exists r.B \vee (B(a) \wedge B(b)), r(a,b)\}$.

We treat the interpretation of the connectives $\wedge$ and $\vee$ as in propositional logic. An interpretation $\mathcal{I}$ *satisfies* a restricted Boolean assertion $\varphi \vee \psi$ (written $\mathcal{I} \models \varphi \vee \psi$) if $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$ and $\mathcal{I} \models \varphi \wedge \psi$ if $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$. An interpretation $\mathcal{I}$ *satisfies a restricted Boolean ABox $\mathcal{A}$* if $\mathcal{I}$ satisfies all role and restricted Boolean assertions in $\mathcal{A}$. There exists a very close connection between $\mathcal{ALC}^@$ ABoxes and restricted Boolean $\mathcal{ALC}$ ABoxes.

Lemma 34.

1. *For every $\mathcal{ALC}^@$ ABox, there exists an equivalent restricted Boolean $\mathcal{ALC}$ ABox.*
2. *For every restricted Boolean $\mathcal{ALC}$ ABox, there exists an equivalent $\mathcal{ALC}^@$ ABox.*

Proof.

(Point 1). Let $\mathcal{A}$ be an $\mathcal{ALC}^@$ ABox and $C(a) \in \mathcal{A}$ such that $@_b D \in \mathsf{sub}(C)$. We construct an $\mathcal{ALC}$ restricted Boolean ABox $\mathcal{A}'$ such that $\mathcal{A} \equiv \mathcal{A}'$. The ABox $\mathcal{A}'$ is obtained by replacing all assertions $C(a) \in \mathcal{A}$ where $@_b D \in \mathsf{sub}(C)$, with $(D(b) \wedge C[\top/@_b D]) \vee (\neg D(b) \wedge C[\bot/@_b D])$. It is easy to see that $\mathcal{A} \equiv \mathcal{A}'$

(Point 2). Let us define a mapping $\cdot^\diamond$ from $\mathcal{ALC}$ restricted Boolean assertions to $\mathcal{ALC}^@$ concepts as follow.

$$
\begin{aligned}
(\varphi \wedge \psi)^\star &:= \varphi^\star \sqcap \psi^\star \\
(\varphi \vee \psi)^\star &:= \varphi^\star \sqcup \psi^\star \\
(a : C)^\star &:= @_a C
\end{aligned}
$$

Now, we are ready to convert a restricted Boolean $\mathcal{ALC}$-ABox $\mathcal{A}$ into an $\mathcal{ALC}^@$ ABox. Let α be any arbitrary individual name and φ an $\mathcal{ALC}$ restricted Boolean assertion.

$$
\begin{aligned}
\mathcal{A}' = \; & \{\alpha : \varphi^\star \mid \varphi \in \mathcal{A}\} \cup \\
. \quad & \{r(a,b) \mid r(a,b) \in \mathcal{A}\} \cup \\
& \{\neg r(a,b) \mid \neg r(a,b) \in \mathcal{A}\}
\end{aligned}
$$

It is easy to see that $\mathcal{A}'$ is equivalent to $\mathcal{A}$.

□

Thus, $\mathcal{ALC}^{@}$ ABoxes have the same expressive power as $\mathcal{ALC}$ restricted Boolean ABoxes. Recall the ABoxes $\mathcal{A}$, $\mathcal{U}$, and $\mathcal{A}'$ in Lemma 21.

$$\begin{aligned}\mathcal{A} &= \{a : \exists r.A, A(b), r(b,b)\},\\ \mathcal{U} &= \{\neg A(b)\}\\ \mathcal{A}' &= \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\}.\end{aligned}$$

To prove that $\mathcal{ALC}^{@}$ does not have ABox syntactic updates, we show that there is no restricted Boolean $\mathcal{ALC}$ ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}'$ and $\mathcal{A}^{\diamond}$ are $\mathcal{ALC}^{@}$-indistinguishable. To show this, it is enough to show that there is no restricted Boolean $\mathcal{ALC}$ ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}'$ and $\mathcal{A}^{\diamond}$ are $\mathcal{ALC}^{@}$-indistinguishable because every $\mathcal{ALC}$ ABox assertion is an $\mathcal{ALC}^{@}$ ABox assertion. Without loss of generality, we can assume that the restricted Boolean ABoxes are always represented in disjunctive normal form as follows.

$$\mathcal{B} = \mathcal{B}|_R \wedge (\mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1})$$

where $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are $\mathcal{ALC}$ ABoxes without role assertions and $\mathcal{B}|_R$ contains only role assertions. We first establish a property of restricted Boolean ABoxes in the following lemma.

Lemma 35. *Let φ be an $\mathcal{ALC}$ concept assertion, $r(a,b)$ a positive role assertion and $\mathcal{B}$ a consistent $\mathcal{ALC}$ restricted Boolean ABox in disjunctive normal form. Then,*

1. $\mathcal{B} \models \varphi$ *iff for all* $0 \leq m \leq n-1, \mathcal{B}|_R \cup \mathcal{B}_m \models \varphi$
2. $\mathcal{B} \models r(a,b)$ *iff* $r(a,b) \in \mathcal{B}|_R$

Proof.

1. (⇒) Assume there exists an $m < n$ such that $\mathcal{B}|_R \cup \mathcal{B}_m \not\models \varphi$. Then there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{B}|_R \cup \mathcal{B}_m$ but $\mathcal{I} \not\models \varphi$. Since $\mathcal{I} \models \mathcal{B}|_R \cup \mathcal{B}_m$ implies $\mathcal{I} \models \mathcal{B}$, we have $\mathcal{B} \not\models \varphi$.

   (⇐) Assume $\mathcal{B} \not\models \varphi$. Then there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{B}$ but $\mathcal{I} \not\models \varphi$. Since $\mathcal{I} \models \mathcal{B}$ then there exists an $m < n$ such that $\mathcal{I} \models \mathcal{B}|_R \cup \mathcal{B}_m$. Hence, for this particular $\mathcal{B}_m$, we have $\mathcal{B}|_R \cup \mathcal{B}_m \not\models \varphi$.

2. (⇒) Assume $\mathcal{B}$ is consistent, $\mathcal{B} \models r(a,b)$ and $r(a,b) \notin \mathcal{B}|_R$. Since $\mathcal{B}$ is consistent, there exists a model $\mathcal{I}$ of $\mathcal{B}$. Thus from the assumption $\mathcal{B} \models r(a,b)$, we have $\mathcal{I} \models r(a,b)$. We construct $\mathcal{I}'$ from $\mathcal{I}$ as follow. Let $A \in \mathsf{N_I}$, $s \in \mathsf{N_R} \setminus \{r\}$ and $\alpha \in \mathsf{N_I}$. Then

$$\begin{aligned}\Delta^{\mathcal{I}'} &:= \Delta^{\mathcal{I}} \cup \{b'\} \text{ where } b' \notin \Delta^{\mathcal{I}}\\ A^{\mathcal{I}'} &:= A^{\mathcal{I}} \cup \{b' \mid b \in A^{\mathcal{I}}\}\\ r^{\mathcal{I}'} &:= r^{\mathcal{I}} \cup \{(b',c) \mid (b^{\mathcal{I}},c) \in r^{\mathcal{I}}\} \cup \{(a^{\mathcal{I}},b')\} \setminus \{(a^{\mathcal{I}},b^{\mathcal{I}})\}\\ s^{\mathcal{I}'} &:= s^{\mathcal{I}} \cup \{(b',c) \mid (b^{\mathcal{I}},c) \in s^{\mathcal{I}}\}\\ \alpha^{\mathcal{I}'} &:= \alpha^{\mathcal{I}}\end{aligned}$$

   The idea of the construction is to replace $b^{\mathcal{I}}$ with its imitation b' and then cut the edge $r(a,b)$. We now claim that the followings hold

(i) Let ψ be a positive role assertion. Then,

$$\mathcal{I} \models \psi \text{ and } \mathcal{I}' \not\models \psi \text{ implies } \psi = r(a,b).$$

(ii) For all $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALC}$ concepts C, $d \in C^{\mathcal{I}'}$ iff $d \in C^{\mathcal{I}'}$.

Point (i) is a direct consequence of the construction of $\mathcal{I}'$ and Point (ii) can be proved easily using structural induction on C. From Point (i), (ii) and the fact that $r(a,b) \notin \mathcal{B}|_R$ we can conclude that $\mathcal{I} \models \mathcal{B}|_R$ implies $\mathcal{I}' \models \mathcal{B}|_R$. And from the fact that there is no role assertion in $\mathcal{B}_m$, we have $\mathcal{I} \models \mathcal{B}_m$ iff $\mathcal{I}$ for $0 \leq m \leq n-1$. Hence we have $\mathcal{I} \models \mathcal{B}|_R \cup \mathcal{B}_m$ implies $\mathcal{I}' \models \mathcal{B}|_R \cup \mathcal{B}_m$ for $0 \leq m \leq n-1$. This implies $\mathcal{I}' \models \mathcal{B}$. But then the fact that $\mathcal{I}' \not\models r(a,b)$ gives us a contradiction to our original assumption $\mathcal{B} \models r(a,b)$.

($\Leftarrow$) This direction is trivial.

□

Using the previous lemma, we are now ready to show the non existence of restricted Boolean $\mathcal{ALC}$ ABox that has the same $\mathcal{ALC}^{@}$ logical consequences as $\mathcal{A}' = \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\}$.

Lemma 36. *Let $\mathcal{A}'$ be the semantically updated ABox described in Lemma 21. Then there exists no restricted Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$ such that $\mathcal{A}'$ and $\mathcal{B}$ are $\mathcal{ALC}^{@}$-indistinguishable.*

Proof. Assume that there exists a restricted Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$ such that $\mathcal{A}'$ and $\mathcal{B}$ are $\mathcal{ALC}^{@}$-indistinguishable. Without loss of generality, we assume that $\mathcal{B}$ is in disjunctive normal form. Let $C_k = \exists r.(A \sqcup \exists r^k.\neg A)$. Then we know that

- for all $k \in \mathbb{N}$, $\mathcal{A}' \models C_k(a)$,
- $\mathcal{A}' \not\models a : \exists r.A$ and
- for all $c \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{A}' \not\models r(a,c)$ and $\mathcal{A}' \not\models r(c,a)$.

From the assumption $\mathcal{A}'$ and $\mathcal{B}$ are $\mathcal{ALC}^{@}$-indistinguishable and Lemma 35, we can conclude that

- for all $m < n$ and $k \in \mathbb{N}$, $\mathcal{B}|_R \cup \mathcal{B}_m \models C_k(a)$,
- there exists an $m \leq n$ such that $\mathcal{B}|_R \cup \mathcal{B}_m \not\models a : \exists r.A$ and
- for all $c \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{B}|_R \not\models r(a,c)$ and $\mathcal{B}|_R \not\models r(c,a)$.

Let us consider the ABox $\mathcal{A}^{\diamond} = \mathcal{B}|_R \cup \mathcal{B}_m$ in the second item. We know that $\mathcal{A}^{\diamond}$ has the properties as follows:

- for all $k \in \mathbb{N}$, $\mathcal{A}^{\diamond} \models C_k(a)$,
- $\mathcal{A}^{\diamond} \not\models a : \exists r.A$ and
- for all $c \in \mathsf{N_I}$ and $r \in \mathsf{N_R}$, $\mathcal{A}^{\diamond} \not\models r(a,c)$ and $\mathcal{A}^{\diamond} \not\models r(c,a)$.

But then the fact that $\mathcal{A}^{\diamond}$ is an $\mathcal{ALC}$ ABox and Lemma 25 guarantee that such an ABox does not exist (contradiction). □

Corollary 37. *$\mathcal{ALC}^{@}$ does not have syntactic ABox updates.*

We have seen that we cannot recover the existence of ABox syntactic update if we restrict the Boolean combinations of ABox assertions only to concept assertions. We now generalize the notion of a restricted Boolean ABox to that of a Boolean ABox. Unlike a restricted Boolean ABox, the connectives in a Boolean ABox can be used to any ABox assertions. The following is an example of a Boolean ABox:

$$\{B(a) \vee r(a,b), (s(a,c) \vee s(b,c)) \wedge a : \exists r.\neg A\}$$

Definition 38 (Boolean $\mathcal{L}$ ABox)**.** A Boolean $\mathcal{L}$ ABox is a finite set of Boolean combinations of $\mathcal{L}$ ABox assertions expressed in terms of the connectives $\wedge$ and $\vee$.

As in a restricted Boolean ABox, the connectives $\wedge$ and $\vee$ are interpreted as in propositional logic. We have seen that restricted Boolean $\mathcal{ALC}$ ABoxes have exactly the same expressive power as $\mathcal{ALC}^{@}$ ABoxes. This is not the case for Boolean ABox. Boolean $\mathcal{ALC}$ ABoxes are more expressive than $\mathcal{ALC}^{@}$ ABoxes. It is easy to see that there is no $\mathcal{ALC}^{@}$ ABox that is equivalent to the Boolean $\mathcal{ALC}$ ABox $\{A(a) \vee r(b,c)\}$. Now we show that even Boolean $\mathcal{ALC}$ ABoxes are still not expressive enough to express syntactically updated ABoxes.

In order to do this, we need a new combination of original and semantically updated ABoxes because the semantically updated ABox $\mathcal{A}'$ in Lemma 21

$$\mathcal{A}' = \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\}$$

can be expressed in Boolean $\mathcal{ALC}$ ABox. It is easy to check the Boolean $\mathcal{ALC}$ ABox

$$\mathcal{B} = \{\neg A(b), r(b,b), a : \exists r.A \vee r(a,b)\}$$

is equivalent to $\mathcal{A}'$.

Lemma 39. *Let $\mathcal{A} = \{a : \exists r^2.A, A(b), r(b,b)\}$, $\mathcal{U} = \{\neg A(b)\}$ and*

$$\mathcal{A}' = \{\neg A(b), r(b,b), a : \exists r^2.(A \sqcup \{b\}\}.$$

Then $\mathcal{A} * \mathcal{U} \equiv \mathcal{A}'$.

Let $\mathcal{A}'$ be the semantically updated ABox as described in Lemma 39. We want to show that there is no Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$ such that for all Boolean $\mathcal{ALC}$ ABox assertions φ, $\mathcal{A}' \models \varphi$ iff $\mathcal{B} \models \varphi$. We will use a similar strategy as the one we used to prove negative result in $\mathcal{ALC}$. The idea now is to find interesting Boolean $\mathcal{ALC}$ assertions that are entailed or not entailed by $\mathcal{A}'$. Let $C_n = \exists r^2.(A \sqcup \exists r^n.\neg A)$. It is easy to see that for all $n \in \mathbb{N}$, $\mathcal{A}' \models C_n(a)$. Now, it remains to find an interesting Boolean $\mathcal{ALC}$ assertion that is not entailed by $\mathcal{A}'$. Let I be a finite subset of $\mathsf{N_I}$ and $\varphi_I = a : \exists r^2.A \vee \bigvee_{c \in I} r(a,c) \vee r(c,a)$. It is easy to see that $\mathcal{A}' \not\models \varphi_{\mathsf{inds}(\mathcal{A}')}$. We summarize the properties above in the following lemma.

Lemma 40. *Let $\mathcal{A}'$ be the ABox as described in Lemma 39. Then*

- *$\mathcal{A}' \models C_n(a)$ for all $n \in \mathbb{N}$ and*

- $\mathcal{A}' \not\models \varphi_{\mathsf{inds}(\mathcal{A}')}$

Like in the case of restricted Boolean ABoxes, we introduce a normal form for Boolean $\mathcal{ALC}$ ABoxes as follows:

$$\mathcal{B} = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$$

where $\mathcal{B}_0 \ldots \mathcal{B}_{n-1}$ are $\mathcal{ALC}$ ABoxes. The following lemma shows an interesting property of Boolean $\mathcal{ALC}$ ABoxes.

Lemma 41. *Let φ be a Boolean $\mathcal{ALC}$ ABox assertion and $\mathcal{B}$ a consistent Boolean $\mathcal{ALC}$ ABox in normal form. Then,*

$$\mathcal{B} \models \varphi \textit{ iff for all } 0 \leq m \leq n-1, \mathcal{B}_m \models \varphi.$$

Hence, from Lemma 41, we know that if there exists a Boolean $\mathcal{ALC}$ ABox $\mathcal{B} = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$ that satisfies all conditions given in Lemma 40, then there exists an $\mathcal{ALC}$ ABox $\mathcal{B}_m$ where $m < n$ such that $\mathcal{B}_m$ satisfies those conditions. Since $\mathcal{B}_m$ is an $\mathcal{ALC}$ ABox and $\mathcal{B}_m \not\models \varphi$, we will show in the following lemma that we can indeed apply Lemma 22 to $\mathcal{B}_m$ and then follow the same strategy as we show negative results for syntactic ABox updates in $\mathcal{ALC}$.

Lemma 42. *Let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox and $a \in \mathsf{N_I}$. If $\mathcal{A} \not\models \bigvee_{b \in \mathsf{inds}(\mathcal{A})} r(a,b) \vee r(b,a)$, then for all $c \in \mathsf{N_I}$, $\mathcal{A} \not\models r(a,c)$ and $\mathcal{A} \not\models r(c,a)$.*

Proof. Assume that $\mathcal{A} \not\models \bigvee_{b \in \mathsf{inds}(\mathcal{A})} r(a,b) \vee r(b,a)$ and there exists a $c \in \mathsf{N_I}$ such that $\mathcal{A} \models r(a,c)$ or $\mathcal{A} \models r(c,a)$. Then we know that $c \notin \mathsf{inds}(\mathcal{A})$. Now, let $\mathcal{I}$ be a model of $\mathcal{A}$. Hence, $\mathcal{I} \models r(a,c) \vee r(c,a)$. From this, we construct a new interpretation $\mathcal{I}'$ by modifying only the interpretation of the individual name c.

$$c^{\mathcal{I}'} = d$$

where $d \in \Delta^{\mathcal{I}'} \setminus \Delta^{\mathcal{I}}$. The concept, role and other individual names are interpreted the same as $\mathcal{I}$. It is easy to see that $\mathcal{I}$ is still a model of $\mathcal{A}$ because we only change the interpretation of c which does not occur in $\mathcal{A}$. And as a direct consequence of the construction of $\mathcal{I}'$ we have $\mathcal{I}' \not\models r(a,c) \vee r(c,a)$. This contradicts the assumption $\mathcal{A} \models r(a,c) \vee r(c,a)$. □

From Lemma 42, the fact that $\mathcal{B}_m$ is an $\mathcal{ALC}$ ABox and $\mathcal{B}_m \not\models \varphi_{\mathsf{inds}(\mathcal{B})}$, we can conclude that $\mathcal{B}_m \models C_n(a)$ iff $\mathcal{B}_m|_a \models C_n(a)$. Without loss of generality, we assume that $\mathcal{B}_m|_a$ contains only a single concept assertion $D(a)$. It is not hard to see that $\mathcal{B}_m \models C_n(a)$ iff $\mathcal{B}_m|_a \models C_n(a)$ iff $D \sqsubseteq C_n$. From the fact that $\mathcal{B}_m \not\models \varphi$, we can also conclude that $\mathcal{B}_m \not\models a : \exists r^2.A$. Then, using the same deduction as above we have $\mathcal{B}_m \not\models a : \exists r^2.A$ iff $D \not\sqsubseteq \exists r^2.A$.

Now, we use again the fact that an $\mathcal{ALC}$ concept cannot "see" deeper than its own role depth. We use this fact to show a size limitation for an $\mathcal{ALC}$ concept C that has the property $C \sqsubseteq C_n$ but $C \not\sqsubseteq \exists r^2.A$. where $n \in \mathbb{N}$. It turns out that $|C|$ has to be greater than n. We will then use this result to show that there is no such concept C that satisfies both property for all $n \in \mathbb{N}$ which then implies that such $\mathcal{ALC}$ ABox $\mathcal{B}_m$ may not exists. This then shows that there is no such Boolean $\mathcal{ALC}$ ABox that can satisfy the conditions stated in Lemma 40.

Lemma 43. *Let C be an $\mathcal{ALC}$ concept and $n \in \mathbb{N}$. If $C \not\sqsubseteq \exists r^2.A$ and $C \sqsubseteq C_n$, then $|C| > n$.*

Proof. This lemma can be proven in a very similar way as in Lemma 24. □

Theorem 44. *Boolean $\mathcal{ALC}$ ABoxes do not have syntactic ABox updates.*

Proof. Consider the $\mathcal{ALC}$ ABox $\mathcal{A}$, the update $\mathcal{U}$ and the $\mathcal{ALCO}$ ABox $\mathcal{A}'$ given in Lemma 39. Assume that there exists a Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$ that has the same Boolean $\mathcal{ALC}$ logical consequences as $\mathcal{A}'$. Without loss of generality, we assume that $\mathcal{B}$ is in disjunctive normal form ($\mathcal{B} = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$). Then from Lemma 40, $\mathcal{B}$ fulfills all of the conditions follow:

$$\begin{aligned} \mathcal{B} &\models C_n(a) \\ \mathcal{B} &\not\models \varphi_{\mathsf{inds}(\mathcal{B})}. \end{aligned}$$

And from Lemma 41 we know that there exists an $m < n$ such that:

$$\begin{aligned} \mathcal{B}_m &\models C_n(a) \\ \mathcal{B}_m &\not\models \varphi_{\mathsf{inds}(\mathcal{B})}. \end{aligned}$$

From $\mathcal{B}_m \not\models \varphi_{\mathsf{inds}(\mathcal{B})}$, Lemma 42 and 22 we have $\mathcal{B}_m \models C_n(a)$ iff $\mathcal{B}_m|_a \models C_n(a)$. Without loss of generality, we assume that $\mathcal{B}_m|_a$ contains only a single assertion $D(a)$. Then, we can conclude the following.

$$\begin{aligned} D &\sqsubseteq C_n \\ D &\not\sqsubseteq \exists r^2.A. \end{aligned}$$

But then Lemma 43 says that $|D| > n$ for all $n \in \mathbb{N}$. This contradicts the fact that $|D|$ is finite. □

Chapter 4

Extended Syntactic Updates

As we have seen in the previous chapter, weakening semantic updates to syntactic updates is not enough to recover ABox updates in the DLs $\mathcal{ALC}$, $\mathcal{ALCO}$ and $\mathcal{ALC}^{@}$. In this chapter, we consider a weaker definition of ABox updates which is the extended syntactic ABox updates. It turns out that extended syntactically updated ABoxes can be expressed in $\mathcal{ALC}$ (with the help of TBoxes) and $\mathcal{ALCO}$. But unfortunately this is not the case for $\mathcal{ALC}^{@}$ and Boolean $\mathcal{ALC}$ ABoxes.

4.1 Extended Syntactic Updates in $\mathcal{ALC}$

In this section, we study the availability of extended syntactic ABox updates in $\mathcal{ALC}$. First, we show that if we do not use the help of TBoxes, the extended syntactically updated ABox cannot be expressed in $\mathcal{ALC}$. Then, we show that we can recover the availability of such ABox updates in $\mathcal{ALC}$ by using TBoxes.

Theorem 45. *$\mathcal{ALC}$ does not have extended syntactic ABox updates.*

Proof. Recall the original ABox $\mathcal{A}$, update $\mathcal{U}$ and semantically updated ABox $\mathcal{A}'$ in Lemma 21.

$$\begin{aligned}\mathcal{A} &= \{a : \exists r.A, A(b), r(b,b)\}\\ \mathcal{U} &= \{\neg A(b)\}\\ \mathcal{A}' &= \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\}\end{aligned}$$

Now assume that there exists an $\mathcal{ALC}$ ABox $\mathcal{A}^{\diamond}$ such that for all $\mathcal{ALC}$ assertions φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^{\diamond} \subseteq \mathsf{sig}(\mathcal{A}')$, $\mathcal{A}' \models \varphi$ iff $\mathcal{A}^{\diamond} \models \varphi$. We now need to find interesting ABox assertions φ that satisfy the precondition $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^{\diamond} \subseteq \mathsf{sig}(\mathcal{A}')$. Let $C_n = \exists r.(A \sqcup \exists r^n.\neg A)$. It is easy to see that that $\mathsf{sig}(C_n) \subseteq \mathsf{sig}(\mathcal{A}')$ which then implies $\mathsf{sig}(C_n) \cap \mathsf{sig}(\mathcal{A}^{\diamond}) \subseteq \mathsf{sig}(\mathcal{A}')$. It is also not hard to see that $r \in \mathsf{sig}(\mathcal{A}')$ which then implies for all $\{a, b\} \subseteq \mathsf{N_I}$, $\mathsf{sig}(r(a,b)) \cap \mathsf{sig}(\mathcal{A}^{\diamond}) \subseteq \mathsf{sig}(\mathcal{A}')$. Hence for the following assertions φ:

- $C_n(a)$ where $n \in \mathbb{N}$,

Strong ⊔ Brave	⊑	Dependable
Secure	⊑	∀has_friend.(¬Strong → Brave)

Figure 4.1: Examples of GCIs

- $a : \exists r.A$ and
- $r(a, b)$ where $\{a, b\} \subseteq \mathsf{N_I}$

we have $\mathcal{A}' \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$.

It is easy to see that $\mathcal{A}' \models C_n(a)$ for all $n \in \mathbb{N}$ but $\mathcal{A}' \not\models a : \exists r.A$. For the role assertions, we know that the only role assertions that is entailed by $\mathcal{A}'$ is $r(b, b)$. But then Lemma 25 ensures that there is no $\mathcal{ALC}$ ABox that can fulfill the properties above (contradiction). □

Description Logic TBox and KB

A general *terminological box* (TBox) is a finite set of *general concept inclusions* (GCI) $C \sqsubseteq D$ where C and D are concepts. A TBox is used to store subsumption relations between concepts in our domain of interest. A TBox that stores only $\mathcal{L}$ concepts is called an *$\mathcal{L}$ TBox*. Similarly, a TBox is called an *$\mathcal{L}^\Sigma$ TBox* if it contains only subsumption relations between $\mathcal{L}^\Sigma$ concepts. The signature $\mathsf{sig}(\mathcal{T})$ is a pair $\langle \mathsf{N}'_\mathsf{C}, \mathsf{N}'_\mathsf{R} \rangle$ where N'_C and N'_R are the set of concept and role names used in $\mathcal{T}$. Figure 4.1 is an example of an $\mathcal{ALC}$ TBox. The *size* $|\mathcal{T}|$ *of a TBox* $\mathcal{T}$ is defined as follow.

$$|\mathcal{T}| := \sum_{C \sqsubseteq D \in \mathcal{T}} |C| + |D|$$

A TBox $\mathcal{T}$ *is in NNF* iff for all GCIs, $C \sqsubseteq D \in \mathcal{T}$, both C and D are in NNF. The set $\mathsf{sub}(\mathcal{T})$ is defined as follows.

$$\mathsf{sub}(\mathcal{T}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(C) \cup \mathsf{sub}(D)$$

An interpretation $\mathcal{I}$ *satisfies* a GCI $C \sqsubseteq D$ (written $\mathcal{I} \models C \sqsubseteq D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$. An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ (written $\mathcal{I} \models \mathcal{T}$) iff for all GCIs $\phi \in \mathcal{T}$, $\mathcal{I} \models \phi$. A TBox is *consistent* iff it has a model.

Combining the two formalisms (TBox and ABox) that we have introduced, we define another formalism called the knowledge base formalism. A *knowledge base* (KB) $\mathcal{K}$ is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox. If $\mathcal{T}$ is an $\mathcal{L}$ TBox and $\mathcal{A}$ an $\mathcal{L}$ ABox then $\mathcal{K}$ is an $\mathcal{L}$ KB. Similarly, if $\mathcal{T}$ is an $\mathcal{L}^\Sigma$ TBox and $\mathcal{A}$ an $\mathcal{L}^\Sigma$ ABox then $\mathcal{K}$ is an $\mathcal{L}^\Sigma$ KB. The signature $\mathsf{sig}(\mathcal{K})$ is defined as $\mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{T})$. The size $|\mathcal{K}|$ of a KB $\mathcal{K}$ is $|\mathcal{T}| + |\mathcal{A}|$. A KB $\mathcal{K}$ is in NNF if both $\mathcal{T}$ and $\mathcal{A}$ are in NNF. And the set $\mathsf{sub}(\mathcal{K}) := \mathsf{sub}(\mathcal{T}) \cup \mathsf{sub}(\mathcal{A})$.

An interpretation $\mathcal{I}$ is a *model* of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$. A KB is *consistent* iff it has a model. An ABox assertion φ is a *logical consequence* of a KB $\mathcal{K}$ (written $\mathcal{K} \models \varphi$) iff for all interpretations $\mathcal{I}$, $\mathcal{I} \models \mathcal{K}$ implies $\mathcal{I} \models \varphi$. A KB $\mathcal{K}$ *entails* an ABox $\mathcal{A}$ (written $\mathcal{K} \models \mathcal{A}$)) if for all assertions $\varphi \in \mathcal{A}$, $\mathcal{K} \models \varphi$.

Computing Updates in $\mathcal{ALC}$

Unlike in the previous chapter, we now try to partially recover extended syntactic ABox updates in $\mathcal{ALC}$. We say a DL $\mathcal{L}$ *has partially extended syntactic ABox updates* if for all original $\mathcal{L}$ ABox and update $\mathcal{U}$, there exists an $\mathcal{L}$ KB $\mathcal{K}$ such that for all $\mathcal{L}$ assertions φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{K}) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{U})$, $\mathcal{K} \models \varphi$ iff $\mathcal{A}' \models \varphi$. This means, given a semantically updated $\mathcal{ALCO}^@$ ABox $\mathcal{A}'$, we construct a KB $\mathcal{K}^\diamond = \langle \mathcal{T}^\diamond, \mathcal{A}^\diamond \rangle$ such that for all $\mathcal{ALC}$ ABox assertions φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{K}^\diamond) \subseteq \mathsf{sig}(\mathcal{A}')$, we have $\mathcal{A}' \models \varphi$ iff $\mathcal{K}^\diamond \models \varphi$.

Before constructing the whole KB, we first deal with the role assertions. We do not need the help of TBoxes here. The idea here is to exploit the fact that given an original $\mathcal{ALC}$ ABox and an update, the semantically updated ABox is always representable in $\mathcal{ALCO}^@$. And, given all role assertions that are entailed by the original ABox and an update, we can generate all the role assertions that are entailed by the semantically updated ABox.

Definition 46 (Role Explicit)**.** An ABox $\mathcal{A}$ is *role explicit* if for all role assertions $r(a, b)$,

- $\mathcal{A} \models r(a, b)$ implies $r(a, b) \in \mathcal{A}$ and
- $\mathcal{A} \models \neg r(a, b)$ implies $\neg r(a, b) \in \mathcal{A}$.

Lemma 47. *Let $\mathcal{A}$ be an original $\mathcal{ALC}$ ABox, $\mathcal{U}$ an update and $\mathcal{A}' \equiv \mathcal{A} * \mathcal{U}$. If $\mathcal{A}$ is role explicit and consistent, then for all role assertions $r(a, b)$,*

(i) $\mathcal{A}' \models r(a, b)$ iff $r(a, b) \in \mathcal{U}$ or $r(a, b) \in \mathcal{A}$ and $\neg r(a, b) \notin \mathcal{U}$.

(ii) $\mathcal{A}' \models \neg r(a, b)$ iff $\neg r(a, b) \in \mathcal{U}$ or $\neg r(a, b) \in \mathcal{A}$ and $r(a, b) \notin \mathcal{U}$.

Proof. We only show Point (i). Point (ii) can be shown in a similar way.

($\Rightarrow$) Assume $\mathcal{A}' \models r(a, b)$. Let $\mathcal{I}'$ be a model of $\mathcal{A}'$. From (U2) in Definition 6, we know that there exists a model $\mathcal{I}$ of the $\mathcal{A}$ such that $\mathcal{I} \Longrightarrow_{\mathcal{U}} \mathcal{I}'$. Then from Definition 5 we know that $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in r^{\mathcal{I}'}$ iff $r(a, b) \in \mathcal{U}$ or $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ and $\neg r(a, b) \notin \mathcal{U}$.

($\Leftarrow$) Let $\mathcal{I}'$ be a model of $\mathcal{A}'$. Then, we know that there is a model $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{I} \Longrightarrow_{\mathcal{U}} \mathcal{I}'$ from (U2). To show $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in r^{\mathcal{I}'}$ we distinguish two cases.

- $r(a, b) \in \mathcal{U}$. From Definition 5 and the assumption that $r(a, b) \in \mathcal{U}$, we know that $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in r^{\mathcal{I}'}$.
- $r(a, b) \in \mathcal{A}$ and $\neg r(a, b) \notin \mathcal{U}$. Since $\mathcal{I} \models \mathcal{A}$, we know that $\mathcal{I} \models r(a, b)$. Then, from Definition 5 and the assumption that $\neg r(a, b) \notin \mathcal{U}$, we have $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in r^{\mathcal{I}'}$.

Since $\mathcal{I}'$ is arbitrary, we can conclude that for all $\mathcal{I}'$, $\mathcal{I}' \models \mathcal{A}'$ implies $\mathcal{I}' \models r(a, b)$. Hence $\mathcal{A}' \models r(a, b)$. □

Lemma 47 says that given a role explicit $\mathcal{ALC}$ ABox and update $\mathcal{U}$, we can derive all role assertions that should be entailed by the semantically updated ABox. Using this information, we can generate a role explicit semantically updated ABox. This then has solved "half" of the problem that needs to be handled because we can easily assume without loss of generality that the semantically updated ABox is role explicit.

Lemma 48. *Let $\mathcal{A}$ be an original role explicit $\mathcal{ALC}$ ABox, $\mathcal{U}$ an update and $\mathcal{A}'$ the $\mathcal{ALCO}^{@}$ semantically udpated ABox. Then we can construct a role explicit $\mathcal{ALCO}^{@}$ ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}^{\diamond} \equiv \mathcal{A}'$.*

Proof.

$$\begin{aligned}\mathcal{A}^{\diamond} &= \mathcal{A}' \cup \{r(a,b) \mid r(a,b) \in \mathcal{U}\} \cup \{\neg r(a,b) \mid \neg r(a,b) \in \mathcal{U}\} \cup \\ &\quad \{r(a,b) \mid r(a,b) \in \mathcal{A} \text{ and } \neg r(a,b) \notin \mathcal{U}\} \cup \\ &\quad \{\neg r(a,b) \mid \neg r(a,b) \in \mathcal{A} \text{ and } r(a,b) \notin \mathcal{U}\}\end{aligned}$$

□

The next thing we need to consider are the concept assertions. The idea now is to construct an $\mathcal{ALC}$ KB $\mathcal{K}(\mathcal{A}')$ from a role explicit semantically updated $\mathcal{ALCO}^{@}$ ABox $\mathcal{A}'$ such that for all $\mathcal{ALC}$ concept assertions $C(a)$ with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A}')) \subseteq sig(\mathcal{A}')$, we have $\mathcal{K}(\mathcal{A}') \models C(a)$ iff $\mathcal{A}' \models C(a)$. Before going into the details of the construction, we first define several notions. Let $\mathcal{A}$ be an $\mathcal{ALCO}^{@}$ ABox. We define:

- $X_s := \{X_a \mid a \in \mathsf{N_I}\}$,
- $Z_s := \{Z_{a,C} \mid a \in \mathsf{N_I}$ and C is an $\mathcal{ALCO}^{@}$ concept$\}$,
- $\mathsf{cl}(\mathcal{A}) := \{C, \neg C \mid C \in \mathsf{sub}(\mathcal{A})\}$ and
- $Z(\mathcal{A}) := \{Z_{a,C} \in Z_s \mid a \in \mathsf{inds}(\mathcal{A})$ and $C \in \mathsf{cl}(\mathcal{A})\}$.

The set X_s and Z_s are sets of *fresh* (i.e. does not appear in the ABox $\mathcal{A}'$) concept names for each individual name and possible concept assertion respectively. The *closure* $\mathsf{cl}(\mathcal{A})$ of $\mathcal{A}$ is the set of all subconcepts of $\mathcal{A}$ and their negations. All of these sets are disjoint each other and also to the set $\mathsf{concs}(\mathcal{A})$. The set $Z(\mathcal{A})$ is a subset of Z_s. Using these sets, we define an $\mathcal{ALC}$ concept $C^{\mathcal{ALC}}$ as the concept obtained from an $\mathcal{ALCO}^{@}$ concept C by replacing all occurrences of nominals $\{a\}$ with $X_a \in X_s$ and $@_a D$ with $Z_{a,D} \in Z_s$.

Definition 49 (Constructing $\mathcal{K}(\mathcal{A})$)**.** Let $\mathcal{A}$ be a role explicit $\mathcal{ALCO}^{@}$ ABox and $u \notin \mathsf{roles}(\mathcal{A})$. We define the KB $\mathcal{K}(\mathcal{A}) := \langle \mathcal{T}', \mathcal{A}' \rangle$ as follows.

$$\begin{aligned}
\mathcal{A}' &:= \{X_a(a) \mid a \in \mathsf{inds}(\mathcal{A}) \text{ and } X_a \in X_s\} \cup && (4.1)\\
&\quad \{r(a,b) \mid r(a,b) \in \mathcal{A}\} \cup \\
&\quad \{\neg r(a,b) \mid \neg r(a,b) \in \mathcal{A}\} \cup \\
&\quad \{u(a,b), u(b,a) \mid \{a,b\} \subseteq \mathsf{inds}(\mathcal{A})\} && (4.2)\\
\mathcal{T}' &:= \{X_a \sqsubseteq C^{\mathcal{ALC}} \mid a : C \in \mathcal{A}\} \cup && (4.3)\\
&\quad \{X_a \sqsubseteq \forall r.\neg X_b \mid \neg r(a,b) \in \mathcal{A}\} \cup && (4.4)\\
&\quad \{X_a \sqcap C^{\mathcal{ALC}} \sqsubseteq Z_{a,C} \mid Z_{a,C} \in Z(\mathcal{A}))\} \cup && (4.5)\\
&\quad \{Z_{a,C} \sqsubseteq \forall r.Z_{a,C} \mid Z_{a,C} \in Z(\mathcal{A}) \text{ and } r \in \mathsf{roles}(\mathcal{A}')\} \cup && (4.6)\\
&\quad \{\exists r.Z_{a,C} \sqsubseteq Z_{a,C} \mid Z_{a,C} \in Z(\mathcal{A}) \text{ and } r \in \mathsf{roles}(\mathcal{A}')\} \cup && (4.7)\\
&\quad \{X_a \sqcap Z_{a,C} \sqsubseteq C^{\mathcal{ALC}} \mid Z_{a,C} \in Z(\mathcal{A}))\} && (4.8)
\end{aligned}$$

Now, we show that if $\mathcal{A}'$ is a role explicit $\mathcal{ALCO}^{@}$ ABox that is equivalent to a semantically updated ABox, then the $\mathcal{ALC}$ KB $\mathcal{K}(\mathcal{A}')$ is the extended syntactically updated ABox.

Lemma 50. *Let $\mathcal{A}$ be a role explicit $\mathcal{ALCO}^{@}$ ABox and $\mathcal{K}(\mathcal{A})$ a KB defined as in Definition 49. Then, for all $\mathcal{ALC}$ assertions φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $\mathcal{K}(\mathcal{A}) \models \varphi$ iff $\mathcal{A} \models \varphi$.*

Proof. Let $\mathcal{K}(\mathcal{A}) = \langle \mathcal{T}', \mathcal{A}' \rangle$. Let φ be an $\mathcal{ALC}$ assertion such that $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$. It is easy to see that if φ is a role assertion, then $\mathcal{A} \models \varphi$ iff $\varphi \in \mathcal{A}$ iff $\varphi \in \mathcal{A}'$ iff $\mathcal{K} \models \varphi$ due to the fact that $\mathcal{A}$ is role explicit and $\mathcal{A}'$ contains all role assertions in $\mathcal{A}$. We now come to a more interesting case where φ is a concept assertion $E(a)$.

($\Leftarrow$) Assume that $\mathcal{K}(\mathcal{A}) \not\models E(a)$. Then there exists a model $\mathcal{I}'$ of $\mathcal{K}(\mathcal{A})$, such that $a^{\mathcal{I}'} \notin E^{\mathcal{I}'}$. We now claim that for all models $\mathcal{I}'$ of $\mathcal{K}(\mathcal{A})$, there exists a model $\mathcal{I}$ of $\mathcal{A}$ such that $a^{\mathcal{I}} = a^{\mathcal{I}'}$ and for all $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALC}$ concepts C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $d \in C^{\mathcal{I}'}$ implies $d \in C^{\mathcal{I}}$. Hence, we have that there exists a $\mathcal{I}$ of $\mathcal{A}$ such that $a^{\mathcal{I}} \notin E^{\mathcal{I}}$ which then implies $\mathcal{A} \not\models E(a)$.

To conclude this direction, we need to show that the claim holds. Let $\mathcal{I}'$ be a model of $\mathcal{K}$ such that $\mathcal{I} \not\models E(a)$. We construct $\mathcal{I}$ as follows.

$$\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'} \setminus \{d \mid d \in X_a^{\mathcal{I}'} \text{ and } d \neq a^{\mathcal{I}'}\} \tag{4.9}$$
$$a^{\mathcal{I}} = a^{\mathcal{I}'} \tag{4.10}$$
$$A^{\mathcal{I}} = A^{\mathcal{I}'} \cap \Delta^{\mathcal{I}} \tag{4.11}$$
$$\begin{aligned} r^{\mathcal{I}} = {} & (r^{\mathcal{I}} \setminus \{(d,e) \mid (d,e) \in r^{\mathcal{I}'} \text{ and } \exists a \in \mathsf{N_I} : e \in X_a^{\mathcal{I}'}\}) \cup \\ & \{(d, a^{\mathcal{I}}) \mid \exists e \in \Delta^{\mathcal{I}'} : (d,e) \in r^{\mathcal{I}'} \text{ and } e \in X_a^{\mathcal{I}'}\} \end{aligned} \tag{4.12}$$

From the construction of $\mathcal{I}$, it is easy to see that $a^{\mathcal{I}} = a^{\mathcal{I}'}$. We still need to show that $\mathcal{I} \models \mathcal{A}$. We show this making a case analysis of $\varphi \in \mathcal{A}$. The first two cases are positive and negative role assertions.

- $\varphi = r(a,b)$. If $r(a,b) \in \mathcal{A}$ then $r(a,b) \in \mathcal{A}'$. Hence $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in r^{\mathcal{I}'}$ and $b^{\mathcal{I}'} \in X_b^{\mathcal{I}'}$. Thus, from (4.12) we have $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.
- $\varphi = \neg r(a,b)$. If $\neg r(a,b) \in \mathcal{A}$ then from (4.4), we have that $X_a \sqsubseteq \forall r.\neg X_b \in \mathcal{T}'$. Since $\mathcal{I}' \models \mathcal{T}'$, we know that there is no r edge going from individuals in $X_a^{\mathcal{I}'}$ to individuals in $X_b^{\mathcal{I}'}$. And since in (4.12) we do not add that kind of edge, we have $\mathcal{I} \models \neg r(a,b)$.

For the concept assertions case and the condition for all $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALC}$ concepts C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $d \in C^{\mathcal{I}'}$ implies $d \in C^{\mathcal{I}}$, it is enough to show the following claim. For all $d \in \Delta^{\mathcal{I}}$ and $\mathcal{ALCO}^{@}$ concepts C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $d \in (C^{\mathcal{ALC}})^{\mathcal{I}'}$ implies $d \in C^{\mathcal{I}}$. Showing this claim is enough to conclude the concept assertions case and the condition mentioned above because:

- every concept assertion $C(a) \in \mathcal{A}$, it holds that C is an $\mathcal{ALCO}^{@}$ concept with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$,
- every $\mathcal{ALC}$ concept is an $\mathcal{ALCO}^{@}$ concept and
- for all $\mathcal{ALC}$ concepts C, $C^{\mathcal{ALC}} = C$.

We show this using structural induction on C. Without loss of generality, we assume that C in NNF and $\mathcal{I}'$ is a forest like model. Thus, (4.2) implies $\mathcal{I}'$ is a *connected model*, i.e. in $\mathcal{I}'$, there is a path from any individual to any other individual using role names that occur in $\mathcal{A}'$.

- $C = A$ or $\neg A$, where $A \in \mathsf{N_C}$. Trivial since $A^{\mathcal{ALC}} = A$ and for all $d \in \Delta^{\mathcal{I}'}, d \in A^{\mathcal{I}'}$ iff $d \in A^{\mathcal{I}}$.
- $C = D \sqcap E$ and $C = D \sqcup E$. These cases follow directly from the induction hypothesis.
- $C = \{a\}$. Then $C^{\mathcal{ALC}} = X_a$. Assume that $d \in X_a^{\mathcal{I}'}$. From (4.10) and (4.9), we know that if $d \in \Delta^{\mathcal{I}}$ and $d \in X_a^{\mathcal{I}'}$, then it has to be the case that $d = a^{\mathcal{I}'} = a^{\mathcal{I}}$. Hence $d \in \{a\}^{\mathcal{I}}$.
- $C = \neg\{a\}$. Then $C^{\mathcal{ALC}} = \neg X_a$. We need to show $d \notin \{a\}^{\mathcal{I}}$. Assume that $d \notin \{a\}^{\mathcal{I}}$ but $d \in X_a^{\mathcal{I}'}$. From (4.10), we get $d \neq a^{\mathcal{I}'}$. Hence, from (4.9) we get that $d \notin \Delta^{\mathcal{I}}$ (contradiction).
- $C = @_a D$. Then $C^{\mathcal{ALC}} = Z_{a,D}$. We need to show $d \in (@_a D)^{\mathcal{I}}$. Assume that $d \in Z_{a,D}^{\mathcal{I}'}$. Then, (4.5), (4.6) and (4.7) assure that $Z_{a,D}$ holds in every individuals that are connected to d. The assumption $\mathcal{I}'$ is a connected model and (4.8) guarantee that if $Z_{a,D}$ has at least one element, it has to be the case that for all element of $e \in X_a^{\mathcal{I}'}$, we have $e \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$. Since $a^{\mathcal{I}'} \in X_a^{\mathcal{I}'}$, we have $a^{\mathcal{I}'} \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$. From the induction hypothesis and the fact that $a^{\mathcal{I}} = a^{\mathcal{I}'}$, we get $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Hence, from the semantics of the @ constructor, we have for all $e \in \Delta^{\mathcal{I}}$, $e \in (@_a D)^{\mathcal{I}}$. Since $d \in \Delta^{\mathcal{I}}$, we can conclude that $d \in (@_a D)^{\mathcal{I}}$.
- $C = \exists r.D$. Then $C^{\mathcal{ALC}} = \exists r.D^{\mathcal{ALC}}$. Assume $d \in (\exists r.D^{\mathcal{ALC}})^{\mathcal{I}'}$. We need to show $d \in (\exists r.D)^{\mathcal{I}}$. This case is interesting only if during the construction, we remove the edge $(d, e) \in r^{\mathcal{I}'}$ where $e \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$ and for all $f \in \Delta^{\mathcal{I}} \setminus \{e\}$, $(d, f) \in r^{\mathcal{I}'}$ implies $f \notin (D^{\mathcal{ALC}})^{\mathcal{I}'}$. Otherwise this case becomes trivial. We only remove the edge $(d, e) \in r^{\mathcal{I}'}$ if there exists an $a \in \mathsf{N_I}$ such that $e \in X_a^{\mathcal{I}'}$, $e \neq a^{\mathcal{I}}$. From (4.12), we know that the existence of r-successor is recovered with $(d, a^{\mathcal{I}})$. To show that $d \in (\exists r.D)^{\mathcal{I}}$, it is enough to show that $a^{\mathcal{I}} \in D^{\mathcal{I}}$. From the fact that $e \in (X_a \sqcap D^{\mathcal{ALC}})^{\mathcal{I}'}$, $D \in \mathsf{sub}(\mathcal{A})$, (4.1), (4.5), (4.6), (4.7) and (4.8), we get that $a^{\mathcal{I}'} \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$. Then from the induction hypothesis, we have $a^{\mathcal{I}'} \in D^{\mathcal{I}}$. Hence, from the fact that $a^{\mathcal{I}} = a^{\mathcal{I}'}$, we have $a^{\mathcal{I}} \in D^{\mathcal{I}}$.
- $C = \forall r.D$. Then $C^{\mathcal{ALC}} = \forall r.D^{\mathcal{ALC}}$. Assume $d \in (\forall r.D^{\mathcal{ALC}})^{\mathcal{I}'}$. We need to show $d \in (\forall r.D)^{\mathcal{I}}$. This case is interesting only if during the construction, we add the edge $(d, a^{\mathcal{I}})$ to $r^{\mathcal{I}'}$. Otherwise this case becomes trivial. We only add the edge $(d, a^{\mathcal{I}})$ if there exists an $e \in \Delta^{\mathcal{I}'}$ such that $(d, e) \in r^{\mathcal{I}'}$ and $e \in X_a^{\mathcal{I}'}$. To show that $d \in (\forall r.D)^{\mathcal{I}}$, it is enough to show that $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Since $d \in (\forall r.D^{\mathcal{ALC}})^{\mathcal{I}'}$ and $(d, e) \in r^{\mathcal{I}'}$, we can conclude that $e \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$. Then, from the fact that $e \in (X_a \sqcap D^{\mathcal{ALC}})^{\mathcal{I}'}$, $D \in \mathsf{sub}(\mathcal{A})$, (4.1), (4.5), (4.6), (4.7) and (4.8), we get that $a^{\mathcal{I}'} \in (D^{\mathcal{ALC}})^{\mathcal{I}'}$. Then from the induction hypothesis, we have $a^{\mathcal{I}'} \in D^{\mathcal{I}}$. Hence, from the fact that $a^{\mathcal{I}} = a^{\mathcal{I}'}$, we have $a^{\mathcal{I}} \in D^{\mathcal{I}}$.

($\Rightarrow$) Assume that $\mathcal{A} \not\models E(a)$. We claim that for all models $\mathcal{I}$ of $\mathcal{A}$, there exists a model $\mathcal{I}'$ of $\mathcal{K}(\mathcal{A})$ such that $a^{\mathcal{I}'} = a^{\mathcal{I}}$ and for $d \in \Delta^{\mathcal{I}'}$ and all $\mathcal{ALC}$ concept C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $d \in C^{\mathcal{I}'}$ implies $d \in C^{\mathcal{I}}$. Then, from the claim we can conclude that there exists a model $\mathcal{I}'$ of $\mathcal{K}(\mathcal{A})$ such that $a^{\mathcal{I}'} \notin E^{\mathcal{I}'}$ which then implies $\mathcal{K}(\mathcal{A}) \not\models E(a)$.

To show the claim, we show the construction of $\mathcal{I}'$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{A} \cup \{\neg E(a)\}$. We define an interpretation $\mathcal{I}'$ as an extension of $\mathcal{I}$ as follows.

$$
\begin{aligned}
X_a^{\mathcal{I}'} &= \{a^{\mathcal{I}}\} \text{ for all } a \in \mathsf{N_I} \\
Z_{a,C}^{\mathcal{I}'} &= \begin{cases} \Delta^{\mathcal{I}'} & \text{if } a^{\mathcal{I}} \in C^{\mathcal{I}}, \\ \emptyset & \text{otherwise.} \end{cases} \\
u^{\mathcal{I}'} &= \{(a^{\mathcal{I}}, b^{\mathcal{I}}), (b^{\mathcal{I}}, a^{\mathcal{I}}) \mid \{a, b\} \subseteq \mathsf{N_I}\}
\end{aligned}
$$

It is easy to see that $\mathcal{I}' \models \mathcal{K}(\mathcal{A})$ and $a^{\mathcal{I}'} = a^{\mathcal{I}}$. For concluding the case, it remains to show that for all $d \in \Delta^{\mathcal{I}'}$ and $\mathcal{ALC}$ concepts C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{K}(\mathcal{A})) \subseteq \mathsf{sig}(\mathcal{A})$, $d \in C^{\mathcal{I}}$ implies $d \in C^{\mathcal{I}'}$. But, this is trivial because during the construction of $\mathcal{I}'$, we do not modify the interpretation of concept and role names that are used to construct C.

□

Theorem 51. *$\mathcal{ALC}$ has partially extended syntactic ABox updates.*

Proof. Let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox, $\mathcal{U}$ an update and $\mathcal{A}' \equiv \mathcal{A} * \mathcal{U}$. Let

$$
\begin{aligned}
\mathcal{B} \;=\; & \mathcal{A} \cup \{r(a,b) \mid \mathcal{A} \models r(a,b) \wedge \{a,b\} \subseteq \mathsf{inds}(\mathcal{A}) \wedge r \in \mathsf{roles}(\mathcal{A})\} \cup \\
& \{\neg r(a,b) \mid \mathcal{A} \models \neg r(a,b) \wedge \{a,b\} \subseteq \mathsf{inds}(\mathcal{A}) \wedge r \in \mathsf{roles}(\mathcal{A})\}.
\end{aligned}
$$

It is easy to see that $\mathcal{B} \equiv \mathcal{A}$ and $\mathcal{B}$ is role explicit. Hence, from Lemma 48, we know that we can construct a role explicit $\mathcal{ALCO}^{@}$ ABox $\mathcal{A}^{\diamond}$ that is equivalent to $\mathcal{A}'$. Lemma 50 concludes the proof. □

4.2 Computing Updates in $\mathcal{ALCO}$

The results obtained in the previous section imply that the availability of extended syntactic ABox updates in $\mathcal{ALC}$ cannot be recovered without using the help of $\mathcal{ALC}$ TBoxes. In this section, we show that this is not the case in the DL $\mathcal{ALCO}$. More precisely, given a semantically updated $\mathcal{ALCO}^{@}$ ABox, we show how to construct the extended syntactically updated ABox in $\mathcal{ALCO}$.

The idea of the construction is to utilize the usage of the expressive power of $\mathcal{ALCO}$, where we can express assertions using concepts, to imitate the @ constructor. Given an $\mathcal{ALCO}^{@}$ ABox $\mathcal{A}$, we replace all occurrences of subconcept $@_a C$ with an $\mathcal{ALCO}$ concept $\exists u.(\{a\} \sqcap C)$ where u is a fresh role name. It is easy to see that we will get an $\mathcal{ALCO}$ ABox. Let us name the $\mathcal{ALCO}$ ABox $\mathcal{A}^{\diamond}$. We show $\mathcal{A}$ and $\mathcal{A}^{\diamond}$ are $\mathcal{L}^{\Sigma}$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_R} \rangle \setminus (\mathsf{sig}(\mathcal{A}^{\diamond}) \setminus \mathsf{sig}(\mathcal{A}))$. We first establish the relation between $\exists u.(\{a\} \sqcap C)$ and $@_a C$ in the following lemma.

Lemma 52. *Let C be an $\mathcal{ALCO}$ concept. Then for all interpretations $\mathcal{I}$, we have the following:*

(i) for all $d \in \Delta^{\mathcal{I}}$, $d \in (\exists u.(\{a\} \sqcap C))^{\mathcal{I}}$ implies $d \in (@_a C)^{\mathcal{I}}$ and

(ii) if $(d, a^{\mathcal{I}}) \in u^{\mathcal{I}}$ and $d \in (@_a C)^{\mathcal{I}}$, then $d \in (\exists u.(\{a\} \sqcap C))^{\mathcal{I}}$

Now, let $\mathcal{A}$ be an $\mathcal{ALCO}^@$ ABox in NNF and u a role name that does not occur in $\mathcal{A}$. We define $\mathcal{A}^\diamond$ as follows.

$$\begin{aligned}\theta &= \{\exists u.(\{a\} \sqcap C)/@_a C \mid @_a C \in \mathrm{subs}(\mathcal{A})\}\\ \mathcal{A}^\diamond &= \{r(a,b) \mid r(a,b) \in \mathcal{A}\} \cup \{\neg r(a,b) \mid \neg r(a,b) \in \mathcal{A}\} \cup\\ &\quad\ \{a : C[\theta] \mid C(a) \in \mathcal{A}\}\end{aligned}$$

where $C[\theta]$ is the result of applying the substitution θ to C. We show in the following lemma that for all $\mathcal{ALCO}$ assertion φ where $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, we have $\mathcal{A} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$. This result implies that if $\mathcal{A}$ is the semantically updated ABox, then $\mathcal{A}^\diamond$ is the extended syntactically updated ABox.

Lemma 53. *For all $\mathcal{ALCO}^@$ ABoxes $\mathcal{A}$, there exists an $\mathcal{ALCO}$ ABox $\mathcal{A}^\diamond$ such that $\mathcal{A}$ and $\mathcal{A}^\diamond$ are $\mathcal{ALCO}^\Sigma$-indistinguishable where $\Sigma = \langle \mathsf{N_C}, \mathsf{N_I}\rangle \setminus (sig(\mathcal{A}^\diamond) \setminus \mathsf{sig}(\mathcal{A}))$.*

Proof. Without loss of generality, we assume that $\mathcal{A}$ is in NNF. We show for all $\mathcal{ALCO}$ assertion φ where $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, we have $\mathcal{A} \models \varphi$ iff $\mathcal{A}^\diamond \models \varphi$.

($\Rightarrow$) For this direction, we show that $\mathcal{A}^\diamond \models \mathcal{A}$. This then implies for all $\mathcal{ALCO}$ assertion φ, $\mathcal{A} \models \varphi$ implies $\mathcal{A}^\diamond \models \varphi$. We do a case analysis on φ. If φ is a role assertion then the case is trivial because we do not add or remove any role assertion during the construction of $\mathcal{A}^\diamond$. The more interesting case is if φ is concept assertion. Let $\mathcal{I}^\diamond$ be a model of $\mathcal{A}^\diamond$ and $\varphi \in \mathcal{A}$. We show that $\mathcal{I}^\diamond \models C(a)$ for all $C(a) \in \mathcal{A}$ by showing the following claim. For all $d \in \Delta^{\mathcal{I}^\diamond}$ and $\mathcal{ALCO}$ concepts C, $d \in (C[\theta])^{\mathcal{I}^\diamond}$ implies $d \in C^{\mathcal{I}^\diamond}$. We prove this using structural induction on C. From the assumption $\mathcal{A}$ is in NNF, we know that C is in NNF.

- $C = A$, $C = \neg A$, $C = \{a\}$ or $C = \neg\{a\}$, where $A \in \mathsf{N_C}$ and $a \in \mathsf{N_I}$. These cases are trivial because $C[\theta] = C$.
- $C = D \sqcup E$, $C = D \sqcap E$, $C = \exists r.D$ or $C = \forall r.D$. These cases follow directly from the induction hypothesis.
- $C = @_a D$. Then $C[\theta] = \exists u.(\{a\} \sqcap D[\theta])$. Assume that $d \in (\exists u.(\{a\} \sqcap D[\theta]))^{\mathcal{I}^\diamond}$. Then $a^{\mathcal{I}^\diamond} \in (D[\theta])^{\mathcal{I}^\diamond}$. From the induction hypothesis, we get $a^{\mathcal{I}^\diamond} \in D^{\mathcal{I}^\diamond}$. Thus, $d \in (\exists u.(\{a\} \sqcap D))^{\mathcal{I}^\diamond}$. Then from Point (i) in Lemma 52, we get $d \in (@_a D)^{\mathcal{I}^\diamond}$.

($\Leftarrow$) Assume that there exists an $\mathcal{ALCO}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, such that $\mathcal{A} \not\models \varphi$. Then there exists a model $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A} \cup \{\neg\varphi\}$. The idea now is to construct an interpretation $\mathcal{I}^\diamond$ from $\mathcal{I}$ such that $\mathcal{I}^\diamond \models \mathcal{A}^\diamond \cup \{\neg\varphi\}$. This will then show that $\mathcal{A}^\diamond \not\models \varphi$.

The interpretation $\mathcal{I}^\diamond$ can be constructed by extending $\mathcal{I}$ as follows.

$$u^{\mathcal{I}^\diamond} = \{(d,e) \mid \{d,e\} \subseteq \Delta^{\mathcal{I}}\}$$

To conclude the proof, we need to show that $\mathcal{I}^\diamond \models \mathcal{A}^\diamond \cup \{\neg\varphi\}$. We first show that $\mathcal{I}^\diamond$ is a model of $\mathcal{A}^\diamond$. We do a case analysis on $\psi \in \mathcal{A}^\diamond$. If $\psi = r(a,b)$, then we know that $r \neq u$ because u does not occur in $\mathcal{A}$ and we never add or remove any role assertion from $\mathcal{A}$ while constructing $\mathcal{A}^\diamond$. Hence, it is easy to see that $\mathcal{I} \models r(a,b)$ iff $\mathcal{I}^\diamond \models r(a,b)$. The case $\psi = \neg r(a,b)$ can be shown in a similar way.

The more interesting case is if ψ is concept assertion. Let $\psi = C(a)$. We show that $\mathcal{I}^\diamond \models C(a)$ by showing the following claim. For all $d \in \Delta^\mathcal{I}$ and $\mathcal{ALCO}^@$ concepts C with $\mathsf{sig}(C) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, $d^\mathcal{I} \in C^\mathcal{I}$ implies $d \in (C[\theta])^{\mathcal{I}^\diamond}$. We show the claim using structural induction on C. Since $\mathcal{A}$ is in NNF, we know that C is also in NNF.

- $C = A$, $C = \neg A$, $C = \{a\}$ or $C = \neg\{a\}$. These cases are trivial because $C[\theta] = C$ and we do not change the interpretation of concept names nor individual names when constructing $\mathcal{I}^\nabla$.
- $C = D \sqcap E$ or $C = D \sqcup E$. These cases follow directly from induction hypothesis.
- $C = \exists r.D$ or $C = \forall r.D$. These cases are trivial because we know that $r \neq u$ and we do not change the interpretation of any other role names except u during the construction of $\mathcal{I}^\diamond$.
- $C = @_a D$. Then $C[\theta] = \exists u.(\{a\} \sqcap D[\theta])$. Assume that $d \in (@_a D)^\mathcal{I}$. Then $a^\mathcal{I} \in D^\mathcal{I}$. From the induction hypothesis and the fact that $a^{\mathcal{I}^\diamond} = a^\mathcal{I}$, we get $a^{\mathcal{I}^\diamond} \in (D[\theta])^{\mathcal{I}^\diamond}$. Thus, $d \in (@_a(D[\theta]))^{\mathcal{I}^\diamond}$. From the construction of $\mathcal{I}^\diamond$, we know $(d, a^{\mathcal{I}^\diamond}) \in u^{\mathcal{I}^\diamond}$. Hence, from Point (ii) in Lemma 52, we have $d \in (\exists u.(\{a\} \sqcap D[\theta]))^{\mathcal{I}^\diamond}$.

To conclude the case, we still need to show that $\mathcal{I}^\diamond \models \neg\varphi$. Since $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^\diamond) \subseteq \mathsf{sig}(\mathcal{A})$, we know that u does not occur in φ. But then this case becomes trivial because we only modify the interpretation of $u^\mathcal{I}$ while constructing $\mathcal{I}^\diamond$. Hence, $\mathcal{I} \models \neg\varphi$ implies $\mathcal{I}^\diamond \models \neg\varphi$.

□

Theorem 54. *$\mathcal{ALCO}$ has extended syntactic ABox updates.*

4.3 Extended Syntactic Updates in $\mathcal{ALC}^@$ and Boolean $\mathcal{ALC}$ ABoxes

From the previous chapter, we know that syntactic ABox updates do not exist in $\mathcal{ALC}^@$. In this section, we will see that this is the case even if we only consider extended syntactic ABox updates. We show that the extended syntactically updated ABox still cannot be expressed in $\mathcal{ALC}^@$.

We also know that Boolean $\mathcal{ALC}$ ABoxes do not have syntactic updates from the previous chapter. Here, we show that this also the case for extended syntactic updates. We show that $\mathcal{ALC}$ has uniform ABox interpolation with respect to Boolean ABoxes (from now on called uniform Boolean ABox interpolation). Then using the this fact, we show that the non-existence result of syntactic updates carries over to non-existence of extended syntactic updates in Boolean $\mathcal{ALC}$ ABoxes.

Theorem 55. *$\mathcal{ALC}^@$ does not have extended syntactic ABox updates.*

Proof. Recall the original ABox $\mathcal{A}$, update $\mathcal{U}$ and semantically updated ABox $\mathcal{A}'$ in Lemma 21.

$$\begin{array}{rcl} \mathcal{A} & = & \{a : \exists r.A, A(b), r(b,b)\} \\ \mathcal{U} & = & \{\neg A(b)\} \\ \mathcal{A}' & = & \{\neg A(b), r(b,b), a : \exists r.(A \sqcup \{b\})\} \end{array}$$

Assume that $\mathcal{ALC}^{@}$ has extended syntactic ABox updates. Then there exists an $\mathcal{ALC}^{@}$ ABox $\mathcal{A}^{\diamond}$ such that $\mathcal{A}^{\diamond}$ is the extended syntactically updated ABox. This means that for all $\mathcal{ALC}^{@}$ assertions φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^{\diamond}) \subseteq \mathsf{sig}(\mathcal{A}')$, we have $\mathcal{A}^{\diamond} \models \varphi$ iff $\mathcal{A}' \models \varphi$.

We use the same logical consequences as in the proof of Theorem 55. Let $C_n = \exists r.(A \sqcup \exists r^n.\neg A)$. It is easy to see that that C_n satisfies the precondtion $\mathsf{sig}(C_n) \subseteq \mathsf{sig}(\mathcal{A}')$ which then implies $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}^{\diamond}) \subseteq \mathsf{sig}(\mathcal{A}')$. It is also not hard to see that for all $\{a, b\} \subseteq \mathsf{N_I}$, $r(a, b)$ satisfies the precondition. Hence, for the following assertions φ:

- $C_n(a)$ where $n \in \mathbb{N}$,
- $a : \exists r.A$ and
- $r(a, b)$ where $\{a, b\} \subseteq \mathsf{N_I}$

we have $\mathcal{A}' \models \varphi$ iff $\mathcal{A}^{\diamond} \models \varphi$. It is not hard to see that $\mathcal{A}' \models C_n(a)$ for all $n \in \mathbb{N}$, $\mathcal{A}' \not\models a : \exists r.A$ and for all $c \in \mathsf{N_I}$, $\mathcal{A}' \not\models r(a, c)$ and $\mathcal{A}' \not\models r(c, a)$. Thus,

(i) $\mathcal{A}^{\diamond} \models C_n(a)$ for all $n \in \mathbb{N}$,

(ii) $\mathcal{A}^{\diamond} \not\models a : \exists r.A$ and

(iii) for all $c \in \mathsf{N_I}$, $\mathcal{A}' \not\models r(a, c)$ and $\mathcal{A}^{\diamond} \not\models r(c, a)$.

From Lemma 34, we know that there exists a restricted Boolean ABox $\mathcal{B}$ such that $\mathcal{B} \equiv \mathcal{A}^{\diamond}$. Without loss of generality we assume that $\mathcal{B}$ is in the normal form as follows.

$$\mathcal{B} = \mathcal{B}|_R \wedge (\mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1})$$

where $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are $\mathcal{ALC}$ ABoxes that contain only concept assertions and $\mathcal{B}|_R$ contains only role assertions. Hence, from Lemma 35, we know that there exists an $m < n$ such that

(i) $\mathcal{B}_m \cup \mathcal{B}|_R \models C_n(a)$ for all $n \in \mathbb{N}$,

(ii) $\mathcal{B}_m \cup \mathcal{B}|_R \not\models a : \exists r.A$ and

(iii) for all $c \in \mathsf{N_I}$, $\mathcal{A}' \not\models r(a, c)$ and $\mathcal{B}_m \cup \mathcal{B}|_R \not\models r(c, a)$.

But then, Lemma 25 ensures that there is no $\mathcal{ALC}$ ABox that has the properties above (contradiction).

□

Uniform Boolean ABox Interpolation in $\mathcal{ALC}$

As we can see in the proof of Lemma 20, $\mathcal{ALC}$ does not have uniform ABox interpolation because it cannot express the assertion $A(a) \vee A(b)$. From this, one may conjecture that we can recover the existence of uniform ABox interpolant if we allow Boolean combination of $\mathcal{ALC}$ assertions to appear in the uniform ABox interpolant.

Definition 56 (Uniform Boolean ABox Interpolation)**.**

1. A Boolean $\mathcal{L}$ ABox $\mathcal{A}^\Sigma$ is the *uniform Boolean ABox interpolant of a Boolean* $\mathcal{L}$ *ABox* $\mathcal{A}$ *w.r.t.* $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$ *if*
   (a) $\mathsf{sig}(\mathcal{A}^\Sigma) \subseteq \Sigma$
   (b) $\mathcal{A} \models \mathcal{A}^\Sigma$
   (c) for all Boolean $\mathcal{L}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, $\mathcal{A} \models \varphi$ implies $\mathcal{A}^\Sigma \models \varphi$.
2. A description logic $\mathcal{L}$ *has uniform Boolean ABox interpolation* if for all Boolean $\mathcal{L}$ ABox $\mathcal{A}$ and $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$, there exists a Boolean $\mathcal{L}$ ABox $\mathcal{A}^\Sigma$ that is the uniform Boolean ABox interpolant of $\mathcal{A}$ with respect to Σ.

Lemma 57. *If a DL* $\mathcal{L}$ *has uniform Boolean ABox interpolation, then Boolean* $\mathcal{L}$ *ABoxes have syntactic ABox updates iff Boolean* $\mathcal{L}$ *ABoxes have extended syntactic updates.*

Proof. This lemma can be shown in a very similar way as Lemma 19. □

It turns out that $\mathcal{ALC}$ indeed has uniform Boolean ABox interpolation. We first introduce the notion of complete $\mathcal{ALC}$ ABox and then show that for all complete $\mathcal{ALC}$ ABoxes $\mathcal{A}$ and signatures $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$, $\mathcal{A}$ has uniform ABox interpolant w.r.t Σ. Using this result, we will show how the uniform Boolean interpolant of a Boolean $\mathcal{ALC}$ ABox $\mathcal{A}$ w.r.t. $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$ can be constructed.

Definition 58 (Complete $\mathcal{ALC}$ ABox)**.** An $\mathcal{ALC}$ ABox $\mathcal{A}$ is *complete* if

- $a : C \sqcup D \in \mathcal{A}$ implies $C(a) \in \mathcal{A}$ or $D(a) \in \mathcal{A}$,
- $a : C \sqcap D \in \mathcal{A}$ implies $\{C(a), D(a)\} \subseteq \mathcal{A}$ and
- $\{a : \forall r.C, r(a,b)\} \subseteq \mathcal{A}$ implies $C(b) \in \mathcal{A}$.

Lemma 59. *For all complete* $\mathcal{ALC}$ *ABoxes* $\mathcal{A}$ *and signatures* $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$*, there exists an* $\mathcal{ALC}$ *ABox* $\mathcal{A}^\Sigma$ *such that* $\mathcal{A}^\Sigma$ *is the uniform interpolant of* $\mathcal{A}$ *with respect to* Σ*.*

Proof. In this proof, we only analyze consistent ABoxes because the case is trivial if the ABoxes are inconsistent ABox. Let $\mathcal{A}$ be a complete and consistent $\mathcal{ALC}$ ABox and $\Sigma \subseteq \mathsf{sig}(\mathcal{A})$. We claim the following ABox is the uniform ABox interpolant of $\mathcal{A}$ with respect to Σ.

$$\begin{aligned} \mathcal{A}^\Sigma \;=\; & \{r(a,b) \mid \mathcal{A} \models r(a,b) \text{ and } r \in \Sigma\} \cup \\ & \{\neg r(a,b) \mid \mathcal{A} \models \neg r(a,b) \text{ and } r \in \Sigma\} \cup \\ & \{a : (\textstyle\bigsqcap_{C(a) \in \mathcal{A}} C)^\Sigma\} \end{aligned}$$

where C^Σ is the uniform concept interpolant of C with respect to Σ. To show the claim, we show that $\mathcal{A}^\Sigma$ fulfills all conditions in Definition 18.

1. $\mathsf{sig}(\mathcal{A}^\Sigma) \subseteq \Sigma$
2. $\mathcal{A} \models \mathcal{A}^\Sigma$
3. for all $\mathcal{L}$ ABox assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, $\mathcal{A} \models \varphi$ implies $\mathcal{A}^\Sigma \models \varphi$.

Without loss of generality, we assume for all $a \in \mathsf{inds}(\mathcal{A})$, there exists a concept assertion $\top(a) \in \mathcal{A}$. Using this assumption, we guarantee that $\mathsf{inds}(\mathcal{A}) = \mathsf{inds}(\mathcal{A}^\Sigma)$.

Point 1 and 2 are direct consequences of the construction of $\mathcal{A}^\Sigma$. To show Point 3, we use a case analysis on φ. If φ is a role assertion, then the case follows directly from the construction of $\mathcal{A}^\Sigma$.

We now show the more interesting case where φ is a concept assertion. We assume that there exists a concept assertion $E(a)$ with $\mathsf{sig}(E) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, such that $\mathcal{A}^\Sigma \not\models E(a)$. Then, we know that $\mathcal{A}^\Sigma \cup \{\neg E(a)\}$ is consistent. Hence, there exists an interpretation $\mathcal{I}^\Sigma$ such that $\mathcal{I}^\Sigma \models \mathcal{A}^\Sigma \cup \{\neg E(a)\}$. We will then construct an interpretation $\mathcal{I}$ from $\mathcal{I}^\Sigma$ such that $\mathcal{I} \models \mathcal{A} \cup \{\neg E(a)\}$. This then concludes Point 3.

Now, let $\alpha \in \mathsf{inds}(\mathcal{A})$. We define the concepts X_α, Y_α and Y_α^Σ as follows.

$$\begin{aligned}
X_\alpha &= \bigsqcap_{\substack{D \in \mathsf{sub}(\neg E) \\ \mathcal{I}^\Sigma \models D(\alpha)}} D \sqcap \bigsqcap_{\substack{D \in \mathsf{sub}(\neg E) \\ \mathcal{I}^\Sigma \not\models D(\alpha)}} \neg D \\
Y_\alpha &= \bigsqcap_{C(\alpha) \in \mathcal{A}} C
\end{aligned}$$

and Y_α^Σ the uniform interpolant of Y_a with respect to Σ. Then, from the fact that $\mathcal{I}^\Sigma \models \mathcal{A}^\Sigma$, we know that for all $\alpha \in \mathsf{inds}(\mathcal{A}^\Sigma)$, $\mathcal{I}^\Sigma \models Y_\alpha^\Sigma(\alpha)$. It is also easy from the definition of X_α that $\mathcal{I}^\Sigma \models X_\alpha(\alpha)$. Thus, for all $\alpha \in \mathsf{inds}(\mathcal{A}^\Sigma)$, we have $X_\alpha \sqcap Y_\alpha^\Sigma$ is satisfiable. Since Y_α^Σ is the uniform interpolant of Y_α with respect to Σ and $\mathsf{sig}(X_\alpha) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, we have for all $\alpha \in \mathsf{inds}(\mathcal{A}^\Sigma) = \mathsf{inds}(\mathcal{A})$, $X_\alpha \sqcap Y_\alpha$ is also satisfiable.

In order to show that $\mathcal{A} \not\models E(a)$, we will construct an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A} \cup \{\neg E(a)\}$. Let $\mathcal{I}_\alpha$ be an interpretation such that $\mathcal{I}_\alpha \models \alpha : X_\alpha \sqcap Y_\alpha$. Without loss of generality, we assume that for all $\alpha \in \mathsf{inds}(\mathcal{A})$, $\Delta^{\mathcal{I}_\alpha}$ are disjoint sets. We then construct $\mathcal{I}$ as follows. Let $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $\alpha \in \mathsf{N_I}$.

$$\begin{aligned}
\Delta^\mathcal{I} &= \bigcup_{\alpha \in \mathsf{inds}(\mathcal{A})} \Delta^{\mathcal{I}_\alpha} \\
\alpha^\mathcal{I} &= \alpha^{\mathcal{I}_\alpha} \\
A^\mathcal{I} &= \bigcup_{\alpha \in \mathsf{inds}(\mathcal{A})} A^{\mathcal{I}_\alpha} \\
r^\mathcal{I} &= \bigcup_{\alpha \in \mathsf{inds}(\mathcal{A})} r^{\mathcal{I}_\alpha} \cup \{(b^\mathcal{I}, c^\mathcal{I}) \mid r(b,c) \in \mathcal{A}\}
\end{aligned}$$

To finish the proof, we still need to show that $\mathcal{I} \models \mathcal{A} \cup \{\neg E(a)\}$. We first show that $\mathcal{I}$ is a model of $\mathcal{A}$. We do case analysis on $\varphi \in \mathcal{A}$. If φ is a role assertion, then the case is trivial. The more interesting case is if φ is concept assertion. We show this by showing the following claim. For all $\alpha \in \mathsf{inds}(\mathcal{A})$ and concept assertions $D(\alpha) \in \mathcal{A}$, $\mathcal{I}_\alpha \models D(\alpha)$ implies $\mathcal{I} \models D(\alpha)$. Without loss of generality, we assume that D is in NNF. We show the claim using structural induction on D.

- $D = A$ or $D = \neg A$ where $A \in \mathsf{N_C}$. These cases are direct consequences of the construction of $\mathcal{I}$.
- $D = F \sqcup G$. Since $\mathcal{A}$ is a complete ABox, we know that $F(\alpha) \in \mathcal{A}$ or $G(\alpha) \in \mathcal{A}$. Then, from the induction hypothesis we get $\mathcal{I} \models F(\alpha)$ or $\mathcal{I} \models G(\alpha)$. Hence, $\mathcal{I} \models D(\alpha)$.
- $D = F \sqcap G$. This case can be shown similarly as the previous case.
- $D = \exists r.F$. This case is trivial because we do not remove any edge from $r^{\mathcal{I}_\alpha}$ during the construction of $r^{\mathcal{I}}$.
- $D = \forall r.F$. During the construction of $\mathcal{I}$, we add an edge $(\alpha^{\mathcal{I}}, b^{\mathcal{I}})$ to the interpretation of r only when $r(\alpha, b) \in \mathcal{A}$. So, to conclude the proof, we need to show that $\mathcal{I} \models F(b)$. Since $\mathcal{A}$ is a complete ABox, we have $F(b) \in \mathcal{A}$. And then, from the induction hypothesis, we get $\mathcal{I} \models F(b)$.

It remains to show that $\mathcal{I} \models \neg E(a)$. We show this by showing the following claim. For all $\alpha \in \mathsf{inds}(\mathcal{A})$ and $D \in \mathsf{sub}(\neg E)$, $\mathcal{I}_\alpha \models D(\alpha)$ implies $\mathcal{I} \models D(\alpha)$. We also show this using structural induction on D. Without loss of generality we assume D is in NNF.

- $D = A$ or $D = \neg A$ where $A \in \mathsf{N_C}$. These cases are direct consequences of the construction of $\mathcal{I}$.
- $D = F \sqcap G$ or $D = F \sqcup G$. These cases follow directly from the semantics and induction hypothesis.
- $D = \exists r.F$. This case is trivial because we do not remove any edge from $r^{\mathcal{I}_\alpha}$ during the construction of $r^{\mathcal{I}}$.
- $D = \forall r.F$. During the construction of $\mathcal{I}$, we add an edge $(\alpha^{\mathcal{I}}, b^{\mathcal{I}})$ to the interpretation of r only when $r(\alpha, b) \in \mathcal{A}$. So, to conclude the proof, we need to show that $\mathcal{I} \models F(b)$. Since $D \in \mathsf{sub}(\neg E)$, it has to be the case that $r(\alpha, b) \in \mathcal{A}^{\Sigma}$. Otherwise, $\mathsf{sig}(\neg E) \cap \mathsf{sig}(\mathcal{A}) \not\subseteq \Sigma$. Since $\mathcal{I}_\alpha \models X_\alpha(\alpha)$ and $\mathcal{I}_\alpha \models D(\alpha)$, we know that $\mathcal{I}^{\Sigma} \models D(\alpha)$. And from the fact that $\mathcal{I}^{\Sigma} \models \mathcal{A}^{\Sigma}$ and $r(\alpha, b) \in \mathcal{A}^{\Sigma}$, we know that $\mathcal{I}^{\Sigma} \models F(b)$. Thus, it is not hard to see that $X_b \sqsubseteq F$ because from the definition of X_b, F has to be one of the conjuncts in X_b. Hence, from the induction hypothesis and the fact that $\mathcal{I}_b \models X_b(b)$, we conclude $\mathcal{I} \models F(b)$.

□

The idea now is to show that for every Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$, there exists an equivalent Boolean $\mathcal{ALC}$ ABox $\mathcal{B}' = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$, where $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are complete $\mathcal{ALC}$ ABoxes. Without loss of generality, we assume that $\mathsf{sig}(\mathcal{B}) = \mathsf{sig}(\mathcal{B}_0) = \ldots = \mathsf{sig}(\mathcal{B}_{n-1})$. Let $\Sigma \subseteq \mathsf{sig}(\mathcal{B})$. Then for each $\mathcal{B}_m$ where $m < n$, we construct the uniform ABox interpolation $\mathcal{B}_m^{\Sigma}$ with respect to Σ. After that, we show that $\mathcal{B}^{\Sigma} = \mathcal{B}_0^{\Sigma} \vee \ldots \vee \mathcal{B}_{n-1}^{\Sigma}$ is the uniform Boolean ABox interpolant of $\mathcal{B}$ with respect to Σ.

Lemma 60. *For every Boolean $\mathcal{ALC}$ ABox $\mathcal{B}$, there exists an equivalent Boolean $\mathcal{ALC}$ ABox $\mathcal{B}' = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$ where $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are complete $\mathcal{ALC}$ ABoxes.*

Proof. Let $\mathcal{B}$ a Boolean $\mathcal{ALC}$ ABox. We assume that $\mathcal{B}$ is in the following normal form.

$$\mathcal{B} = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{m-1}$$

where $\mathcal{B}_0 \ldots \mathcal{B}_{m-1}$ are $\mathcal{ALC}$ ABoxes. To conclude the proof, it is enough to show the following claim. We claim that for all $\mathcal{ALC}$ ABoxes $\mathcal{A}$, there exists an equivalent Boolean $\mathcal{ALC}$ ABox $\mathcal{A}' = \mathcal{A}'_0 \vee \ldots \vee \mathcal{A}'_{k-1}$ where $\mathcal{A}'_0, \ldots, \mathcal{A}'_{k-1}$ are complete $\mathcal{ALC}$ ABoxes.

An ABox $\mathcal{A}^{\mathsf{comp}}$ is a *completion* of $\mathcal{A}$ if for all $C \in \mathsf{sub}(\mathcal{A})$ and $a \in \mathsf{inds}(\mathcal{A})$, $C(a) \in \mathcal{A}$ or $\neg C(a) \in \mathcal{A}$. Let $\mathbb{A}$ be the set of all possible completions of $\mathcal{A}$. Then it is not hard to see that $\mathcal{A}' = \bigvee_{\mathcal{A}^{\mathsf{comp}} \in \mathbb{A}} \mathcal{A}^{\mathsf{comp}}$ is equivalent to $\mathcal{A}$. It is also not hard to see that every $\mathcal{A}^{\mathsf{comp}}$ is a complete $\mathcal{ALC}$ ABox. □

Theorem 61. *$\mathcal{ALC}$ has uniform Boolean ABox interpolation.*

Proof. Let $\mathcal{B}$ be a Boolean $\mathcal{ALC}$-ABox. From Lemma 60, we know that there exists an equivalent Boolean $\mathcal{ALC}$ ABox

$$\mathcal{B}' = \mathcal{B}_0 \vee \ldots \vee \mathcal{B}_{n-1}$$

where $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are complete $\mathcal{ALC}$ ABoxes. Without loss of generality, we assume $\mathsf{sig}(\mathcal{B}) = \mathsf{sig}(\mathcal{B}') = \mathsf{sig}(\mathcal{B}_1) = \ldots = \mathsf{sig}(\mathcal{B}_{n-1})$.

Let $\Sigma \subseteq \mathsf{sig}(\mathcal{B})$. Now, we claim the following Boolean $\mathcal{ALC}$ ABox $\mathcal{B}^\Sigma$ is the uniform Boolean ABox interpolant of $\mathcal{B}$ w.r.t. Σ. Let $\mathcal{A}^\Sigma$ be a uniform interpolant of $\mathcal{A}$ with respect to signature Σ.

$$\mathcal{B}^\Sigma = \bigvee_{\mathcal{A} \in \{\mathcal{B}_1 \ldots \mathcal{B}_m\}} \mathcal{A}^\Sigma$$

It is easy to see that $\mathsf{sig}(\mathcal{B}^\Sigma) \subseteq \Sigma$ and $\mathcal{B} \models \mathcal{B}^\Sigma$. It remains to show for all Boolean $\mathcal{ALC}$ assertion φ with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, $\mathcal{B} \models \varphi$ implies $\mathcal{B}^\Sigma \models \varphi$. Let φ be a Boolean $\mathcal{ALC}$ assertion with $\mathsf{sig}(\varphi) \cap \mathsf{sig}(\mathcal{B}) \subseteq \Sigma$. Assume $\mathcal{B}^\Sigma \not\models \varphi$. Then there is an $\mathcal{ALC}$ ABox $\mathcal{B}_k^\Sigma$ where $k \leq m$ such that $\mathcal{B}_k^\Sigma \not\models \varphi$. Since $\mathcal{B}_k^\Sigma$ is the uniform ABox interpolant of $\mathcal{B}_k$ w.r.t. Σ, we know that $\mathcal{B}_k \not\models \varphi$. Hence, $\mathcal{B} \not\models \varphi$. □

Corollary 62. *Boolean $\mathcal{ALC}$ ABoxes do not have extended syntactic ABox updates.*

Proof. Follows directly from Theorem 44, Theorem 61 and Lemma 57. □

Chapter 5

Conclusion

We have revisited the ABox update problem in several basic description logics. Some of the results presented in this thesis strengthen the results given in [4]. Here, we have shown that weakening the definition of ABox update from semantic update to syntactic update is not enough to recover the existence of ABox updates in the DLs $\mathcal{ALC}$, $\mathcal{ALCO}$ and $\mathcal{ALC}^{@}$. We show the negative results by giving combinations of original $\mathcal{ALC}$ (resp. $\mathcal{ALCO}$) ABox, update and semantically updated ABox, and then show that there is no $\mathcal{ALC}$ (resp. $\mathcal{ALCO}$) ABox that has the same $\mathcal{ALC}$ (resp. $\mathcal{ALCO}$) logical consequences as the semantically updated ABox. For $\mathcal{ALC}^{@}$, we show the negative result indirectly. We first show that $\mathcal{ALC}^{@}$ ABoxes have the same expressive power as restricted Boolean $\mathcal{ALC}$ ABoxes. And then we show that there is no restricted Boolean $\mathcal{ALC}$ ABox that has the same $\mathcal{ALC}$ logical consequences as the semantically updated ABox given in the section where we show the negative result for $\mathcal{ALC}$. We can use the same combination of ABoxes because every $\mathcal{ALC}$ ABox is also a restricted Boolean $\mathcal{ALC}$ ABox. While studying the restricted Boolean ABox, we also explored the availability of syntactic ABox update in Boolean $\mathcal{ALC}$ ABox. It turned out even Boolean $\mathcal{ALC}$ ABoxes, which are more expressive than $\mathcal{ALC}^{@}$ ABoxes, are still not expressive enough to express semantically updated ABox. Unlike the case for restricted Boolean $\mathcal{ALC}$ ABoxes, we show this negative result using a different combination of ABoxes. We needed a new combination because there is a Boolean $\mathcal{ALC}$ ABox that is equivalent to the semantically updated ABox in the combination that we used to show the negative results for $\mathcal{ALC}$ and $\mathcal{ALC}^{@}$. So, in summary, we conclude that weakening semantic ABox updates to syntactic ABox updates does not help in recovering the existence of ABox updates in standard DLs.

To recover the existence of ABox updates, we tried to weaken the definition of ABox update even further from syntactic update to extended syntactic update. These two definitions of ABox updates coincide in DLs that admit uniform ABox interpolation.

We have seen in Chapter 4 that we are able to recover the existence of ABox updates in $\mathcal{ALCO}$ by this weakening. Unfortunately, we are not able to fully recover the existence of ABox updates in $\mathcal{ALC}$. This negative result is shown in a very similar way as we show the negative result for $\mathcal{ALC}$ concerning the syntactic ABox updates. The recovery of ABox update in $\mathcal{ALC}$ can be done if we allow $\mathcal{ALC}$ KB as the result of the update instead of having just

	$\mathcal{ALC}$	$\mathcal{ALCO}$	$\mathcal{ALC}^{@}$
Semantic Updates	[4]($\times$)	[4]($\times$)	[4]($\times$)
Syntactic Updates	Th.27($\times$)	Th.33($\times$)	Cor.37($\times$)
Extended Syntactic Updates	Th.51($\otimes$)	Th.54($\surd$)	Th.55($\times$)
B. Semantic Update	[4]($\times$)	[4]($\surd$)	[4]($\times$)
B. Syntactic Update	Th.44($\times$)	[4]($\surd$)	?
B. ext. Syntactic Update	Cor.62($\times$)	[4]($\surd$)	?

Figure 5.1: Existence of ABox updates in some DLs

an $\mathcal{ALC}$ ABox. If we allow this, the updated $\mathcal{ALC}$ KB can be computed. We also studied extended syntactic ABox updates for $\mathcal{ALC}^{@}$ and Boolean $\mathcal{ALC}$ ABoxes. The result that we obtained is that this weakening is not enough to fully recover the existence of ABox updates in $\mathcal{ALC}^{@}$ and Boolean $\mathcal{ALC}$ ABoxes. We showed the negative result for $\mathcal{ALC}^{@}$ by using a similar strategy as we showed the negative result for $\mathcal{ALC}^{@}$ concerning the syntactic ABox updates. For the negative result of Boolean $\mathcal{ALC}$ ABoxes, we showed that $\mathcal{ALC}$ has uniform Boolean ABox interpolation. This then implies that the non-existence of syntactic ABox updates result carries over to Boolean $\mathcal{ALC}$ ABoxes.

A summary of the results is presented in Figure 5.1. "B.", "Th." and "Cor." are abbreviations for Boolean, Theorem and Corollary, respectively. This states where we show the results in this thesis. If the cell has a citation remark, it means that the result is obtained from that citation. The symbol $\surd$ means that ABox updates exist, $\times$ means that ABox updates do not exist, $\otimes$ means that ABox updates exist if we allow KBs instead of ABoxes as the results of updates and ? means that we did not know whether it exists or not.

As stated in the introduction, we only consider simple ABoxes as updates. One obvious future work is to develop a theory on how to consider a more general ABox as the update. If we try to apply this, then we will need to define a new way to update an interpretation. One idea is to have a set of interpretations instead of just an interpretation as the result of updating an interpretation.

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